Demonstration that the equation $a^x + b^y = c^z$ has no solution in positive integers if $a$, $b$, $c$ not have a common prime factor.

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Abstract

Beal's conjecture is studied, which arises from investigations Andrew Beal, on Fermat's Last Theorem in 1993.

Beal’s conjecture, proposed to the equation $a^x + b^y = c^z$ where $A$, $B$, $C$, $x$, $y$, $z$ positive integers $x$, $y$, $z > 2$ so that the equation has a solution $A$, $B$, and $C$ must have a common prime factor.

Given the vain attempts to find a counterexample to the conjecture (which has been proven by the help of modular arithmetic, for all values of the six variables to a value of 1000) values for the six variables. To advance the theory of numbers.

Given the relationship that has Beal's conjecture with Fermat's last theorem; it is considered important to number theory, demonstration of this conjecture.

To solve Beal's conjecture, using Fermat's last theorem and the remainder theorem is proposed.

Keywords
Beal’s conjecture, Fermat
Introduction

Beal's conjecture is a conjecture in theory proposed by Andrew Beal on 1993 numbers; A similar conjecture was suggested independently by Andrew Granville those dates.

While researching generalizations of Fermat’s Last Theorem in 1993, Beal made the following conjecture.

Beal's conjecture states that if \( A^x + B^y = C^z \), where \( x, y \) and \( z \) greater than 2; solution is then positive integers \( A, B \) and \( C \) must have a common prime factor.

The equation \( A^x + B^y = C^z \) has infinite solutions if \( A, B \) and \( C \) have a common prime factor, using a computer and modular arithmetic, the conjecture has been verified for all values of 6 variables until 1000.

Fermat’s last theorem says that the equation \( A^x + B^y = C^z \) do not have solutions integer other than 0 for \( x = y = z \) where \( x>2 \). Then there will be no solutions in positive integers, so that Beal’s Conjecture is true, for this particular case.

For demonstration it requires the following theorems.

Theorem 1 Fermat’s last theorem

If \( n \) is a positive integer such that \( n \geq 3 \) then the equation \( x^n + y^n = z^n \) has no solution in positive integers.

Proof.

Demonstrated by Andrew Wallis and Richard Taylor. [1]

Theorem 2

In a full and rational polynomial in \( x \) which is divisible by a binomial of the form \( x-a \), to replace the \( x \) by \( a \) in the polynomial, the polynomial is zero.

Proof.

Be the \( Ax^m + Bx^{m-1} + Cx^{m-2} + \cdots + Mx + N \) a polynomial which is divisible by \( x-a \). Dividing the polynomial by \( x-a \), as it is divisible, its residue is 0. Let \( Q \) the quotient obtained of this division.

As in any exact division the dividend is equal to the product of the divisor by the quotient.

\[
Ax^m + Bx^{m-1} + Cx^{m-2} + \cdots + Mx + N = Q(x - a)
\]

This equality is true for all values of \( x \). substituting \( x \) by \( a \), is obtained

\[
Aa^m + Ba^{m-1} + Ca^{m-2} + \cdots + Ma + N = Q(a - a)
\]

\[
Aa^m + Ba^{m-1} + Ca^{m-2} + \cdots + Ma + N = Q(0)
\]
Proof of the conjecture

Theorem 3 (Beal’s Conjecture)

If the equation \( A^x + B^y = C^z \) with \( x, y, z > 2 \) has positive integers \( A, B, \) and \( C \) must have a common prime factor.

Proof

The equation \( A^x + B^y = C^z \) has infinite solutions if \( A, B \) and \( C \) have a common prime factor, so to prove the theorem is sufficient to prove that the equation has no solution if \( A, B \) and \( C \) not have a common prime factor.

First it is proved that the equation \( A^x + B^y = C^z \) \( x, y, z \) are equal to 4 or an odd prime number.

If an exponent \( x \) is not divisible by any odd prime, then it is a power of 2, and \( x > 2 \), then it is equal to 4 or divisible by 4, is \( x = 4x' \) by substituting in equation \( A^x = (A^{x'})^4 \) and being \( A^{x'} \) another positive integer, the equation is satisfied for \( x = 4 \).

If an exponent \( x \) is divisible by any odd prime, then is \( x = px' \) by substituting in equation \( A^x = (A^{x'})^p \) and being \( A^{x'} \) another positive integer, the equation is satisfied for \( x = p \).

The proof will be by cases.

1) Case 1 \( x = y = z = 4 \) \( a^4 + b^4 = c^4 \)

By Theorem 1 it has no solution in positive integers.

2) Case 2 \( x = y = z \) \( a^x + b^x = c^x \)

By Theorem 1 it has no solution in positive integers.

\[ a^x = c^x - b^x \]

3) Case 3 \( x = y \neq z \) \( a^x + b^x = c^z \)

Solving for \( a^x \)

\[ a^x = c^x \]

Assuming \( a, b \) and \( c \) numbers co-prime positive integers; \( a^x \) is a positive integer, then there is a positive integer such that \( a^x \) is divisible by \( e \). then there is a integer \( d \) such that \( e = c - d \).

Then \( c^x - b^x \) is divisible by \( c-d \). By theorem 2.

\[ d^x - b^x = 0 \]
Solving d

\[ d^x = b^x \]
\[ d = b^{\frac{x}{z}} \]

x, z are co-prime \( b^\frac{x}{z} \) is an irrational number, so that d is not integer. And then e cannot be a positive integer. Then \( a^x \) has no divisors positive integers, which is a contradiction; therefore the equation \( a^x + b^x = c^z \) has no solution in positive integers if a, b and c are co-prime.

4) Case 4 \( x = z \neq y \)

\( a^x + b^y = c^x \)

Solving for \( a^x \)

\[ a^x = c^x - b^y \]

Assuming a, b and c numbers co-prime positive integers; \( a^x \) is a positive integer, then there is a positive integer such that \( a^x \) is divisible by e. then there is a integer d such that e = c-d.

Then \( c^x - b^y \) is divisible by c-d. By theorem 2.

\[ d^x - b^y = 0 \]

Solving d

\[ d^x = b^y \]
\[ d = b^{\frac{y}{x}} \]

x, y are co-prime \( b^{\frac{y}{x}} \) is an irrational number, so that d is not integer. And then e cannot be a positive integer. Then \( a^x \) has no divisors positive integers, which is a contradiction; therefore the equation \( a^x + b^y = c^z \) has no solution in positive integers if a, b and c are co-prime.

5) Case 5 \( x \neq y \neq z \)

\( a^x + b^y = c^z \)

Solving for \( a^x \)

\[ a^x = c^z - b^y \]

Assuming a, b and c numbers co-prime positive integers; \( a^x \) is a positive integer, then there is a positive integer such that \( a^x \) is divisible by e. then there is a integer d such that e = c-d.

Then \( c^z - b^y \) is divisible by c-d. by theorem 2.

\[ d^z - b^y = 0 \]

Solving d
\[ d^z = b^y \]
\[ d = \frac{b^y}{e} \]

\( y, z \) are co-prime \( \frac{b^y}{e} \) is an irrational number, so that \( d \) is not integer. And then \( e \) can not be a positive integer. Then \( a^x \) has no divisors positive integers, which is a contradiction; therefore the equation \( a^x + b^y = c^z \) has no solution in positive integers if \( a, b \) and \( c \) are co-prime.

6) Case \( 6 \) \( x = 4 \)

\[ a^4 + b^y = c^z \]

Solving for \( b^y \)

\[ b^y = c^z - a^4 \]

Assuming \( a, b \) and \( c \) numbers co-prime positive integers; \( b^y \) is a positive integer, then there is a positive integer such that \( b^y \) is divisible by \( e \). Then there is an integer \( d \) such that \( e = c - d \).

Then \( c^z - a^4 \) is divisible by \( c - d \). by theorem 2

\[ d^z - a^4 = 0 \]

Solving \( d \)

\[ d^z = a^4 \]
\[ d = a^\frac{4}{z} \]

\( 4, z \) are co-prime \( a^\frac{4}{z} \) is an irrational number, so that \( d \) is not integer. And then \( e \) can not be a positive integer. Then \( b^y \) has no divisors positive integers, which is a contradiction; therefore the equation \( a^4 + b^y = c^z \) has no solution in positive integers if \( a, b \) and \( c \) are co-prime.

7) Case \( 7 \) \( z = 4 \)

\[ a^x + b^y = c^4 \]

Solving for \( b^y \)

\[ b^y = c^4 - a^x \]

Assuming \( a, b \) and \( c \) numbers co-prime positive integers; \( b^y \) is a positive integer, then there is a positive integer such that \( b^y \) is divisible by \( e \). Then there is an integer \( d \) such that \( e = c - d \).

Then \( c^4 - a^x \) is divisible by \( c - d \). by theorem 2

\[ d^4 - a^x = 0 \]

Solving \( d \)

\[ d^4 = a^x \]
4, x are co-prime $a^x$ is an irrational number, so that d is not integer. And then e can not be a positive integer. Then $b^y$ has no divisors positive integers, which is a contradiction; therefore the equation $a^x + b^y = c^z$ has no solution in positive integers if a, b and c are co-prime.

8) Case 8 $x = y = 4$ \hspace{1cm} $a^4 + b^4 = c^z$

Solving for $a^4$

$$a^4 = c^z - b^4$$

Assuming a, b and c numbers co-prime positive integers; $a^4$ is a positive integer, then there is a positive integer such that $a^4$ is divisible by e. then there is a integer d such that $e = c - d$.

Then $c^z - b^4$ is divisible by c-d. By theorem 2

$$d^2 - b^4 = 0$$

Solving d

$$d^2 = b^4$$

$$d = b^{2\frac{4}{z}}$$

4, z are co-prime $a^\frac{4}{z}$ is an irrational number, so that d is not integer. And then e can not be a positive integer. Then $a^4$ has no divisors positive integers, which is a contradiction; therefore the equation $a^x + b^4 = c^z$ has no solution in positive integers if a, b and c are co-prime.

By the cases the equation $a^x + b^y = c^z$ has no solution in positive integers if a, b and c have no common prime factor.

Q.A.D.

Conclusions

Beal’s conjecture is a generalization of Fermat’s last theorem, so the proof is important to the development of the theory of numbers. It establishes a condition to see if the equation $a^x + b^y = c^z$ has a solution in positive integers.

To the proof Beal’s conjecture stating that for the equation $a^x + b^y = c^z$ with $x, y, z > 2$ has a solution in positive integers if A, B, and C have a common prime factor.

Bibliography
