Abstract: In the first part of this paper, a neoclassical framework is proposed which places the Marxian conceptions of both Constant Capital and Variable Capital into a Cobb-Douglas production function like model in order to obtain the mathematical formulations of Marx labour value function \[ Q = b_c C^\alpha V^{1-\beta} \] and Marx surplus value function \[ M = b_c C^\alpha V^{1-\beta} \] as well as Marx production function \[ Y = a_c C^\alpha V^{1-\alpha} \], which leads to the Marxian 1st theorem about technical progress: \[ m = \alpha \dot{w} + (1 - \alpha) \dot{w} + \rho \dot{\alpha} / \beta \]. In the second part, the general equilibrium properties of the quantitative Marxian productivity theories are investigated by using variation method. The Marxian 2nd theorem about dynamic equilibrium asserts, there is a input-output equilibrium existed in the reproduction process between Two Departments \[ \dot{y}^* = \dot{c}^* \]; The Marxian 3rd theorem states that only equilibrium growth leads to the positive value of the productivity parameter which is defined as the product of the change rate of the organic composite of capital with the labor output elasticity of Cobb-Douglas production function \[ F = (1 - \alpha) \dot{\gamma} \], as well as the rising rate of profit. The present paper is also a
generalization of the precise conditions under which the profit rate rises or falls. Only when an economic system achieves the Marxian equilibrium including its each production Department, there would be no business cycle; otherwise there exists some potential crisis. At last, an econo-sociological Marxism model is proposed as a criterion for a regional optimal economic growth.

*Key words:* transformation problem, roundabout production, Solow residue, Okishio theorem, competitive equilibrium

*JEL:* E11, O47

I. Marx Productivity Economics

Marx's labor theory of value pointed out that the value of each commodity (Q) contained three sources: the first part is “constant capital (C)”, representing the value transferred from raw materials and machinery used up, the second part is “variable capital (V)” replacing the value of the labor power, and the third part is the surplus value (M) including net profit (P) and taxation (T). Therefore, the total value Q is expressed as a linear production function:

\[ Q = C + V + M = C + V + p'(C + V) = P'(C + V) = P'CV \]
\[ M = P + T = p'(C + V) = p'CV = m'V \]
\[ CV = C + V, \quad P' = \frac{Q}{C + V}, \quad p' = \frac{M}{CV} = \frac{m'}{1 + g} \]

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1. 曾尔曼、王gráfico. 《马克思生产力经济学导引》[M], 厦门大学出版社, 2013.09.
Where

\[ C = nK \ \text{(K: capital, n: capital turnover rate), constant capital;} \]

\[ V = wL \ \text{(L: labor, w: per-capita wages), variable capital} \]

\( P' \): the productivity rate, \( P' = Q / (C + V) = p' + 1 \)

\( M \): the surplus value

\( p' \): the rate of profit, \( p' = M / (C + V) = m' / (g + 1) = P' - 1 \)

\( m' \): the rate of surplus value, \( m' = M / V \)

\( g \): the organic surplus value, \( g = C / V = nK / (wL) = nk / w \)

Differentiating \( Q \) with respect to time \( t \) yields:

\[
\frac{dQ}{dt} = \frac{dp'}{dt} + \frac{dc}{dt} = \frac{dp'}{dt} + \frac{C}{Cv} \frac{dc}{dt} + \frac{V}{V} \frac{dV}{dt}
\]

\[
\beta = \frac{C}{Cv} \frac{g}{g+1} - \beta = \frac{V}{Cv} = \frac{1}{g+1}, \frac{dp'}{dt} \equiv f
\]

Re-integration gives the Labor Value Function\(^3\) as:

\[ Q = B_0 e^{\beta C} V^{\gamma-\beta} \]

\( f \): the productivity growth rate, and the cost function is:

\[ Cv = C + V = c_0 C^{\beta} V^{\gamma-\beta}. \]

Similarly, differentiates \( M \) with time \( t \): \( M = p' Cv = p'(C + V) \)

\[
\frac{dM}{dt} = \frac{dp'}{dt} + \frac{dc}{dt} = \frac{dp'}{dt} + \frac{C}{Cv} \frac{dc}{dt} + \frac{V}{V} \frac{dV}{dt}
\]

\[
\frac{dp'}{dt} \equiv p = g \gamma
\]

After integration, Marx surplus value function\(^4\) \( M \) is obtained as:

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\(^3\) 曾尔曼.《厦门科技》2014 (3) 31-35.

\(^4\) 曾尔曼.《厦门科技》2015 (2) 27-29; 第四届中国经济学年会会议论文.
\[ M = b_s e^{n} C^{\beta} V^{1-\beta}, \]
\[ m' = b_s e^{n} g^{\beta}, \]
\[ p' = \frac{M}{Cv} = \frac{m'}{g + 1} = (1 - \beta) m' = b_s e^{n} \frac{g^{\beta}}{g + 1} = b \frac{g^{\beta}}{g + 1} \]

As well as the function of the rate of surplus value \( m' \), \( p \) is the growth rate of profit.

Using the similar mathematical process, the Cobb-Douglas production function\(^5\) can be rewritten as:

\[ Y = V + M = V + p' Cv = p'(C + \frac{p'}{p'}) \equiv p'(C + V') = p' Cv' \]
\[ \frac{dY}{dt} = \frac{dp'}{dt} \quad \frac{dCv'}{dt} = \frac{dp'}{dt} + \frac{C}{Cv'} \frac{dC}{dt} + \frac{V'}{Cv'} \frac{dV'}{dt} = p + \alpha \frac{dC}{dt} + (1 - \alpha) \frac{dV'}{dt}, \]
\[ V' = \gamma w L \quad \alpha \equiv C \frac{g}{Cv} < \beta \quad \gamma = \frac{g}{g + \gamma} > 1 - \beta \]

After integration, the production function became:

\[ Y = a_s e^{n} C^{\alpha} V^{1-\alpha} = a_s e^{n} C^{\alpha} V^{1-\alpha} \gamma^{1-\alpha} = a_s e^{n} (nK)^{\alpha} (\gamma w L)^{1-\alpha}, \]
\[ Y = AK^{\alpha} L^{1-\alpha} = A_s e^{n} K^{\alpha} L^{1-\alpha} \]
\[ \Rightarrow p = m - \alpha \dot{n} - (1 - \alpha) \dot{w} = m - \alpha \dot{n} - (1 - \alpha) p' \dot{w} + \dot{\gamma}, \]
\[ p = f' ; \dot{\gamma} = \dot{p}' - \dot{p} = f' - p \]
\[ m = \alpha \dot{n} + (1 - \alpha) \dot{w} + [1 - (1 - \alpha)(1 + p')^{-1}] p = \alpha \dot{n} + (1 - \alpha) \dot{w} + \frac{p' + \alpha}{p' + 1} p \]

Therefore, it’s obvious that the rate of technical change (\( m \)) could be characterized as the linear combination of the growth rate of profit (\( p' \)), the wage (\( w \)), and the capital circulation (\( n \)) with respect to the labor output elasticity of C-D production function (1-\( \gamma \)). Technological progress ought to improve the rate of profit, wage rates and cash flow. Technical change stems from the division of labor, exacerbated by the division of

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\(^5\) Cobb C.W., Douglas P.H. "A Theory of Production", Amer. Econ. Rev. 1928,8(1), Spp1.139-165.
labor. The rate of technological progress (Solow residue\(^6\)) is proportional to the growth rate of profit \(p\) (thus productivity growth rate \(f\)).

Okishio Theorem\(^7\) asserts that if real wages remain unchanged, the rate of profit necessarily rises in consequence of an cost-saving technology innovation. Then after transformation, the relationship between the profit rate and the organic composition of capital can be obtained:

\[
p = \frac{m - \alpha \dot{n} - (1 - \alpha) \dot{w}}{(1 - \alpha + \alpha \gamma) / \gamma} = \frac{\ddot{y} - \alpha \dot{g} - \dot{w}}{(1 - \alpha + \alpha \gamma) / \gamma} = (\ddot{y} - \alpha \dot{g} - \dot{w}) \frac{p^{\prime + 1}}{p^{\prime + \alpha}}
\]

the growth rate of profit is determined by the growth rate of labor productivity \((y)\) positively only; it seems that Marx was right about that the rate of profit tends to fall due to the rise of the OCC \((g)\).

Differentiates \(p^{\prime}\) with \(t\):

\[
p^{\prime} = \frac{M}{Cv} = \frac{m^\prime}{g + 1} = \frac{Y - V}{C + V} = \frac{y - w}{nk + w} = \frac{y - 1}{g + 1}
\]

\[
p = \frac{d p^{\prime}}{dt} = \frac{d (y / w - 1)}{(y / w - 1) dt} = \frac{d (1 + g)}{(1 + g) dt} = \frac{\ddot{y} - \dot{w}}{1 - w / y} - \beta \dot{g} = (\ddot{y} - \dot{w} - \alpha \dot{g}) \frac{p^{\prime + 1}}{p^{\prime + \alpha}}
\]

\[
\Rightarrow \frac{\beta}{\alpha} = \frac{p^{\prime + 1}}{p^{\prime + \alpha}} = \frac{1}{1 - w / y} = \frac{Y}{M}, \frac{1 - \alpha}{1 - \beta} = \frac{Q}{Y} - \beta = \frac{p^{\prime + 1} - \alpha}{m^\prime + 1} = \frac{p^{\prime + 1}}{m^\prime + 1}
\]

\[
p^{\prime} = \frac{\alpha(1 - \beta)}{\beta - \alpha} = \frac{1}{\beta} \frac{1}{1 - \beta}, p^{\prime + 1} = \frac{\beta(1 - \alpha)}{\beta - \alpha} = \frac{1 - \alpha}{\alpha - \beta}
\]

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\[ m = \alpha \dot{n} + (1 - \alpha) \dot{w} + \frac{p^{1+\alpha}}{p^{1}} \rho \]

And the following relationships about the equilibrium state are obtained:

\[ m = \alpha \dot{n} + (1 - \alpha) \dot{w} + \frac{\alpha}{\beta} \rho \]

Therefore, the Marx production function is obtained as:

\[ Y = A_0 e^{\alpha} K^{\alpha} L^{1-\alpha} = \frac{A_0}{n_0 w_0} e^{\frac{(\alpha \dot{n} + (1 - \alpha) \dot{w})}{\beta}} K^{\alpha} n_0 w_0^{1-\alpha} L^{1-\alpha} \]

\[ = a_o e^{\frac{\alpha}{\beta}} C^\alpha V^{1-\alpha} = a_o e^{\frac{1-\alpha}{\beta}} C^\alpha V^{1-\alpha} (a_o = \frac{A_0}{n_0 w_0^{1-\alpha}}) \]

\[ = a_o e^{(1-\alpha) \dot{n} + \dot{w}} C^\alpha V^{1-\alpha} \equiv a_o e^{(1-\alpha) \dot{n} + \dot{w}} C^\alpha V^{1-\alpha} \]

Since the above derivation is based on the equilibrium condition, the symbol "*" is put as a label, and the productivity development parameter \( F \) is characterized as:

\[ F = (1 - \alpha) \dot{g} * \]

Thus, the Solow residue combined with Okishio theorem could be rewritten as Marxian 1st theorem about technical change, which depends upon the combination of the growth rates of capital circulating, the wage,
and the profit rate: \( m = \alpha \dot{n} + (1 - \alpha) \dot{w} + \frac{\alpha}{\beta} p = \alpha \dot{n} + (1 - \alpha) \dot{w} + F \).

The division of labor by Adam Smith\(^8\) can be characterized as the labor output elasticity \((1 - \alpha)\) of the Cobb-Douglas production function: \( d_\perp := 1 - \alpha = \frac{\gamma}{g + \gamma} = \frac{p' + 1}{m + 1} \); similarly, the degree of the roundabout production by Allyn Young\(^9\) can be characterized as the variable capital output elasticity \((1 - \beta)\) of the Marx production function:

\[
d_\parallel := 1 - \beta = \frac{1}{1 + g} = \frac{p'}{m + 1} \gamma = 1 + \frac{1}{\beta} = \frac{1 - \alpha}{\alpha} = \frac{1}{(1 - \beta)} - 1,
\]

\[
\dot{y} = \frac{d (P' / p')}{(P' / p') dt} = f - p = f(1 - \gamma) = (1 - \beta) \dot{g}(1 - \frac{\beta(1 - \alpha)}{(1 - \beta)\alpha}) = \dot{g}(1 - \frac{\beta}{\alpha}) < 0
\]

\[
\dot{d}_\perp = \dot{y} - \frac{d (g + \gamma)}{(g + \gamma) dt} = \frac{\gamma}{g + \gamma} - \dot{g} - \frac{\gamma}{g + \gamma} = \frac{\gamma}{g + \gamma} (\dot{y} - \dot{g}) = \gamma R = p,
\]

\[
\dot{d}_\parallel = -\frac{d (1 + g)}{(1 + g) dt} = -\frac{g}{1 + g} \dot{g} = -\beta \dot{g} = \dot{d}_\perp
\]

\[
\dot{\alpha} = \dot{g} - \frac{g}{g + \gamma} \dot{\gamma} = \frac{\gamma}{g + \gamma} (\dot{g} - \dot{\gamma}) = \gamma \dot{y} = p,
\]

\[
\dot{\beta} = \dot{g} - \frac{g}{g + 1} \dot{\gamma} = (1 - \beta) \dot{g} = f
\]

\[
\frac{Y}{C + V} = \frac{A e^{\beta} m}{a \beta V^{1 - \beta}} C^{a} V^{1 - \alpha} = \frac{Y}{C V} = \frac{A e^{\beta} m}{a \beta V^{1 - \beta}} g^{a - \beta} = \frac{A e^{\beta} m}{a \beta} \frac{g^{a - \beta}}{\beta} \rightarrow \frac{Y - \dot{V}}{C - V}, \alpha = \frac{C - V}{M - V}
\]

Schumpeterian innovation\(^10\) function/degree can be characterized by \( \frac{\gamma}{g} \)

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\(^{\text{1}}\) 亚当·斯密著，郭大力 王亚南译. 国民财富的性质和原因的研究[M]. 北京：商务印书馆，1972.


\(^{\text{3}}\) Schumpeter, J.A. The theory of economic development: an inquiry into profits, capital, credit, interest, and the business cycle translated from the German by Redvers Opie (1961) New York: OUP
since \( \frac{1}{g} = \frac{1 - \beta}{\beta} \) and the labor division coefficient \( d_p \) has the same change rate as the degree of the roundabout production \( d_p \):

\[
S := \frac{\gamma}{g} = \frac{1 - \alpha}{\alpha} = \frac{Q}{M} - \frac{V}{C} = \frac{b_0 e^{\frac{\beta}{g}}}{b_0} = \frac{\beta}{\alpha} \cdot \frac{g^\beta}{g^\alpha}\]
\[
\dot{S} = \dot{y} - \dot{g} = \frac{\beta}{\alpha} \dot{y}^* = -\dot{m},
\]

\[
\begin{align*}
S & = \frac{Q}{M} - \frac{V}{C} = \frac{Y + C}{C} - \frac{M}{M} = \frac{(Y + C) / C}{V} = \frac{Y + 1}{V - 1} \\
\dot{S} & = \dot{Q} - \dot{M} - \dot{Y} - \dot{V} - \dot{C} - \dot{M}
\end{align*}
\]

\[
\frac{\partial S}{\partial C} = -\frac{C^2}{Y} < 0,
\]

\[
\frac{\partial S}{\partial V} = \frac{1}{V} < 0,
\]

\[
\frac{\partial S}{\partial Y} = \frac{V}{CM} > 0
\]

All the equations are:

\[\text{LTV} : Q = C + V + M = C + Y = b_0 e^{\frac{\beta}{Y}} V^{1-\beta} = b_0 e^{\frac{\beta}{Y}} V^{1-\beta}\]

\[\text{MIPF} : Y = M + V = a_0 e^{\frac{\alpha}{V}} C^{1-\alpha} = a_0 e^{\frac{\alpha}{V}} \alpha^{1-\alpha} C^{1-\alpha}\]

\[\text{STV} : M = b_0 e^{\frac{\beta}{Y}} V^{1-\beta} = b_0 e^{\frac{\beta}{Y}} V^{1-\beta}\]

\[\text{CostFun} : C v = C + V = c_0 C^{1\beta} V^{1-\beta} = c_0 g^\beta\]

\[\text{Productivity} : \dot{P} = Q / C v = b_0 e^{\frac{\beta}{Y}} V^{1-\beta} = b_0 e^{\frac{\beta}{Y}} V^{1-\beta}\]

\[\text{Surplus Value} : \dot{m} = M / V = b_0 e^{\frac{\beta}{Y}} V^{1-\beta} = b_0 e^{\frac{\beta}{Y}} V^{1-\beta}\]

\[\text{Profit Rate} : \dot{p} = M / C v = \frac{\dot{m}}{g + 1} = \dot{m}(1 - \beta) = b_0 e^{\frac{\beta}{Y}} V^{1-\beta}\]

II. Transformation Problem
If there is no currency inflation, and the values of commodities keep invariant, then we have:

\[ C + V = C V = N = \text{const.} , \]

\[ dN = 0 = dCV = d(C + V) ; \]

\[ dC = dV = 0 \]

**Total value (C + V) equal total production price (P, C + P, V):**

\[ dN = d(P_1 C + P_2 V) = C dP_1 + P_1 dC + P_2 dV + V dP_2 = C dP_1 + V dP_2 \]

\[ = CV \{ \beta dP_1 (1 + (1 - \beta) dP_2) \} = CV \{ \beta \overline{dP_1} + (1 - \beta) \overline{dP_2} \} \]

\[ \equiv CV \{ \beta \ln(1 + \overline{dP_1}) + (1 - \beta) \ln(1 + \overline{dP_2}) \} = CV \{ \beta \ln P_1 + (1 - \beta) \ln P_2 \} \]

\[ = CV \ln(\frac{P_1^\beta P_2^{1-\beta}}{0}) = 0 \Rightarrow \]

\[ P_1^\beta P_2^{1-\beta} = 1 \Longleftrightarrow P_1^\beta P_2^{1-\beta} = 1 \Rightarrow \]

\[ \beta \overline{dP_1} + (1 - \beta) \overline{dP_2} = 0, \text{ or :} \]

\[ \beta \overline{dP_1} + (1 - \beta) \overline{dP_2} = 0 \]

let (I denotes the inflation index and

\[ \overline{dP_1} := PPI - 1, \overline{dP_2} := CPI - 1 \]

\[ \beta = \frac{\overline{dP_2}}{\overline{dP_2} - \overline{dP_1}} = \frac{CPI - 1}{CPI - PPI} , \]

\[ Q' = C' + V' + M' = P_1 C + P_2 V + P_3 M = B(P_1 C)^\beta (P_2 V)^{1-\beta} = Q P_1^\beta P_2^{1-\beta} = Q \]

\[ M' = P_3 M = b(P_1 C)^\beta (P_2 V)^{1-\beta} = MP_1^\beta P_2^{1-\beta} = M , P_3 = 1 ; \]

\[ P_1 C + P_2 V = C + V ; \]

\[ M' = Q' (P_1 C + P_2 V) = P_1 M = M = r(C + V) = r' (P_1 C + P_2 V) \]

**Total profit equals total surplus values.**

For a Marxian two production departments system:

\[ (1 + r_1)(P_1 C_1 + P_2 V_1) = Q_1 = P_1(C_1 + C_2) = P_1 C, \]

\[ (1 + r_2)(P_1 C_2 + P_2 V_2) = Q_2 = P_2 V + M , \]

\[ r_1 = r_2 = r = \frac{P_2 V - V_1}{P_1 C + V_1} = \frac{M}{P_1 C + P_2 V} = \frac{M}{C + V} = \frac{M}{Cv} = \frac{P}{C_2 - r C} \]
Or a three production departments system described by J. Winternitz:\(^\text{11}\):

\[(1) : P_1C_1 + P_2V_1 + P_3M_1 = (P_1C_1 + P_2V_1)(1 + \rho') = P_1C = P_1(C_1 + C_2 + C_3)\]

\[(2) : P_1C_2 + P_2V_2 + P_3M_2 = (P_1C_2 + P_2V_2)(1 + \rho') = P_2V = P_2(V_1 + V_2 + V_3)\]

\[(3) : P_1C_3 + P_2V_3 + P_3M_3 = (P_1C_3 + P_2V_3)(1 + \rho') = P_3M = M = M_1 + M_2 + M_3\]

\[p^* = \frac{M}{P_1C + P_2V}, \quad P = \frac{(1 + \rho')V_1}{C_1 + C_2 + C_3 - \rho'C_1}, \quad P_1 = \frac{CV - P_2V}{C + V}, \quad P_2 = \frac{CV}{C + V}.\]

\[p_1^*p_2^{1-\beta} = 1 \iff \ln P_2 = \ln I + \beta \ln \frac{P_1}{P_1}, \quad \beta_1 + (1 - \beta) P_2 = \delta, \quad \text{or} : \quad \beta \delta P_1 + (1 - \beta) \delta P_2 = \delta I\]

\[p^\beta = \frac{1}{p_2} \text{e}^{-\frac{\beta}{p_2} \frac{C + V}{p_2 + V}} \implies p^\beta = p^\beta + 1 - \beta\]

1. \(e^{\beta ln p} = 1 + \beta \ln p + \frac{\beta^2}{2} \ln^2 p + \ldots = 1 + \beta(p - 1) \implies p = e^{\beta - 2}\)

\[\begin{align*}
1) \quad & \frac{\beta}{2} y^2 = e^y - 1 - y = \frac{y^2}{2} + \frac{y^3}{6}, \quad y = 3(\beta - 1) = \ln P, \quad P = e^{3(\beta - 1)} \\
2) \quad & \frac{\beta}{2} y^2 + \frac{\beta^2}{6} y^3 + \frac{\beta^3}{24} y^4 - 2 = \frac{y^2}{2} + \frac{y^3}{6}, \quad y = \frac{\beta}{1 + \beta} \implies P = e^{\frac{\beta}{2 + 2\beta^2}} \\
3) \quad & \frac{\beta}{6} y^2 + \frac{\beta^2}{24} y^3 + \frac{\beta^3}{24} y^4 - 2 = \frac{y^2}{2} + \frac{y^3}{6}, \quad y = \frac{\beta}{1 + \beta} \implies P = e^{\frac{\beta}{2 + 2\beta^2}} \\
4) \quad & \frac{\beta}{2} y^2 + \frac{\beta^2}{6} y^3 + \frac{\beta^3}{24} y^4 - 2 = \frac{y^2}{2} + \frac{y^3}{6}, \quad y = \frac{\beta}{1 + \beta} \implies P = e^{\frac{\beta}{2 + 2\beta^2}}
\end{align*}\]

\[\frac{d}{d\beta} e^{\beta} = \frac{e^\beta}{1 + \beta} \quad \text{for} \quad e^{\beta} = \frac{1}{1 + \beta} \approx 1 - \frac{\beta}{4}, \quad \beta \delta P_1 + (1 - \beta) \delta P_2 = \delta I\]

Values and Prices: a solution to the so-called transformation problem, Econ. Jour. 1948, 58, 276-280.

\[\frac{d}{d\beta} e^{\beta} = \frac{e^\beta}{1 + \beta} \quad \text{for} \quad e^{\beta} = \frac{1}{1 + \beta} \approx 1 - \frac{\beta}{4}, \quad \beta \delta P_1 + (1 - \beta) \delta P_2 = \delta I\]

\[\frac{d}{d\beta} e^{\beta} = \frac{e^\beta}{1 + \beta} \quad \text{for} \quad e^{\beta} = \frac{1}{1 + \beta} \approx 1 - \frac{\beta}{4}, \quad \beta \delta P_1 + (1 - \beta) \delta P_2 = \delta I\]
Taking into consideration of the prices, the change rate of OCC would be:

\[ g_p = \frac{P_c}{P_V} = p_g \Rightarrow \dot{g}_p = \dot{p} + \dot{g} \]

\[ \dot{p} = \frac{dP}{Pdt} = \frac{d \ln Pd}{d \beta dt} = \pm \frac{6 \beta}{\sqrt{6 \beta - 2}} \beta = \pm \frac{6 \beta(1 - \beta)}{\sqrt{6 \beta - 2}} \dot{g}; \dot{g}_p = (1 \pm \frac{6 \beta(1 - \beta)}{\sqrt{6 \beta - 2}}) \dot{g} \]

III. General Equilibrium

In *Das Kapital*, Marx\[12\] defined the reproduction schemes as abstract, two-sector models of the production and circulation of capital. Department one produces means of production, the value of its output\( (Q_1) \) is made up of \( C_1 + V_1 + M_1 = Q_1 \); where \( C_1 \) is the constant capital and \( V_1 \) the variable capital used up in production, \( M_1 \) is the surplus value produced. Department two produces means of consumption and the value of its output\( (Q_2) \) is likewise made up of \( C_2 + V_2 + M_2 = Q_2 \). Simple reproduction requires that capitalists in Department two acquire means of production to the value \( C_2 \) from Department one in order to be able to

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produce again, namely: \( C_2 = V_1 + M_1 = Y_1 \), therefore—the Marxian 1st theorem: \( \dot{C}_2 = \dot{Y}_1 \); Then, Marx’s theory about crisis is:

\[
Y_i = Y_i^0 e^{\lambda t} = C_2 = C_2^0 e^{\lambda t}
\]

\[
\Rightarrow t = \frac{\ln Y_i^0 - \ln C_2^0}{\dot{C}_2 - Y_1} = \begin{cases} \lambda, \dot{Y}_1 = \dot{C}_2; \\ 0, Y_i^0 = C_2^0 \end{cases}
\]

Namely, if an economic system achieves the Marxian equilibrium including its each production Department \( (\dot{Y} = \dot{Y}_1 = \dot{C}_2, \dot{Y}_2 = \dot{C}_2 = \dot{C}_2^0) \), there would be no business cycle; otherwise there exists some potential crisis:

\[
\dot{\beta} = \frac{\beta}{\beta} \Rightarrow \dot{\beta} = \frac{d\beta}{\beta dt} \Rightarrow \dot{\beta} = \beta \beta' \beta - \dot{\beta} = \beta \beta' \beta - \dot{\beta} = \dot{\beta} = \dot{\beta} \approx \dot{\beta}
\]

\[
\beta = (1 + \beta)(1 - \beta) \beta' \beta' \beta' \beta = -\beta (g - 1) \beta \beta \beta \beta = -\beta \omega
\]

\[
\omega \equiv \sqrt{\frac{g - 1}{1 + g}} = \sqrt{(2 \beta - 1)(1 - \beta)}
\]

\[
\Rightarrow \beta \equiv A_\beta \sin(\omega t) + B_\beta \cos(\omega t) = \beta \omega (\omega t + \theta)
\]

\[
\beta \equiv \sqrt{A_\beta^2 + B_\beta^2} \leq 1, \theta \equiv \arctg \frac{A_\beta}{B_\beta}
\]

\[
\Rightarrow T = \frac{2\pi}{\frac{\omega}{g - 1}} = \frac{2\pi}{\frac{\omega}{g}} \frac{2}{\sqrt{g - 1}} = \frac{4\sqrt{2\pi}}{\sqrt{g}}
\]

\[
\dot{\beta} = \frac{\omega}{\beta dt} \Rightarrow \dot{\beta} = \frac{\omega}{\sqrt{g - 1}(\omega t + \theta)} = \frac{\sqrt{(2 \beta - 1)(1 - \beta)} \beta}{\sqrt{g (\omega t + \theta)}}
\]

\[
\Rightarrow \sqrt{\frac{g - 1}{1 - \beta}} = \frac{2 \beta - 1}{g - 1}, \sin(\omega t + \theta) = \frac{1}{\beta}
\]
Moreover, according to Leontief’s input-output theory and the steady state analysis of the CD production function, there should be a dynamic input-output equilibrium between Department I (input: C) and Department II (output: Y=V+M)

\[
\dot{\alpha} = p = \frac{\alpha'}{\alpha} \Rightarrow \frac{\dot{p} = \ddot{\alpha}}{\alpha' \ dt} - \dot{\alpha} = \frac{\alpha''}{\alpha \ p} - p = \dot{S} + \beta + \ddot{\alpha}.
\]

\[
\Rightarrow \alpha'' = (p - \frac{\beta}{\alpha} + 1 - \beta) \ddot{\alpha} = -(2 \beta -1) S \beta \ddot{\gamma} \alpha = -(2 \beta -1) \ddot{\gamma}(1 - \beta) \ddot{\gamma} \alpha = -\ddot{\gamma} \alpha.
\]

\[
\Rightarrow \alpha'' = (p - \frac{\beta}{\alpha} + 1 - \beta) \ddot{\gamma} \alpha = -(2 \beta -1) S \beta \ddot{\gamma}(1 - \beta) \ddot{\gamma} \alpha = -\ddot{\gamma} \alpha.
\]

\[
\Rightarrow \alpha'' = \frac{2 \pi}{\ddot{\gamma} \sqrt{(2 \beta -1)}} \leq 1, \ \bar{\theta} = \frac{4 \sqrt{2 \pi}}{\ddot{\gamma} \sqrt{Y}}
\]

\[
\Rightarrow Y = A_0 \sin(\omega t) + B_0 \cos(\omega t), \ \omega = \sqrt{Y(\ddot{C} - \ddot{Y})};
\]

or: \[Y'' = (\ddot{C} - \ddot{Y}) Y' = \sqrt{Y(C - Y)}
\]

Moreover,

\[Q_1 = C_1 + V_1 + M_1 = C_1 + Y_1 = C_1 + C_2 = C \]
\[Q_2 = C_2 + V_2 + M_2 = C_2 + Y_2 = Y_2 + Y_1 = Y \]
\[Q = Q_1 + Q_2 = C + Y \Leftrightarrow Q \dot{Q} = C \dot{C} + YY \Leftrightarrow (1 - \alpha) \dot{Q} = (\beta - \alpha) \dot{C} + (1 - \beta) \dot{Y} \]

\[\dot{\gamma} = k^{ss}, \ \ddot{\gamma} = 0 \Leftrightarrow \dot{Y}^{ss} = k^{ss} + \dot{L}^{ss} + \dot{n}^{ss} = C^{ss} = \dot{O}^{ss}; \]

Marxian 2nd theorem about reproducibility states: there is a dynamic input-output equilibrium inside an economic system. The regression analysis of the USA manufacture industry data from 1958-1996 supported the above results:

\[
\begin{align*}
Y / C &= b_1 e^{(1-\alpha)\hat{g}} - 1, \\
\dot{Y} - \dot{C} &= (1 - \alpha) \hat{g}^* + (\alpha - 1) \hat{g} = (1 - \alpha)(\hat{g}^* - \hat{g}) \\
Q / C &= B_2 e^{(1-\beta)\hat{g}} - 1, \\
\dot{Q} - \dot{C} &= (1 - \beta) \hat{g}^* + (\beta - 1) \hat{g}^* = (1 - \beta)(\hat{g}^* - \hat{g})
\end{align*}
\]

\[\therefore \hat{g}^* = \dot{C}^* = \dot{Q}^* \iff \hat{g}^* = \hat{g} \]

The Marxian 3rd theorem about productivity development asserts: only Marxian equilibrium leads to productivity development and a rising profit rate.

\[
\begin{align*}
F &= \dot{Y} - \alpha \dot{C} - (1 - \alpha) \dot{V} = (1 - \alpha) \hat{g}^* > 0 \iff Y = \dot{C}, \hat{g}^* > 0; \\
F &= \dot{Q} - \beta \dot{C} - (1 - \beta) \dot{V} = (1 - \beta) \hat{g}^* > 0 \iff Q = \dot{C}, \hat{g}^* > 0
\end{align*}
\]

\[\therefore \hat{g}^* = \hat{g} = \frac{f}{1 - \beta} = \frac{\beta}{1 - \beta}, \beta = \frac{g}{g + 1} = \frac{C}{Cv}, \beta = \dot{C} - \dot{C}v = \dot{Q} - \dot{Q}v
\]

\[Q = C + V + M = Cv + M
\]

\[\dot{Q} - \dot{C}v = \frac{M (M - \dot{Q})}{Cv} = p'(-\gamma) > 0
\]

\[\therefore \hat{g}^* = \hat{g} = \frac{f}{1 - \beta} = \frac{\beta}{1 - \beta} > 0
\]

IV. Marxian Optimal Growth

The tendency of the rate of profit depends on the OCC(g) and the output elasticity of the constant capital(), similar to the conclusion obtained by D.H. Dickinson:

\[ p' = b_o c^{\beta} \left( \frac{g}{g + 1} \right) \]

\[ \frac{\partial p'}{\partial g} = \frac{p'}{g} (\beta - \frac{g}{g + 1}) \geq 0 \]

\[ \Rightarrow \beta \geq \frac{g}{g + 1}, \]

\[ \frac{\partial^2 p'}{\partial g^2} = -\frac{p' \beta}{g^2 (g + 1)} < 0 \]

\[ p'_{\text{max}} = b \beta^\beta (1 - \beta)^{1-\beta} \Rightarrow (\beta \rightarrow 0 \text{ or } 1) b > p'_{\text{max}} \geq \frac{b}{2} (\beta = \frac{1}{2}) \]

By means of variation, Marx was right about the falling rate of the profit under the competitive equilibrium situation:

\[ p' = \frac{M}{C_v} = b_o c^{\beta} C^{V^{1-\beta}} = \frac{b_o c^{(1-\alpha)\frac{\beta}{\alpha}} C^{\beta} V^{1-\beta}}{a_o \beta^{V^{1-\beta}}} = \frac{b_o c^{\beta(1-\alpha)\frac{\beta}{\alpha}}}{a_o} \]

\[ \frac{\partial p'}{\partial g} = p' - \beta g^t (1 - \alpha) = -\frac{p' \beta - \alpha}{g} \beta g \]

\[ \frac{d}{dt} \left( \frac{\partial p'}{\partial g} \right) = \frac{d}{dt} \left( p' \frac{\beta_t}{g (1 - \alpha)} \right) = \frac{p' \beta (1 - \alpha)}{g} \beta (t + 1 - i_g) \]

\[ \Rightarrow p_t = -1 < 0 \]

\[ \Rightarrow p' = p'_{\text{opt}} t^{-\alpha} \]

---

The exploitation rate will decrease under the equilibrium state:

\[
\text{if } \frac{d}{dt} \left( \frac{\partial p^*}{\partial g^*} \right) = \frac{d}{dt} \left( \frac{p^* \beta_t}{g} \right) = \frac{p^*}{g} \beta (\varphi + 1 + \dot{c} + \dot{e} - t \dot{g}) = -\frac{p^*}{g} \beta \dot{g} \Rightarrow \\
\varphi + \dot{c} + \dot{e} - t = \varphi + \dot{c} + \dot{e} - t \Rightarrow \beta > \frac{1}{2};
\]

or:

\[
\varphi t = \frac{1 - \alpha}{\alpha} \beta \dot{g} t = \frac{s \beta}{2 \beta - 1} = \frac{\gamma}{g - 1} > 0 \Leftrightarrow g > 1 \Leftrightarrow C > V
\]

Moreover, the rate of variable capital accumulation will increase under equilibrium state:

\[
m' = \frac{M}{V} = \frac{d}{dt} \left( \frac{\partial m'}{\partial g^*} \right) = \frac{d}{dt} \left( \frac{p^* \beta_t}{g} \right) = \frac{p^*}{g} \beta (\varphi + 1 + \dot{c} + \dot{e} - t \dot{g}) = -\frac{p^*}{g} \beta \dot{g} \Rightarrow \\
\varphi + \dot{c} + \dot{e} - t = \varphi + \dot{c} + \dot{e} - t \Rightarrow \beta > \frac{1}{2};
\]

or:

\[
\varphi t = \frac{1 - \alpha}{\alpha} \beta \dot{g} t = \frac{s \beta}{2 \beta - 1} = \frac{\gamma}{g - 1} > 0 \Leftrightarrow g > 1 \Leftrightarrow C > V
\]
A maximum production would achieve under equilibrium state together with a minimum input requirement under equilibrium state:

\[ V = b_2 e^{\frac{\beta}{\alpha} t} \]

\[ \frac{\partial V}{\partial g} = -\frac{\beta}{\alpha} \frac{g'}{g} V = a - \frac{\beta}{\alpha} \frac{g'}{g} V \]

\[ \frac{d}{dt} \left( \frac{\partial V}{\partial g'} \right) = \frac{d}{dt} \left[ V \left( \frac{\beta}{\alpha} - 1 \right) \right] = \left( \frac{\beta}{\alpha} - 1 \right) \left( \frac{V}{g} \right) \frac{g'}{g} + \frac{V}{g} \frac{g'}{g} \implies \]

\[ \dot{V}_t = \frac{\beta}{\alpha} \frac{g'}{g} t - 1 = \dot{m}' t - 1 > 0 \quad \text{for} \quad \frac{1}{1 - \alpha} - \frac{1}{1 - \frac{\beta}{\alpha}} \iff \frac{g + \gamma}{\gamma} - \frac{g + \gamma}{g} > 1 \]

\[ \Leftrightarrow g > (1 + \gamma) \gamma > 2 \implies g > \sqrt{2} \approx 1.414 > 1; \]

\[ 1 - \beta = \frac{1}{1 + g} < \frac{1}{g} \approx 0.414 \quad (2 \alpha > \beta > 0.5) ; \]

\[ \alpha^2 + (2 g - 1) \alpha - g > 0 \implies (\beta > \alpha) > \frac{1 + \sqrt{4 g^2 + 1 - 2 g}}{2} > 0.5 \]

with a minimum input requirement under equilibrium state:

\[ Y = A_2 e^{\frac{\beta}{\alpha} t} \]

\[ \frac{\partial Y}{\partial g} = Y \left( \frac{\alpha}{g} - \frac{1 - \alpha}{g^2} \right) = Y \left[ \alpha - (1 - \alpha) \frac{g'}{g} \right] \]

\[ \frac{d}{dt} \left( \frac{\partial Y}{\partial g'} \right) = \frac{d}{dt} \left( Y \frac{1 - \alpha}{g} \right) = Y \frac{1 - \alpha}{g} \left( \dot{y}_t + \dot{g}_t t + 1 - \dot{g}_t \right) \]

\[ \implies \dot{Y}_t = S^{-1} - 1 > 0 \iff \alpha > 0.5, S < 1, Y = Y_0 t S^{-1}, \]

or \( \dot{Y} - \beta \dot{g}_t = S^{-1} - 1 \equiv b - \beta \dot{m}_t \ln Y = b t + b \ln t + c \geq 0 \)

\[ \implies \dot{Y}_t = \beta \dot{g}_t + S^{-1} - 1 \geq 0 \iff S^{-1} = \frac{\alpha}{1 - \alpha} \geq 1 - \beta \dot{g}_t \iff \alpha \geq \frac{1 - \beta \dot{g}_t}{2 - \beta \dot{g}_t} \]
V. The econo-sociological Marxism

The first-rate reaction of natural decomposition of some Persistent Organic Pollutants is: \( C = C_0 e^{-kt} \), \( k \): reaction rate, \( C \): concentration of certain POPs; the economic growth could be related to the amount of the POPs\(^{17} \) as: \( \ln Y = a + b \ln C \),

\(^{17}\) Private communication with Prof. QQ Wang (Chem. Coll., Xiamen Univ.)
a, b are both positive coefficients; therefore, the total change of the POPs is:

\[ \dot{C} = \frac{d}{dt} [\ln Y - a/b] - k = \frac{d}{dt} \ln Y - k = \frac{\dot{Y}}{b} - k \]

The environment couldn't be worse meaning the amount of POPs wouldn't increase:

\[ \dot{C} \leq 0 \iff \frac{\dot{Y}}{b} \leq k \iff \dot{Y} \leq bk (\text{if } b < 1) \]

namely, the growth rate of GDP should not be over a critic value.

VI. Conclusion

In short, this study is aimed to obtain a quantitative description of Marxian capital theory including Marx labour value function and Marx surplus value function as well as Marx production function. The labor
output elasticity \((1 - \alpha)\) of Cobb-Douglas production function is defined as the parameter for the division of labor. The productivity parameter in Marx production function is defined as the product of the change rate of the organic composite of capital with the coefficient of the division of labor. Furthermore, three Marxian theorems are proposed, which assert that there is a dynamic equilibrium existed in reproduction between the Two Departments, only equilibrium growth leads to the positive value of the productivity parameter (Productivity Development Theorem) and also the rate of profit, of which the change rate combined that of the wage and capital circulating with respect to the capital output elasticity of Cobb-Douglas production function characterizes the technological progress rate or as called the Solow residue. By means of variation, the tendency of the profit rate to fall is proved under the situation that the degree of the labor division remains unchanged.