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BIPOLAR INTERVAL NEUTROSOPHIC SET AND ITS APPLICATION IN MULTICRITERIA DECISION MAKING

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ABSTRACT. In this paper we combine Bipolar valued fuzzy sets and interval neutrosophic set to introduce the concept of bipolar interval neutrosophic sets. We define the union, complement, intersection and containement of bipolar interval neutrosophic sets. Also we defined BINWA operator and BINWG and estiblished a MADM method for BINSs.

1. INTRODUCTION

The concept of fuzzy sets was introduced by L. A. Zadeh in 1965[19]. After the introduction of fuzzy sets, fuzzy set and fuzzy logic have been applied in many real applications to handle uncertainty. The traditional fuzzy set uses one single value $\mu_A \in [0,1]$ to represent the grade of membership of the fuzzy set A defined on a universe. But sometimes it is difficult to define degree of membership by a single point. So Turksen presented the concept of interval valued fuzzy sets[9]. After the introduction of interval valued fuzzy sets Attanassov[1] introduced the concept intuitionistic fuzzy sets as a geralization of fuzzy sets. Vague set are introduced by Gau and Bueherer[5]. Later on Bustince[2] pointed out that vague sets and intuitionistic fuzzy sets are mathematically equivalent. K. M. Lee[6] introduced the concept of bipolar fuzzy sets, as an extension of fuzzy sets. In bipolar fuzzy sets the degree of membership is enlarged from [0,1] to [-1,1].If the degree of membership in bipolar valued fuzzy set is equal to zero then we say that the element are unrelated and the membership degree (0,1] indicates the element somewhat satisfy the property , and the membership degree [-1,0) indicate that element satisfy some what implicit counter property. Then Samarandache[7] proposed the concept of neutrosophic sets. Neutrosopy is a branch of philosophy which studies the nature and scope of neutralities, as well as their interaction with different ideational spectra.The neutrosophic set is a powerful general prescribed framework, Which generalizes all the above sets from philosophical point of view. In neutrosophic set the elements of a universal set has a degree of truth-membership, indeterminancy membership and a falsity membership respectively which lies in $]0^-, 1^+[$ the non-standard unit interval. Without specification Samarandache neutrposophic set is difficult to apply in real and scientific problems. To overcome this difficulty Haibin wang et.al[10, 11] introduced the concept of single valued neutrosophic set (SVNS) and interval neutrosophic sets (INS). In this types of sets instead of non-standard unit interval they take real standard interval and define set theoretic operators for single valued neutrosophic sets and interval neutrosophic sets. Jun Ye[12, 13] also introduced the correlation coefficient of single valued neutrosophic sets and

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also presented another form of correlation and applied it to multi-criteria ecision making prolems.Ye [14, 15], defined the Hamming and Euclidean distance of the INS presented the similirity measure between single valued neutrosophic set and interval neutrosophic sets based on the relationship between similirity measures and distance. Another generalization of neutronsophic set was made by Jun Ye[16] and named this generalization simplified neutrosophic sets. Simplified neutrosophic sets contain the concept of interval and single valued neutrosophic sets. Jun Ye[17, 18] presented vector similirity measure and improved cosine similirity measure for simplified neutrosophic set and apllied it to multicriteria dcesion making and medical diagnosis. Ridvan sahin[8] give the concept of multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment.Irfen Deli et.al[4] introduced bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In this paper we combine bipolar valued fuzzy set and neutrosophic set and gave its operations. This article is arranged as follows.In section 2 we present a brief overview of Bipolar valued fuzzy set, neutrosophic set single valued neutrosophic set, Interval neutrosophic sets. In section 3 we define bipolar interval neutrosophic set and its operations, and in section 4 we define some aggregation operator for bipolar interval neutrosophic set (BINS). In section 5 we give a numerical example to illustrate the effectiveness of the proposed method, at the end of the article conclusion, future work and references are given.

2. Preliminaries

In this section we defined some basic definitions of bipolar valued fuzzy set, neutrosophic set, single valued neutrosophic sets and interval neutrosophic set, Which will be used in the rest of the paper. For futher information the reader should refer to read [6, 7, 10, 11]

2.1. **DEFINITION**[6]**.** Let M be the universe of discourse. Then a bipolar valued fuzzy set B on M is defined by positive membership function μ_B^+, that is $\mu_B^+ : M \longrightarrow [0,1]$,and a negative membership function μ_B^-, that is $\mu_B^- : M \longrightarrow [-1,0]$.

Mathematically a bipolar valued fuzzy set is represented by

$$B = \{\langle a, \mu_B^+(a), \mu_B^-(a)\rangle | for all a \in M\}.$$

2.2. **Definition**[6]**.** Let A and B be two bipolar valued fuzzy sets then their union, intersection and complement are defined as follow:

$$\begin{aligned}
A \cup B &= \{\langle a, \mu_{A\cup B}(a)\rangle | a \in M\} \\
\mu_{A\cup B} &= \{\langle \mu_{A\cup B}^+(a), \mu_{A\cup B}^-(a)\rangle\} \\
\mu_{A\cup B}^+(a) &= \max\{\langle(\mu_A^+(a)\rangle, \mu_B^+(a)\} \\
\mu_{A\cup B}^-(a) &= \min\{\langle \mu_A^-(a), \mu_A^-(a)\rangle\}
\end{aligned}$$

$$\begin{aligned} A \cap B &= \{\langle a, \mu_{A\cap B}(a)\rangle | a \in M\} \\ \mu_{A\cap B} &= \{\langle \mu^+_{A\cap B}(a), \mu^-_{A\cap B}(a)\rangle\} \\ \mu^+_{A\cap B}(a) &= \min\{\langle(\mu^+_A(a)\rangle, \mu^+_B(a)\} \\ \mu^-_{A\cap B}(a) &= \max\{\langle \mu^-_A(a), \mu^-_A(a)\rangle\} \end{aligned}$$

$$\begin{aligned} A^C &= \{\langle a, \mu_{A^C}(a)\rangle\}, \\ \mu_{A^C} &= \{\langle \mu^+_{A^C}(a), \mu^-_{A^C}(a)\rangle\} \\ \mu^+_{A^C} &= 1 - \mu^+_A(a), \mu^-_{A^C} = 1 - \mu^-_A(a) \end{aligned}$$

for all $a \in M$.

2.3. **Definition**[7]**.** Let M be a universal set. Then a neutrosophic set R is an object of the form

$$R = \{\langle b, T_R(b), I_R(b), F_R(b)\rangle | b \in M\}$$

Which is characterized by a truth-membership function $T_R(b) : M \longrightarrow]0^-, 1^+[$, an indeterminancy-membership funtion $I_R(b) : M \longrightarrow]0^-, 1^+[$ and falsity-membership function $F_R(b) : M \longrightarrow]0^-, 1^+[$. There is no restriction on the sum of $T_R(b), I_R(b)$ and $F_R(b)$,so $0^- \le \sup T_R(b) + \sup I_R(b) + \sup F_R(b) \le 3^+$.

2.4. **Definition**[10]**.** Let Let M be a universal set. Then a single valued neutrosophic set R is an object of the form

$$R = \{\langle b, T_R(b), I_R(b), F_R(b)\rangle | b \in M\}$$

Which is characterized by a truth-membership function $T_R(b) : M \longrightarrow [0, 1]$, an indeterminancy-membership funtion $I_R(b) : M \longrightarrow [0, 1]$ and falsity-membership function $F_R(b) : M \longrightarrow [0, 1]$. There is no restriction on the sum of $T_R(b), I_R(b)$ and $F_R(b)$,so $0 \le T_R(b) + I_R(b) + F_R(b) \le 3$.

2.5. **Definition**[11]**.** Let M be a space of points (objects) with a generic element $b \in M$.Then interval neutrosophic set is an object of the form
$R = \{\langle b, T_R(b), I_R(b), F_R(b)\rangle | b \in M\}$

Which is characterized by a truth-membership function $T_R(b) : M \longrightarrow [0, 1]$, an indeterminancy-membership funtion $I_R(b) : M \longrightarrow [0, 1]$ and falsity-membership function $F_R(b) : M \longrightarrow [0, 1]$. There is no restriction on the sum of $T_R(b), I_R(b)$ and $F_R(b) \subseteq [0, 1]$,so $0 \le \sup T_R(b) + \sup I_R(b) + \sup F_R(b) \le 3$.

3. BIPOLAR INTERVAL NEUTROSOPHIC SET

3.1. **Definition.** Let X be the universe of discourse. A bipolar interval neutrosophic set B^* in X is characterized by positive and negative truth-membership,T^+, T^- indeterminancy-membership, I^+, I^- and falsity-membership, F^+, F^- functions respectively.For any $x \in X, T^+(x), I^+(x), F^+(x) \subseteq [0, 1]$ and also $T^-(x), I^-(x), F^-(x) \subseteq [-1, 0]$.

The condition that $0 \le \sup T^+(x) + \sup I^+(x) + \sup F^+(x) \le 3$ and $-3 \le \sup T^-(x) + \sup I^-(x) + \sup F^-(x) \le 0$.

3.2. **Example.** Let $X = \{a, b, c\}$. Then bipolar interval neutrosophic set B^* in X is given by

$$B^* = \left\{ \begin{array}{c} \langle a, [0.7, 0.8], [0.1, 0.2], [0.2, 0.3], [-0.6, -0.5], [-0.4, -0.3], [-0.5, -0.4] \rangle, \\ \langle b, [0.5, 0.6], [0.2, 0.3], [0.3, 0.4], [-0.8, -0.7], [-0.3, -0.2], [-0.4, -0.3] \rangle, \\ \langle c, [0.3, 0.4], [0.3, 0.4], [0.4, 0.5], [-0.9, -0.8], [-0.2, -0.1], [-0.3, -0.2] \rangle \end{array} \right\}$$

3.3. **Definition.** Let A^* and B^* be two bipolar interval neutrosophic sets defined over a universe of discourse X. We say that $A^* \subseteq B^*$ if and only if

$$\inf T^+_{A^*}(x) \leq \inf T^+_{B^*}(x), \sup T^+_{A^*}(x) \leq \sup T^+_{B^*}(x)$$

$$\inf F^+_{A^*}(x) \geq \inf F^+_{B^*}(x), \sup F^+_{A^*}(x) \geq \sup F^+_{B^*}(x)$$

$$\inf T^-_{A^*}(x) \geq \inf T^-_{B^*}(x), \sup T^-_{A^*}(x) \geq \sup T^-_{B^*}(x)$$

$$\inf I^-_{A^*}(x) \leq \inf I^-_{B^*}(x), \sup I^-_{A^*}(x) \leq \sup I^-_{B^*}(x)$$

$$\inf F^-_{A^*}(x) \leq \inf F^-_{B^*}(x), \sup F^-_{A^*}(x) \leq \sup F^+_{B^*}(x)$$

for all $x \in X$.

3.4. **Example.** Let $U = \{a_1, a_2, a_3\}$be a universe set. Then B_1^* and B_2^* are two bipolar interval neutrosophic set defined as

$$B_1^* = \left\{ \begin{array}{c} \langle a_1, [0.4, 0.5], [0.3, 0.4], [0.5, 0.6], [-0.6, -0.4], [-0.3, -0.2], [-0.4, -0.3] \rangle, \\ \langle a_2, [0.5, 0.6], [0.2, 0.3], [0.4, 0.5], [-0.5, -0.3], [-0.2, -0.1], [-0.3, -0.2] \rangle, \\ \langle a_3, [0.6, 0.7], [0.1, 0.2], [0.3, 0.4], [-0.4, -0.2], [-0.1, -0.2], [-0.2, -0.3] \rangle \end{array} \right\}$$

$$B_2^* = \left\{ \begin{array}{c} \langle a_1, [0.5, 0.6], [0.2, 0.3], [0.4, 0.5], [-0.7, -0.5], [-0.2, -0.1], [-0.3, -0.2] \rangle, \\ \langle a_2, [0.6, 0.7], [0.2, 0.2], [0.3, 0.4], [-0.6, -0.4], [-0.1, -0.01], [-0.2, -0.1] \rangle, \\ \langle a_3, [0.7, 0.8], [0.01, 0.1], [0.2, 0.3], [-0.5, -0.3], [-0.01, -0.1], [-0.1, -0.2] \rangle \end{array} \right\}$$

Clearly B_1^* is cotained in B_2^*. that is $B_1^* \subseteq B_2^*$.

3.5. **Definition.** Let A^* and B^* be two bipolar interval neutrosophic sets defined over a universe of discourse X. We say that $A^* = B^*$ if and only if
$A^* \subseteq B^*$ and $B^* \subseteq A^*$.

3.6. **Definition.** Let A^* be a bipolar interval neutrosophic sets defined over a universe of discourse X. Then the complement of A^* is denoted by ${A^*}^C$ and defined by

$$\inf T^+_{{A^*}^C}(x) = 1 - \sup T^+_{A^*}, \sup T^+_{{A^*}^C}(x) = 1 - \inf T^+_{A^*}(x)$$

$$\inf I^+_{{A^*}^C}(x) = 1 - \sup I^+_{A^*}, \sup I^+_{{A^*}^C}(x) = 1 - \inf I^+_{A^*}(x)$$

$$\inf F^+_{{A^*}^C}(x) = 1 - \sup F^+_{A^*}, \sup F^+_{{A^*}^C}(x) = 1 - \inf F^+_{A^*}(x)$$

$$\inf T^-_{{A^*}^C}(x) = -1 - \sup T^-_{A^*}, \sup T^-_{{A^*}^C}(x) = -1 - \inf T^-_{A^*}(x)$$

$$\inf I^{-}_{A^{*C}}(x) = -1 - \sup I^{-}_{A^*}, \sup I^{-}_{A^{*C}}(x) = -1 - \inf I^{-}_{A^*}(x)$$

$$\inf F^{-}_{A^{*C}}(x) = 1 - \sup F^{-}_{A^*}, \sup F^{-}_{A^{*C}}(x) = -1 - \inf F^{-}_{A^*}(x)$$

for all $x \in X$.

3.7. **Example.** Let B^* the bipolar interval neutrosophic set of the 3.2 then its complement is given as

$$B^{*C} = \left\{ \begin{array}{c} \langle a, [0.2, 0.3], [0.8, 0.9], [0.7, 0.8], [-0.5, -0.4], [-0.7, -0.6], [-0.6, -0.5] \rangle, \\ \langle b, [0.4, 0.5], [0.7, 0.8], [0.6, 0.7], [-0.3, -0.2], [-0.8, -0.7], [-0.7, -0.6] \rangle, \\ \langle c, [0.6, 0.7], [0.6, 0.7], [0.5, 0.6], [-0.2, -0.1], [-0.9, -0.8], [-0.8, -0.7] \rangle \end{array} \right\}$$

3.8. **Definition.** Let A^* and B^* be two bipolar interval neutrosophic sets defined over a universe of discourse X. Then their union is denoted by $D = A^* \cup B^*$ and is defined as

$$\begin{aligned}
\inf T^{+}_{D} &= \left\{\max(\inf T^{+}_{A^*}(x), \inf T^{+}_{B^*}(x)\right\} \\
\sup T^{+}_{D} &= \left\{\max(\sup T^{+}_{A^*}(x), \sup T^{+}_{B^*}(x)\right\} \\
\inf I^{+}_{D} &= \left\{\min(\inf I^{+}_{A^*}(x), \inf I^{+}_{B^*}(x)\right\} \\
\sup I^{+}_{D} &= \left\{\min(\sup I^{+}_{A^*}(x), \sup I^{+}_{B^*}(x)\right\} \\
\inf F^{+}_{D} &= \left\{\min(\inf F^{+}_{A^*}(x), \inf F^{+}_{B^*}(x)\right\} \\
\sup F^{+}_{D} &= \left\{\min(\sup F^{+}_{A^*}(x), \sup F^{+}_{B^*}(x)\right\}
\end{aligned}$$

$$\begin{aligned}
\inf T^{-}_{D} &= \{\min(\inf T^{-}_{A^*}(x), \inf T^{-}_{B^*}(x) \\
\sup T^{-}_{D} &= \left\{\min(\sup T^{-}_{A^*}(x), \sup T^{-}_{B^*}(x)\right\} \\
\inf I^{-}_{D} &= \left\{\max(\inf I^{-}_{A^*}(x), \inf I^{-}_{B^*}(x)\right\} \\
\sup I^{-}_{D} &= \left\{\max(\sup I^{-}_{A^*}(x), \sup I^{-}_{B^*}(x)\right\} \\
\inf F^{-}_{D} &= \left\{\max(\inf F^{-}_{A^*}(x), \inf F^{-}_{B^*}(x)\right\} \\
\sup F^{-}_{D} &= \left\{\max(\sup F^{-}_{A^*}(x), \sup F^{-}_{B^*}(x)\right\}
\end{aligned}$$

for all $x \in X$.

3.9. **Example.** Let B_1^* and B_2^* be two bipolar interval neutrosophic set defined in 3.4 then its union is given as

$$B_1^* \cup B_2^* = D = \left\{ \begin{array}{c} \langle a_1, [0.5, 0.6], [0.2, 0.3], [0.4, 0.5], [-0.7, -0.5], [-0.2, -0.1], [-0.3, -0.2] \rangle, \\ \langle a_2, [0.6, 0.7], [0.2, 0.2], [0.3, 0.4], [-0.6, -0.4], [-0.1, -0.01], [-0.2, -0.1] \rangle, \\ \langle a_3, [0.7, 0.8], [0.01, 0.1], [0.2, 0.3], [-0.5, -0.3], [-0.01, -0.1], [-0.1, -0.2] \rangle \end{array} \right\}$$

3.10. **Definition.** Let A^* and B^* be two bipolar interval neutrosophic sets defined over a universe of discourse X. Then their union is denoted by $H = A^* \cap B^*$ and is defined as

$$\begin{aligned}
\inf T_H^+ &= \left\{\min(\inf T_{A^*}^+(x), \inf T_{B^*}^+(x)\right\} \\
\sup T_H^+ &= \left\{\min(\sup T_{A^*}^+(x), \sup T_{B^*}^+(x)\right\} \\
\inf I_H^+ &= \left\{\max(\inf I_{A^*}^+(x), \inf I_{B^*}^+(x)\right\} \\
\sup I_H^+ &= \left\{\max(\sup I_{A^*}^+(x), \sup I_{B^*}^+(x)\right\} \\
\inf F_H^+ &= \left\{\max(\inf F_{A^*}^+(x), \inf F_{B^*}^+(x)\right\} \\
\sup F_H^+ &= \left\{\max(\sup F_{A^*}^+(x), \sup F_{B^*}^+(x)\right\}
\end{aligned}$$

$$\begin{aligned}
\inf T_H^- &= \left\{\max(\inf T_{A^*}^-(x), \inf T_{B^*}^-(x)\right. \\
\sup T_H^- &= \left\{\max(\sup T_{A^*}^-(x), \sup T_{B^*}^-(x)\right\} \\
\inf I_H^- &= \left\{\min(\inf I_{A^*}^-(x), \inf I_{B^*}^-(x)\right\} \\
\sup I_H^- &= \left\{\min(\sup I_{A^*}^-(x), \sup I_{B^*}^-(x)\right\} \\
\inf F_H^- &= \left\{\min(\inf F_{A^*}^-(x), \inf F_{B^*}^-(x)\right\} \\
\sup F_H^- &= \left\{\min(\sup F_{A^*}^-(x), \sup F_{B^*}^-(x)\right\}
\end{aligned}$$

for all $x \in X$.

3.11. **Examples.** Let B_1^* and B_2^* be two bipolar interval neutrosophic set defined in 3.4 then its intersection is given as

$$B_1^* \cap B_2^* = H = \left\{ \begin{array}{c} \langle a_1, [0.4, 0.5], [0.3, 0.4], [0.5, 0.6], [-0.6, -0.4], [-0.3, -0.2], [-0.4, -0.3] \rangle, \\ \langle a_2, [0.5, 0.6], [0.2, 0.3], [0.4, 0.5], [-0.5, -0.3], [-0.2, -0.1], [-0.3, -0.2] \rangle, \\ \langle a_3, [0.6, 0.7], [0.1, 0.2], [0.3, 0.4], [-0.4, -0.2], [-0.1, -0.2], [-0.2, -0.3] \rangle \end{array} \right\}$$

The set of all bipolar interval neutrosophic sets $(BINSs)$ is denoted by ξ. A bipolar interval neutrosophic number $(BINN)$ is denoted $a^* = \langle T^+, I^+, F^+, T^-, I^-, F^- \rangle$ for convenience.

3.12. **Definition.** Let $a_1^* = (T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-)$ and $a_2^* = (T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^-)$ be two bipolar interval neutrosophic numbers.

Then the operations for bipolar interval neutrosophic numbers are defined below:

$$
\begin{aligned}
\gamma a_1^* &= \langle [1-(1-\inf T_1^+)^\gamma, 1-(1-\sup T_1^+)^\gamma], [(\inf I_1^+)^\gamma, (\sup I_1^+)^\gamma],\\
&\quad [(\inf F_1^+)^\gamma, (\sup F_1^+)^\gamma], [-(-\inf T_1^-)^\gamma, -(-\sup T_1^-)^\gamma],\\
&\quad [-(1-(1-(-\inf I_1^-))^\gamma), -(1-(1-(-\sup I_1^-))^\gamma)],\\
&\quad [-(1-(1-(-\inf F_1^-))^\gamma), -(1-(1-(-\sup F_1^-))^\gamma)],\\
a_1^{*\gamma} &= [(\inf T_1^+)^\gamma, (\sup T_1^+)^\gamma], [(1-(1-\inf I_1^+))^\gamma), (1-(1-\sup I_1^+))^\gamma)],\\
&\quad [(1-(1-\inf F_1^+))^\gamma), (1-(1-\sup F_1^+))^\gamma)],\\
&\quad [-(1-(1-(-\inf T_1^-))^\gamma), -(1-(1-(-\sup T_1^-))^\gamma)],\\
&\quad [-(-\inf I_1^-)^\gamma, -(-\sup I_1^-)^\gamma], [-(-\inf F_1^-)^\gamma, -(-\sup F_1^-)^\gamma].\\
a_1^*+a_2^* &= \langle [\inf T_1^+ + \inf T_2^+ - \inf T_1^+ . \inf T_2^+, \sup T_1^+ + \sup T_2^+ - \sup T_1^+ . \sup T_2^+],\\
&\quad [\inf I_1^+ . \inf I_2^+, \sup I_1^+ . \sup I_2^+], [\inf F_1^+ . \inf F_2^+, \sup F_1^+ . \sup F_2^+],\\
&\quad [-((-\inf T_1^-).(-\inf T_2^-)), -(-\sup T_1^-).(-\sup T_2^-))],\\
&\quad [-((-\inf I_1^-)+(-\inf I_2^-)-(-\inf I_1^-).(-\inf I_2^-)),\\
&\quad -((-\sup I_1^-)+(-\sup I_2^-)-(-\sup I_1^-).(-\sup I_2^-))],\\
&\quad [-((-\inf F_1^-)+(-\inf F_2^-)-(-\inf F_1^-).(-\inf F_2^-)),\\
&\quad -((-\sup F_1^-)+(-\sup F_2^-)-(-\sup F_1^-).(-\sup F_2^-))].\\
a_1^*.a_2^* &= \langle [\inf T_1^+ . \inf T_2^+, \sup T_1^+ . \sup T_2^+],\\
&\quad [\inf I_1^+ + \inf I_2^+ - \inf I_1^+ . \inf I_2^+, \sup I_1^+ + \sup I_2^+ - \sup I_1^+ . \sup I_2^+],\\
&\quad [\inf F_1^+ + \inf F_2^+ - \inf F_1^+ . \inf F_2^+, \sup F_1^+ + \sup F_2^+ - \sup F_1^+ . \sup F_2^+],\\
&\quad [-((-\inf T_1^-)+(-\inf T_2^-)-(-\inf T_1^-).(-\inf T_2^-)),\\
&\quad -((-\sup T_1^-)+(-\sup T_2^-)-(-\sup T_1^-).(-\sup T_2^-))],\\
&\quad [-((-\inf I_1^-).(-\inf I_2^-), -((-\sup I_1^-).(-\sup I_2^-))],\\
&\quad [-((-\inf F_1^-).(-\inf F_2^-)), -((-\sup F_1^-).(-\sup F_2^-))]\rangle.
\end{aligned}
$$

Where $\gamma > 0$.

3.13. **Example.** Let $a_1^* = \langle [0.6, 0.7], [0.1, 0.2], [0.2, 0.3], [-0.2, -0.1], [-0.6, -0.5], [-0.4, -0.3]\rangle$ and

$a_2^* = \langle [0.4, 0.5], [0.3, 0.4], [0.5, 0.6], [-0.6, -0.5], [-0.3, -0.2], [-0.2, -0.1]\rangle$ be two bipolar interval neutrosophic numbers.Then for $\gamma = 2$, we have

(1) $2a_1^* = \langle [1-(1-0.6)^2, 1-(1-0.7)^2], [(0.1)^2, (0.2)^2], [(0.2)^2, (0.3)^2], [-((-(-0.2))^2), -((-(-0.1))^2)],$
$[-(1-(1-(-(-0.6)))^2), -(1-(1-(-(-0.5)))^2)], [-(1-(1-(-(-0.4)))^2), -(1-(1-(-(-0.3)))^2)]\rangle$
$= \langle [0.84, 0.91], [0.01, 0.04], [0.04, 0.09], [-0.04, -0.01], [-0.84, -0.75], [-0.64, -0.51]\rangle.$

(2) $a_1^{*2} = \langle [(0.6)^2, (0.7)^2], [1-(1-0.1)^2, 1-(1-0.2)^2], [1-(1-0.2)^2, 1-(1-0.3)^2],$
$[-(1-(1-(-(-0.2)))^2), -(1-(1-(-(-0.1)))^2)],$
$[-(-(-0.6))^2, -(-(-0.5))^2], [-(-(-0.4))^2, -(-(-0.3))^2]\rangle$
$= \langle [0.36, 0.49], [0.19, 0.36], [0.36, 0.51], [-0.36, -0.19], [-0.36, -0.25], [-0.16, -0.09]\rangle.$

(3) $a_1^*+a_2^* = \langle [0.6+0.4-(0.6).(0.4), 0.7+0.5-(0.7).(0.5)], [(0.1).(0.3), (0.2).(0.4)],$
$[(0.2)(0.5), (0.3).(0.6)], [-(-(-0.2)).-(-0.6)), -(-(-0.1).-(-0.5))],$
$[-\{-(-0.6)+(-(-0.3))-(-(-0.6)).(-(-0.3))\}, -\{-(-0.5)+(-(-0.2))-(-(0.5)).(-(-0.2))\},$
$-\{-(-0.4)+(-(-0.2))-(-(-0.4)).(-(-0.2)), -\{-(-0.3)+(-(-0.1))-(-(-0.3)).(-(-0.1)\}]\rangle$

$= \langle[0.76, 0.85], [0.03, 0.08], [0.1, 1.8], [-0.12, -0.05], [-0.72, -0.6], [-0.52, -0.37]\rangle.$
(4) $a_1^*.a_2^* = \langle[(0.6).(0.4), (0.7).(0.5)], [0.1+0.3-(0.1).(0.3), 0.2+0.4-(0.2).(0.4)],$
$[0.2 + 0.5 - (0.2).(0.5), 0.3 + 0.6 - (0.3).(0.6)],$
$[-\{-(-0.2)+(-(-0.6))-(-(0.2)).(-(-0.6))\}, -\{-(-0.1)+(-(-0.5))-(-(-0.1)).(-(-0.5))],$
$[-\{-(-0.6).(-(-0.3))\}, -\{(-(-0.5)).(-(-0.2))\}],$
$[-\{-(-0.4).(-(-0.2))\}, -\{-(-0.3).(-(-0.2))]\rangle$
$= \langle[0.24, 0.35], [0.37, 0.52], [0.6, 0.72], [-0.68, -0.55], [-0.18, -0.1], [-0.08, -0.06]\rangle.$

3.14. **Definition.** Let $a_1^* = (T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-)$ be a bipolar interval neutrosophic number. Then the sore function $S(a_1^*)$, accuracy function $A(a_1^*)$ and certainty function $C(a_1^*)$ of a bipolar interval number are defined as follows:
(1) $S(a_1^*) = \inf T_1^+ + 1 - \sup I_1^+ + 1 - \sup F_1^+ + \sup T_1^+ + 1 - \inf I_1^+ + 1 - \inf F_1^+ - 1 - (-\sup T_1^-) + \inf I_1^- + \inf F_1^- - 1 - (-\inf T_1^-) + \sup I_1^- + \sup F_1^-/6$
(2) $A(a_1^*) = \inf T_1^+ - \inf F_1^+ + \sup T_1^+ - \sup F_1^+ + \inf T_1^- - \inf F_1^- + \sup T_1^- - \sup F_1^-$
(3) $C(a_1^*) = \inf T_1^+ - \inf F_1^- + \sup T_1^+ - \sup F_1^-$.

3.15. **Definition.** Let $a_1^* = (T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-)$ and $a_2^* = (T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^-)$ be two bipolar interval neutrosophic numbers. Then the comparison method can be defined as follows:
(i) If $S(a_1^*) > S(a_2^*)$, then $a_1^* > a_2^*, i.e$ a_1^* is greater than a_2^*.
(ii) If $S(a_1^*) = S(a_2^*)$ and $A(a_1^*) > A(a_2^*)$, then $a_1^* > a_2^*, i.e$ a_1^* is greater than a_2^*.
(iii) If $S(a_1^*) = S(a_2^*)$, $A(a_1^*) = A(a_2^*)$ and $C(a_1^*) > C(a_2^*)$ then $a_1^* > a_2^*, i.e$ a_1^* is greater than a_2^*.
Obviously $S(a^*)$, $A(a^*)$ and $C(a_1^*) \in [-3, 3]$.

3.16. **Example.** Let $a_1^* = \langle[0.6, 0.7], [0.1, 0.2], [0.2, 0.3], [-0.2, -0.1], [-0.6, -0.5], [-0.4, -0.3]\rangle$ and $a_2^* = \langle[0.4, 0.5], [0.3, 0.4], [0.5, 0.6], [-0.6, -0.5], [-0.3, -0.2], [-0.2, -0.1]\rangle$ be two bipolar interval neutrosophic numbers. Then teir score, accuracy and certainity functions are calculate as
$S(a_1^*) = 0.6 + 1 - 0.2 + 1 - 0.3 + 0.7 + 1 - 0.1 + 1 - 0.2 - 1 - (-(-0.1)) + (-0.6) + (-0.4) - 1 - (-(-0.2)) + (-0.5) + (-0.3)/6$
$= 0.65$
$S(a_2^*) = 0.4 + 1 - 0.4 + 1 - 0.6 + 0.5 + 1 - 0.3 + 1 - 0.5 + 1 - (-(-0.5)) + (-0.3) + (-0.2) + 1 - (-(-0.6)) + (-0.2) + (-0.1)/6$
$= -0.8$
Which implies that $a_1^* > a_2^*$
$A(a_1^*) = 0.6 - 0.2 + 0.7 - 0.3 + (-0.2) - (-0.4) + (-0.1) - (-0.3) = 1.2$
$A(a_2^*) = 0.4 - 0.5 + 0.5 - 0.6 + (-0.6) - (-0.2) + (-0.5) - (-0.1) == -1$
Which implies that $a_1^* > a_2^*$.
$C(a_1^*) = 0.6 - (-0.4) + 0.7 - (-0.3) = 2$
$C(a_2^*) = 0.4 - (-0.2) + 0.5 - (-0.1) = 1.2$
Which implies that $a_1^* > a_2^*$.

4. SOME AGGREGATION WEIGHTED OPERATORS RELATED TO BIPOLAR INTERVAL NEUTROSOPHIC SETS.

4.1. **Definition.** Let $a_k^* = \langle T_k^+, I_k^+, F_k^+, T_k^-, I_k^-, F_k^-\rangle(k = 1, 2, 3, ..., m)$ be a family of $BINNs$.Then a mapping $BINNWA : BINN^m \longrightarrow BINN$ is called a bipolar interval neutrosophic weighted awerage operator if it satisfies:

$$\begin{aligned} BINNWA_{\omega}(a_1^*, a_2^*, ...a_m^*) &= \omega_1 a_1^* + \omega_2 a_2^* + ... + \omega_m a_m^* \\ &= \Sigma_{k=1}^{m} \omega_k a_k^* \end{aligned}$$

Where $W = (\omega_{1,}\omega_2, ..., \omega_k)$ is the weight vector of $a_k^*(k = 1, 2, 3, ..., m)$ and $\Sigma_{k=1}^{m}\omega_k = 1$.

4.2. **Theorem.** Let $a_k^* = \langle T_k^+, I_k^+, F_k^+, T_k^-, I_k^-, F_k^- \rangle (k = 1, 2, 3, ..., m)$ be a family of $BINNs$ and $W = (\omega_{1,}\omega_2, ..., \omega_k)$ is the weight vector of $a_k^*(k = 1, 2, 3, ..., m)$ and $\Sigma_{k=1}^{m}\omega_k = 1$. Then their aggregated result using the $BINNWA_{\omega}$ is also a $BINN.and$

$BINNWA_{\omega}(a_1^*, a_2^*, ...a_m^*) =$

$$\left\langle \begin{array}{c} [1 - \Pi_{k=1}^{m}(1 - \inf T_{a_k}^{+})^{\omega_k}, 1 - \Pi_{k=1}^{m}(1 - \sup T_{a_k}^{+})^{\omega_k}], [\Pi_{k=1}^{m} \inf I_{a_k}^{+\omega_k}, \Pi_{k=1}^{m} \sup I_{a_k}^{+\omega_k}], \\ [\Pi_{k=1}^{m} \inf F_{a_k}^{+\omega_k}, \Pi_{k=1}^{m} \sup F_{a_k}^{+\omega_k}], [-(\Pi_{k=1}^{m} - \inf T_{a_k}^{-\omega_k}), -(\Pi_{k=1}^{m} - \sup T_{a_k}^{-\omega_k})], \\ [-(1 - \Pi_{k=1}^{m}(1 - (-\inf I_{a_k}^{-\omega_k})), -(1 - \Pi_{k=1}^{m}(1 - (-\sup I_{a_k}^{-\omega_k}))] \\ [-(1 - \Pi_{k=1}^{m}(1 - (-\inf F_{a_k}^{-\omega_k})), -(1 - \Pi_{k=1}^{m}(1 - (-\sup F_{a_k}^{-\omega_k}))]. \end{array} \right\rangle$$

We prove the result by mathematical induction, for $k = 2$

$$
\begin{aligned}
BINNWA(a_1^*, a_2^*) &= \omega_1 a_1^* + \omega_2 a_2^* \\
&= \langle [1-(1-\inf T^{+}_{a_1^*})^{\omega_1}, 1-(1-\sup T^{+}_{a_1^*})^{\omega_1}], [\inf I^{+\omega_1}_{a_1^*}, \sup I^{+\omega_1}_{a_1^*}], \\
&\quad [\inf F^{+\omega_1}_{a_1^*}, \sup F^{+\omega_1}_{a_1^*}], [-(-\inf T^{-\omega_1}_{a_1^*}), -(-\sup T^{-\omega_1}_{a_1^*})], \\
&\quad [-(1-(1-(-\inf I^{-\omega_1}_{a_1^*})), -(1-(1-(-\sup I^{-\omega_1}_{a_1^*}))] \\
&\quad [-(1-(1-(-\inf F^{-\omega_1}_{a_1^*})), -(1-(1-(-\sup F^{-\omega_1}_{a_1^*}))] \rangle + \\
&\quad \langle [1-(1-\inf T^{+}_{a_2^*})^{\omega_2}, 1-(1-\sup T^{+}_{a_2^*})^{\omega_2}], [\inf I^{+\omega_2}_{a_2^*}, \sup I^{+\omega_2}_{a_2^*}], \\
&\quad [\inf F^{+\omega_2}_{a_2^*}, \sup F^{+\omega_2}_{a_2^*}], [-(-\inf T^{-\omega_2}_{a_2^*}), -(-\sup T^{-\omega_2}_{a_2^*})], \\
&\quad [-(1-(1-(-\inf I^{-\omega_2}_{a_2^*})), -(1-(1-(-\sup I^{-\omega_2}_{a_2^*}))], \\
&\quad [-(1-(1-(-\inf F^{-\omega_2}_{a_2^*})), -(1-(1-(-\sup F^{-\omega_2}_{a_2^*}))] \rangle \\
&\quad \langle [1-(1-\inf T^{+}_{a_1^*})^{\omega_1} + 1-(1-\inf T^{+}_{a_2^*})^{\omega_2} - \\
&\quad (1-(1-\inf T^{+}_{a_1^*})^{\omega_1})(1-(1-\inf T^{+}_{a_2^*})^{\omega_2}), \\
&\quad 1-(1-\sup T^{+}_{a_1^*})^{\omega_1} + 1-(1-\sup T^{+}_{a_2^*})^{\omega_2} - \\
&\quad (1-(1-\sup T^{+}_{a_1^*})^{\omega_1})(1-(1-\sup T^{+}_{a_2^*})^{\omega_2})] \\
&\quad [\inf I^{+\omega_1}_{a_1^*} . \inf I^{+\omega_2}_{a_2^*}, \sup I^{+\omega_1}_{a_1^*} . \sup I^{+\omega_2}_{a_2^*}], [\inf F^{+\omega_1}_{a_1^*} . \inf F^{+\omega_2}_{a_2^*}, \sup F^{+\omega_1}_{a_1^*} . \sup F^{+\omega_2}_{a_2^*}] \\
&\quad [-\{(-\inf T^{-\omega_1}_{a_1^*}).(-\inf T^{-\omega_2}_{a_2^*})\}, -\{(-\sup T^{-\omega_1}_{a_1^*}).(-\sup T^{-\omega_2}_{a_2^*})] \\
&\quad [-\{1-(1-(-\inf I^{-\omega_1}_{a_1^*})) + 1-(1-(-\inf I^{-\omega_2}_{a_2^*})) \\
&\quad -(1-(1-(-\inf I^{-\omega_1}_{a_1^*}))).(1-(1-(-\inf I^{-\omega_2}_{a_2^*}))\}, \\
&\quad -\{1-(1-(-\sup I^{-\omega_1}_{a_1^*})) + (1-(1-(-\sup I^{-\omega_2}_{a_2^*})) - \\
&\quad (1-(1-(-\sup I^{-\omega_1}_{a_1^*})) + (1-(1-(-\sup I^{-\omega_2}_{a_2^*}))\}] \\
&\quad [-\{(1-(1-(-\inf F^{-\omega_1}_{a_1^*})) + (1-(1-(-\inf F^{-\omega_2}_{a_2^*})) \\
&\quad -((1-(1-(-\inf F^{-\omega_1}_{a_1^*}))(1-(1-(-\inf F^{-\omega_2}_{a_2^*}))\}, \\
&\quad -\{(1-(1-(-\sup F^{-\omega_1}_{a_1^*})) + (1-(1-(-\sup F^{-\omega_2}_{a_2^*})) - \\
&\quad ((1-(1-(-\sup F^{-\omega_1}_{a_1^*})).(1-(1-(-\sup F^{-\omega_2}_{a_2^*}))] \rangle \\
&= \langle 1-(-\inf T^{-\omega_1}_{a_1^*}).(-\inf T^{-\omega_2}_{a_2^*}), 1-(-\sup T^{-\omega_1}_{a_1^*}).(-\sup T^{-\omega_2}_{a_2^*})], \\
&\quad [\inf I^{+\omega_1}_{a_1^*} . \inf I^{+\omega_2}_{a_2^*}, \sup I^{+\omega_1}_{a_1^*} . \sup I^{+\omega_2}_{a_2^*}], \\
&\quad [\inf F^{+\omega_1}_{a_1^*} . \inf F^{+\omega_2}_{a_2^*}, \sup F^{+\omega_1}_{a_1^*} . \sup F^{+\omega_2}_{a_2^*}], \\
&\quad [-\{(-\inf T^{-\omega_1}_{a_1^*}).(-\inf T^{-\omega_2}_{a_2^*})\}, -\{(-\sup T^{-\omega_1}_{a_1^*}).(-\sup T^{-\omega_2}_{a_2^*})], \\
&\quad [-\{1-((-\inf I^{-\omega_1}_{a_1^*})(-\inf I^{-\omega_2}_{a_2^*}))\}, -\{1-((-\sup I^{-\omega_1}_{a_1^*})(-\sup I^{-\omega_2}_{a_2^*}))], \\
&\quad [-\{1-((-\inf F^{-\omega_1}_{a_1^*})(-\inf F^{-\omega_2}_{a_2^*}))\}, -\{1-((-\sup F^{-\omega_1}_{a_1^*})(-\sup F^{-\omega_2}_{a_2^*}))] \rangle.
\end{aligned}
$$

Hence the result is true for $k = 2$. Now assume that the result is true for $k = m$. We would like to prove that the result is true for $k = m + 1$.

$BINNWA_\omega(a_1^*, a_2^*, ...a_m^*, a_{m+1}^*) =$

$$\begin{gathered}
[1 - \Pi_{k=1}^{m}(1 - \inf T_{a_k}^{+})^{\omega_k} + 1 - (1 - \inf T_{a_{m+1}}^{+})^{\omega_{m+1}} - (1 - \Pi_{k=1}^{m}(1 - \inf T_{a_k}^{+})^{\omega_k})(1 - (1 - \inf T_{a_{m+1}}^{+})^{\omega_{m+1}}),\\
1 - \Pi_{k=1}^{m}(1 - \sup T_{a_k}^{+})^{\omega_k} + 1 - (1 - \sup T_{a_{m+1}}^{+})^{\omega_{m+1}} - (1 - \Pi_{k=1}^{m}(1 - \sup T_{a_k}^{+})^{\omega_k})(1 - (1 - \sup T_{a_{m+1}}^{+})^{\omega_{m+1}}\\
[\Pi_{k=1}^{m} \inf I_{a_k}^{+\omega_k} + \inf I_{a_{m+1}}^{+\omega_{m+1}}, \Pi_{k=1}^{m} \sup I_{a_k}^{+\omega_k} + \sup I_{a_{m+1}}^{+\omega_{m+1}}],\\
[\Pi_{k=1}^{m} \inf F_{a_k}^{+\omega_k} + \inf F_{a_{m+1}}^{+\omega_{m+1}}, \Pi_{k=1}^{m} \sup F_{a_k}^{+\omega_k} + \sup F_{a_{m+1}}^{+\omega_{m+1}}],\\
[-\{(\Pi_{k=1}^{m} - \inf T_{a_k}^{-\omega_k}) + (-\inf T_{a_{m+1}}^{-\omega_{m+1}}), -\{(\Pi_{k=1}^{m} - \sup T_{a_k}^{-\omega_k}) + (-\sup T_{a_{m+1}}^{-\omega_{m+1}})],\\
[-\{1 - \Pi_{k=1}^{m}(1 - (-\inf I_{a_k}^{-\omega_k})) + (1 - (1 - (-\inf I_{a_{m+1}}^{-\omega_{m+1}})))\\
-(1 - \Pi_{k=1}^{m}(1 - (-\inf I_{a_k}^{-\omega_k}))(1 - (1 - (-\inf I_{a_{m+1}}^{-\omega_{m+1}}))\},\\
-\{1 - \Pi_{k=1}^{m}(1 - (-\sup I_{a_k}^{-\omega_k})) + (1 - (1 - (-\sup I_{a_{m+1}}^{-\omega_{m+1}})))\\
-(1 - \Pi_{k=1}^{m}(1 - (-\sup I_{a_k}^{-\omega_k}))(1 - (1 - (-\sup I_{a_{m+1}}^{-\omega_{m+1}}))\}]\\
-\{1 - \Pi_{k=1}^{m}(1 - (-\inf F_{a_k}^{-\omega_k})) + 1 - (1 - (-\inf F_{a_{m+1}}^{-\omega_{m+1}}))\\
-(1 - \Pi_{k=1}^{m}(1 - (-\inf F_{a_k}^{-\omega_k}))(1 - (1 - (-\inf F_{a_{m+1}}^{-\omega_{m+1}})),\\
-\{1 - \Pi_{k=1}^{m}(1 - (-\sup F_{a_k}^{-\omega_k})) + 1 - (1 - (-\sup F_{a_{m+1}}^{-\omega_{m+1}}))\\
-(1 - \Pi_{k=1}^{m}(1 - (-\sup F_{a_k}^{-\omega_k}))(1 - (1 - (-\sup F_{a_{m+1}}^{-\omega_{m+1}}))].
\end{gathered}$$

$$= \begin{gathered}
[1 - \Pi_{k=1}^{m+1}(1 - \inf T_{a_k}^{+})^{\omega_k}, 1 - \Pi_{k=1}^{m+1}(1 - \sup T_{a_k}^{+})^{\omega_k}], [\Pi_{k=1}^{m+1} \inf I_{a_k}^{+\omega_k}, \Pi_{k=1}^{m+1} \sup I_{a_k}^{+\omega_k}],\\
[\Pi_{k=1}^{m+1} \inf F_{a_k}^{+\omega_k}, \Pi_{k=1}^{m+1} \sup F_{a_k}^{+\omega_k}], [-(\Pi_{k=1}^{m+1} - \inf T_{a_k}^{-\omega_k}), -(\Pi_{k=1}^{m+1} - \sup T_{a_k}^{-\omega_k})],\\
[-(1 - \Pi_{k=1}^{m+1}(1 - (-\inf I_{a_k}^{-\omega_k})), -(1 - \Pi_{k=1}^{m+1}(1 - (-\sup I_{a_k}^{-\omega_k}))]\\
-(1 - \Pi_{k=1}^{m+1}(1 - (-\inf F_{a_k}^{-\omega_k})), -(1 - \Pi_{k=1}^{m+1}(1 - (-\sup F_{a_k}^{-\omega_k}))].
\end{gathered}$$

This completes the proof.

4.3. **Theorem.** Let $a_k^* = \langle T_k^+, I_k^+, F_k^+, T_k^-, I_k^-, F_k^- \rangle (k = 1, 2, 3, ..., m)$ be a family of $BINNs$. Then,

(1) if $a_k^* = a^*$ for all $k = 1, 2, ..., m$ then $WA_\omega(a_1^*, a_2^*, ..., a_m^*) = a^*$

(2) if $\min_{k=1,2,..,m}\{a_k^*\} \le WA_\omega(a_1^*, a_2^*, ..., a_m^*) \le \max_{k=1,2,..,m}\{a_k^*\}$

(3) if $a_k^* \le a_k^{**}$ for all $k = 1, 2, ..., m$ then $WA_\omega(a_1^*, a_2^*, ..., a_m^*) \le WA_\omega(a_1^{**}, a_2^{**}, ..., a_m^{**})$.

Proof. Straight forward. □

4.4. **Definition.** Let $a_k^* = \langle T_k^+, I_k^+, F_k^+, T_k^-, I_k^-, F_k^- \rangle (k = 1, 2, 3, ..., m)$ be a family of $BINNs$.Then a mapping $BINNWG : BINN^m \longrightarrow BINN$ is called a bipolar interval neutrosophic weighted geometric($BINNWG_\omega$) operator if it satisfies:

$$\begin{aligned}
BINNWG_\omega(a_1^*, a_2^*, ...a_m^*) &= a_1^{\omega_1 *} + a_2^{\omega_2 *} + ... + a_m^{\omega_m *}\\
&= \Sigma_{k=1}^{m} a_k^{\omega_k *}
\end{aligned}$$

Where $W = (\omega_1, \omega_2, ..., \omega_k)^T$ is the weight vector of $a_k^*(k = 1, 2, 3, ..., m)$ and $\Sigma_{k=1}^{m} \omega_k = 1$.

4.5. **Theorem.** Let $a_k^* = \langle T_k^+, I_k^+, F_k^+, T_k^-, I_k^-, F_k^- \rangle (k = 1, 2, 3, ..., m)$ be a family of $BINNs$. Then,

(1) if $a_k^* = a^*$ for all $k = 1, 2, ..., m$ then $WG_\omega(a_1^*, a_2^*, ..., a_m^*) = a^*$

(2) if $\min_{k=1,2,..,m}\{a_k^*\} \le WG_\omega(a_1^*, a_2^*, ..., a_m^*) \le \max_{k=1,2,..,m}\{a_k^*\}$

(3) if $a_k^* \le a_k^{**}$ for all $k = 1, 2, ..., m$ then $WG_\omega(a_1^*, a_2^*, ..., a_m^*) \le GA_\omega(a_1^{**}, a_2^{**}, ..., a_m^{**})$.

4.6. **Theorem.** Let $a_k^* = \langle T_k^+, I_k^+, F_k^+, T_k^-, I_k^-, F_k^- \rangle (k = 1, 2, 3, ..., m)$ be a family of $BINNs$ and $W = (\omega_{1,}\omega_2, ..., \omega_k)$ is the weight vector of $a_k^* (k = 1, 2, 3, ..., m)$ and $\Sigma_{k=1}^m \omega_k = 1$. Then their aggregated result using the $BINNWA_\omega$ is also a $BINN.and$

$$BINNWA_\omega(a_1^*, a_2^*, ...a_m^*) =$$

$$\left\langle \begin{array}{c} [\Pi_{k=1}^m (\inf T_{a_k}^+)^{\omega_k}, \Pi_{k=1}^m (\sup T_{a_k}^+)^{\omega_k}], [1 - \Pi_{k=1}^m (1 - \inf I_{a_k}^{+\omega_k}), 1 - \Pi_{k=1}^m (1 - \sup I_{a_k}^{+\omega_k})], \\ [1 - \Pi_{k=1}^m (1 - \inf F_{a_k}^{+\omega_k}), 1 - \Pi_{k=1}^m (1 - \sup F_{a_k}^{+\omega_k}], \\ [-(1 - \Pi_{k=1}^m (1 - (-\inf T_{a_k}^{-\omega_k})), -(1 - \Pi_{k=1}^m (1 - (-\sup T_{a_k}^{-\omega_k}))], \\ [-(\Pi_{k=1}^m (-\inf I_{a_k}^{-\omega_k})), -(\Pi_{k=1}^m (-\sup I_{a_k}^{-\omega_k}))], [-(\Pi_{k=1}^m (-\inf F_{a_k}^{-\omega_k})), -(\Pi_{k=1}^m ((-\sup F_{a_k}^{-\omega_k}))]. \end{array} \right\rangle$$

Proof. same as 4.2 □

5. BINNs- DECISION MAKING Method

In this section we propose a multi-criteria decision making method for bipolar interval neutrosophic numbers with weights known.

Suppose that $a_k^* = (a_1^*, a_2^*, ...a_m^*)$ be a set of alternatives and $C^* = (c_1^*, c_2^*, ...c_n^*)$ be a set of attributes or criteria. Let $W = (\omega_{1,}\omega_2, ..., \omega_n)$ be the weight vectors of the criteria, such that $\Sigma_{i=1}^n \omega_i = 1, \omega_i \geq 0 (i = 1, 2, 3, ..., n)$ and ω_i mentions to the weight of criteria c_i^*. An alternative is evaluated by the decision makers on criterions and the values evaluated by the decision makers are represented by bipolar interval neutrosophic numbers. Let us suppose that the weight of the attributes C_l $(l = 1, 2, 3, ..., n)$ by the decesiom makers is $\omega_l, \omega_l \in [0, 1]$. Thus representative of the alternative a_k^* $(k = 1, 2, 3, ..., m)$ is given by the following $BINNs$ respectively.

$a_k^* = \{\langle C_l, [\inf T_{a_k^*}^+(C_l), \sup T_{a_k^*}^+(C_l)], [\inf I_{a_k^*}^+(C_l), \sup I_{a_k^*}^+(C_l)], [\inf F_{a_k^*}^+(C_l), \sup F_{a_k^*}^+(C_l)],$
$[\inf T_{a_k^*}^-(C_l), \sup T_{a_k^*}^-(C_l)], [\inf I_{a_k^*}^-(C_l), \sup I_{a_k^*}^-(C_l)], [\inf F_{a_k^*}^-(C_l), \sup F_{a_k^*}^-(C_l)]\rangle$

Where $0 \leq \sup T_{a_k^*}^+(C_l) + \sup I_{a_k^*}^+(C_l) + \sup F_{a_k^*}^+(C_l) \leq 3$ and $-3 \leq \sup T_{a_k^*}^-(C_l) + \sup I_{a_k^*}^-(C_l) + \sup F_{a_k^*}^-(C_l) \leq 0$. The $BINN$ is a six-tuple for C_l is denoted by $\beta_{kl} = ([r_{kl}, r_{kl}^*], [s_{kl}, s_{kl}^*], [t_{kl}, t_{kl}^*], [u_{kl}, u_{kl}^*], [v_{kl}, v_{kl}^*], [w_{kl}, w_{kl}^*])$.

Where $[r_{kl}, r_{kl}^*]$ represent the degree of truth-membership for alternative a_k^* for the attributes C_l, and similarly the other represents thier degree respectively. so that we can represent a decision making matrix $(a_{kl}^*)_{m \times n}$.

Now we represent an algoritham as follows:

Step (1)

Construct the decision matrix provided by the decision maker as;

$(a_{kl}^*)_{m \times n} = (\langle T_{kl}^+, I_{kl}^+, F_{kl}^+, T_{kl}^-, I_{kl}^-, F_{kl}^- \rangle)_{m \times n}$

Step (2)

Compute $a_k^* = WA_\omega(a_{k1}^*, a_{k2}^*, ..., a_{kn}^*)$ for all $k = 1, 2, ..., m$.

Step(3)

Calculate the scores values of $S(a_k^*)$, or accuracy function $A(a_k^*)$ or certainty function $C(a_k^*)(k = 1, 2, ..., m)$ for the collective overall $BINNs$ of a_k^*.

Step (4)

Rank the alternative according to their scores values.

Now we give a numerical example.

5.1. **Example.** Let us consider a decision making problem adapted from [17]. suppose that there is a panel with four possible alternatives to invest money;

(1) a_1^* is a food company; (2) a_2^* is a car company; (3) a_3^* is an arm company; (4) a_4^* is computer company. The investment company must make a decision according to the three criteria given below:

(1) C_1 is the growth analysis; (2) C_2 is risk analysis; (3) C_3 is the enviromental impact analysis. By using the bipolar interval neutrosophic information , the decision maker has evaluated the four possible alternatives under the above three criteria and his listed in the following matrix:

$$\begin{array}{cc} & C_1 \\ \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_4^* \end{pmatrix} & \begin{bmatrix} ([0.4,0.5],[0.2,0.3],[0.3,0.4],[-0.3,-0.2],[-0.4,-0.3],[-0.5,-0.4] \\ [0.6,0.7],[0.1,0.2],[0.2,0.3],[-0.2,-0.1],[-0.3,-0.2],[-0.7,-0.6] \\ [0.3,0.6],[0.2,0.3],[0.3,0.4],[-0.3,-0.2],[-0.4,-0.3],[-0.6,-0.3] \\ [0.7,0.8],[0.0,0.1],[0.1,0.2],[-0.1,0.0],[-0.2,-0.1],[-0.8,-0.7] \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & C_2 \\ \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_4^* \end{pmatrix} & \begin{bmatrix} [0.4,0.6],[0.1,0.3],[0.2,0.4],[-0.3,-0.1],[-0.4,-0.2],[-0.6,-0.4] \\ [0.6,0.7][0.1,0.2],[0.2,0.3],[-0.2,-0.1],[-0.3,-0.2],[-0.7,-0.6] \\ [0.5,0.6],[0.2,0.3],[0.3,0.4],[-0.3,-0.2],[-0.4,-0.3],[-0.6,-0.5] \\ [0.6,0.7],[0.1,0.2],[0.1,0.3],[-0.2,-0.1],[-0.3,-0.1],[-0.7,-0.6] \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & C_3 \\ \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_4^* \end{pmatrix} & \begin{bmatrix} [0.7,0.9],[0.2,0.3],[0.4,0.5],[-0.3,-0.2],[-0.5,-0.4],[-0.9,-0.7] \\ [0.3,0.6],[0.3,0.5],[0.8,0.9],[-0.5,-0.3],[-0.9,-0.8],[-0.6,-0.3] \\ [0.4,0.5],[0.2,0.4],[0.7,0.9],[-0.4,-0.2],[-0.9,-0.7],[-0.5,-0.4] \\ [0.6,0.7],[0.3,0.4],[0.8,0.9],[-0.4,-0.3],[-0.9,-0.8],[-0.7,-0.6] \end{bmatrix} \end{array}$$

Suppose the weights of C_1, C_2 and C_3 are $0.35, 0.25$ and 0.4. Then we use the approach developed to obtain the most desirable alternative.

Step(2)

$\beta_1 = \langle [0.5453, 0.9742], [0.1681, 0.2999], [0.3041, 0.4176],$
$[-0.2999, -0.1681], [-0.4423, -0.3196], [-0.7516, -0.5453]\rangle.$

$\beta_2 = \langle [0.9960, 0.9980], [0.1551, 0.2885], [0.3481, 0.4654],$
$[-0.2885, -0.1551], [-0.9961, -0.9875], [-0.9815, -0.9925]\rangle$

$\beta_3 = \langle [0.9832, 0.9952], [0.3141, 0.3365], [0.4209, 0.5531],$
$[-0.3365, -0.3141], [-0.9976, -0.9881], [-0.9952, -0.9832]\rangle$

$\beta_4 = \langle [0.9974, 0.9991], [0, 0.206], [0.2296, 0.4038],$
$[-0.2070, 0], [-0.9953, -0.979], [-0.9991, -0.9974]\rangle.$

Step(3)

$S(\beta_1) = 0.5453 + 1 - 0.2999 + 1 - 0.4176 + 0.9742 + 1 - 0.1681 + 1 - 0.3041 - 1 - (-(-0.1681)) + (-0.4423) + (-0.7516) - 1 - (-(-0.2999)) + (-0.3196) + (-0.5453)/6$
$= 0.2574$

$S(\beta_2) = 0.9960 + 1 - 0.2885 + 1 - 0.4654 + 0.9980 + 1 - 0.1551 + 1 - 0.3481 - 1 - (-(-0.1551)) + (-0.9961) + (-0.9815) - 1 - (-(-0.2885)) + (-0.9875) + (-0.9925)/6$
$= -0.837$

$S(\beta_3) = 0.9832 + 1 - 0.3365 + 1 - 0.5531 + 0.9952 + 1 - 0.3141 + 1 - 0.4209 - 1 - (-(-0.3141)) + (-0.9976) + (-0.9952) - 1 - (-(-0.3365)) + (-0.9881) + (-0.9832)/6$
$= -0.454$

$S(\beta_4) = 0.9974 + 1 - 0.206 + 1 - 0.4038 + 0.9991 + 1 - 0 + 1 - 0.2296 - 1 - (-(0)) + (-0.9953) + (-0.9991) - 1 - (-(-0.2070)) + (-0.979) + (-0.9974)/6$
$= 0.805$

Step (4)

Rank all the alternatives according to their score function to Select the best alternative. According to score best alternative is

$a_4^* \succ a_1^* \succ a_3^* \succ a_2^*$

THus a_4^* is the most desirable based on Weighted arithematic average operator.

Similarly we show for Weighted geometric operator.

6. Conclusion

In this paper we combine Bipolar valued fuzzy sets and interval neutrosophic set to introduce the concept of bipolar interval neutrosophic sets. We define the union, complement, intersection and containement of bipolar interval neutrosophic sets. We also define some basic operation on bipolar interval neutrosophic set (BINNs). Also we defined bipolar interval neutrosophic weighted averaging operator (BINWA) and bipolar interval neutrosophic weighted geometric operator (BINWG) and estiblished a multi-attribute decision making method (MADM) for bipolar interval neutrosophic numbers (BINSs). At the end we give a numerical example to show the affectiveness of the proposed method for bipolar interval neutrosophic numbers (BINNs).In future we shall work to define some new operators for BINSs and applied it to multi-attribute decision making (MADM).

References

[1] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy sets and Systems, 20(1), 87-96.

[2] Bustince, H., & Burillo, P. (1996). Vague sets are intuitionistic fuzzy sets. Fuzzy sets and systems, 79(3), 403-405.

[3] Broumi, S., & Smarandache, F. (2013, December). Correlation coefficient of interval neutrosophic set. In Applied Mechanics and Materials (Vol. 436, pp. 511-517).

[4] Deli, I., Ali, M., & Smarandache, F. (2015). Bipolar Neutrosophic Sets And Their Application Based On Multi-Criteria Decision Making Problems. arXiv preprint arXiv:1504.02773.

[5] Gau, W. L., & Buehrer, D. J. (1993). Vague sets. IEEE transactions on systems, man, and cybernetics, 23(2), 610-614.

[6] Lee, K. M. (2000). Bipolar-valued fuzzy sets and their operations. In Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (Vol. 307312).

[7] Smarandache, F. (1999). A Unifying Field in Logics: Neutrosophic Logic. Philosophy, 1-141.

[8] Şahin, R. (2014). Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment. arXiv preprint arXiv:1412.5202.

[9] Turksen, I. B. (1986). Interval valued fuzzy sets based on normal forms. Fuzzy sets and systems, 20(2), 191-210.

[10] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single valued neutrosophic sets. Review, 10.

[11] Wang, H., Smarandache, F., Sunderraman, R., & Zhang, Y. Q. (2005). Interval Neutrosophic Sets and Logic: Theory and Applications in Computing: Theory and Applications in Computing (Vol. 5).

[12] Ye, J. (2013). Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems, 42(4), 386-394.

[13] Ye, J. (2013). Another form of correlation coefficient between single valued neutrosophic sets and its multiple attribute decision-making method. Neutrosophic Sets and Systems, 1(1), 8-12.

[14] Ye, J. (2014). Single valued neutrosophic cross-entropy for multicriteria decision making problems. Applied Mathematical Modelling, 38(3), 1170-1175.

[15] Ye, J. (2014). Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. Journal of Intelligent and fuzzy Systems, 26(1), 165-172.

[16] Ye, J. (2014). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology, 26(5), 2459-2466.
[17] Ye, J. (2014). Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making. International Journal of Fuzzy Systems, 16(2), 204-215.
[18] Ye, J. (2015). Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. Artificial intelligence in medicine, 63(3), 171-179.
[19] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3), 338-353.

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