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Cross-entropy measure on interval neutrosophic sets and its applications in Multicriteria decision making

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ABSTRACT: The neutrosophic set model is an important tool for dealing with real scientific and engineering applications because it can handle not only incomplete information but also the inconsistent information and indeterminate information which exist commonly in real situations. In this paper, we firstly propose two practical methods to convert the interval neutrosophic sets into fuzzy sets and single valued neutrosophic sets, respectively. Then we define the interval neutrosophic cross-entropy in two different ways, which are based on extension of fuzzy cross-entropy and the single valued neutrosophic cross-entropy. Additionally, a multi-criteria decision-making method (MCDM) using the interval neutrosophic cross-entropy measure between an alternative and the ideal alternative is developed in order to determine the order of the alternatives and choose most preferred one(s). Finally, an illustrative example is presented to verify the proposed approach and to demonstrate its effectiveness and practicality.

1. Introduction

Because of different types of uncertainties in real world, there are many of mathematics tools for handling incomplete, indeterminate and inconsistent information. Zadeh ([30]) proposed theory of fuzzy set which is applied successfully in various fields. Turksen ([21]) introduced the interval valued fuzzy set to overcome the difficulty to define the grade of membership of fuzzy set in some cases. Atanassov ([1]) proposed the concept of intuitionistic fuzzy set, which is a generalization of the concept of fuzzy set. Atanassov ([2],[3]) further generalized the intuitionistic fuzzy set theory and introduced the notion of interval-valued intuitionistic fuzzy set by combining the intuitionistic fuzzy set with interval valued fuzzy set, which is characterized by a membership function and a non-membership function whose values are intervals rather than real numbers. Then Smarandache ([18],[19]) defined the neutrosophic sets which are more general than the aforementioned sets. In a neutrosophic set, each element of the universe has a degree of truth, indeterminacy and falsity respectively. The indeterminacy value of neutrosophic sets is independent of truth and falsity values while the indeterminacy or hesitation margin is dependent of the degree of belongingness and degree of non-belongingness in the case intuitionistic fuzzy set. Neutrosophic sets have been successful used in image processing field, such as image thresholding and image segmentation ([7],[9]). But, a neutrosophic set will be difficult to apply in real scientific and engineering fields. Therefore, Wang et al. ([24],[25]) proposed the concepts of single valued neutrosophic set and interval neutrosophic set, which are an instance of a neutrosophic set, and provided them set-theoretic operators and various properties. Then, single valued neutrosophic sets present uncertainty, imprecise, inconsistent and incomplete information existing in real world. Also, it would be more suitable to handle indeterminate information and inconsistent information. Recently, some researchers have shown great interest in the single valued neutrosophic set theory and applied it to the field of decision making. Ye ([26]) proposed correlation coefficients between single valued neutrosophic sets and applied them to multiple attribute decision-making problems with single valued neutrosophic information. Furthermore, Ye ([29]) introduced the concept of a simplified

1
2 neutrosophic set, which is a subclass of a neutrosophic set and includes the concepts of interval neutrosophic
3 set and single valued neutrosophic set, and defined some operational laws of simplified neutrosophic sets,
4 and then he proposed simplified neutrosophic weighted averaging operator and simplified neutrosophic
5 weighted geometric operator and applied them to multi-criteria decision-making problems under the
6 simplified neutrosophic environment. Peng et al. ([13]) point out that some operations defined by Ye ([29])
7 may also be invalid. They defined the novel operations and aggregation operators and applied them to multi
8 criteria decision making problems. Liu and Wang ([10]) defined a single-valued neutrosophic normalized
9 weighted Bonferroni mean operator. Şahin and Küçük ([17]) proposed the concept of neutrosophic
10 subsethood based on distance measure for single valued neutrosophic sets. Recently, Peng et al.
11 ([14]) defined the multi-valued neutrosophic sets and proposed two aggregation operators such that
12 weighted average operator and the weighted geometric operator for solving a multi criteria decision making
13 problem.
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17 On the other hand, since the degree of truth, falsity and indeterminacy about a certain statement can not be
18 defined exactly in the real situations, Wang et al. ([25]) generalized the concept of single valued
19 neutrosophic set to the interval neutrosophic set which is characterized by the degree of truth, falsity and
20 indeterminacy whose values are intervals rather than real numbers. After the pioneering work of Wang et al.
21 ([25]), the interval neutrosophic set has received much attention in the literature. Ye ([27]) introduced the
22 Hamming and Euclidean distances between interval neutrosophic sets and the distances-based similarity
23 measures and applied them to multiple attribute decision-making problems with interval neutrosophic
24 information. Broumi and Smarandache ([4]) proposed correlation coefficients between interval valued
25 neutrosophic sets. Liu and shi ([11]) combined the interval neutrosophic sets and interval-valued hesitant
26 fuzzy sets, and propose the concept of the interval neutrosophic hesitant sets. Also they developed some new
27 aggregation operators for the interval neutrosophic hesitant fuzzy information, including interval
28 neutrosophic hesitant fuzzy generalized weighted operator, interval neutrosophic hesitant fuzzy generalized
29 ordered weighted operator, and interval neutrosophic hesitant fuzzy generalized hybrid weighted operator,
30 and discussed some properties. Zhang et al. ([32]) introduced two interval neutrosophic aggregation
31 operators and applied them to multicriteria decision-making problems with interval neutrosophic
32 information.
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37 Entropy, as a very important notion for measuring fuzziness degree or uncertain information in fuzzy set
38 theory, has received great attention in the past decades. In 1968 ([31]), Zadeh first introduced the entropy of
39 fuzzy event. Later, Deluca and Termini ([8]) presented some axioms to describe the fuzziness degree of
40 fuzzy set, with which a fuzzy entropy based on Shannon's function ([16]) was proposed. After that, many
41 other researchers have studied the fuzzy entropy in different ways. Bhandari and Pal ([6]) made a survey on
42 information measures on fuzzy sets and gave some new measures of fuzzy entropy. Burillo and Bustince
43 ([5]) presented an entropy on interval-valued fuzzy sets and intuitionistic fuzzy sets. Szmidt and Kacprzyk
44 ([20]) proposed an entropy measure for intuitionistic fuzzy set by employing a geometric interpretation of
45 intuitionistic fuzzy sets. Wei et al. ([23]) discussed an entropy measure for interval-valued intuitionistic
46 fuzzy sets, which generalizes entropy measures defined for intuitionistic fuzzy sets. Majumdar et al. ([12])
47 used the entropy to solve a multi criteria decision making problem under single valued neutrosophic
48 environment. Cross-entropy is used to measure the discrimination information between objects, according to
49 Shannon's inequality ([16]). Shang and Jiang ([15]) defined the fuzzy cross-entropy between two fuzzy
50 sets. Vlachos and Sergiadis ([22]) gave a definition of intuitionistic fuzzy cross entropy and proved a
51 mathematical connection between the notions of entropy for fuzzy sets and intuitionistic fuzzy sets in terms
52 of fuzziness and intuitionism. For interval-valued fuzzy set, Zhang ([33]) defined the concepts of entropy
53 and cross-entropy for interval valued intuitionistic sets and discussed its some properties. Ye ([28])
54 proposed the cross-entropy on single valued neutrosophic sets and defined the concept of single valued
55 neutrosophic cross-entropy.
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2 Although some distance measures, similarity measures and correlation measures of interval neutrosophic sets
3 have been presented in ([4], [12], [27], [32]) recently, there are no investigation on the entropy and cross-
4 entropy measures of interval neutrosophic sets. As we know, characterization of uncertainty for interval
5 neutrosophic sets are important issues that affect the management of uncertainty information in most real-
6 world system models involving imperfect information with interval neutrosophic set values. Thus, it focuses
7 us to study an useful cross-entropy measure of interval neutrosophic set for the possible applications in many
8 real-life areas such as pattern recognition, cluster analysis, image segment. In this paper, we firstly propose
9 two practical methods to convert the interval neutrosophic sets into fuzzy sets and single valued neutrosophic
10 sets. In addition, a cross-entropy of interval neutrosophic set is presented to determine the information
11 measure for discrimination between two interval neutrosophic sets. Finally, an application of the proposed
12 cross-entropy of interval neutrosophic sets is established in a decision making problem.
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16 2. Preliminaries

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18 In the following we give a brief review of some preliminaries.

19 2.1 Neutrosophic set

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21 **Definition 2.1** ([19]) Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by
22 a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership
23 function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]0^-, 1^+[$. That is
24 $T_A(x): X \rightarrow]0^-, 1^+[$, $I_A(x): X \rightarrow]0^-, 1^+[$ and $F_A(x): X \rightarrow]0^-, 1^+[$. There is not restriction on the sum of
25 $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) \leq \sup I_A(x) \leq \sup F_A(x) \leq 3^+$.
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30 **Definition 2.2** ([19]) The complement of a neutrosophic set A is denoted by A^c and is defined as $T_A^c(x) =$
31 $\{1^+\} \ominus T_A(x)$, $I_A^c(x) = \{1^+\} \ominus I_A(x)$ and $F_A^c(x) = \{1^+\} \ominus F_A(x)$ for all $x \in X$.
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33 **Definition 2.3** ([19]) A neutrosophic set A is contained in the other neutrosophic set B , $A \subseteq B$ iff $\inf T_A(x) \leq$
34 $\inf T_B(x)$, $\sup T_A(x) \leq \sup T_B(x)$, $\inf I_A(x) \geq \inf I_B(x)$, $\sup I_A(x) \geq \sup I_B(x)$ and $\inf F_A(x) \geq \inf F_B(x)$,
35 $\sup F_A(x) \geq \sup F_B(x)$ for all $x \in X$.
36

37 In the following, we will adopt the representations $u_A(x)$, $p_A(x)$ and $v_A(x)$ instead of $T_A(x)$, $I_A(x)$ and
38 $F_A(x)$, respectively.
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40 2.2 Single valued neutrosophic sets

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42 A single valued neutrosophic set has been defined in ([24]) as follows:

43 **Definition 2.4** ([24]) Let X be a universe of discourse. A single valued neutrosophic set A over X is an object
44 having the form
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$$46 A = \{(x, u_A(x), p_A(x), v_A(x)): x \in X\},$$

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48 where $u_A(x): X \rightarrow [0,1]$, $p_A(x): X \rightarrow [0,1]$ and $v_A(x): X \rightarrow [0,1]$ with $0 \leq u_A(x) + p_A(x) + v_A(x) \leq 3$ for
49 all $x \in X$. The values $u_A(x)$, $p_A(x)$ and $v_A(x)$ denote the truth- membership degree, the indeterminacy-
50 membership degree and the falsity membership degree of x to A , respectively.
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53 **Definition 2.5** ([24]) The complement of a single valued neutrosophic set A is denoted by A^c and is defined
54 as $u_A^c(x) = v(x)$, $p_A^c(x) = 1 - p_A(x)$, and $v_A^c(x) = u(x)$ for all $x \in X$. That is,
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$$56 A^c = \{(x, v_A(x), 1 - p_A(x), u_A(x)): x \in X\}.$$

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58 **Definition 2.6** ([24]) A single valued neutrosophic set A is contained in the other single valued neutrosophic
59 set B , $A \subseteq B$, iff $u_A(x) \leq u_B(x)$, $p_A(x) \geq p_B(x)$ and $v_A(x) \geq v_B(x)$ for all $x \in X$.
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Definition 2.7 ([24]) Two single valued neutrosophic sets A and B are equal, written as $A = B$, iff $A \subseteq B$ and $B \subseteq A$. Moreover, we denote the family of all the single valued neutrosophic sets by $SVNS(X)$.

2.3 Interval neutrosophic sets

An interval neutrosophic set is a model of a neutrosophic set, which can be used to handle uncertainty in fields of scientific, environment and engineering. We introduce the definition of an interval neutrosophic set as follows.

Definition 2.7 ([25]) Let X be a space of points (objects) and $\text{Int}[0,1]$ be the set of all closed subsets of $[0,1]$. An interval neutrosophic A in X is defined with the form

$$A = \{ \langle x, u_A(x), p_A(x), v_A(x) \rangle : x \in X \}$$

where $u_A(x): X \rightarrow \text{int}[0,1]$, $p_A(x): X \rightarrow \text{int}[0,1]$ and $v_A(x): X \rightarrow \text{int}[0,1]$ with $0 \leq \sup u_A(x) + \sup p_A(x) + \sup v_A(x) \leq 3$ for all $x \in X$. The intervals $u_A(x), p_A(x)$ and $v_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A , respectively.

For convenience, if let $u_A(x) = [u_A^L(x), u_A^U(x)]$, $p_A(x) = [p_A^L(x), p_A^U(x)]$ and $v_A(x) = [v_A^L(x), v_A^U(x)]$, then

$$A = \{ \langle x, [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)] \rangle : x \in X \}$$

with the condition, $0 \leq \sup u_A^U(x) + \sup p_A^U(x) + \sup v_A^U(x) \leq 3$ for all $x \in X$. Here, we only take the sub-unitary interval of $[0,1]$. Therefore, an interval neutrosophic set is clearly neutrosophic set.

Definition 2.8 ([25]) Let $INS(X)$ denote the family of all the interval neutrosophic sets in universe X , assume $A, B \in INS(X)$ such that

$$\begin{aligned} A &= \{ \langle x, [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)] \rangle : x \in X \} \\ B &= \{ \langle x, [u_B^L(x), u_B^U(x)], [p_B^L(x), p_B^U(x)], [v_B^L(x), v_B^U(x)] \rangle : x \in X \} \end{aligned}$$

then some operations can be defined as follows:

- (1) $A \cup B = \{ \langle x, [\max\{u_A^L(x), u_B^L(x)\}, \max\{u_A^U(x), u_B^U(x)\}], [\min\{p_A^L(x), p_B^L(x)\}, \min\{p_A^U(x), p_B^U(x)\}], [\min\{v_A^L(x), v_B^L(x)\}, \min\{v_A^U(x), v_B^U(x)\}] \rangle : x \in X \}$;
- (2) $A \cap B = \{ \langle x, [\min\{u_A^L(x), u_B^L(x)\}, \min\{u_A^U(x), u_B^U(x)\}], [\max\{p_A^L(x), p_B^L(x)\}, \max\{p_A^U(x), p_B^U(x)\}], [\max\{v_A^L(x), v_B^L(x)\}, \max\{v_A^U(x), v_B^U(x)\}] \rangle : x \in X \}$;
- (3) $A^c = \{ \langle x, [v_A^L(x), v_A^U(x)], [1 - p_A^U(x), 1 - p_A^L(x)], [u_A^L(x), u_A^U(x)] \rangle : x \in X \}$;
- (4) $A \subseteq B$, iff $u_A^L(x) \leq u_B^L(x)$, $u_A^U(x) \leq u_B^U(x)$, $p_A^L(x) \geq p_B^L(x)$, $p_A^U(x) \geq p_B^U(x)$ and $v_A^L(x) \geq v_B^L(x)$, $v_A^U(x) \geq v_B^U(x)$ for all $x \in X$.
- (5) $A = B$, iff $A \subseteq B$ and $B \subseteq A$.

Now, we define the cross-entropy measure of interval neutrosophic sets.

3. Cross-entropy for interval neutrosophic sets

In this section we present two different approaches to calculate the cross-entropy of interval neutrosophic sets, which are based on the fuzzy cross-entropy and interval neutrosophic cross entropy. We first introduce the concepts of cross-entropy and discrimination information measures between two fuzzy sets which were proposed by Shang and Jiang ([15]), and Bhandari and Pal ([6]), respectively.

Definition 3.1 ([6]) Let A and B are two fuzzy sets in universe $X = \{x_1, x_2, \dots, x_n\}$. The fuzzy information for discrimination of A from B is defined as follows:

$$K_{FS}(A, B) = \sum_{i=1}^n \left(\mu_A(x_i) \ln \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1 - \mu_A(x_i)) \ln \frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)} \right). \quad (1)$$

To overcome some drawbacks in formula (1), Shang and jiang ([15]) proposed a modified version of it called as fuzzy cross entropy.

Definition 3.3 Assume that $A = \langle \mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n) \rangle$ and $B = \langle \mu_B(x_1), \mu_B(x_2), \dots, \mu_B(x_n) \rangle$ are two fuzzy sets in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$. The fuzzy cross entropy of A from B is defined as follows:

$$H_{FS}(A, B) = \sum_{i=1}^n \left(\mu_A(x_i) \log_2 \frac{\mu_A(x_i)}{\frac{1}{2}(\mu_A(x_i) + \mu_B(x_i))} + (1 - \mu_A(x_i)) \log_2 \frac{1 - \mu_A(x_i)}{1 - \frac{1}{2}(\mu_A(x_i) + \mu_B(x_i))} \right) \quad (2)$$

which indicates the degree of discrimination of A from B .

However, $H_{FS}(A, B)$ is does not have symmetric property with respect to its arguments. Shang and Jiang ([15]) defined a symmetric discrimination information measure as $I_{FS}(A, B) = H_{FS}(A, B) + H_{FS}(B, A)$. Moreover, there are $I_{FS}(A, B) \geq 0$ and $I(A, B) = 0$ iff $A = B$.

The information carried by the truth-membership, the indeterminacy-membership, and the falsity-membership in single valued neutrosophic sets A and B can be considered as fuzzy spaces with the three elements. Recently, Ye ([28]) extended the cross-entropy and symmetric discrimination information measures between two fuzzy sets to single valued neutrosophic sets and defined the concept of single valued neutrosophic cross entropy, which is a generalization of fuzzy cross entropy as follows:

Definition 3.4 Assume that $A = \langle u_A(x_i), p_A(x_i), v_A(x_i) \rangle$ and $B = \langle u_B(x_i), p_B(x_i), v_B(x_i) \rangle$ are two single valued neutrosophic sets in the universe of universe $X = \{x_1, x_2, \dots, x_n\}$. Then the single valued neutrosophic cross entropy is defined by

$$\begin{aligned} H_{SVNS}(A, B) = & \sum_{i=1}^n \left[u_A(x_i) \log_2 \frac{u_A(x_i)}{\frac{1}{2}(u_A(x_i) + u_B(x_i))} + (1 - u_A(x_i)) \log_2 \frac{1 - u_A(x_i)}{1 - \frac{1}{2}(u_A(x_i) + u_B(x_i))} \right] \\ & + \left[p_A(x_i) \log_2 \frac{p_A(x_i)}{\frac{1}{2}(p_A(x_i) + p_B(x_i))} + (1 - p_A(x_i)) \log_2 \frac{1 - p_A(x_i)}{1 - \frac{1}{2}(p_A(x_i) + p_B(x_i))} \right] \\ & + \left[v_A(x_i) \log_2 \frac{v_A(x_i)}{\frac{1}{2}(v_A(x_i) + v_B(x_i))} + (1 - v_A(x_i)) \log_2 \frac{1 - v_A(x_i)}{1 - \frac{1}{2}(v_A(x_i) + v_B(x_i))} \right] \quad (3) \end{aligned}$$

which measures the degree of discrimination of A from B . By considering Shannon's inequality ([16]), one can easily show that $H_{SVNS}(A, B) \geq 0$ and $H_{SVNS}(A, B) = 0$ iff $u_A(x_i) = u_B(x_i)$, $p_A(x_i) = p_B(x_i)$ and $v_A(x_i) = v_B(x_i)$ for $x_i \in X$. Additionally, $H_{SVNS}(A, B)$ doesn't have symmetry property. So it should be transformed to a symmetric discrimination information measure for single valued neutrosophic sets as $I_{SVNS}(A, B) = H_{SVNS}(A, B) + H_{SVNS}(B, A)$. Thus, if $I_{SVNS}(A, B)$ is the larger, then the difference between A and B is the larger.

3.1 Interval neutrosophic cross entropy based on fuzzy cross-entropy

In the section, we propose a method to convert any interval neutrosophic set into a fuzzy set

Definition 3.5 Let A be an interval neutrosophic set in universe $X = \{x_1, x_2, \dots, x_n\}$, where

$$A = \langle [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)] \rangle$$

Then the average possible membership degree of element x to interval neutrosophic set A can be defined as

$$\begin{aligned} \tilde{\mu}_A(x) &= \frac{1}{3} \left[\frac{u_A^L(x) + u_A^U(x)}{2} + 1 - \frac{p_A^L(x) + p_A^U(x)}{2} + 1 - \frac{v_A^L(x) + v_A^U(x)}{2} \right] \\ &= \frac{u_A^L(x) + u_A^U(x) + 4 - p_A^L(x) - p_A^U(x) - v_A^L(x) - v_A^U(x)}{6}. \end{aligned} \quad (4)$$

Obviously, $F = \{\langle x, \tilde{\mu}_A(x) \rangle : x \in X\}$ is a fuzzy set in universe X .

Example 3.6 Assume that $X = \{x_1, x_2\}$ is a universe set including only two elements and $A = \langle (x_1, [0.6, 0.7], [0.1, 0.2], [0.2, 0.3]), (x_2, [0.4, 0.5], [0.2, 0.4], [0.7, 0.9]) \rangle : x_1, x_2 \in X$.

Then $F = \{\langle (x_1, 0.75), (x_2, 0.45) \rangle : x \in X\}$ is a fuzzy set in universe X corresponding to interval neutrosophic set A .

Similar to the cross entropy of fuzzy sets, we give the definition of the cross-entropy between two interval neutrosophic sets A and B as follows.

Definition 3.7 Let A and B be two interval neutrosophic sets in universe X and $x \in X$, such that

$$\begin{aligned} A &= \langle [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)] \rangle, \\ B &= \langle [u_B^L(x), u_B^U(x)], [p_B^L(x), p_B^U(x)], [v_B^L(x), v_B^U(x)] \rangle. \end{aligned}$$

Then the average possible membership degree of element x to interval neutrosophic sets A and B can be, respectively, defined as;

$$\begin{aligned} \tilde{\mu}_A(x) &= \frac{1}{3} \left[\frac{u_A^L(x) + u_A^U(x)}{2} + 1 - \frac{p_A^L(x) + p_A^U(x)}{2} + 1 - \frac{v_A^L(x) + v_A^U(x)}{2} \right] \\ &= \frac{u_A^L(x) + u_A^U(x) + 4 - p_A^L(x) - p_A^U(x) - v_A^L(x) - v_A^U(x)}{6}, \\ \tilde{\mu}_B(x) &= \frac{1}{3} \left[\frac{u_B^L(x) + u_B^U(x)}{2} + 1 - \frac{p_B^L(x) + p_B^U(x)}{2} + 1 - \frac{v_B^L(x) + v_B^U(x)}{2} \right] \\ &= \frac{u_B^L(x) + u_B^U(x) + 4 - p_B^L(x) - p_B^U(x) - v_B^L(x) - v_B^U(x)}{6}. \end{aligned}$$

So, the interval neutrosophic cross-entropy based on fuzzy cross-entropy is defined by

$$\begin{aligned} H_{INS}(A, B) &= \sum_{i=1}^n \left(\left(\frac{u_A^{L_i} + u_A^{U_i} + 4 - p_A^{L_i} - p_A^{U_i} - v_A^{L_i} - v_A^{U_i}}{6} \right) \times \right. \\ &\log_2 \left(\frac{2(u_A^{L_i} + u_A^{U_i} + 4 - p_A^{L_i} - p_A^{U_i} - v_A^{L_i} - v_A^{U_i})}{u_A^{L_i} + u_A^{U_i} + 4 - p_A^{L_i} - p_A^{U_i} - v_A^{L_i} - v_A^{U_i} + u_B^{L_i} + u_B^{U_i} + 4 - p_B^{L_i} - p_B^{U_i} - v_B^{L_i} - v_B^{U_i}} \right) + \\ &\left. \left(\frac{p_A^{L_i} + p_A^{U_i} + v_A^{L_i} + v_A^{U_i} + 2 - u_A^{L_i} - u_A^{U_i}}{6} \right) \times \right. \\ &\left. \log_2 \left(\frac{2(p_A^{L_i} + p_A^{U_i} + v_A^{L_i} + v_A^{U_i} + 2 - u_A^{L_i} - u_A^{U_i})}{p_A^{L_i} + p_A^{U_i} + v_A^{L_i} + v_A^{U_i} + 2 - u_A^{L_i} - u_A^{U_i} + p_B^{L_i} + p_B^{U_i} + v_B^{L_i} + v_B^{U_i} + 2 - u_B^{L_i} - u_B^{U_i}} \right) \right), \end{aligned} \quad (5)$$

which indicates discrimination degree of A from B . Here u_A^{Li} denotes the $u_A^L(x_i)$, the same as other notions such as $u_A^{Ui}, p_A^{Li}, p_A^{Ui}, v_A^{Li}, v_A^{Ui}, u_B^{Li}, u_B^{Ui}, p_B^{Li}, p_B^{Ui}, v_B^{Li}, v_B^{Ui}$.

Then we simply can give the cross-entropy as

$$H_{INS}(A, B) = \sum_{i=1}^n \left(\tilde{\mu}_A(x_i) \log_2 \frac{\tilde{\mu}_A(x_i)}{\frac{1}{2}(\tilde{\mu}_A(x_i) + \tilde{\mu}_B(x_i))} + (1 - \tilde{\mu}_A(x_i)) \log_2 \frac{1 - \tilde{\mu}_A(x_i)}{1 - \frac{1}{2}(\tilde{\mu}_A(x_i) + \tilde{\mu}_B(x_i))} \right), \quad (6)$$

which $H_{INS}(A, B)$ indicates the degree of discrimination of A from B .

However, it doesn't have symmetry property. So, it should be transformed to a symmetric form of cross entropy for interval neutrosophic sets as follows: $I_{INS}(A, B) = H_{INS}(A, B) + H_{INS}(B, A)$.

Based on Shannon's inequality ([16]), one can easily show that $H_{INS}(A, B) \geq 0$ and $H_{INS}(A, B) = 0$ iff $u_A(x) = p_A(x) = v_A(x)$ that is, $[u_A^L(x), u_A^U(x)] = [u_B^L(x), u_B^U(x)]$, $[p_A^L(x), p_A^U(x)] = [p_B^L(x), p_B^U(x)]$ and $[v_A^L(x), v_A^U(x)] = [v_B^L(x), v_B^U(x)]$, for $x_i \in X$. Moreover, it is clear that $I_{INS}(A, B) = I_{INS}(A^c, B^c)$ and $I_{INS}(A, B) = I_{INS}(B, A)$.

On the other hand, the weighted cross-entropy measure between an alternative A_i and the ideal alternative A^* can be expressed as

$$H_{INS\omega}(A^*, A_i) = \sum_{j=1}^n \omega_j \left(\tilde{\mu}_{A^*}(x_j) \log_2 \frac{\tilde{\mu}_{A^*}(x_j)}{\frac{1}{2}(\tilde{\mu}_{A^*}(x_j) + \tilde{\mu}_{A_i}(x_j))} + (1 - \tilde{\mu}_{A^*}(x_j)) \log_2 \frac{1 - \tilde{\mu}_{A^*}(x_j)}{1 - \frac{1}{2}(\tilde{\mu}_{A^*}(x_j) + \tilde{\mu}_{A_i}(x_j))} \right), \quad (7)$$

whose weight vector is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$.

It is easily proved that $I_{INS\omega}(A, B) = H_{INS\omega}(A, B) + H_{INS\omega}(B, A)$ for the interval neutrosophic sets A and B .

3.2 Interval neutrosophic cross entropy based on single valued neutrosophic cross-entropy

In this subsection, we develop another technic to compute the cross entropy of interval valued neutrosophic sets. The approach is based on the reduction of the interval neutrosophic sets. To transform the interval neutrosophic sets into single valued neutrosophic sets, we propose the following reduction operator.

Definition 3.8 Assume that $A = \langle [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)] \rangle$ is an interval neutrosophic set and $f_\lambda: INS(X) \rightarrow SVNS(X)$ is a mapping given by

$$f_\lambda(A) = \langle u_A^L(x) + \lambda \Delta u_A(x), p_A^L(x) + (1 - \lambda) \Delta p_A(x), v_A^L(x) + (1 - \lambda) \Delta v_A(x) \rangle, \quad (8)$$

where $\Delta u_A(x) = u_A^U(x) - u_A^L(x)$, $\Delta p_A(x) = p_A^U(x) - p_A^L(x)$ and $\Delta v_A(x) = v_A^U(x) - v_A^L(x)$ for $x \in X$ and $\lambda \in [0, 1]$. Then f_λ is called a reduction operator which is used to assign a single valued neutrosophic set to an interval neutrosophic set.

Obviously, A_λ is a single valued neutrosophic set in universe X .

Example 3.9 Assume that $X = \{x_1\}$ is a universe set including only one element and $A = \langle (x_1, [0.2, 0.5], [0.1, 0.3], [0.2, 0.6]) : x_1 \in X \rangle$ and $B = \langle (x_1, [0.3, 0.5], [0.1, 0.4], [0.3, 0.4]) : x_1 \in X \rangle$. For $\lambda = 0.5$, we can compute the single valued neutrosophic sets $f_{0.5}(A)$ and $f_{0.5}(B)$ as follows

$$\begin{aligned} f_{0.5}(A) &= \langle (x_1, 0.35, 0.20, 0.40) : x_1 \in X \rangle \\ f_{0.5}(B) &= \langle (x_1, 0.40, 0.25, 0.35) : x_1 \in X \rangle \end{aligned}$$

Proposition 3.10 Let A and B two interval neutrosophic sets in universe $X = \{x_1, x_2, \dots, x_n\}$, where

$$A = \langle [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)] \rangle,$$

$$B = \langle [u_B^L(x), u_B^U(x)], [p_B^L(x), p_B^U(x)], [v_B^L(x), v_B^U(x)] \rangle$$

Assume $f_\lambda: INS(X) \rightarrow SVNS(X)$ is a reduction operator and $\lambda, \delta \in [0,1]$. Then

- (1) if $0 \leq \lambda \leq \delta$, then $f_\lambda(A) \subseteq f_\delta(A)$;
- (2) if $A \subseteq B$, then $f_\lambda(A) \subseteq f_\lambda(B)$;
- (3) $f_\lambda(f_\delta(A)) = f_\delta(A)$;
- (4) $(f_\lambda(A^c))^c = f_{1-\lambda}(A)$.

Proof:

- (1) Assume that $0 \leq \lambda \leq \delta$ for $\lambda, \delta \in [0,1]$. Then we have

$$f_\lambda(A) = \langle u_A^L(x) + \lambda \Delta u_A(x), p_A^L(x) + (1 - \lambda) \Delta p_A(x), v_A^L(x) + (1 - \lambda) \Delta v_A(x) \rangle,$$

$$f_\delta(A) = \langle u_A^L(x) + \delta \Delta u_A(x), p_A^L(x) + (1 - \delta) \Delta p_A(x), v_A^L(x) + (1 - \delta) \Delta v_A(x) \rangle,$$

where,

$$u_A^L(x) + \lambda \Delta u_A(x) \leq u_A^L(x) + \delta \Delta u_A(x),$$

$$p_A^L(x) + (1 - \delta) \Delta p_A(x) \leq p_A^L(x) + (1 - \lambda) \Delta p_A(x),$$

$$v_A^L(x) + (1 - \delta) \Delta v_A(x) \leq v_A^L(x) + (1 - \lambda) \Delta v_A(x).$$

So, $f_\lambda(A) \subseteq f_\delta(A)$.

- (2) Assume that $A \subseteq B$. Then we have $u_A^L(x) \leq u_B^L(x)$, $u_A^U(x) \leq u_B^U(x)$, $p_A^L(x) \geq p_B^L(x)$, $p_A^U(x) \geq p_B^U(x)$ and $v_A^L(x) \geq v_B^L(x)$, $v_A^U(x) \geq v_B^U(x)$ for all $x \in X$. It implies that

$$u_A^L(x) + \lambda \Delta u_A(x) \leq u_B^L(x) + \lambda \Delta u_B(x),$$

$$p_B^L(x) + (1 - \lambda) \Delta p_B(x) \leq p_A^L(x) + (1 - \lambda) \Delta p_A(x),$$

$$v_B^L(x) + (1 - \lambda) \Delta v_B(x) \leq v_A^L(x) + (1 - \lambda) \Delta v_A(x).$$

So, $f_\lambda(A) \subseteq f_\lambda(B)$.

- (3) It is easily proved that $f_\lambda(f_\delta(A)) = f_\delta(A)$.

- (4) Let $A \in INS(X)$ and so $A^c = \langle [v_A^L(x), v_A^U(x)], [1 - p_A^U(x), 1 - p_A^L(x)], [u_A^L(x), u_A^U(x)] \rangle$ for each $x \in X$. Then we have the following

$$f_\lambda(A^c) = \langle v_A^L(x) + \lambda \Delta v_A(x), 1 - p_A^U(x) + (1 - \lambda) \Delta p_A(x), u_A^L(x) + (1 - \lambda) \Delta u_A(x) \rangle$$

for each $x \in X$. Hence, it follows that

$$\begin{aligned} (f_\lambda(A^c))^c &= \langle u_A^L(x) + (1 - \lambda) \Delta u_A(x), 1 - (1 - p_A^U(x) + (1 - \lambda) \Delta p_A(x)), v_A^L(x) + \lambda \Delta v_A(x) \rangle \\ &= \langle u_A^L(x) + (1 - \lambda) \Delta u_A(x), 1 - (1 - p_A^U(x) + (p_A^U(x) - p_A^L(x)) - \lambda \Delta p_A(x)), v_A^L(x) \\ &\quad + \lambda \Delta v_A(x) \rangle \\ &= \langle u_A^L(x) + (1 - \lambda) \Delta u_A(x), p_A^L(x) + \lambda \Delta p_A(x), v_A^L(x) + \lambda \Delta v_A(x) \rangle \\ &= f_{1-\lambda}(A). \end{aligned}$$

Similar to the cross entropy of single valued neutrosophic sets, we define the concept of the cross entropy based on the reduction of the interval neutrosophic sets, which is a generalization of single valued neutrosophic cross entropy, as follows:

Definition 3.11 Let $A, B \in INS(X)$ and $X = \{x_1, x_2, \dots, x_n\}$. Then interval neutrosophic cross entropy based on single valued neutrosophic cross entropy is defined by

$$E_{INS}(A, B) = \sum_{i=1}^n \left[\bar{u}_A(x_i) \log_2 \frac{\bar{u}_A(x_i)}{\frac{1}{2}(\bar{u}_A(x_i) + \bar{u}_B(x_i))} + (1 - \bar{u}_A(x_i)) \log_2 \frac{1 - \bar{u}_A(x_i)}{1 - \frac{1}{2}(\bar{u}_A(x_i) + \bar{u}_B(x_i))} \right]$$

$$\begin{aligned}
& + \left[\bar{p}_A(x_i) \log_2 \frac{\bar{p}_A(x_i)}{\frac{1}{2}(\bar{p}_A(x_i) + \bar{p}_B(x_i))} + (1 - \bar{p}_A(x_i)) \log_2 \frac{1 - \bar{p}_A(x_i)}{1 - \frac{1}{2}(\bar{p}_A(x_i) + \bar{p}_B(x_i))} \right] \\
& + \left[\bar{v}_A(x_i) \log_2 \frac{\bar{v}_A(x_i)}{\frac{1}{2}(\bar{v}_A(x_i) + \bar{v}_B(x_i))} + (1 - \bar{v}_A(x_i)) \log_2 \frac{1 - \bar{v}_A(x_i)}{1 - \frac{1}{2}(\bar{v}_A(x_i) + \bar{v}_B(x_i))} \right] \quad (9)
\end{aligned}$$

where $\bar{u}_A(x_i) = u_A^L(x) + \lambda \Delta u_A(x)$, $\bar{p}_A(x_i) = p_A^L(x) + \lambda \Delta p_A(x)$ and $\bar{v}_A(x_i) = v_A^L(x) + \lambda \Delta v_A(x)$, and similarly, $\bar{u}_B(x_i) = u_B^L(x) + \lambda \Delta u_B(x)$, $\bar{p}_B(x_i) = p_B^L(x) + \lambda \Delta p_B(x)$ and $\bar{v}_B(x_i) = v_B^L(x) + \lambda \Delta v_B(x)$ for all for $x \in X$ and $\lambda \in [0,1]$, which λ is the threshold value.

Unless otherwise stated, we choose the center value, i.e., $\lambda = 0,5$.

Based on Shannon's inequality ([16]), one can easily show that $E_{INS}(A, B) \geq 0$ and $E_{INS}(A, B) = 0$ iff $u_A(x) = p_A(x) = v_A(x)$ that is, $[u_A^L(x), u_A^U(x)] = [u_B^L(x), u_B^U(x)]$, $[p_A^L(x), p_A^U(x)] = [p_B^L(x), p_B^U(x)]$ and $[v_A^L(x), v_A^U(x)] = [v_B^L(x), v_B^U(x)]$ for all $x \in X$. Moreover, $E_{INS}(A, B)$ doesn't have symmetry property. So it should be transformed to a symmetric discrimination information measure for interval neutrosophic sets as $J_{INS}(A, B) = E_{INS}(A, B) + E_{INS}(B, A)$. Thus, if $J_{SVNS}(A, B)$ is the larger, then the difference between A and B is the larger.

On the other hand, the weighted cross-entropy measure between an alternative A_i and the ideal alternative A^* can be expressed as

$$\begin{aligned}
E_{INS\omega}(A^*, A_i) = & \sum_{j=1}^n \omega_j \left[\bar{u}_{A^*}(x_j) \log_2 \frac{\bar{u}_{A^*}(x_j)}{\frac{1}{2}(\bar{u}_{A^*}(x_j) + \bar{u}_{A_i}(x_j))} + (1 - \bar{u}_{A^*}(x_j)) \log_2 \frac{1 - \bar{u}_{A^*}(x_j)}{1 - \frac{1}{2}(\bar{u}_{A^*}(x_j) + \bar{u}_{A_i}(x_j))} \right] \\
& + \left[\bar{p}_{A^*}(x_j) \log_2 \frac{\bar{p}_{A^*}(x_j)}{\frac{1}{2}(\bar{p}_{A^*}(x_j) + \bar{p}_{A_i}(x_j))} + (1 - \bar{p}_{A^*}(x_j)) \log_2 \frac{1 - \bar{p}_{A^*}(x_j)}{1 - \frac{1}{2}(\bar{p}_{A^*}(x_j) + \bar{p}_{A_i}(x_j))} \right] \\
& + \left[\bar{v}_{A^*}(x_j) \log_2 \frac{\bar{v}_{A^*}(x_j)}{\frac{1}{2}(\bar{v}_{A^*}(x_j) + \bar{v}_{A_i}(x_j))} + (1 - \bar{v}_{A^*}(x_j)) \log_2 \frac{1 - \bar{v}_{A^*}(x_j)}{1 - \frac{1}{2}(\bar{v}_{A^*}(x_j) + \bar{v}_{A_i}(x_j))} \right], \quad (10)
\end{aligned}$$

whose weight vector is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \in [0,1]$, $\sum_{j=1}^n \omega_j = 1$.

It is easily proved that $J_{INS\omega}(A, B) = E_{INS\omega}(A, B) + E_{INS\omega}(B, A)$ for the interval neutrosophic sets A and B .

4. Multi-criteria neutrosophic decision-making method based on the cross entropy

As a new branch of neutrosophic sets, interval neutrosophic set can be used to solve problems including uncertain, imprecise, inconsistent and incomplete information existing in real scientific, environment and engineering applications. It has drawn the attention of many researchers for handling uncertainty. In this section, an illustrative example adopted by ([32]) is given to verify the proposed approach and to demonstrate its practicality and effectiveness.

Let $A = \{A_1, A_2, \dots, A_m\}$ be the set of alternatives and $C = \{C_1, C_2, \dots, C_n\}$ the set of attributes, whose weight vector is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \in [0,1]$, $\sum_{j=1}^n \omega_j = 1$. Thus, the characteristic of the alternative A_i ($i = 1, 2, \dots, m$) is expressed by an interval neutrosophic set:

$$A_i = \{ \langle x, [u_{A_i}^L(C_j), u_{A_i}^U(C_j)], [p_{A_i}^L(C_j), p_{A_i}^U(C_j)], [v_{A_i}^L(C_j), v_{A_i}^U(C_j)] \rangle : C_j \in C \}$$

where $0 \leq u_{A_i}^U(C_j) + p_{A_i}^U(C_j) + v_{A_i}^U(C_j) \leq 3$, $u_{A_i}^L(C_j) \geq 0$, $p_{A_i}^U(C_j) \geq 0$, $v_{A_i}^L(C_j) \geq 0$, $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, m$. Here, $[u_{A_i}^L(C_j), u_{A_i}^U(C_j)]$ indicates the degree that the alternative A_i satisfies the criterion C_j and $[p_{A_i}^L(C_j), p_{A_i}^U(C_j)]$ indicates the degree that the alternative A_i is indeterminacy on the criterion C_j , where as $[v_{A_i}^L(C_j), v_{A_i}^U(C_j)]$ indicates the degree that the alternative A_i does not satisfy the criterion C_j given by the decision-maker. So we can express a decision matrix $A = (\alpha_{ij})_{m \times n}$. Additionally, in multicriteria decision making process, the concept of ideal point has been used to help determine the best alternative in the decision set. Although the ideal alternative that does not exist in real life, it does provide a useful theoretical construct against which to evaluate alternatives. Here, we can define an ideal fuzzy value and an ideal single valued neutrosophic value, respectively as $A^* = \langle (x, 1) \rangle$ and $A^* = \langle (x, 1, 0, 0) \rangle$ for all $x \in X$. Also, the positive-ideal solution A^+ for single valued neutrosophic sets is a single valued neutrosophic value defined by

$$A^+ = \langle (u_A^+(C_j), p_A^+(C_j), v_A^+(C_j)) \rangle$$

where $u_{A^+}(C_j) = \max_i u_{A_i}(C_j)$, $p_{A^+}(C_j) = \min_i p_{A_i}(C_j)$ and $v_{A^+}(C_j) = \min_i v_{A_i}(C_j)$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. It is clear that the positive-ideal solution A^+ for fuzzy sets is a fuzzy value defined by $A^+ = \langle (\mu_A^+(C_j)) \rangle$, where $\mu_{A^+}(C_j) = \max_i \mu_{A_i}(C_j)$.

Therefore, the smaller the value of $I_{INS}(A^*, A_i)$ (or $J_{INS}(A^*, A_i)$) is, the better the alternative A_i is. In this case, the alternative A_i is closer to the ideal alternative A^* . Through the weighted cross entropy $I_{INS\omega}(A^*, A_i)$ (or $J_{INS\omega}(A^*, A_i)$) ($i = 1, 2, \dots, m$) between each alternative and the ideal alternative, we can determine the ranking order of all alternatives and can easily select the best one. Moreover, we mean similar thoughts for the positive-ideal solution A^+ .

In the following, we can summary the decision procedure for two developed methods to compute the cross-entropy of interval neutrosophic sets, which are called the interval neutrosophic cross entropy based on extension of single valued neutrosophic cross entropy and fuzzy cross-entropy, respectively.

Method (1) Interval neutrosophic cross entropy based on fuzzy cross-entropy

- Step 1. Obtain the fuzzy sets with the average possible membership corresponding to interval neutrosophic sets by using Eq. (4).
- Step 2. Calculate the symmetric form of cross entropy $I_{INS\omega}(A^*, A_i)$ (or $I_{INS\omega}(A^+, A_i)$) for interval neutrosophic set A_i ($i = 1, 2, \dots, m$) by using Eq. (7).
- Step 3. Rank the alternatives A_i ($i = 1, 2, \dots, m$) and select to best one(s) in according to $I_{INS\omega}(A^*, A_i)$ (or $I_{INS\omega}(A^+, A_i)$) for $i = 1, 2, \dots, m$.

Method (2) Interval neutrosophic cross entropy based on single valued neutrosophic cross-entropy

- Step 1. Obtain the single valued neutrosophic sets based on the threshold value corresponding to interval neutrosophic sets by using Eq. (8).
- Step 2. Calculate the symmetric form of cross entropy $J_{INS\omega}(A^*, A_i)$ (or $J_{INS\omega}(A^+, A_i)$) for single valued neutrosophic set A_i ($i = 1, 2, \dots, m$) by using Eq. (10).
- Step 3. Rank the alternatives A_i ($i = 1, 2, \dots, m$) and select to best one(s) in according to the symmetric form $J_{INS\omega}(A^*, A_i)$ (or $J_{INS\omega}(A^+, A_i)$) for $i = 1, 2, \dots, m$.

4.1. Numerical example

Example 4.1 Let us consider decision making problem adapted from ([32]). Suppose that there is a panel with four possible alternatives to invest the money: (1) A_1 is a food company; (2) A_2 is a car company; (3) A_3 is an arms company; (4) A_4 is a computer company. The investment company must make a decision according to three criteria given below: (1) C_1 is the growth analysis; (2) C_2 is the risk analysis; (3) C_3 is the environmental impact analysis. By using the interval-valued intuitionistic fuzzy information, the decision-maker has evaluated the four possible alternatives under the above three criteria and has listed in the following matrix:

	C_1	C_2	C_3
A_1	$([0.4,0.5], [0.2,0.3], [0.3,0.4])$	$([0.4,0.6], [0.1,0.3], [0.2,0.4])$	$([0.7,0.9], [0.2,0.3], [0.4,0.5])$
A_2	$([0.6,0.7], [0.1,0.2], [0.2,0.3])$	$([0.6,0.7], [0.1,0.2], [0.2,0.3])$	$([0.3,0.6], [0.3,0.5], [0.8,0.9])$
A_3	$([0.3,0.6], [0.2,0.3], [0.3,0.4])$	$([0.5,0.6], [0.2,0.3], [0.3,0.4])$	$([0.4,0.5], [0.2,0.4], [0.7,0.9])$
A_4	$([0.7,0.8], [0.0,0.1], [0.1,0.2])$	$([0.6,0.7], [0.1,0.2], [0.1,0.3])$	$([0.6,0.7], [0.3,0.4], [0.8,0.9])$

Suppose that the weights of C_1 , C_2 and C_3 are 0.35, 0.25 and 0.40. Then, we use the approach proposed to obtain the most desirable alternative(s).

Method (1) Interval neutrosophic cross entropy based on fuzzy crosss-entropy:

Step 1. We can obtain the fuzzy sets with the average possible membership corresponding to interval neutrosophic sets by using Eq. (4) as follows:

$$\begin{aligned} A_1 &= \{(C_1, 0.6166), (C_2, 0.6666), (C_3, 0.7000)\}, \\ A_2 &= \{(C_1, 0.7500), (C_2, 0.7500), (C_3, 0.4000)\}, \\ A_3 &= \{(C_1, 0.6166), (C_2, 0.6500), (C_3, 0.4500)\}, \\ A_4 &= \{(C_1, 0.8500), (C_2, 0.7666), (C_3, 0.4833)\}. \end{aligned}$$

Step 2. By using Eq. (7), we compute $I_{INS\omega}(A^*, A_i)$ ($i = 1, 2, \dots, m$) as

$$I_{INS\omega}(A^*, A_1) = 0.3881, I_{INS\omega}(A^*, A_2) = 0.4822, I_{INS\omega}(A^*, A_3) = 0.5401, I_{INS\omega}(A^*, A_4) = 0.3792.$$

Step 3. Rank all alternatives according to the symmetric form $I_{INS\omega}(A^*, A_i)$ ($i = 1, 2, \dots, m$): $A_4 > A_1 > A_2 > A_3$.

Thus the alternative A_4 is the most desirable alternative based on the interval neutrosophic cross entropy based on extension, which is also consistent with the result obtain in ([32]).

Now, assuming the same weights for C_1 , C_2 and C_3 , we use the interval neutrosophic cross-entropy based on reduction.

Method (2) Interval neutrosophic cross entropy based on single valued neutrosophic cross entropy:

Step (1) We can the single valued neutrosophic sets based on the threshold value corresponding to interval neutrosophic sets by using Eq. (8) as follows:

$$\begin{aligned} A_1 &= \{(C_1, 0.45, 0.25, 0.35), (C_2, 0.50, 0.20, 0.30), (C_3, 0.80, 0.25, 0.45)\}, \\ A_2 &= \{(C_1, 0.65, 0.15, 0.25), (C_2, 0.65, 0.15, 0.25), (C_3, 0.45, 0.40, 0.85)\}, \\ A_3 &= \{(C_1, 0.45, 0.45, 0.25), (C_2, 0.55, 0.25, 0.35), (C_3, 0.45, 0.30, 0.80)\}, \\ A_4 &= \{(C_1, 0.75, 0.05, 0.15), (C_2, 0.65, 0.15, 0.20), (C_3, 0.65, 0.35, 0.85)\}. \end{aligned}$$

Step (2) By using Eq. (10), we compute $J_{INS\omega}(A^*, A_i)$ ($i = 1, 2, \dots, m$) as

$$J_{INS\omega}(A^*, A_1) = 1.2924, J_{INS\omega}(A^*, A_2) = 1.5861, J_{INS\omega}(A^*, A_3) = 1.8694, J_{INS\omega}(A^*, A_4) = 1.1252.$$

Step (3) Rank all alternatives according to the symmetric form $I_{INS\omega}(A^*, A_i)$ ($i = 1, 2, \dots, m$): $A_4 > A_1 > A_2 > A_3$.

Thus the alternative A_4 is also the most desirable alternative based on the interval neutrosophic cross entropy, which is also consistent with the result obtain in Method1 and ([32]).

Remark: Ranking the alternatives in accordance with the positive-ideal solution A^+ in Method1, we can compute as

$$I_{INS\omega}(A^+, A_1) = 0.1005, I_{INS\omega}(A^+, A_2) = 0.1564, I_{INS\omega}(A^+, A_3) = 0.2020, I_{INS\omega}(A^+, A_4) = 0.0817$$

and so we deduce that the priority ranking of all the four alternatives is the same as $A_4 > A_1 > A_2 > A_3$. Similarity, by employing the positive-ideal solution A^+ in Method2, we can obtain as

$$J_{INS\omega}(A^+, A_1) = 0.3128, J_{INS\omega}(A^+, A_2) = 0.4136, J_{INS\omega}(A^+, A_3) = 0.6211, J_{INS\omega}(A^+, A_4) = 0.1860.$$

Then we have the same ranking order for $\lambda = 0.5$; $A_4 > A_1 > A_2 > A_3$ and thus the most desirable alternative is A_4 .

From the examples, since the proposed neutrosophic decision-making method can handle not only incomplete information but also the inconsistent information and indeterminate information existing in real situations, it is more suitable for real scientific and engineering applications. The technique proposed in this paper presents a generalization of existing decision making methods and provides a new way for decision makers.

7. Conclusions

Interval neutrosophic set, as a combining concepts of single valued neutrosophic set and interval fuzzy set, provides the additional capability to deal with uncertainty, inconsistent, incomplete and imprecise information by including a truth-membership interval, an indeterminacy-membership interval and a falsity membership interval. Therefore, it has played a significant role in the uncertain system. In this paper, we first proposed two useful methods which are reduce the interval neutrosophic information to single valued neutrosophic information and fuzzy information, respectively. Based on the single valued neutrosophic cross-entropy and the fuzzy cross-entropy, we define the concept of interval neutrosophic cross-entropy by using the two reduction methods. Then the proposed cross-entropy is established in multi criteria decision making problem, which the alternatives on criteria are characterized by interval neutrosophic sets. Finally, a practical example was presented to illustrate the application of the proposed multi criteria decision making.

Since the interval neutrosophic set is a generalization of the aforementioned sets, the proposed method extends previous approaches because of the consideration of indeterminacy information as well as truth and falsity information in the selection process of the alternatives according to criteria. Therefore it is more effective and more remarkable for handling imprecise, inconsistent and incomplete information in multi criteria decision making. Hence, one can easily apply the method to make the optimal choice and to solve multi criteria decision making problems located in other areas such as pattern recognition, clustering analysis, image process and medical diagnoses. In the future, we will continue working on entropy of neutrosophic sets, and so the single valued neutrosophic sets and the interval neutrosophic sets.

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