The Generalized Bode's Law

This paper introduces a generalized Bode's Law. This law predicts the mean distances of all the planets (including the asteroid/dwarf planet Ceres) from the sun with a maximum relative error of, approximately 16% corresponding to Mars. For the rest of the planets the error is, approximately, 10 % or less. Thus, the new formula eliminates the inaccuracies of the original Law with respect to Neptune and Pluto. The formula also works extremely well for Mercury with n = -1, eliminating the need of using arbitrary numbers such as n = -∞. The generalized formula is suitable to predict the existence of undetected exoplanets.

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1. Introduction

It is not clear who was the discoverer of the so called Titius-Bodes's Law but several mathematicians have been involved with it during the course of history: David Gregory, Christian Wolff, Johann Daniel Titius and Johann Elert Bode, just to mention a few.

In 1801 Ceres, a large spherical asteroid (or dwarf planet) in the asteroid belt, was discovered exactly at the Bode's predicted distance. This discovery boosted the importance of Bode's Law and, consequently, this law was widely accepted by the scientific community. But this success would only last until 1846 when Neptune was discovered. Because Neptune did not obey this Law, the Law was "demoted" and since then it was considered just a mathematical coincidence. What's even worse was that when Pluto was discovered, it was found that this planet did not satisfy Bode's Law either. Thus, Bode's Law's status went from bad to worse. But one may ask: what if Bode's Law were part of a more general law. Would that new and more complete law change the status of Bode's law from coincidence to law again? Nobody knows. However more important questions are these ones: can we, at least, modify Bode's law so that it can predict the distances of all planets (without exceptions) from the Sun with reasonable accuracy? And: is it worth trying even if the modified law is regarded as being a numeric coincidence? I think the answer is yes. The reason is that the modified law could be used to predict the positions of mysterious undetected exoplanets.

2. The Generalized Law

The following formula is the generalized Bode's Law

$$a = \left(0.3 \times 2^{n} + 0.003\, n^{4} + 0.6 - 0.18 \times 2^{-n} - 0.1\, n^{2} \times 2^{n-5.4} - 0.0000004\, n^{8.2} \right) AU \tag{2.1}$$

where

a = mean predicted distance of the planet from the Sun (major semi axis) in astronomical units (AU)
n = integer (this is in fact an “allowed” gravitational number) (-1, 0, 1, 2, etc.)

The mean predicted distance of the planet from the sun, *a*, is given in Astronomical Units (denoted by AU) and the variable *n* represents an integer number varying form -1 for Mercury to 8 for Pluto. When we move from one planet to the next, we have to increase *n* by one. Table 1 shows the predicted values for the average major semi-axis of the planet according to the generalized Bode's Law.

PLANET	*n*	PREDICTED DISTANCE (AU)	OBSERVED AVERAGE DISTANCE (AU)	RELATIVE ERROR (e%)	QUALITY OF THE PREDICTION
Mercury	-1	0.392	0.39	0.51	excellent
Venus	0	0.720	0.73	-1.37	excellent
Earth	1	1.108	1.00	10.08	good
Mars	2	1.765	1.52	16.12	acceptable
(Ceres)	3	3.047	2.77	10.00	good
Jupiter	4	5.516	5.22	5.67	excellent
Saturn	5	9.959	9.57	4.07	excellent
Uranus	6	17.267	19.26	-10.02	good
Neptune	7	27.945	30.17	-7.31	excellent
Pluto	8	40.713	39.60	3.07	excellent

Table 1: Predicted values for the average major semi-axis of the planet according to the generalized Bode's Law. Quality is assigned as follows: excellent if the absolute value of the relative error, $|e\%|$, is less than 10%, good if $10 < |e\%| < 10.1$, and acceptable if $10.1 < |e\%| < 20$.

The absolute value of the relative error (see **Appendix 1**) for the generalized Bode's law is less than 10 % for 6 planets, approximately 10 % for 3 planets and greater than 10.1 % for only one planet (Mars). Thus, the overall accuracy of the law is very good.

It is worthwhile to emphasize that the formula given above corrects the problem of the original Bode's law with Mercury. To have a look at this old problem let's us consider the original Bode's law [1]

$$a = \left(0.3 \times 2^n + 0.4\right) AU \qquad (2.2)$$

The problem was that to obtain the correct value for the average major semi axis of Mercury, we were forced to use an infinitely large negative number ($n = -\infty$), so that the major semi axis for this planet turned out to be right

$$a\left(Mercury\right) = \left(0.3 \times 2^{-\infty} + 0.4\right) AU = 0.4\ AU$$

which is approximately equal to the observed value of 0.39 AU. However, nobody was able to explain the reason of using ($-\infty$) other than the fact that $-\infty$ worked quite well. This was totally arbitrary as it did not followed the same rule of adding one when moving to the next planet located further away from the Sun (or subtracting 1 if we move in the opposite direction). On the other hand, the generalized law yields the correct value for Mercury with a finite exponent ($n = -1$) and, thus, the generalized law uses the same rule for all the planets, with no exceptions:

"Allowed" gravitational numbers rule

we add one when we move to the next planet located further away from the Sun, and we subtract one when we move to the next planet located closer to the Sun. The "allowed" gravitational numbers are:

-1, 0, 1, 2, 3, 4, 5, 6, 7, 8

3. Conclusions

In summary, the absolute value of the relative error for the generalized Bode's law is less than 10 % for 6 planets, approximately 10 % for 3 planets and greater than 10.1 % for only one planet (Mars). Thus, the overall accuracy of the law is very good.

Another important aspect of this law is that it can be applied to other solar systems to find out whether the positions of their planets (exoplanets for us) obey this law. If the law were able to predict the average distances of the exoplanets from their "suns", then the law could be used to discover undetected exoplanets. In the particular case or our solar system the generalized law can be extrapolated beyond Pluto's orbit to make a prediction which is shown on Table 2. The hypothetical new planet is planet X which was predicted a long time ago but it was never found.

PLANET	n	PREDICTED DISTANCE (AU)
Planet X	9	48.944

Table 2: The generalized Bode's law was used to predict the existence of another planet beyond Pluto's orbit. Is this predicted planet Planet X?

Thus, according to the generalized Bode's law presented in this paper an undetected planet is orbiting the Sun at an average distance of 49 AU, approximately. However, we have to keep in mind that all laws have limitations. Consequently, it is possible that this law does not work beyond the orbit of the dwarf planet Pluto.

Finally, if we compare formulas (2.1) and (2.2), which are, respectively

1) *Generalized Bode's Law*

$$a = \left(0.3\times 2^{n} + 0.6 + 0.003\, n^{4} - 0.18\times 2^{-n} - 0.1\, n^{2}\times 2^{n-5.4} - 0.0000004\, n^{8.2} \right) AU$$

2) *Original Bode's Law*

$$a = \left(0.3\times 2^{n} + 0.4 \right) AU$$

we find that the original Law is a particular case of the general Law. In fact, if we add the terms: $0.2 + 0.003\, n^{4} - 0.18\times 2^{-n} - 0.1\, n^{2}\times 2^{n-5.4} - 0.0000004\, n^{8.2}$ to the original Law, we will get the generalized formula. We now understand why these terms (or any other equivalent ones) were not present in the original formula. At the time Bode's Law was discovered, Neptune and Pluto were not known.

Appendix 1
Absolute and Relative Errors

Absolute Error

The absolute error is the difference between the measured value and the true value

$$E_{\text{abs}} = X_m - X_t$$

Relative Error

The relative error is the absolute error expressed in parts of the true value

$$E_{rel} = \frac{X_m - X_t}{X_t}$$

Relative Error as a Percentage

$$e\,\% = E_{rel}\,\% = \frac{X_m - X_t}{X_t} \times 100$$

$E_{\text{abs}} =$ absolute error

$E_{rel} =$ relative error

$e\,\% = E_{rel}\,\% =$ relative error as percentage

$X_m =$ measured value

$X_t =$ true value, observed value, standard value, or the value thought to be the true value (for example a NIST standard value)

REFERENCES

[1] R. A. Frino, *The Role of Powers of 2 in Physics*, vixra.org: vixra 1507.0047, (2015).