Quantum system symmetry is not the source of unitary information in wave mechanics — context quantum randomness

Homogeneity of space is non-unitary

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Abstract  The homogeneity symmetry is re-examined and shown to be non-unitary, with no requirement for the imaginary unit. This removes symmetry, as reason, for imposing unitarity (or self-adjointness) — by Postulate. The work here is part of a project researching logical independence in quantum mathematics, for the purpose of advancing a full and complete theory of quantum randomness.

Keywords  foundations of quantum theory, quantum physics, quantum mechanics, wave mechanics, Canonical Commutation Relation, symmetry, homogeneity of space, unitary, non-unitary, unitarity, mathematical logic, formal system, elementary algebra, information, axioms, mathematical propositions, logical independence, quantum indeterminacy, quantum randomness.

1 Introduction

In classical physics, experiments of chance, such as coin-tossing and dice-throwing, are deterministic, in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. The ‘randomness’ stems from ignorance of physical information in the initial toss or throw.

In diametrical contrast, in the case of quantum physics, the theorems of Kocken and Specker [7], the inequalities of John Bell [3], and experimental evidence of Alain Aspect [1,2], all indicate that quantum randomness does not stem from any such physical information.

As response, Tomasz Paterek et al offer explanation in mathematical information. They demonstrate a link between quantum randomness and logical independence in (Boolean) mathematical propositions [8,9]. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. In experiments measuring photon polarisation, Tomasz Paterek et al demonstrate statistics correlating predictable outcomes with logically dependent mathematical propositions, and random outcomes with propositions that are logically independent.

While those Boolean propositions do convey definitive information about quantum randomness, any insight they offer is obscure. In order to advance a full and complete theory of quantum randomness, understanding is needed of how this Boolean logical independence connects with standard textbook quantum theory.

In a related article by this author [4], logical independence in Elementary Algebra is discussed. This is the very familiar algebra upon which Applied Mathematics and Mathematical Physics rest. Logical independence in this algebra is well-known to Mathematical Logic [12]. Of particular interest is logical independence of the imaginary scalars, seen in contrast to logical dependence of the rational scalars — and – the possible prospect that these two types of logical information might pass into quantum mathematics. As it happens, the passage of that logical information is blocked. It is prevented, not by unitarity per se, but by the fact that unitarity (or self-adjointness) is imposed axiomatically — by Postulate.
Historically, the reason for unitarity is the universal need for preserved invariance of probability amplitude. Interpretationally, this universality is seen to imply that all fundamental symmetries in Nature are ontologically unitary [5, p109][6, p34]. This would indicate that unitarity should be a blanket condition covering the whole of quantum theory — and should be regarded as **axiomatic**. In short, unitarity is never in absence.

The findings of this paper contradicts the belief that symmetries are intrinsically unitary. It shows that the homogeneity symmetry, generally purported to imply the Canonical Commutation Relation [5, p115][11, p44], is not itself unitary, and does not imply the Canonical Commutation Relation, indicating that the Relation’s unitarity originates elsewhere.

This removes symmetry, as reason, for imposing unitarity — **by Postulate**.

### 2 The basic symmetry of wave mechanics: homogeneity of space

The **Canonical Commutation Relation**

\[ px - xp = -i\hbar \]

embodies core algebra at the heart of wave mechanics. The professed significance of this relation is that it represents the homogeneity of space, with general acceptance by quantum theorists, as being unitary. In this paper, I re-examine and scrutinise the Canonical Relation’s derivation and establish that the homogeneity symmetry is itself _not_ unitary. And in consequence establish that the Canonical Commutation Relation does not, itself, faithfully represent homogeneity, but contains other (unitary) information also.

Imposing homogeneity on a system is identical to imposing a null physical effect, under arbitrary translation of reference frame. To formulate this arbitrary translation, resulting in null effect, the principle we invoke is **Form Invariance**. This is the concept, from relativity, that symmetry transformations leave (physical) formulae fixed in _form_, though _values_ may alter [10]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position:

\[ x |f_x(x)\rangle = x |f_x(x)\rangle . \]  

(1)

The san-serif _x_, here, is a label for _f_x_ whose eigenvalue is _x_. The variable _x_ (curly) is the function domain. The use of two different variables, here, may seem unusual but pointless. In fact, logically they are different. _x_ is quantified _universally_ but _x_ is quantified _existentially_.

With form held fixed, as the reference system is displaced, variation in the position operator _x_ determines a group relation, representing the homogeneity symmetry. Under arbitrarily small displacements, this group corresponds to a linear algebra representing homogeneity locally (Lie group and Lie algebra). To maintain the form of (1), under translation, the basis \[ |f_x\rangle \] is cleverly managed: while the translation transforms the basis from \[ |f_x\rangle \] to \[ |f_{x+\epsilon}\rangle \], a similarity transformation is also applied, chosen to revert \[ |f_{x+\epsilon}\rangle \] back to \[ |f_x\rangle \]. In this way \[ |f_x\rangle \] is held static. We see below that, actually, similarity transforms can be found only for a certain class of functions: \{ _\psi_ \in L^1 \} \subseteq \{ _f_x \}.

The similarity transformations are the one-parameter subgroup of the general linear group, \( S(\epsilon) \subset S \in GL(F) \), with the transformation parameter \( \epsilon \) coinciding with the displacement parameter. The overall scheme of transformations is depicted in Figure 1.

In standard theory, textbook understanding is that \( S(\epsilon) \) is intrinsically and necessarily unitary, and it is in _that_ unitarity where the Canonical Commutation Relation
Relation finds its unitary origins. And so, because its presence is thought \textit{intrinsically necessary}, unitarity is imposed axiomatically on the theory, \textit{by Postulate}. The upshot is that standard theory imposes Hilbert space on vectors \(|\psi_x\rangle\). This imposed unitarity is added information, extra to the information of homogeneity. In consequence, the underlying symmetry beneath wave mechanics is not homogeneity of space, but instead, a unitary subgroup.

As an experiment, we proceed, in this paper, by treating unitarity as a purely separate issue from homogeneity, allowing \(S(\epsilon)\) it’s widest generality, so that the \textit{whole information} of homogeneity (upto the general linear similarity transformation) is faithfully and genuinely conveyed through the theory.

The experiment begins with the eigenvalue equation for position (1) being rewritten, as the eigenformula in the quantified proposition (2). From here on, all informal assumptions are to be shed, with the Dirac notation dropped to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or equipped with a scalar product; none of these is implied.

Consider the eigenformula for position operator \(x\), eigenfunctions \(f_x\) and eigenvalues \(\epsilon\), seen from the reference frame \(O_x\):

\[
\forall x \exists x' \exists \epsilon \\exists f_x | \ x f_x (x) = \epsilon f_x (x) \tag{2}
\]

\textbf{Translation:} Applying the translation first. Under translation, homogeneity demands existence of an equally relevant reference frame \(O_{x'}\) displaced arbitrarily through \(\epsilon\). See Figure 2. \textit{Form Invariance} guarantees a formula for \(O_{x'}\) of the same form as that for \(O_x\) in (2), thus:

\[
\forall x' \exists x' \exists \epsilon' \exists f'_{x'} | \ x' f'_{x'} (x') = \epsilon' f'_{x'} (x') \tag{3}
\]

A relation for \(x\) is to be evaluated, so \(x\) is held static for all reference frames. The transformation translates position, thus:

\[
\forall \epsilon \forall x' \exists x | \ x \mapsto x' = x + \epsilon \tag{4}
\]

and transforms the function, thus:

\[
\forall \epsilon \forall x' \forall f' \exists f_{x-\epsilon} \exists x | \ f_{x-\epsilon} \mapsto f'_{x-\epsilon} = f_{x-\epsilon} (x - \epsilon) \tag{5}
\]

Substituting (4) and (5) into (3) gives the translated formula:

\[
\forall x \forall \epsilon \exists x' \exists \epsilon' \exists f_{x-\epsilon} | \ x f_{x-\epsilon} (x - \epsilon) = (x + \epsilon) f_{x-\epsilon} (x - \epsilon) \tag{6}
\]

\textbf{Substitution involving quantified variables}

\[
\forall \beta \forall \gamma \exists \alpha | \ \alpha = \beta + \gamma \Rightarrow \forall \beta \forall \gamma | \ \gamma = \beta + \beta \Rightarrow \forall \beta \forall \gamma | \ \alpha = \beta + \beta + \beta
\]

For logically dependent substitution, an existential quantifier of one proposition should be matched with a universal quantifier of the other. This is because, for this type of substitution coincidence is certain and not accidental. In the example given, matching quantifiers are underlined.

\textbf{Similarity:} Applying the similarity transformation. This involves the one parameter linear operator \(S(\epsilon)\). Such an \(S(\epsilon)\) exists only if there exists a space of functions \(\psi_x\), that is complete, normalisable, and not restricted to separable\(^1\) functions, that is also a subset of the translatable functions \(f_x\). Logically, the act of assuming such an \(S(\epsilon)\) hypothesises that such a class of functions does indeed exist. No such function space is guaranteed. Accordingly, the assertion of proposition (7) is newly assumed information entering the system.

\[
\forall x \forall \epsilon \forall \psi \exists S(\epsilon) | \ S^{-1}(\epsilon) \psi (x) = \psi_{x-\epsilon} (x - \epsilon) \tag{7}
\]

\(^1\) Separable means countable, as are the integers, as opposed to continuous, like the reals.
In standard theory, $S'(g)$ is set unitary by the mathematician. Doing that restricts the space of functions $\psi$ to the Hilbert space $L^2$. Here, $S'(g)$ is a member of the one parameter subgroup of the infinite dimensional, (non-unitary) general linear group GL($\mathbb{F}$). This restricts $\psi$ not to the Hilbert space $L^2$ but to the Banach space $L^1$.

The similarity transformation is formed, thus:

$$\forall x \exists S(\eta)x \exists \psi_k S \mid S'(g)xS^{-1}(g)\psi_k(x) = (x + \epsilon)\psi_k(x).$$

Introducing the trivial eigenformula: $\forall \psi_k \forall \epsilon \epsilon \psi_k(x) = \epsilon \psi_k(x)$ and subtracting:

$$\forall x \exists S(\eta)x \exists \psi_k S \mid \left( S'(g)xS^{-1}(g) - \epsilon I \right) \psi_k(x) = x\psi_k(x).$$

Now comparing the original position formula (2) against the transformed one (8), we deduce the group relation for similarity transformed homogeneity:

$$\forall x \exists \psi_k S \mid x\psi_k(x) = \left( S'(g)xS^{-1}(g) - \epsilon I \right) \psi_k(x).$$

From this group relation, the commutator for the Lie algebra is now computed. Because $S'(g)$ is a one-parameter subgroup of the Lie algebra, there exists a unique linear operator $g$ for real parameters $\epsilon$, such that:

$$\forall S \exists g \mid S'(g) = e^{\epsilon g}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor $\eta$ is extracted, thus: $\forall \eta \forall \exists k : g = \eta k$, implying:

$$\forall \eta \forall \exists k \mid S'(g) = e^{\eta k}$$

$$\forall \eta \forall \exists k \mid S'(\eta) = e^{-\eta k}$$

Substitution of (11) and (12) into (9) gives:

$$\forall x \forall \eta \exists \psi_k \exists \exists k \mid \exp\left(\eta k\right) x \exp\left(-\eta k\right) \psi_k(x) = \left[x + \epsilon I\right] \psi_k(x)$$

$$\Rightarrow \forall x \forall \eta \exists \psi_k \exists \exists k \mid \left[x + \eta k + O\left(x^2\right)\right] \psi_k(x) = \left[x + \epsilon I\right] \psi_k(x)$$

$$\Rightarrow \forall x \forall \eta \exists \psi_k \exists \exists k \mid \left[x + \eta k + O\left(x^2\right)\right] \psi_k(x) = \left[x + \epsilon I\right] \psi_k(x)$$

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$$\Rightarrow \forall x \forall \eta \exists \psi_k \exists \exists k \mid \left[x + \eta k + O\left(x^2\right)\right] \psi_k(x) = \left[x + \epsilon I\right] \psi_k(x)$$

At the limit, as $\epsilon \to 0$, we have:

$$\forall x \forall \eta \exists \psi_k \exists \exists k \mid \left[k, x\right] \psi_k(x) = \eta^{-1} \psi_k(x)$$

And by a similar proof, conditional on the existence of eigenfunctions $\chi(k)$, of $k$

$$\forall k \forall \eta \exists \chi_k \exists \exists k \mid \left[x, k\right] \chi_k(k) = \theta^{-1} \chi_k(k).$$

Importantly, we see that (13) is quantified $\forall \eta$, and similarly (14) is $\forall \theta$ rather than the particular case of $\eta^{-1} = -i$ or $\theta^{-1} = +i$ that we see in the unitary subalgebra we know as the Canonical Commutation Relation:

$$\left[k, x\right] = -i \mathbb{1} \quad \text{or} \quad \left[p, x\right] = -i \mathbb{1}$$

Conclusion

The above establishes that the homogeneity symmetry is not a source of unitary information in wave mechanics. And therefore, if the reason given for postulating that quantum theory should be unitary or self-adjoint, is that symmetries in Nature are ontologically unitary, then a different reason must be found, and the postulate must be withdrawn.

This opens up the possibility of a logical modification to quantum theory, where quantum theory remains open to unitarity, but where, that unitarity (or self-adjointness) is not axiomatically imposed by Postulate.

As a further possibility, that modified quantum theory would be open to logical independence, well-known to Mathematical Logic, entering from elementary algebra into quantum mathematics.
References