BEAL’s Conjecture: A Complete Proof

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Abstract

In 1997, Andrew Beal [1] announced the following conjecture: Let $A, B, C, m, n,$ and $l$ be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then $A, B,$ and $C$ have a common factor. We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with $p, q$ integers depending of $A^m, B^n$ and $C^l$. We resolve $x^3 - px + q = 0$ and we obtain the three roots $x_1, x_2, x_3$ as functions of $p, q$ and a parameter $\theta$. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that $x_1, x_2, x_3$ are integers. Three numerical examples are given.

Keywords: Prime numbers, divisibility, roots of polynomials of third degree.

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O my Lord! Increase me further in knowledge.

(Holy Quran, Surah Ta Ha, 20:114.)

To my wife Wahida

1. Introduction

In 1997, Andrew Beal [1] announced the following conjecture:

Conjecture 1. Let $A, B, C, m, n,$ and $l$ be positive integers with $m, n, l > 2$. If:

$$A^m + B^n = C^l$$

(1)

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then $A, B,$ and $C$ have a common factor.

In this paper, we give a complete proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots $A^m, B^n$ and $-C^l$ with the condition $[1]$. The paper is organized as follows. In Section 2 of preliminaries, we begin with the trivial case where $A^m = B^n$. Then we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. We express the three roots of $P(x) = x^3 - px + q = 0$ in function of two parameters $\rho, \theta$ that depend of $A^m, B^n, C^l$. The Section 3 is the main part of the paper. We write that $A^{2m} = \frac{4p\cos^2\theta}{3}$. As $A^{2m}$ is an integer, it follows that $\cos^2\theta$ must be written as $\frac{a}{b}$ where $a, b$ are two positive coprime integers. We discuss the conditions of divisibility of $p, a, b$ so that the expression of $A^{2m}$ is an integer. Depending on each individual case, we obtain that $A, B, C$ have or not a common factor. In the last Section, three numerical examples are presented. We finish with the conclusion.

2. Preliminaries

We begin with the trivial case when $A^m = B^n$. The equation $[1]$ becomes:

$$2A^m = C^l$$

(2)

then $2|C^l$ $\Rightarrow$ $2|C$ $\Rightarrow$ $\exists c \in N^*/C = 2c$, it follows $2A^m = 2^l c^l$ $\Rightarrow$ $A^m = 2^{l-1} c^l$. As $l > 2$, then $2|A^m$ $\Rightarrow$ $2|A$ $\Rightarrow$ $2|B^n$ $\Rightarrow$ $2|B$. The conjecture (??) is verified.

We suppose in the following that $A^m > B^n$.

2.1. General Case

Let $m, n, l \in N^* > 2$ and $A, B, C \in N^*$ such:

$$A^m + B^n = C^l$$

(3)
We call:

\[ P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \]
\[ + x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \]  

Using the equation (3), \( P(x) \) can be written:

\[ P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n (A^m + B^n) \]  

We introduce the notations:

\[ p = (A^m + B^n)^2 - A^m B^n \]  
\[ q = A^m B^n (A^m + B^n) \]  

As \( A^m \neq B^n \), we have :

\[ p > (A^m - B^n)^2 > 0 \]  

Equation (5) becomes:

\[ P(x) = x^3 - px + q \]  

Using the equation (4), \( P(x) = 0 \) has three different real roots : \( A^m, B^n \) and \( -C^l \).

Now, let us resolve the equation:

\[ P(x) = x^3 - px + q = 0 \]  

To resolve (10) let:

\[ x = u + v \]  

Then \( P(x) = 0 \) gives:

\[ P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \implies u^3 + v^3 + (u+v)(3uv-p) + q = 0 \]  

To determine \( u \) and \( v \), we obtain the conditions:

\[ u^3 + v^3 = -q \]  
\[ uv = p/3 > 0 \]
Then \( u^3 \) and \( v^3 \) are solutions of the second order equation:

\[
X^2 + qX + p^3/27 = 0 \tag{15}
\]

Its discriminant \( \Delta \) is written as:

\[
\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27} \tag{16}
\]

Let:

\[
\bar{\Delta} = 27q^2 - 4p^3 = 27(A^mB^n(A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^mB^n]^3
\]

\[
= 27A^{2m}B^{2n}(A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^mB^n]^3 \tag{17}
\]

Noting:

\[
\alpha = A^mB^n > 0 \tag{18}
\]

\[
\beta = (A^m + B^n)^2 \tag{19}
\]

we can write (17) as:

\[
\bar{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3 \tag{20}
\]

As \( \alpha \neq 0 \), we can also rewrite (20) as:

\[
\bar{\Delta} = \alpha^3 \left( 27 \frac{\beta}{\alpha} - 4 \left( \frac{\beta}{\alpha} - 1 \right)^3 \right) \tag{21}
\]

We call \( t \) the parameter:

\[
t = \frac{\beta}{\alpha} \tag{22}
\]

\( \bar{\Delta} \) becomes:

\[
\bar{\Delta} = \alpha^3(27t - 4(t - 1)^3) \tag{23}
\]

Let us calling:

\[
y = y(t) = 27t - 4(t - 1)^3 \tag{24}
\]

Since \( \alpha > 0 \), the sign of \( \bar{\Delta} \) is also the sign of \( y(t) \). Let us study the sign of \( y \).

We obtain \( y'(t) \):

\[
y'(t) = y' = 3(1 + 2t)(5 - 2t) \tag{25}
\]
Figure 1: The table of variation

\( y' = 0 \implies t_1 = -1/2 \) and \( t_2 = 5/2 \), then the table of variations of \( y \) is given below:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( -\infty )</th>
<th>(-1/2)</th>
<th>( 5/2)</th>
<th>( 4)</th>
<th>( +\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y' )</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( y )</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

The table of the variations of the function \( y \) shows that \( y < 0 \) for \( t > 4 \). In our case, we are interested for \( t > 0 \). For \( t = 4 \) we obtain \( y(4) = 0 \) and for \( t \in [0, 4] \implies y > 0 \). As we have \( t = \frac{\beta}{\alpha} > 4 \) because as \( A^m \neq B^n \):

\[
(A^m - B^n)^2 > 0 \implies \beta = (A^m + B^n)^2 > 4\alpha = 4A^mB^n \quad (26)
\]

Then \( y < 0 \implies \Delta < 0 \implies \Delta < 0 \). Then, the equation (15) does not have real solutions \( u^3 \) and \( v^3 \). Let us find the solutions \( u \) and \( v \) with \( x = u + v \) is a positive or a negative real and \( u.v = p/3 \).

2.2. Demonstration

**Proof.** The solutions of (15) are:

\[
X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \quad (27)
\]

\[
X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \quad (28)
\]

We may resolve:

\[
u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \quad (29)
\]

\[
v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \quad (30)
\]

Writing \( X_1 \) in the form:

\[
X_1 = \rho e^{i\theta} \quad (31)
\]
with:

\[ \rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{\rho}}{3\sqrt{3}} \]  (32)

and

\[ \sin \theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \]  (33)

\[ \cos \theta = -\frac{q}{2\rho} < 0 \]  (34)

Then \[ \theta \in \left[ \frac{\pi}{2}, +\pi \right], \] let:

\[ \frac{\pi}{6} < \theta < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \cos \frac{\theta}{3} < \frac{\sqrt{3}}{2} \]  (35)

and:

\[ \frac{1}{4} < \cos^2 \frac{\theta}{3} < \frac{3}{4} \]  (36)

hence the expression of \( X_2 \):

\[ X_2 = \rho e^{-i\theta} \]  (37)

Let:

\[ u = re^{i\psi} \]  (38)

and

\[ j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \]  (39)

\[ j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j} \]  (40)

\( j \) is a complex cubic root of the unity \( \iff j^3 = 1 \). Then, the solutions \( u \) and \( v \) are:

\[ u_1 = re^{i\psi_1} = \sqrt[3]{\rho} e^{i\frac{\pi}{6}} \]  (41)

\[ u_2 = re^{i\psi_2} = \sqrt[3]{\rho j} e^{i\frac{\pi}{3}} = \sqrt[3]{\rho} e^{i\frac{\pi}{2} + i\frac{\pi}{3}} \]  (42)

\[ u_3 = re^{i\psi_3} = \sqrt[3]{\rho j^2} e^{i\frac{\pi}{2} + i\frac{\pi}{3}} = \sqrt[3]{\rho} e^{i\frac{\pi}{2} + i\frac{5\pi}{6}} \]  (43)

and similarly:

\[ v_1 = re^{-i\psi_1} = \sqrt[3]{\rho} e^{-i\frac{\pi}{6}} \]  (44)

\[ v_2 = re^{-i\psi_2} = \sqrt[3]{\rho j} e^{-i\frac{\pi}{3}} = \sqrt[3]{\rho} e^{i\frac{\pi}{2} - i\frac{\pi}{3}} \]  (45)

\[ v_3 = re^{-i\psi_3} = \sqrt[3]{\rho j} e^{-i\frac{\pi}{2}} = \sqrt[3]{\rho} e^{i\frac{\pi}{2} - i\frac{2\pi}{3}} \]  (46)
We may now choose \( u_k \) and \( v_h \) so that \( u_k + v_h \) will be real. In this case, we have necessary:

\[
\begin{align*}
v_1 &= u_1 \quad (47) \\
v_2 &= u_2 \quad (48) \\
v_3 &= u_3 \quad (49)
\end{align*}
\]

We obtain as real solutions of the equation (12):

\[
\begin{align*}
x_1 &= u_1 + v_1 = 2\sqrt{\rho}\cos\frac{\theta}{3} > 0 \quad (50) \\
x_2 &= u_2 + v_2 = 2\sqrt{\rho}\cos\frac{2\pi}{3} = -\sqrt{\rho}\left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) < 0 \quad (51) \\
x_3 &= u_3 + v_3 = 2\sqrt{\rho}\cos\frac{4\pi}{3} = \sqrt{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) > 0 \quad (52)
\end{align*}
\]

We compare the expressions of \( x_1 \) and \( x_3 \), we obtain:

\[
\begin{align*}
2\sqrt{\rho}\cos\frac{\theta}{3} &> \sqrt{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \\
3\cos\frac{\theta}{3} &> \sqrt{3}\sin\frac{\theta}{3} \quad (53)
\end{align*}
\]

As \( \frac{\theta}{3} \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \), then \( \sin\frac{\theta}{3} \) and \( \cos\frac{\theta}{3} \) are > 0. Taking the square of the two members of the last equation, we get:

\[
\frac{1}{4} < \cos^2\frac{\theta}{3} \quad (54)
\]

which is true since \( \frac{\theta}{3} \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \) then \( x_1 > x_3 \). As \( A^m, B^n \) and \( -C^l \) are the only real solutions of (10), we consider, as \( A^m \) is supposed great than \( B^n \), the expressions:

\[
\begin{align*}
A^m &= x_1 = u_1 + v_1 = 2\sqrt{\rho}\cos\frac{\theta}{3} \\
B^n &= x_3 = u_3 + v_3 = 2\sqrt{\rho}\cos\frac{\theta + 4\pi}{3} = \sqrt{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \\
-C^l &= x_2 = u_2 + v_2 = 2\sqrt{\rho}\cos\frac{\theta + 2\pi}{3} = -\sqrt{\rho}\left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right)
\end{align*}
\]
3. Proof of the Main Theorem

**Main Theorem:** Let $A, B, C, m, n, \text{ and } l$ be positive integers with $m, n, l > 2$. If:

$$A^m + B^n = C^l$$

then $A, B$, and $C$ have a common factor.

**Proof.** $A^m = 2\sqrt[3]{p\cos \theta}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{p^2\cos^2 \theta} = 3$ is an integer. But:

$$\sqrt[3]{p^2} = \frac{p}{3}$$

Then:

$$A^{2m} = 4\sqrt[3]{p^2\cos^2 \theta} = 4\frac{p}{3}\cos^2 \theta = \frac{4p}{3}\cos^2 \theta$$

As $A^{2m}$ is an integer, and $p$ is an integer then $\cos^2 \frac{\theta}{3}$ must be written in the form:

$$\cos^2 \frac{\theta}{3} = \frac{1}{b} \text{ or } \cos^2 \frac{\theta}{3} = \frac{a}{b}$$

with $b \in N^*$, for the last condition $a \in N^*$ and $a, b$ coprime.

3.1. **Case** $\cos^2 \frac{\theta}{3} = \frac{1}{b}$

We obtain:

$$A^{2m} = \frac{4p}{3}\cos^2 \frac{\theta}{3} = \frac{4p}{3b}$$

As $\frac{1}{4} < \cos^2 \frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3$.

3.1.1. **Case** $b = 1$

$$b = 1 \Rightarrow 4 < 3 \text{ which is impossible.}$$
3.1.2. Case $b = 2$

$$b = 2 \Rightarrow A^{2m} = p \cdot \frac{4}{3}\left(\frac{1}{2}\right) = \frac{2p}{3} \Rightarrow 3|p \Rightarrow p = 3p'$$ with $p' \neq 1$ because $3 \ll p$.

and $b = 2$, we obtain:

$$A^{2m} = \frac{2p}{3} = 2p'$$  \hspace{1cm} (61)

But :

$$B^n C^l = \sqrt[n]{p^2 \left(3 - 4 \cos^2 \frac{\theta}{3}\right)} = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3}\right) = \frac{p}{3} = \frac{3p'}{3} = p'$$ \hspace{1cm} (62)

On the one hand:

$$A^{2m} = (A^m)^2 \Rightarrow 2|p' \Rightarrow p' = 2p'^2 \Rightarrow A^{2m} = 4p'^2$$

$$\Rightarrow A'' = 2p'' \Rightarrow 2|A'' \Rightarrow 2|A$$

On the other hand:

$$B^n C^l = p' = 2p'^2 \Rightarrow 2|B^n \text{ or } 2|C^l.$$ If $2|B^n \Rightarrow 2|B$. As $C^l = A^m + B^n$ and $2|A$ and $2|B$, it follows $2|A''$ and $2|B^n$ then $2|(A^m + B^n) \Rightarrow 2|C^l \Leftrightarrow 2|C$.

Then, we have : $A, B$ and $C$ solutions of (3) have a common factor. Also if $2|C^l$, we obtain the same result : $A, B$ and $C$ solutions of (3) have a common factor.

3.1.3. Case $b = 3$

$$b = 3 \Rightarrow A^{2m} = p \cdot \frac{4}{3}\left(\frac{1}{3}\right) = \frac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p'$$ with $p' \neq 1$ since $9 \ll p$ then $A^{2m} = 4p' \Rightarrow p'$ is not a prime. Let $\mu$ a prime with $\mu|p' \Rightarrow \mu|A^{2m} \Rightarrow \mu|A$.

On the other hand:

$$B^n C^l = \frac{p}{3} \left(3 - 4 \cos \frac{\theta}{3}\right) = 5p'$$

Then $\mu|B^n$ or $\mu|C^l$. If $\mu|B^n \Rightarrow \mu|B.$ As $C^l = A^m + B^n$ and $\mu|A$ and $\mu|B$, it follows $\mu|A^n$ and $\mu|B^n$ then $\mu|(A^m + B^n) \Rightarrow \mu|C^l \Rightarrow \mu|C$.

Then, we have : $A, B$ and $C$ solutions of (3) have a common factor. Also if $\mu|C^l$, we obtain the same result : $A, B$ and $C$ solutions of (3) have a common factor.
3.2. **Case** $a > 1$.  
\[ \cos^2 \frac{\theta}{3} = \frac{a}{b} \]
That is to say:

\[ \cos^2 \frac{\theta}{3} = \frac{a}{b} \quad (63) \]
\[ A^{2m} = p \cdot \frac{4}{3} \cos^2 \frac{\theta}{3} = \frac{4p \cdot a}{3b} \quad (64) \]

and $a, b$ verify one of the two conditions:

\[
\begin{align*}
\{3|p \text{ and } b|4p\} & \quad \text{or} \quad \{3|a \text{ and } b|4p\} \\
\end{align*}
\]

(65)

and using the equation (36), we obtain a third condition:

\[ b < 4a < 3b \quad (66) \]

In these conditions, respectively, $A^{2m} = 4\sqrt{p^2 \cos^2 \frac{\theta}{3}} = 4\frac{p}{3} \cos^2 \frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (65).

3.2.1. **Hypothesis:** \{3|p \text{ and } b|4p\}

3.2.1.1. **Case** $b = 2$ and $3|p$:

$3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

\[ A^{2m} = \frac{4p \cdot a}{3b} = \frac{4.3p'.a}{3b} = \frac{4p'.a}{2} = 2p'.a \quad (67) \]

As:

\[ \frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow a < 2 \Rightarrow a = 1 \quad (68) \]

But $a > 1$ then the case $b = 2$ and $3|p$ is impossible.

3.2.1.2. **Case** $b = 4$ and $3|p$:

We have $3|p \Rightarrow p = 3p'$ with $p' \in N^*$, it follows:

\[ A^{2m} = \frac{4p \cdot a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a \quad (69) \]

and:

\[ \frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2 \quad (70) \]

But $a, b$ are coprime. Then the case $b = 4$ and $3|p$ is impossible.
3.2.1.3. Case: \( b \neq 2, b \neq 4, b|p \) and \( 3|p \). As \( 3|p \) then \( p = 3p' \) and :

\[
A^{2m} = \frac{4p}{3} \cos^2 \theta = \frac{4p}{3} \frac{a}{b} = \frac{4 \times 3p' a}{3b} = \frac{4p' a}{b} \quad (71)
\]

We consider the case: \( b|p' \implies p' = bp'' \) and \( p'' \neq 1 \) (if \( p'' = 1 \), then \( p = 3b \), see sub-paragraph II. Case \( k'=1 \) of paragraph 3.2.1.8). Hence :

\[
A^{2m} = \frac{4bp'' a}{b} = 4ap'' \quad (72)
\]

Let us calculate \( B^nC^l \):

\[
B^nC^l = \frac{p}{3} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) = p' \left( 3 - 4 \frac{a}{b} \right) = b.p''. \frac{3b - 4a}{b} = p''(3b - 4a) \quad (73)
\]

Finally, we have the two equations:

\[
A^{2m} = \frac{4bp'' a}{b} = 4ap'' \quad (74)
\]

\[
B^nC^l = p''(3b - 4a) \quad (75)
\]

I. Case \( p'' \) is prime:

From (74), \( p''|A^{2m} \Rightarrow p''|A^m \Rightarrow p''|A \). From (75), \( p''|B^n \) or \( p''|C^l \). If \( p''|B^n \Rightarrow p''|B \), as \( C^l = A^m + B^n \Rightarrow p''|C^l \Rightarrow p''|C \). If \( p''|C^l \Rightarrow p''|C \), as \( B^n = C^l - A^m \Rightarrow p''|B^n \Rightarrow p''|B \).

Then \( A,B \) and \( C \) solutions of (3) have a common factor.

II. Case \( p'' \) is not prime:

Let \( \lambda \) one prime divisor of \( p'' \). From (74), we have :

\[
\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as \( \lambda \) is prime then} \quad \lambda|A \quad (76)
\]

From (75), as \( \lambda|p'' \) we have:

\[
\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or} \quad \lambda|C^l \quad (77)
\]

If \( \lambda|B^n \), \( \lambda \) is prime \( \lambda|B \), and as \( C^l = A^m + B^n \) then we have also :

\[
\lambda|C^l \quad \text{as \( \lambda \) is prime, then} \quad \lambda|C \quad (78)
\]
By the same way, if $\lambda|C^l$, we obtain $\lambda|B$.

Then: $A, B$ and $C$ solutions of \ref{eq:3} have a common factor.

Let us verify the condition \ref{eq:66} given by:

$$b < 4a < 3b$$

In our case, the last equation becomes:

$$p < 3A^{2m} < 3p \quad \text{with} \quad p = A^{2m} + B^{2n} + A^m B^n$$

The condition $3A^{2m} < 3p \implies A^{2m} < p$ is verified.

If:

$$p < 3A^{2m} \implies 2A^{2m} - A^m B^n - B^{2n} > 0$$

We put $Q(Y) = 2Y^2 - B^n Y - B^{2n}$, the roots of $Q(Y) = 0$ are $Y_1 = -\frac{B^n}{2}$ and $Y_2 = B^n$. $Q(Y) > 0$ for $Y < Y_1$ and $Y > Y_2 = B^n$. In our case, we take $Y = A^m$. As $A^m > B^n$ then $p < 3A^{2m}$ is verified. Then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify $A^m > B^n$ which is true.

3.2.1.4. Case $b = 3$ and $3|p$. As $3|p \implies p = 3p'$ and we write:

$$A^{2m} = \frac{4p}{3} \cos^2 \theta = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4 \times 3p' a}{3} = \frac{4p'a}{3}$$

(80)

As $A^{2m}$ is an integer and that $a$ and $b$ are coprime and $\cos^2 \theta$ can not be one in reference to the equation \ref{eq:35}, then we have necessary $3|p' \implies p' = 3p''$ with $p'' \neq 1$, if not $p = 3p' = 3 \times 3p'' = 9$ but $p = A^{2m} + B^{2n} + A^m B^n > 9$, the hypothesis $p'' = 1$ is impossible, then $p'' > 1$. hence:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p'' a}{3} = 4p'' a$$

(81)

$$B^n C^l = \frac{p}{3} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) = p' \left( 3 - 4 \frac{a}{b} \right) = \frac{3p''(9 - 4a)}{3} = p''(9 - 4a)$$

(82)
As $\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} < \frac{3}{4} \implies 3 < 4a < 9 \implies a = 2$ as $a > 1$. $a = 2$, we obtain:

$$A^{2m} = \frac{4p' a}{3} = \frac{4 \times 3p'' a}{3} = 4p'' a = 8p''$$

(83)

$$B'' C' = \frac{p}{3} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) = p' \left( 3 - 4 \frac{a}{b} \right) = \frac{3p'' (9 - 4a)}{3} = p''$$

(84)

The two last equations give that $p''$ is not prime. Then we use the same methodology described above for the case 3.2.1.3., and we have: $A, B$ and $C$ solutions of (3) have a common factor.

3.2.1.5. Case $3 \mid p$ and $b = p$ : We have:

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$$

and :

$$A^{2m} = \frac{4p' a}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4a}{3}$$

(85)

As $A^{2m}$ is an integer, this implies that $3 \mid a$, but $3 \mid p \implies 3 \mid b$. As $a$ and $b$ are coprime, hence the contradiction. Then the case $3 \mid p$ and $b = p$ is impossible.

3.2.1.6. Case $3 \mid p$ and $b = 4p$ : $3 \mid p \implies p = 3p', p' \neq 1$ because $3 \ll p$. Hence $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p a}{3} = \frac{a}{3} \implies 3 \mid a$$

(86)

because $A^{2m}$ is an integer. But $3 \mid p \implies 3 \mid 4p = b$, that is in contradiction with the hypothesis $a, b$ are coprime. Then the case $b = 4p$ is impossible.

3.2.1.7. Case $3 \mid p$ and $b = 2p$ : $3 \mid p \implies p = 3p', p' \neq 1$ because $3 \ll p$. Hence $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p a}{3} = \frac{2a}{3} \implies 3 \mid a$$

(87)

because $A^{2m}$ is an integer. But $3 \mid p \implies 3 \mid 2p \implies 3 \mid b$, that is in contradiction with the hypothesis $a, b$ are coprime. Then the case $b = 2p$ is impossible.
3.2.1.8. Case 3|p and b ≠ 3 is a divisor of p . We have b = p' ≠ 3, and p is written as:

\[ p = kp' \quad \text{with} \quad 3|k \implies k = 3k' \quad (88) \]

and:

\[ A^{2m} = \frac{4p}{3} \cos \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4 \times 3.k'p' \cdot a}{p'} = 4ak' \quad (89) \]

We calculate \( B^n C^l \):

\[ B^n C^l = \frac{p}{3} \left( 3 - 4\cos^2 \frac{\theta}{3} \right) = k'(3p' - 4a) \quad (90) \]

**I. Case \( k' \neq 1 \):**

We suppose \( k' \neq 1 \), we use the same methodology described for the case 3.1.2.3., and we obtain: \( A, B \) and \( C \) solutions of \( (3) \) have a common factor.

**II. Case \( k' = 1 \):**

We have \( k' = 1 \implies p = 3b \), then we have:

\[ A^{2m} = 4a \implies a \quad \text{is even} \quad (91) \]

and:

\[ A^m B^n = 2\sqrt{3} \cos \frac{\theta}{3} \sqrt{3} \sin \frac{\theta}{3} - \cos \frac{\theta}{3} = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a \quad (92) \]

let:

\[ A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2b\sqrt{3} \sin \frac{2\theta}{3} \quad (93) \]

The left member of \( (93) \) is an integer and \( b \) also, then \( 2\sqrt{3} \sin \frac{2\theta}{3} \) can be written in the form:

\[ 2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2} \quad (94) \]

where \( k_1, k_2 \) are two coprime integers and \( k_2 | b \implies b = k_2 k_3 \).

**II.1. Case \( k_3 \neq 1 \):**

We suppose \( k_3 \neq 1 \). Hence:

\[ A^{2m} + 2A^m B^n = k_3 k_1 \quad (95) \]
Let $\mu$ be a prime integer such that $\mu | k_3$. If $\mu = 2 \Rightarrow 2 | b$, but $2 | a$ that is contradiction with $a, b$ coprime. We suppose $\mu \neq 2$ and $\mu | k_3$, then:

$$\mu | A^m(A^m + 2B^n) \Rightarrow \mu | A^m \text{ or } \mu | (A^m + 2B^n) \quad (96)$$

II.1.1. Case $\mu | A^m$:

If $\mu | A^m \Rightarrow \mu | 2^m \Rightarrow \mu | a$. As $\mu | k_3 \Rightarrow \mu | b$ and that $a,b$ are coprime hence the contradiction.

II.1.2. Case $\mu | (A^m + 2B^n)$:

If $\mu | (A^m + 2B^n) \Rightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu | (A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu t' \quad t' \in N^* \quad (97)$$

It follows:

$$A^m + B^n = \mu t' - B^n \Rightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of $p$, we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \quad (98)$$

As $p = 3b = 3k_2k_3$ and $\mu | k_3$ hence $\mu | p \Rightarrow p = \mu \mu'$, so we have:

$$\mu' \mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m) \quad (99)$$

then:

$$\mu | B^n (B^n - A^m) \Rightarrow \mu | B^n \text{ or } \mu | (B^n - A^m) \quad (100)$$

II.1.2.1. Case $\mu | B^n$:

If $\mu | B^n \Rightarrow \mu | B$ which is in contradiction with case II.1.2. above.

II.1.2.2. Case $\mu | (B^n - A^m)$:
If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\mu|3B^n$$

(101)

**II.1.2.2.1. Case $\mu|B^n$:**

If $\mu|B^n$, using the result above of II.1.2.1. of this paragraph, it is impossible.

**II.1.2.2.2. Case $\mu = 3$:**

If $\mu = 3 \implies 3|k_3 \implies k_3 = 3k'_3$, and we have $b = k_2k_3 = 3k_2k'_3$, it follows

$$p = 3b = 9k_2k'_3$$

then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then :

$$9k_2k'_3 - 3A^mB^n = (A^m - B^n)^2$$

we write it as :

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2$$

(102)

hence :

$$3|(3k_2k'_3 - A^mB^n) \implies 3|A^mB^n \implies 3|A^m \text{ or } 3|B^n$$

(103)

**II.1.2.2.2.1. Case $3|A^m$:**

If $3|A^m \implies 3|A$ and we have also $3|A^{2m}$, but $A^{2m} = 4a \implies 3|4a \implies 3|a$. As $b = 3k_2k'_3$ then $3|b$, but $a, b$ are coprime hence the contradiction. Then $3 \nmid A$.

**II.1.2.2.2.2. Case $3|B^n$:**

If $3|B^n \implies 3|B$, but the (102) gives $3|(A^m - B^n)^2 \implies 3|(A^m - B^n) \implies 3|A^m \implies 3|(A^{2m} = 4a) \implies 3|a$. As $3|b$ then the contradiction with $a, b$ coprime.

Then the hypothesis $k_3 \neq 1$ is impossible.

**III. Case $k_3 = 1$:**

Now we suppose that $k_3 = 1 \implies b = k_2$ and $p = 3b = 3k_2$. We have then:

$$2\sqrt{3}\sin\frac{2\theta}{3} = \frac{k_3}{b}$$

(104)
with $k_1, b$ coprime. We write (104) as:

$$4\sqrt{3} \sin \frac{\theta}{3} \cos \frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $\cos^2 \frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2 \cdot a(b - a) = k_1^2$$

which implies that:

$$3|a \quad \text{or} \quad 3|(b - a)$$

(105)

III.1. Case $3|a$:

If $3|a$, as $A^{2m} = 4a \implies 3|A^{2m} \implies 3|A$ and $3|a$. But $p = (A^m - B^n)^2 + 3A^m B^n$ and that $3|p \implies 3|(A^m - B^n)^2 \implies 3|(A^m - B^n)$. But $3|A$ hence $3|B \implies 3|B$, as $m \geq 3 \implies 3^2|p$, it follows $3|b$ then the contradiction with $a, b$ coprime.

III.2. Case $3|(b - a)$:

Considering now that $3|(b - a)$. As $k_1 = A^m(A^m + 2B^n)$ by the equation (95) and that $3|k_1 \implies 3|A^m(A^m + 2B^n) \implies 3|A^m$ or $3|(A^m + 2B^n)$.

(106)

III.2.1. Case $3|A^m$:

If $3|A^m \implies 3|A \implies 3|A^{2m}$ then $3|4a \implies 3|a$. But $3|(b - a) \implies 3|b$ hence the contradiction with $a, b$ are coprime.

III.2.2. Case $3|(A^m + 2B^n)$:

If:

$$3|(A^m + 2B^n) \implies 3|(A^m - B^n)$$

(107)

But $p = A^{2m} + B^{2n} + A^m B^n = (A^m - B^n)^2 + 3A^m B^n$ then $p - 3A^m B^n = (A^m - B^n)^2 \implies 9|(p - 3A^m B^n)$ or $9|(3b - 3A^m B^n)$, then $3|(b - A^m B^n)$ but $3|(b - a) \implies 3|(a - A^m B^n)$.

As $A^{2m} = 4a = (A^m)^2 \implies \exists a' \in \mathbb{N}^*$ and $a = a'^2 \implies A^m = 2a'$. We arrive to:

$$3|(a'^2 - 3a' B^n) \implies 3|a'(a' - 2B^n) \implies 3|a' \quad \text{or} \quad 3|(a' - 2B^n)$$

(108)
III.2.2.1. Case $3|a'$:
If $3|a' \Rightarrow 3|a'^2 \Rightarrow 3|a$, but $3|(b - a) \Rightarrow 3|b$, then the contradiction with $a, b$ coprime.

III.2.2.2. Case $3|(a' - 2B^n)$:
Now if $3|(a' - 2B^n) \Rightarrow 3|(2a' - 4B^n) \Rightarrow 3|(A^m - 4B^n) \Rightarrow 3|(A^m - B^n)$, we refine the case III.2.2., equation (107), that has a solution given by the case 2.2.1. above.

Then, the study of the case 3.2.1.8. is finished.

3.2.1.9 Case $3|p$ and $b|4p$: As $3|p \Rightarrow p = 3p'$ and $b|4p \Rightarrow \exists k_1 \in N^*$ and $4p = 12p' = k_1b$.

I. Case $k_1 = 1$:
If $k_1 = 1$, then $b = 12p'$, ($p' \neq 1$ if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \frac{4p}{3} \cos\frac{\theta}{3} = \frac{12p'}{3} \cdot \frac{a}{b} = \frac{4p'.a}{12p'} = \frac{a}{3} \Rightarrow 3|a$ because $A^{2m}$ is an integer, then the contradiction with $a, b$ coprime.

II. Case $k_1 = 3$:
If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \frac{4p}{3} \cos\frac{\theta}{3} = \frac{k_1.a}{3} = a$.
Let us calculate $A^m B^n$:

$$A^m B^n = 2\sqrt{3} \cos\theta \left(\sqrt{3} \sin\frac{\theta}{3} - \cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3} \sin\frac{2\theta}{3} - \frac{a}{2}$$ (109)

Let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin\frac{2\theta}{3} = 2p'\sqrt{3} \sin\frac{2\theta}{3}$$ (110)

The left member of the equation (110) is an integer and also $p'$, then $2\sqrt{3} \sin\frac{2\theta}{3}$ can be written as :

$$2\sqrt{3} \sin\frac{2\theta}{3} = \frac{k_2}{k_3}$$ (111)

where $k_2, k_3$ are two coprime integers and:

$$k_3|p' \Rightarrow \exists k_4 \in N^* \text{ and } p' = k_3k_4$$ (112)
II.1. Case $k_4 \neq 1$:

We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2 k_4$$  \hspace{1cm} (113)

Let $\mu$ one prime integer with:

$$\mu | k_4$$  \hspace{1cm} (114)

Then :

$$\mu | A^m (A^m + 2B^n) \implies \mu | A^m \text{ or } \mu | (A^m + 2B^n)$$  \hspace{1cm} (115)

II.1.1. Case $\mu | A^m$:

If $\mu | A^m \implies \mu | A^{2m} \implies \mu | a$. As $\mu | k_4 \implies \mu | p' \Rightarrow \mu | (4p' = b)$. But $a, b$ are coprime then the contradiction.

II.1.2. Case $\mu | (A^m + 2B^n)$:

If $\mu | (A^m + 2B^n) \implies \mu | A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu | (A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu t' \hspace{0.5cm} t' \in \mathbb{N}^* $$  \hspace{1cm} (116)

It follows:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of $p$, we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) $$  \hspace{1cm} (117)

As $p = 3p'$ and $\mu | p' \Rightarrow \mu | (3p') \Rightarrow \mu | p$, we can write $\exists \mu' \in \mathbb{N}^* \text{ and } p = \mu \mu'$, then we obtain :

$$\mu' \mu = \mu (\mu t'^2 - 2t' B^n) + B^n (B^n - A^m) $$  \hspace{1cm} (118)

and:

$$\mu | B^n (B^n - A^m) \implies \mu | B^n \text{ or } \mu | (B^n - A^m) $$  \hspace{1cm} (119)
II.1.2.1. Case $\mu | B^n$:
If $\mu | B^n \Rightarrow \mu | B$ which is in contradiction with the case II.1.2. above.

II.1.2.2. Case $\mu | (B^n - A^m)$:
If $\mu | (B^n - A^m)$ and using $\mu | (A^m + 2B^n)$, we obtain:
\[
\mu | 3B^n \quad (120)
\]

II.1.2.2.1. Case $\mu | B^n$:
If $\mu | B^n$ it is impossible, see the case II.1.2.1. above.

II.1.2.2.2 Case $\mu = 3$:
If $\mu = 3 \Rightarrow 3 | k_4 \Rightarrow k_4 = 3k_4'$, and we obtain $p' = k_3k_4 = 3k_3k_4'$; it follows
$p = 3p' = 9k_3k_4'$ then $9 | p$, but $p = (A^m - B^n)^2 + 3A^mB^n$, then:
\[
9k_4k_5' - 3A^mB^n = (A^m - B^n)^2
\]
that we write :
\[
3(3k_4k_5' - A^mB^n) = (A^m - B^n)^2 \quad (121)
\]
then $3 | (3k_4k_5' - A^mB^n) \Rightarrow 3 | A^mB^n \Rightarrow 3 | A^m \ or \ 3 | B^n$

II.1.2.2.2.1. Case $3 | A^m$:
If $3 | A^m \Rightarrow 3 | A^{2m} \Rightarrow 3 | a$, but $3 | p' \Rightarrow 3 | (4p') \Rightarrow 3 | b$, then the contradiction with $a, b$ coprime. Then $3 \nmid A$.

II.1.2.2.2. Case $3 | B^n$:
If $3 | B^n$ and using (116), we have $A^m = \mu t' - 2B^n = 3t' - 2B^n \Rightarrow 3 | A^m \Rightarrow 3 | A^{2m} \Rightarrow 3 | a$, but $3 | p' \Rightarrow 3 | (4p') \Rightarrow 3 | b$, then the contradiction with $a, b$ coprime.

Then the hypothesis $k_4 \neq 1$ is impossible.

II.2. Case $k_4 = 1$:
We suppose that $k_4 = 1 \implies p' = k_3k_4 = k_3$. Then we obtain:

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_2}{p'}$$

(122)

with $k_2, p'$ coprime, we write (122) as:

$$4\sqrt{3} \sin \frac{\theta}{3} \cos \frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $\cos^2 \frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3a(b - a) = k_2^2$$

(123)

that implies:

$$3|a \quad \text{or} \quad 3|(b - a)$$

(124)

II.2.1. Case $3|a$:

If $3|a \implies 3|A^{2m} \implies 3|A$, as $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \implies 3|(A^m - B^n)^2 \implies 9|(A^m - B^n)^2$. But $(A^m - B^n)^2 = p - 3A^mB^n = 3b - 3A^mB^n \implies 3|(b - A^mB^n)$. As $3|A^m \implies 3|b \implies$ the contradiction with $a, b$ coprime.

II.2.2. Case $3|(b - a)$:

We consider that $3|(b - a)$. As $k_2 = A^m(A^m + 2B^n)$ given by the equation (113) and that $3|k_2 \implies 3|A^m(A^m + 2B^n) \implies 3|A^m \quad \text{or} \quad 3|(A^m + 2B^n)$

II.2.2.1. Case $3|A^m$:

If $3|A^m \implies 3|A^{2m} \implies 3|a$, but $3|(b - a) \implies 3|b$ then the contradiction with $a, b$ coprime.

II.2.2.2. Case $3|(A^m + 2B^n)$:

If:

$$3|(A^m + 2B^n) \implies 3|(A^m - B^n)$$

(125)

but $p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n$ then $p - 3A^mB^n = (A^m - B^n)^2 \implies 9|(p - 3A^mB^n)$ or $9|(3p' - 3A^mB^n)$, then $3|(p' - A^mB^n)$ ⇒
3|4(p′ − 4A^mB^n) ⇒ 3|(b − 4A^mB^n) but 3|(b − a) ⇒ 3|(a − A^mB^n). As
3|(A^{2m} − 4A^mB^n) ⇒ 3|A^m(A^m − 4B^n).

II.2.2.2.1. Case 3|A^m:
If 3|A^m ⇒ 3|A^{2m} ⇒ 3|a, but 3|(b − a) ⇒ 3|b then the contradiction with
a, b coprime.

II.2.2.2.2. Case 3|(A^m − 4B^n):
Now if 3|(A^m − 4B^n) ⇒ 3|(A^m − B^n), we refine the hypothesis of the beginning
(125) above, that has a solution II.2.2.1..

III. Case k_1 ≠ 3 and 3|k_1:
We suppose k_1 ≠ 3 and 3|k_1 ⇒ k_1 = 3k_1' with k_1' ≠ 1. We have 4p = 12p' =
k_1b = 3k_1'b ⇒ 4p' = k_1'b. A^{2m} can be written as:

\[ A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{3k_1'b}{3} \frac{a}{b} = k_1'a \] (126)

and B^nC^l:

\[ B^nC^l = \frac{p}{3} (3 - 4\cos^2 \frac{\theta}{3}) = \frac{k_1'}{4}(3b - 4a) \] (127)

As B^nC^l is an integer, we must have \( 4|(3b - 4a) \) or \( 4|k_1' \).

III.1. Case 4|(3b − 4a):
We suppose that 4|(3b − 4a) ⇒ \( \frac{3b − 4a}{4} = c \in N^* \), and we obtain:

\[ A^{2m} = k_1'a \]

\[ B^nC^l = k_1'c \]

III.1.1. Case k_1' is prime:
If k_1' is prime, then k_1'|A^{2m} ⇒ k_1'|A and k_1'|B^nC^l ⇒ k_1'|B^n or k_1'|C^l. If
k_1'|B^n ⇒ k_1'|B, then k_1'|C^l ⇒ k_1'|C. With the same method if k_1'|C^l, we ar-
rive to k_1'|B.
We obtain: $A, B$ and $C$ solutions of (3) have a common factor.

### III.1.2. Case $k_1'$ not a prime:

We suppose $k_1'$ not a prime. Let $\mu$ a prime divisor of $k_1'$, as described in III.1.1. above, we obtain: $A, B$ and $C$ solutions of (3) have a common factor.

### III.2. Case $4|k_1'$:

Now, we suppose that $4|k_1'$.

#### III.2.1. Case $k_1' = 4$:

We suppose $k_1' = 4$, then $A^{2m} = 4a$ and $B^nC^l = 4c$. It is easy to verify that 2 is a common factor of $A, B, C$.

We obtain: $A, B$ and $C$ solutions of (3) have a common factor.

#### III.2.2. Case $k_1' = 4k^{\nu}_1$:

If $k_1' = 4k^{\nu}_1$ with $k^{\nu}_1 > 1$. Then, we have:

$$A^{2m} = 4k^{\nu}_1a$$  \hspace{1cm} (128)

$$B^nC^l = k^{\nu}_1(3b - 4a)$$  \hspace{1cm} (129)

#### III.2.2.1. Case $k^{\nu}_1$ prime:

If $k^{\nu}_1$ is prime, then $k^{\nu}_1|A^{2m} \Rightarrow k^{\nu}_1|A$ and $k^{\nu}_1|B^nC^l \Rightarrow k^{\nu}_1|B^n$ or $k^{\nu}_1|C^l$. If $k^{\nu}_1|B^n \Rightarrow k^{\nu}_1|B$, then $k^{\nu}_1|C^l \Rightarrow k^{\nu}_1|C$. With the same method if $k^{\nu}_1|C^l$, we arrive to $k^{\nu}_1|B$.

We obtain: $A, B$ and $C$ solutions of (3) have a common factor.

#### III.2.2.2. Case $k^{\nu}_1$ not a prime:

If $k^{\nu}_1$ not a prime. Let $\mu$ a prime divisor of $k^{\nu}_1$, as described in case III.2.2.1. above, we obtain: $A, B$ and $C$ solutions of (3) have a common factor.
3.2.2. **Hypothesis**: \{3|a \quad and \quad b|4p\}

We have:

\[ 3|a \implies \exists a' \in \mathbb{N}^* / a = 3a' \]

(130)

3.2.2.1. Case \( b = 2 \) and \( 3|a : \) \( A^{2m} \) is written as:

\[ A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4p}{3} \cdot \frac{a}{2} = \frac{2p.a}{3} \]

(131)

Using the equation (130), \( A^{2m} \) becomes:

\[ A^{2m} = \frac{2p.3a'}{3} = 2p.a' \]

(132)

But \( \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1 \) which is impossible, then \( b \neq 2 \).

3.2.2.2. Case \( b = 4 \) and \( 3|a : \) \( A^{2m} \) is written as:

\[ A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4p}{3} \cdot \frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \]

(133)

and \( \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4} < \left( \frac{\sqrt{3}}{2} \right)^2 = \frac{3}{4} \implies a' < 1 \)

(134)

which is impossible.

Then the case \( b = 4 \) is impossible.

3.2.2.3. Case \( b = p \) and \( 3|a : \) Then:

\[ \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \]

(135)

and:

\[ A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{3a'}{p} = 4a' = (A^m)^2 \]

(136)

\[ \exists a'' \in \mathbb{N}^* / a' = a''^2 \]

(137)

We calculate \( A^m B^n \), hence:

\[ A^m B^n = p.\frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a' \]

or \( A^m B^n + 2a' = p.\frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} \)

(138)
The left member of (138) is an integer and \( p \) is also, then \( 2 \sqrt{\frac{3}{3}} \sin \frac{2\theta}{3} \) will be written as:

\[
2 \sqrt{\frac{3}{3}} \sin \frac{2\theta}{3} = \frac{k_1}{k_2}
\]

(139)

where \( k_1, k_2 \) are two coprime integers and \( k_2 | p \Rightarrow p = k_2.k_3, k_3 \in N^* \).

I. Case \( k_3 \neq 1 \):

We suppose that \( k_3 \neq 1 \). We obtain:

\[
A^m(A^m + 2B^n) = k_1.k_3
\]

(140)

Let us \( \mu \) a prime integer with \( \mu | k_3 \), then \( \mu | b \) and \( \mu | A^m(A^m + 2B^n) \). Hence:

\[
\mu | A^m \quad \text{or} \quad \mu | (A^m + 2B^n)
\]

(141)

I.1. Case \( \mu | A^m \):

If \( \mu | A^m \Rightarrow \mu | A \) and \( \mu | A^{2m} \), but \( A^{2m} = 4a' \Rightarrow \mu | 4a' \Rightarrow (\mu = 2 \text{ but } 2|a') \) or \( \mu | a' \). Then \( \mu | a \) hence the contradiction with \( a, b \) coprime.

I.2. Case \( \mu | (A^m + 2B^n) \):

If \( \mu | (A^m + 2B^n) \Rightarrow \mu \nmid A^m \) and \( \mu \nmid 2B^n \) then \( \mu \neq 2 \) and \( \mu \nmid B^n \). We write \( \mu | (A^m + 2B^n) \) as:

\[
A^m + 2B^n = \mu.t' \quad t' \in N^*
\]

(142)

It follows:

\[
A^m + B^n = \mu t' - B^n \Rightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}
\]

Using the expression of \( p \):

\[
p = t'^2 \mu^2 - 2t' B^n \mu + B^n(B^n - A^m)
\]

(143)

Since \( p = b = k_2.k_3 \) and \( \mu | k_3 \) then \( \mu | b \Rightarrow \exists \mu' \in N^* \) and \( b = \mu' \), so we can write:

\[
\mu' \mu = \mu(\mu t'^2 - 2t' B^n) + B^n(B^n - A^m)
\]

(144)
From the last equation, we get $\mu | B^n (B^n - A^m) \implies \mu | B^n$ or $\mu | (B^n - A^m)$.

**I.2.1. Case $\mu | B^n$:**

If $\mu | B^n$ which is contradiction with $\mu \nmid B^n$.

**I.2.2. Case $\mu | (B^n - A^m)$:**

If $\mu | (B^n - A^m)$ and using $\mu | (A^m + 2B^n)$, we arrive to:

$$\mu | 3B^n \implies \begin{cases} 
\mu | B^n \\
\text{or} \\
\mu = 3 
\end{cases}$$

(145)

**I.2.2.1. Case $\mu | B^n$:**

If $\mu | B^n$ which is contradiction with $\mu \nmid B$ from **I.2. Case $\mu | (A^m + 2B^n)$**.

**I.2.2.2. Case $\mu = 3$:**

If $\mu = 3$, then $b = 3\mu'$, but $3|a$ then the contradiction with $a, b$ coprime.

**II. Case $k_3 = 1$:**

We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^m B^n = k_1$$

(146)

$$b = k_2$$

(147)

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{b}$$

(148)

Taking the square of the last equation, we obtain:

$$\frac{4}{3} \sin \frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} \sin \frac{2\theta}{3} \cos \frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} \sin \frac{2\theta}{3} \cdot \frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2 a'(p - a) = k_1^2$$

(149)
but $a' = a''^2$ then $p - a$ is a square. Let us:

$$\lambda^2 = p - a$$  \quad (150)$$

The equation (149) becomes:

$$4a''^2\lambda^2 = k_1^2 \implies k_1 = 4a''\lambda$$  \quad (151)$$

taking the positive square root. Using (146), we get:

$$k_1 = 4a''\lambda$$  \quad (152)$$

But $k_1 = A^m(A^m + 2B^n) = 2a''(A^m + 2B^n)$, it follows:

$$A^m + 2B^n = 2\lambda$$  \quad (153)$$

Let $\lambda_1$ prime $\neq 2$, a divisor of $\lambda$ (if not, $\lambda_1 = 2\mid\lambda \implies 2\mid\lambda^2 \implies 2\mid(p - a)$ but $a$ is even, then $2\mid p \implies 2\mid b$ which is contradiction with $a, b$ coprime).

We consider $\lambda_1 \neq 2$ and:

$$\lambda_1\mid\lambda \implies \lambda_1\mid\lambda^2 \quad \text{and} \quad \lambda_1\mid(A^m + 2B^n)$$  \quad (154)$$

$$\lambda_1\mid(A^m + 2B^n) \implies \lambda_1 \nmid A^m \quad \text{if not} \quad \lambda_1\mid2B^n $$  \quad (155)$$

But $\lambda_1 \neq 2$ hence $\lambda_1\mid B^n \implies \lambda_1\mid B$, it follows:

$$\lambda_1\mid(p = b) \quad \text{and} \quad \lambda_1\mid A^m \implies \lambda_1\mid2a'' \implies \lambda_1\mid a$$  \quad (156)$$

hence the contradiction with $a, b$ coprime.

**II.1. Case $\lambda_1 \nmid A^m$ and $\lambda_1\mid(A^m + 2B^n)$:**

We assume now $\lambda_1 \nmid A^m$. $\lambda_1\mid(A^m + 2B^n) \implies \lambda_1\mid(A^m + 2B^n)^2$ that is $\lambda_1\mid(A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1\mid(p + 3A^mB^n + 3B^{2n}) \implies \lambda_1\mid(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1\mid(A^m + 2B^n) \implies \lambda_1\mid(p - 3B^{2n})$, as $\lambda_1\mid(p - a)$ hence by difference, we obtain $\lambda_1\mid(a - 3B^{2n})$ or $\lambda_1\mid(3a' - 3B^{2n}) \implies \lambda_1\mid3(a' - B^{2n})$. Then:

$$\lambda_1 = 3 \quad \text{or} \quad \lambda_1\mid(a' - B^{2n})$$  \quad (157)$$

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II.1.1. Case $\lambda_1 = 3$:
If $\lambda_1 = 3$ but $3 \mid (p-a)$ hence the contradiction with $a, b$ coprime.

II.1.2. Case $\lambda_1 \mid (a' - B^{2n})$:
If $\lambda_1 \mid (a' - B^{2n}) \Rightarrow \lambda_1 \mid (a'' - B^{2n}) \Rightarrow \lambda_1 \mid (a'' + B^{2n})$ or $\lambda_1 \mid (a'' - B^{2n})$, because $(a'' - B^{2n}) \neq 1$, if not, we obtain $a'' - B^{2n} = a'' + B^{n} \Rightarrow a'' - a'' = B^{n} - B^{2n}$. The left member is positive and the right member is negative, then the contradiction.

II.1.2.1. Case $\lambda_1 \mid (a'' - B^{n})$:
If $\lambda_1 \mid (a'' - B^{n}) = \Rightarrow \lambda_1 \mid 2(a'' - B^{n}) = \Rightarrow \lambda_1 \mid (2a'' - 2B^{n}) \Rightarrow \lambda_1 \mid (A^{m} + 2B^{n})$ hence $\lambda_1 \mid 2A^{m} \Rightarrow \lambda_1 \mid A^{m}$ as $\lambda_1 \neq 2$, it follows $\lambda_1 \mid A^{m}$ hence the contradiction with (155).

II.1.2.2. Case $\lambda_1 \mid (a'' + B^{n})$:
If $\lambda_1 \mid (a'' + B^{n}) \Rightarrow \lambda_1 \mid 2(a'' + B^{n}) \Rightarrow \lambda_1 \mid (2a'' + 2B^{n}) \Rightarrow \lambda_1 \mid (A^{m} + 2B^{n})$. We find the case II.1. that has solutions.

Then the case $k_3 = 1$ is impossible.

3.2.2.4. Case $b \mid p \Rightarrow p = b.p', p' > 1, b \neq 2, b \neq 4$ and $3 \mid a.$:

\[
A^{2m} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4b.p'.3.a'}{3b} = 4.p'a'
\]  \hspace{1cm} (158)

We calculate $B^nC'$:

\[
B^nC' = \sqrt[3]{p^2} \left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{p^2} \left(3 - 4cos^2\frac{\theta}{3}\right)
\]  \hspace{1cm} (159)

But $\sqrt[3]{p^2} = \frac{p}{3}$, hence using $cos^2\frac{\theta}{3} = \frac{3.a'}{b}$:

\[
B^nC' = \sqrt[3]{p^2} \left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3} \left(3 - 4\frac{3.a'}{b}\right) = p. \left(1 - \frac{4.a'}{b}\right) = p'(b - 4a')
\]  \hspace{1cm} (160)
As \( p = b.p' \), and \( p' > 1 \), we have then:

\[
B^nC^l = p'(b - 4a') \\
\text{and} \quad A^{2m} = 4p'.a'
\]  

(161)

(162)

I. Case \( \lambda \) a prime divisor of \( p' \):

Let \( \lambda \) a prime divisor of \( p' \) (we suppose \( p' \) not prime ). From (162), we have:

\[
\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as \( \lambda \) is a prime, then} \quad \lambda|A
\]  

(163)

From (161), as \( \lambda|p' \) we have:

\[
\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or} \quad \lambda|C^l
\]  

(164)

If \( \lambda|B^n \), \( \lambda \) is a prime \( \lambda|B \), but \( C^l = A^m + B^n \), then we have also :

\[
\lambda|C^l \quad \text{as \( \lambda \) is a prime, then} \quad \lambda|C
\]  

(165)

By the same way, if \( \lambda|C^l \), we obtain \( \lambda|B \). then : \( A, B \) and \( C \) solutions of (3) have a common factor.

II. Case \( p' \) is a prime number:

We suppose now that \( p' \) is prime, from the equations (161) and (162), we obtain that:

\[
p'|A^{2m} \Rightarrow p'|A^m \Rightarrow p'|A
\]  

(166)

and:

\[
p'|B^nC^l \Rightarrow p'|B^n \quad \text{or} \quad p'|C^l
\]  

(167)

If \( p'|B^n \Rightarrow p'|B \)

As \( C^l = A^m + B^n \) and that \( p'|A, p'|B \Rightarrow p'|A^m, p'|B^n \Rightarrow p'|C^l \)

\[
\Rightarrow p'|C
\]  

(169)

By the same way, if \( p'|C^l \), we arrive to \( p'|B \).

Hence: \( A, B \) and \( C \) solutions of (3) have a common factor.
3.2.2.5. Case $b = 2p$ and $3|a$ : We have:

$$\cos^{2} \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \implies A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3} \cdot \frac{3a'}{2p} = 2a' \implies 2|A^{m} \implies 2|a \implies 2|a'$$

Then $2|a$ and $2|b$ which is contradiction with $a, b$ coprime.

3.2.2.6. Case $b = 4p$ and $3|a$ : We have:

$$\cos^{2} \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \implies A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3} \cdot \frac{3a'}{4p} = a'$$

Calculate $A^{m}B^{n}$, we obtain:

$$A^{m}B^{n} = \frac{p\sqrt{3}}{3}. \sin \frac{2\theta}{3} - \frac{2p}{3} \cos^{2} \frac{\theta}{3} = \frac{p\sqrt{3}}{3}. \sin \frac{2\theta}{3} - \frac{a'}{2} \implies A^{m}B^{n} + A^{2m} = \frac{p\sqrt{3}}{3}. \sin \frac{2\theta}{3}$$

(170)

let:

$$A^{2m} + 2A^{m}B^{n} = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3}$$

(171)

The left member of (171) is an integer and $p$ is an integer, then $\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3}$ will be written:

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_{1}}{k_{2}}$$

(172)

where $k_{1}, k_{2}$ are two coprime integers and $k_{2}|p \implies p = k_{2}.k_{3}$.

I. Case $k_{3} \neq 1$:

Firstly, we suppose that $k_{3} \neq 1$. Hence:

$$A^{2m} + 2A^{m}B^{n} = k_{3}.k_{1}$$

(173)

Let $\mu$ a prime integer and $\mu|k_{3}$, then $\mu|A^{m}(A^{m}+2B^{n}) \implies [\mu|A^{m} \text{ or } \mu|(A^{m}+2B^{n})].$

I.1. Case $\mu|A^{m}$:

If $\mu|A^{m} \implies \mu|(A^{2m} = a') \implies \mu|(3a' = a)$. As $\mu|k_{3} \implies \mu|p \implies \mu|(4p = b)$. Then the contradiction with $a, b$ coprime.
I.2. Case \( \mu|(A^n + 2B^n) \):
If \( \mu|(A^n + 2B^n) \implies \mu \nmid A^n \) and \( \mu \nmid 2B^n \) then:
\[
\mu \neq 2 \quad \text{and} \quad \mu \nmid B^n
\] (174)
\( \mu|(A^n + 2B^n) \), we write:
\[
A^n + 2B^n = \mu t' \quad t' \in \mathbb{N}^*
\] (175)
Then :
\[
A^n + B^n = \mu t' - B^n \implies A^{2n} + B^{2n} + 2A^nB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}
\]
\[
\implies p = t'^2\mu^2 - 2t'\mu B^n + B^n(B^n - A^n)
\] (176)
As \( b = 4p = 4k_2k_3 \) and \( \mu|k_3 \) then \( \mu|b \implies \exists \mu' \in \mathbb{N}^* \) that \( b = \mu\mu' \), we obtain:
\[
\mu'\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^n)
\] (177)
The last equation implies \( \mu|4B^n(B^n - A^n) \), but \( \mu \neq 2 \) then \( \boxed{\mu|B^n \text{ or } \mu|(B^n - A^n)} \).

I.2.1. Case \( \mu|B^n \):
If \( \mu|B^n \) then the contradiction with (174).

I.2.2. Case \( \mu|(B^n - A^n) \):
If \( \mu|(B^n - A^n) \) and using \( \mu|(A^n + 2B^n) \), we obtain:

\[
\boxed{\mu|3B^n \implies \mu|B^n \text{ or } \mu = 3}
\] (178)

I.2.2.1. Case \( \mu|B^n \):
If \( \mu|B^n \) it is contradiction with (174).

I.2.2.2. Case \( \mu = 3 \):
If \( \mu = 3 \), then \( b = 3\mu' \), but \( 3|a \) which is contradiction with \( a, b \) coprime.

II. Case \( k_3 = 1 \):
We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^mB^n = k_1 \quad (179)$$

$$p = k_2 \quad (180)$$

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = k_1 \quad (181)$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} \sin \frac{2\theta}{3} = \frac{k_1^2}{p^2} \quad (182)$$

Finally:

$$a'(4p - 3a') = k_1^2$$

but $a' = a^{n/2}$ then $4p - 3a'$ is a square. Let us:

$$\lambda^2 = 4p - 3a' = 4p - a = b - a \quad (183)$$

The equation (182) becomes:

$$a^{n/2}\lambda^2 = k_1^2 \implies k_1 = a^n\lambda \quad (184)$$

taking the positive square root. Using (179), we get:

$$k_1 = a^n\lambda \quad (185)$$

But $k_1 = A^n(A^m + 2B^n) = a^n(A^m + 2B^n)$, it follows:

$$(A^m + 2B^n) = \lambda \quad (186)$$

Let $\lambda_1$ prime $\neq 2$, a divisor of $\lambda$ (if not $\lambda_1 = 2|\lambda \implies 2|\lambda^2$. As $2|(b = 4p) \implies 2|(a = 3a')$ which is contradiction with $a, b$ coprime).
We consider $\lambda_1 \neq 2$ and:

$$\lambda_1 | \lambda \implies \lambda_1 | (A^m + 2B^n) \quad (187)$$

$$\implies \lambda_1 \nmid A^m \quad \text{if not} \quad \lambda_1 | 2B^n \quad (188)$$

But $\lambda_1 \neq 2$ hence $\lambda_1 | B^n \implies \lambda_1 | B$, it follows:

$$\lambda_1 | (b = 4p) \quad \text{and} \quad \lambda_1 | A^m \implies \lambda_1 | 2a' \implies \lambda_1 | a \quad (189)$$

hence the contradiction with $a, b$ coprime.

II.1. Case $\lambda_1 \nmid A^m$, $\lambda_1 \nmid B^n$ and $\lambda_1 | (A^m + 2B^n)$:

We assume now $\lambda_1 | A^m$, $\lambda_1 | B^n$. $\lambda_1 | (A^m + 2B^n) \implies \lambda_1 | (A^m + 2B^n)^2$ that is $\lambda_1 | (A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1 | (p + 3A^mB^n + 3B^{2n}) \implies \lambda_1 | (p + 3B^{2n}(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1 | (A^m + 2B^n) \implies \lambda_1 | (p - 3B^{2n})$, as $\lambda_1 | (4p - a)$ hence by difference, we obtain $\lambda_1 | (a - 3(B^{2n} + p))$ or $\lambda_1 | (3a' - 3(B^{2n} + p)) \implies \lambda_1 | 3(a' - B^{2n} - p) \implies \lambda_1 = 3$ or $\lambda_1 | (a' - (B^{2n} + p))$.

II.1.1. Case $\lambda_1 = 3$:

If $\lambda_1 = 3 | \lambda \Rightarrow 3 | \lambda^2 \Rightarrow 3 | b - a$ but $3 | a \implies 3 | p = b$ hence the contradiction with $a, b$ coprime.

II.1.2. Case $\lambda_1 | (a' - (B^{2n} + p))$:

If $\lambda_1 \neq 3$ and $\lambda_1 | (a' - B^{2n} - p) \implies \lambda_1 | (A^mB^n + B^{2n}) \implies \lambda_1 | B^n(A^m + 2B^n) \implies \lambda_1 | B^n \quad \text{or} \quad \lambda_1 | (A^m + 2B^n)$.

II.1.2.1. Case $\lambda_1 | B^n$:

If $\lambda_1 | B^n$ that is in contradiction with the hypothesis $\lambda_1 \nmid B$ cited above case II.1.

II.1.2.2. Case $\lambda_1 | (A^m + 2B^n)$:

If $\lambda_1 | (A^m + 2B^n)$. We refine this condition in the case II.1.

Then the case $k_3 = 1$ is impossible.
3.2.2.7. Case 3|a and b = 2p', b ≠ 2 with p'|p : 3|a → a = 3a', b = 2p' with p = k.p', hence:

\[ A^{2m} = \frac{4p}{3} a \frac{6p'}{b} = 2.k.a' \]  \hspace{1cm} (190)

Calculate \( B^nC^l \):

\[ B^nC^l = \sqrt[3]{\rho^2} \left( 3sin^2 \frac{\theta}{3} - cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left( 3 - 4cos^2 \frac{\theta}{3} \right) \] \hspace{1cm} (191)

But \( \sqrt[3]{\rho^2} = \frac{p}{3} \) hence en using \( cos^2 \frac{\theta}{3} = \frac{3a'}{b} \):

\[ B^nC^l = \sqrt[3]{\rho^2} \left( 3 - 4cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left( 3 - 4 \frac{3a'}{b} \right) = p \left( 1 - \frac{4a'}{b} \right) = k(p' - 2a') \] \hspace{1cm} (192)

As \( p = b.p' \), and \( p' > 1 \), we have then:

\[ B^nC^l = k(p' - 2a') \] \hspace{1cm} (193)

and \( A^{2m} = 2k.a' \) \hspace{1cm} (194)

I. Case \( \lambda \) is a prime divisor of \( k \):

We suppose that \( \lambda \) is a prime divisor of \( k \) (we suppose \( k \) not a prime ). From (194), we have:

\[ \lambda | A^{2m} \Rightarrow \lambda | A^n \quad \text{as \( \lambda \) is prime then \( \lambda | A \)} \] \hspace{1cm} (195)

From (193), as \( \lambda | k \), we have:

\[ \lambda | B^nC^l \Rightarrow \lambda | B^n \quad \text{or \( \lambda | C^l \)} \] \hspace{1cm} (196)

If \( \lambda | B^n \), \( \lambda \) is prime \( \lambda | B \), and as \( C^l = A^m + B^n \) then we have also:

\[ \lambda | C^l \quad \text{as \( \lambda \) is prime then \( \lambda | C \)} \] \hspace{1cm} (197)

By the same way, if \( \lambda | C^l \), we obtain \( \lambda | B \). Then : \( A, B \) and \( C \) solutions of (3) have a common factor.

II. Case \( k \) is prime:
Now, we suppose now that $k$ is prime, from the equations (193) and (194), we obtain:

\[ k | A^{2m} \Rightarrow k | A^{m} \Rightarrow k | A \quad (198) \]

and:

\[ k | B^n C^1 \Rightarrow k | B^n \text{ or } k | C^1 \quad (199) \]

\[ \text{if } k | B^n \Rightarrow k | B \quad (200) \]

as $C^1 = A^m + B^n$ and that $k | A, k | B \Rightarrow k | A^m, k | B^n \Rightarrow k | C^1$

\[ \Rightarrow k | C \quad (201) \]

By the same way, if $k | C^1$, we arrive to $k | B$.

Hence: $A, B$ and $C$ solutions of (3) have a common factor.

3.2.2.8. Case $3 | a$ and $b = 4p' \quad b \neq 2$ with $p' | p \quad \therefore 3 | a \Rightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$, if not, $b = 4p$ a case that has been studied (paragraph 3.2.2.6), then we have:

\[ A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a' \quad (202) \]

Writing $B^n C^1$:

\[ B^n C^1 = \sqrt[3]{\rho^2} \left( 3 \sin^2 \theta - \cos^2 \theta \right) = \sqrt[3]{\rho^2} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) \quad (203) \]

But $\sqrt[3]{\rho^2} = \frac{p}{3}$ hence en using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

\[ B^n C^1 = \sqrt[3]{\rho^2} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left( 3 - 4 \cdot \frac{3.a'}{b} \right) = p \cdot \left( 1 - \frac{4.a'}{b} \right) = k(p' - a') \quad (204) \]

As $p = b.p'$, and $p' > 1$, we have:

\[ B^n C^1 = k(p' - 2a') \quad (205) \]

and

\[ A^{2m} = 2k.a' \quad (206) \]

I. Case $\lambda$ a prime divisor of $k$:
Let $\lambda$ a prime divisor of $k$ (we suppose $k$ not a prime). From (206), we have:

$$\lambda | A^{2m} \Rightarrow \lambda | A^m \quad \text{as } \lambda \text{ is prime then } \lambda | A$$

From (205), as $\lambda | k$ we obtain:

$$\lambda | B^n C^l \Rightarrow \lambda | B^n \quad \text{or } \lambda | C^l$$

I.1 Case $\lambda | B^n$ or $\lambda | C^n$:

If $\lambda | B^n$, $\lambda$ is a prime, then $\lambda | B$, and as $\lambda | A \Rightarrow \lambda | (A^m + B^n = C^l) \Rightarrow \lambda | C$. By the same way if $\lambda | C^l$, we obtain $\lambda | B$. Then : $A$, $B$ and $C$ solutions of (3) have a common factor.

II. Case $k$ is prime:

We suppose now that $k$ is prime, from the equations (205) and (206), we have:

$$k | A^{2m} \Rightarrow k | A^m \Rightarrow k | A$$

and:

$$k | B^n C^l \Rightarrow k | B^n \quad \text{or } k | C^l$$

if $k | B^n \Rightarrow k | B$

as $C^l = A^m + B^n$ and that $k | A$, $k | B \Rightarrow k | A^m$, $k | B^n \Rightarrow k | C^l$

$$\Rightarrow k | C$$

By the same way if $k | C^l$, we arrive to $k | B$.

Hence: $A$, $B$ and $C$ solutions of (3) have a common factor.

3.2.2.9. Case 3(a and b|4p).

: $a = 3a'$ and $4p = k_1 b$ with $k_1 \in N^*$. As $A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{3a'}{b} = k_1 a'$ and $B^n C^l$:

$$B^n C^l = \sqrt[4]{p} \left( \frac{3 \sin \frac{\theta}{3} - \cos \frac{\theta}{3}}{3} \right) = \frac{p}{3} \left( 3 - 4 \cos \frac{\theta}{3} \right) = \frac{p}{3} \left( 3 - \frac{3a'}{b} \right) = \frac{k_1}{4} (b - 4a')$$

As $B^n C^l$ is an integer, we must have $4 | k_1$ or $4 | (b - 4a')$. 

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I. Case $k_1 = 1$:
If $k_1 = 1 \Rightarrow b = 4p$ : it is the case (3.2.6) above.

II. Case $k_1 = 4$:
If $k_1 = 4 \Rightarrow p = b$ : it is the case (3.2.3) above.

III. Case $4 | k_1$:
We suppose that $4 | k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k'_1$, then we have:

\[ A^{2m} = 4k'_1a' \]
\[ B^nC^l = k'_1(b - 4a') \]

By discussing $k'_1$ is a prime integer or not, we arrive easily to: $A, B$ and $C$ solutions of (3) have a common factor.

III.1. Case $4 \nmid (b - 4a')$ and $4 \nmid k'_1$:
If $4 \nmid (b - 4a')$ and $4 \nmid k'_1$ it is impossible.

III.2. Case $4 | (b - 4a')$:
If $4 | (b - 4a') \Rightarrow (b - 4a') = 4c$, with $c \in N^*$, then we obtain:

\[ A^{2m} = k_1a' \]
\[ B^nC^l = k_1c \]

By discussing $k_1$ is a prime integer or not, we arrive easily to: $A, B$ and $C$ solutions of (3) have a common factor.

The main theorem is proved.

4. Numerical Examples

4.1. Example 1:
We consider the example:

\[ 6^3 + 3^3 = 3^5 \] (214)
with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

\begin{align*}
  p &= 3^6 \times 73, \quad \text{(215)} \\
  q &= 8 \times 3^{11}, \quad \text{(216)} \\
  \bar{\Delta} &= 4 \times 3^{18} (3^7 \times 4^2 - 73^3) < 0, \quad \text{(217)} \\
  \rho &= \frac{p\sqrt{p}}{3\sqrt{3}} = \frac{3^8 \times 73\sqrt{73}}{3}, \quad \text{(218)} \\
  \cos\theta &= \frac{-4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \quad \text{(219)}
\end{align*}

As $A^{2m} = \frac{4p}{3} \cos^2\frac{\theta}{3} \Rightarrow \cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \Rightarrow a = 3 \times 2^4, \ b = 73$; then:

\begin{align*}
  \cos\frac{\theta}{3} &= \frac{4\sqrt{3}}{\sqrt{73}} \quad \text{(220)} \\
  p &= 3^6b \quad \text{(221)}
\end{align*}

Let us verify the equation (219) using the equation (220):

\begin{align*}
  \cos\theta &= \cos(3(\theta/3)) = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3} = 4 \left( \frac{4\sqrt{3}}{\sqrt{73}} \right)^3 - 3 \frac{4\sqrt{3}}{\sqrt{73}} = \frac{-4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \quad \text{(222)}
\end{align*}

That’s OK. For this example, we can use the two conditions of (65) as $3|p, b|4p$ and $3|a$. The cases 3.2.1.3 and 3.2.2.4 are respectively used. We find for both cases that $A^m, B^n$ and $C^l$ of the equation (214) have a common prime factor which is true.

4.2. Example 2:

Let the second example:

\begin{equation}
  7^4 + 7^3 = 14^3 \Rightarrow 2401 + 343 = 2744 \quad \text{(223)}
\end{equation}

With the notations of the paper, we take:

\begin{align*}
  A^m &= 7^4 \quad \text{(224)} \\
  B^n &= 7^3 \quad \text{(225)} \\
  C^l &= 14^3 \quad \text{(226)}
\end{align*}
We obtain:

\[ p = 57 \times 7^6 = 3 \times 19 \times 7^6 \quad (227) \]
\[ q = 8 \times 7^{10} \quad (228) \]
\[ \Delta = 27q^2 - 4p^3 = 27 \times 4 \times 7^{18}(16 \times 49 - 19^3) \]
\[ = -27 \times 4 \times 7^{18} \times 6075 < 0 \quad (229) \]

\[ \rho = p^{\sqrt{p}} \quad (230) \]
\[ \cos \theta = \frac{-q}{2\rho} = \frac{-4 \times 7}{19\sqrt{19}} \quad (231) \]

As \( A^{2m} = \frac{4p}{3} \cos^2 \theta \frac{3}{3} \Rightarrow \cos^2 \theta \frac{3}{3} = \frac{3A^{2m}}{4p} = \frac{7^2}{4 \times 19} = \frac{a}{b} \Rightarrow a = 7^2, b = 4 \times 19; \]
then:
\[ \cos \frac{\theta}{3} = \frac{7}{2\sqrt{19}} \quad (232) \]
\[ 3|p \text{ and } b(4p) \quad (233) \]

Let us verify the equation (231) using the equation (232):
\[ \cos \theta = \cos 3(\theta/3) = 4\cos^3 \frac{\theta}{3} - 3\cos \frac{\theta}{3} = 4 \left( \frac{7}{2\sqrt{19}} \right)^3 - 3 \frac{7}{2\sqrt{19}} = -\frac{4 \times 7}{19\sqrt{19}} \quad (234) \]
It is the same value of (231)!

Now, from (233), we have 3|p \Rightarrow p = 3p', b(4p) with b \neq 2, 4 then 12p' = k_1b = 3 \times 7^6b. It concerns the paragraph 3.2.1.9. of the first hypothesis. As \( k_1 = 3 \times 7^6 = 3k'_1 \) with \( k'_1 = 7^6 \neq 1 \). It is the case III, with the two conditions: \( 4|(3b - 4a) \) or \( 4|k'_1 \). We take \( 4|(3b - 4a) \). Let us calculate \( 3b - 4a \):
\[ 3b - 4a = 3 \times 4 \times 19 - 4 \times 7^2 = 32 \Rightarrow 4|(3b - 4a) \quad (235) \]

Then it is the sous-case III.1. with \( A^{2m} = 7^8 = 7^6 \times 7^2 = k'_1a \) with \( k'_1 \) not a prime, we find the sous-case III.1.2 with the result that \( A, B \) and \( C \) have a common factor namely the prime number 7 a divisor of \( k'_1 = 7^6! \).

4.3. Example 3:

Let the third example:
\[ 7^2 + 2^5 = 3^4 \quad (236) \]
with:

\[ A^m = 7^2; B^n = 2^5; C^l = 3^4 \]

We obtain:

\[ p = 4999 \quad \text{a prime number} \quad (237) \]
\[ q = 2^5 \times 7^2 \times 3^4 = 127008 \gg p \quad (238) \]

As \( q \gg p \), we find that:

\[ \Delta = 27q^2 - 4p^3 > 0 \quad (239) \]

Then we cannot use the results of our proof because in this example, \( m = 2 < 3 \).

We remark that in all the proof, we don’t encountered that \( m, n \) or \( l \) must be great than 2. Then the condition that \( m, n, l > 2 \) is important in \([1]\).

5. Conclusion

As seen above, the examples confirm the results of the proof. In conclusion, we can announce the theorem:

**Theorem 1. (A. Ben Hadj Salem, A. Beal, 2016):** Let \( A, B, C, m, n, \) and \( l \) be positive integers with \( m, n, l > 2 \). If:

\[ A^m + B^n = C^l \quad (240) \]

then \( A, B, \) and \( C \) have a common factor.

References