A Complete Proof of BEAL Conjecture

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Abstract
In 1997, Andrew Beal [1] announced the following conjecture: Let $A, B, C, m, n,$ and $l$ be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then $A, B,$ and $C$ have a common factor. We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with $p, q$ integers depending of $A^m, B^n$ and $C^l$. We resolve $x^3 - px + q = 0$ and we obtain the three roots $x_1, x_2, x_3$ as functions of $p, q$ and a parameter $\theta$. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that $x_1, x_2, x_3$ are integers. A numerical example is given.

Keywords: Prime numbers, divisibility, roots of polynomials of third degree.

O my Lord! Increase me further in knowledge.
(Holy Quran, Surah Ta Ha, 20:114.)

to my wife Wahida

1 Introduction
In 1997, Andrew Beal [1] announced the following conjecture:

Conjecture 1.1. Let $A, B, C, m, n,$ and $l$ be positive integers with $m, n, l > 2$. If:
\[ A^m + B^n = C^l \] (1.1)
then $A, B,$ and $C$ have a common factor.

In this paper, we give a complete proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots $A^m, B^n$ and $-C^l$ with the condition (1.1). In the next section, we do some preliminaries calculus to give the expressions of the three roots of $P(x) = 0$. The proof of the conjecture (1.1) is the subject of the section 3. At the end, a numerical example is presented.

We begin with the trivial case when $A^m = B^n$. The equation (1.1) becomes:
\[ 2A^m = C^l \] (1.2)
2 Preliminaries Calculs

Let \( m, n, l \in \mathbb{N}^* > 2 \) and \( A, B, C \in \mathbb{N}^* \) such:

\[
A^m + B^n = C^l
\]  

(2.1)

We call:

\[
P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l)
+ x[A^mB^n - C^l(A^m + B^n)] + C^lA^mB^n
\]  

(2.2)

Using the equation (2.1), \( P(x) \) can be written:

\[
P(x) = x^3 + x[A^mB^n - (A^m + B^n)^2] + A^mB^n(A^m + B^n)
\]  

(2.3)

We introduce the notations:

\[
p = (A^m + B^n)^2 - A^mB^n
\]

(2.4)

\[
q = A^mB^n(A^m + B^n)
\]

(2.5)

As \( A^m \neq B^n \), we have:

\[
p > (A^m - B^n)^2 > 0
\]  

(2.6)

Equation (2.3) becomes:

\[
P(x) = x^3 - px + q
\]  

(2.7)

Using the equation (2.2), \( P(x) = 0 \) has three different real roots: \( A^m, B^n \) and \(-C^l\).

Now, let us resolve the equation:

\[
P(x) = x^3 - px + q = 0
\]  

(2.8)

To resolve (2.8) let:

\[
x = u + v
\]  

(2.9)

Then \( P(x) = 0 \) gives:

\[
P(x) = P(u + v) = (u + v)^3 - p(u + v) + q = 0 \quad \Rightarrow \quad u^3 + v^3 + (u + v)(3uv - p) + q = 0
\]  

(2.10)

To determine \( u \) and \( v \), we obtain the conditions:

\[
u^3 + v^3 = -q
\]

(2.11)

\[uv = p/3 > 0
\]  

(2.12)

Then \( u^3 \) and \( v^3 \) are solutions of the second order equation:

\[
X^2 + qX + p^3/27 = 0
\]  

(2.13)
Its discriminant $\Delta$ is written as:

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \bar{\Delta}/27$$  \hspace{1cm} (2.14)

Let:

$$\bar{\Delta} = 27q^2 - 4p^3 = 27(A^mB^n(A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^mB^n]^3$$

$$= 27A^{2m}B^{2n}(A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^mB^n]^3$$ \hspace{1cm} (2.15)

Noting:

$$\alpha = A^mB^n > 0 \hspace{1cm} (2.16)$$

$$\beta = (A^m + B^n)^2 \hspace{1cm} (2.17)$$

we can write (2.15) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3 \hspace{1cm} (2.18)$$

As $\alpha \neq 0$, we can also rewrite (2.18) as:

$$\bar{\Delta} = \alpha^3\left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3\right) \hspace{1cm} (2.19)$$

We call $t$ the parameter:

$$t = \frac{\beta}{\alpha} \hspace{1cm} (2.20)$$

$\bar{\Delta}$ becomes:

$$\bar{\Delta} = \alpha^3(27t - 4(t - 1)^3) \hspace{1cm} (2.21)$$

Let us calling:

$$y = y(t) = 27t - 4(t - 1)^3 \hspace{1cm} (2.22)$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the signe of $y(t)$. Let us study the sign of $y$. We obtain $y'(t)$:

$$y'(t) = y' = 3(1 + 2t)(5 - 2t) \hspace{1cm} (2.23)$$

$y' = 0 \implies t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of $y$ is given below:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$-\infty$</th>
<th>$-1/2$</th>
<th>$5/2$</th>
<th>$4$</th>
<th>$+\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(t)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

**Fig. 1:** The table of variation

The table of the variations of the function $y$ shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in [0,4] \implies y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \implies \beta = (A^m + B^n)^2 > 4\alpha = 4A^mB^n \hspace{1cm} (2.24)$$
Then \( y < 0 \Rightarrow \Delta < 0 \Rightarrow \Delta < 0 \). Then, the equation \([2.13]\) does not have real solutions \( u^3 \) and \( v^3 \). Let us find the solutions \( u \) and \( v \) with \( x = u + v \) is a positive or a negative real and \( u.v = \frac{p}{3} \).

2.1 Demonstration

Proof. The solutions of \([2.13]\) are:

\[
X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{2.25}
\]
\[
X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \tag{2.26}
\]

We may resolve:

\[
u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{2.27}
\]
\[
v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \tag{2.28}
\]

Writing \( X_1 \) in the form:

\[
X_1 = \rho e^{i\theta} \tag{2.29}
\]

with:

\[
\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{\rho}}{3\sqrt{3}} \tag{2.30}
\]

and \( \sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \)

\[
\cos\theta = \frac{-q}{2\rho} < 0 \tag{2.31}
\]

Then \( \theta [2\pi] \in ] + \frac{\pi}{2} , +\pi[ \), let:

\[
\begin{array}{c}
\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \cos \frac{\theta}{3} < \frac{\sqrt{3}}{2} \\
\end{array} \tag{2.33}
\]

and

\[
\frac{1}{4} < \cos^2 \frac{\theta}{3} < \frac{3}{4} \tag{2.34}
\]

hence the expression of \( X_2 \):

\[
X_2 = \rho e^{-i\theta} \tag{2.35}
\]

Let:

\[
u = re^{i\psi} \tag{2.36}
\]

and \( j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{2.37}\)

\[
\frac{j^2}{2} = e^{i\frac{2\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} \tag{2.38}
\]
$j$ is a complex cubic root of the unity $\iff j^3 = 1$. Then, the solutions $u$ and $v$ are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{pe^{i\frac{\theta}{3}}} (2.39)$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{pe^{i\frac{2\pi + \theta}{3}}} (2.40)$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{pe^{i\frac{4\pi - \theta}{3}}} (2.41)$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{pe^{-i\frac{\theta}{3}}} (2.42)$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{pe^{i\frac{4\pi}{3}}} e^{-i\frac{\theta}{3}} = \sqrt[3]{pe^{i\frac{4\pi + \theta}{3}}} (2.43)$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{pe^{-i\frac{4\pi}{3}}} e^{-i\frac{\theta}{3}} = \sqrt[3]{pe^{i\frac{4\pi - \theta}{3}}} (2.44)$$

We may now choose $u_k$ and $v_h$ so that $u_k + v_h$ will be real. In this case, we have

$$v_1 = u_1 (2.45)$$

$$v_2 = u_2 (2.46)$$

$$v_3 = u_3 (2.47)$$

We obtain as real solutions of the equation (2.10):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{p}\cos\frac{\theta}{3} > 0 \quad (2.48)$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{p}\cos\frac{2\pi + \theta}{3} = -\sqrt[3]{p}\left(\cos\frac{\theta}{3} + \sqrt[3]{3}\sin\frac{\theta}{3}\right) < 0 \quad (2.49)$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{p}\cos\frac{4\pi + \theta}{3} = \sqrt[3]{p}\left(-\cos\frac{\theta}{3} + \sqrt[3]{3}\sin\frac{\theta}{3}\right) > 0 \quad (2.50)$$

We compare the expressions of $x_1$ and $x_3$, we obtain:

$$2\sqrt[3]{p}\cos\frac{\theta}{3} > \sqrt[3]{p}\left(-\cos\frac{\theta}{3} + \sqrt[3]{3}\sin\frac{\theta}{3}\right)$$

$$3\cos\frac{\theta}{3} > \sqrt[3]{3}\sin\frac{\theta}{3} \quad (2.51)$$

As $\frac{\theta}{3} \in ] + \frac{\pi}{6}, + \frac{\pi}{3} [$, then $\sin\frac{\theta}{3}$ and $\cos\frac{\theta}{3}$ are $> 0$. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < \cos^2\frac{\theta}{3} \quad (2.52)$$

which is true since $\frac{\theta}{3} \in ] + \frac{\pi}{6}, + \frac{\pi}{3} [$ then $x_1 > x_3$. As $A^m, B^n$ and $-C^l$ are the only real solutions of (2.8), we consider, as $A^m$ is supposed great than $B^n$, the expressions:

$$\begin{cases}
A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{p}\cos\frac{\theta}{3} \\
B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{p}\cos\frac{\theta + 4\pi}{3} = \sqrt[3]{p}\left(-\cos\frac{\theta}{3} + \sqrt[3]{3}\sin\frac{\theta}{3}\right) \\
-C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{p}\cos\frac{\theta + 2\pi}{3} = -\sqrt[3]{p}\left(\cos\frac{\theta}{3} + \sqrt[3]{3}\sin\frac{\theta}{3}\right)
\end{cases} \quad (2.53)$$
3 Proof of the Main Theorem

Main Theorem: Let $A, B, C, m, n, l$ be positive integers with $m, n, l > 2$. If:

$$A^m + B^n = C^l$$

then $A, B,$ and $C$ have a common factor.

Proof. $A^m = 2\sqrt{\rho \cos \theta}^3$ is an integer $\Rightarrow A^2m = 4\sqrt{\rho \cos^2 \theta}^3$ is an integer. But:

$$\sqrt{\rho^2} = \frac{p}{3}$$

Then:

$$A^{2m} = 4\sqrt{\rho^2 \cos^2 \theta}^3 = 4\frac{p}{3} \cos^2 \theta^3 = \frac{4}{3} \cos^2 \theta^3$$

As $A^{2m}$ is an integer, and $p$ is an integer then $\cos^2 \theta^3$ must be written in the form:

$$\cos^2 \theta^3 = \frac{1}{b} \text{ or } \cos^2 \theta^3 = \frac{a}{b}$$

with $b \in \mathbb{N}^*$, for the last condition $a \in \mathbb{N}^*$ and $a, b$ co-primes.

3.1 Case $\cos^2 \theta^3 = \frac{1}{b}$

we obtain:

$$A^{2m} = \frac{4}{3} \cos^2 \theta^3 = \frac{4}{3} \frac{p}{b}$$

As $\frac{1}{4} < \cos^2 \theta^3 < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

3.1.1 $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

3.1.2 $b = 2$

$b = 2 \Rightarrow A^{2m} = p \frac{4}{3} \frac{1}{2} = \frac{2p}{3} \Rightarrow 3 | p \Rightarrow p = 3p' \text{ with } p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{2p}{3} = 2p'$$

But :

$$B^n C^l = \sqrt{\rho^2} \left( 3 - 4 \cos^2 \theta^3 \right) = \frac{p}{3} \left( 3 - 4 \frac{1}{2} \right) = \frac{p}{3} = \frac{3p'}{3} = p'$$

On the one hand:

$$A^{2m} = (A^m)^2 = 2p' \Rightarrow 2 | p' \Rightarrow p' = 2p' \Rightarrow A^{2m} = 4p'^2 \Rightarrow A^m = 2p' \Rightarrow 2 | A^m \Rightarrow 2 | A$$
On the other hand:

\[ B^nC^l = p' = 2p^{n^2} \Rightarrow 2|B^n \text{ or } 2|C^l. \]

If \( 2|B^n \Rightarrow 2|B \). As \( C^l = A^m + B^n \) and \( 2|A \) and \( 2|B \), it follows \( 2|A^m \) and \( 2|B^n \) then \( 2|(A^m + B^n) \Rightarrow 2|C^l \Leftrightarrow 2|C. \)

Then, we have: \( A, B \) and \( C \) solutions of (2.1) have a common factor. Also if \( 2|C^l \), we obtain the same result: \( A, B \) and \( C \) solutions of (2.1) have a common factor.

### 3.1.3 \( b = 3 \)

\( b = 3 \Rightarrow A^{2m} = p \cdot \frac{4}{3} \cdot \frac{1}{3} = \frac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p' \) with \( p' \neq 1 \) since \( 9 \ll p \) then \( A^{2m} = 4p' \Rightarrow p' \) is not a prime. Let \( \mu \) a prime with \( \mu|p' \Rightarrow \mu|A^{2m} \Rightarrow \mu|A. \)

On the other hand:

\[ B^nC^l = \frac{p}{3} \left( 3 - 4\cos^2 \theta \right) = 5p' \]

Then \( \mu|B^n \) or \( \mu|C^l. \) If \( \mu|B^n \Rightarrow \mu|B. \) As \( C^l = A^m + B^n \) and \( \mu|A \) and \( \mu|B \), it follows \( \mu|A^m \) and \( \mu|B^n \) then \( \mu|(A^m + B^n) \Rightarrow \mu|C^l \Rightarrow \mu|C. \)

Then, we have: \( A, B \) and \( C \) solutions of (2.1) have a common factor. Also if \( \mu|C^l \), we obtain the same result: \( A, B \) and \( C \) solutions of (2.1) have a common factor.

### 3.2 Case \( a > 1, \) \( \cos^2 \frac{\theta}{3} = \frac{a}{b} \)

That is to say:

\[ \cos^2 \frac{\theta}{3} = \frac{a}{b} \quad (3.8) \]

\[ A^{2m} = p \cdot \frac{4}{3} \cdot \frac{\cos^2 \theta}{3} = \frac{4.p.a}{3.b} \quad (3.9) \]

and \( a, b \) verify one of the two conditions:

\[ \{3|p \text{ and } b|4p\} \quad \text{or} \quad \{3|a \text{ and } b|4p\} \quad (3.10) \]

and using the equation (2.34), we obtain a third condition:

\[ b < 4a < 3b \quad (3.11) \]

In these conditions, respectively, \( A^{2m} = 4\sqrt{p^2 \cos^2 \frac{\theta}{3}} = \frac{4p'}{3} \cdot \cos^2 \frac{\theta}{3} \) is an integer.

Let us study the conditions given by the equation (3.10).

#### 3.2.1 Hypothesis: \( \{3|p \text{ and } b|4p\} \)

##### 3.2.1.1 Case \( b = 2 \) and \( 3|p : \) \( 3|p \Rightarrow p = 3p' \) with \( p' \neq 1 \) because \( 3 \ll p \), and \( b = 2, \) we obtain:

\[ A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4p'.a}{2} = 2.p'.a \quad (3.12) \]
As:
\[
\frac{1}{4} < \cos^2 \theta = \frac{\cos^2 \theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow a < 2 \Rightarrow a = 1
\]  
(3.13)
But \( a > 1 \) then the case \( b = 2 \) and \( 3|p \) is impossible.

3.2.1.2. Case \( b = 4 \) and \( 3|p \): We have \( 3|p \implies p = 3p' \) with \( p' \in \mathbb{N}^* \), it follows:
\[
A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a
\]  
(3.14)
and:
\[
\frac{1}{4} < \cos^2 \theta = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2
\]  
(3.15)
But \( a, b \) are co-primes. Then the case \( b = 4 \) and \( 3|p \) is impossible.

3.2.1.3. Case: \( b \neq 2, b \neq 4, b|p \) and \( 3|p \): We consider the case:
\[
\frac{3}{b} \mid p \implies p = 3p' \text{ and } p' \neq 1 \text{ (if } p' = 1, \text{ then } p = 3b, \text{ see sub-paragraph 2.2.3 souse-case equation (3.36)). Hence :}
\]
\[
A^{2m} = \frac{4b^2 p'' a}{b} = 4a p''
\]  
(3.17)
Finally, we have the two equations:
\[
A^{2m} = \frac{4b^2 p'' a}{b} = 4a p''
\]  
(3.19)
\[
B^n C^l = p''.(3b - 4a)
\]  
(3.20)

**Sous-case 1**: \( p'' \) is prime. From (3.19), \( p''|A^{2m} \Rightarrow p''|A^m \Rightarrow p''|A \). From (3.26), \( p''|B^n \) or \( p''|C^l \). If \( p''|B^n \Rightarrow p''|B \), as \( C^l = A^m + B^n \Rightarrow p''|C^l \Rightarrow p''|C \). If \( p''|C^l \Rightarrow p''|C \), as \( B^n = C^l - A^m \Rightarrow p''|B^n \Rightarrow p''|B \).

Then \( A, B \) and \( C \) solutions of (2.1) have a common factor.

**Sous-case 2**: \( p'' \) is not prime. Let \( \lambda \) one prime divisor of \( p'' \). From (3.19), we have:
\[
\lambda|A^{2m} \Rightarrow \lambda|A^m \text{ as } \lambda \text{ is prime then } \lambda|A
\]  
(3.21)
From (3.20), as \( \lambda|p'' \) we have:
\[
\lambda|B^n C^l \Rightarrow \lambda|B^n \text{ or } \lambda|C^l
\]  
(3.22)
If \( \lambda | B^n \), \( \lambda \) is prime \( \lambda | B \), and as \( C^t = A^m + B^n \) then we have also:

\[
\lambda | C^t \quad \text{as} \quad \lambda \text{ is prime, then} \quad \lambda | C \quad \text{(3.23)}
\]

By the same way, if \( \lambda | C^t \), we obtain \( \lambda | B \).

Then: \( A, B \) and \( C \) solutions of \((2.1)\) have a common factor.

Let us verify the condition \((3.11)\) given by:

\[
p < 3A^{2m} < 3p \quad \text{with} \quad p = A^{2m} + B^{2n} + A^m B^n \quad \text{(3.24)}
\]

The \( 3A^{2m} < 3p \implies A^{2m} < p \) is verified.

If:

\[
p < 3A^{2m} \implies 2A^{2m} - A^m B^n - B^{2n} > 0
\]

We put \( Q(Y) = 2Y^2 - B^n Y - B^{2n} \), the roots of \( Q(Y) = 0 \) are \( Y_1 = \frac{B^n}{2} \) and \( Y_2 = B^n \). \( Q(Y) > 0 \) for \( Y < Y_1 \) and \( Y > Y_2 = B^n \). In our case, we take \( Y = A^m \).

As \( A^m > B^n \) then \( p < 3A^{2m} \) is verified. Then the condition \( b < 4a < 3b \) is true.

In the following of the paper, we verify easily that the condition \( b < 4a < 3b \) implies to verify \( A^m > B^n \) which is true.

### 3.2.1.4. Case \( b = 3 \) and \( 3|p \):

As \( 3|p \implies p = 3p' \) and we write:

\[
A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p a}{3 b} = \frac{4 \times 3p' a}{3} = \frac{4p' a}{3} \quad \text{(3.25)}
\]

As \( A^{2m} \) is an integer and that \( a \) and \( b \) are co-primes and \( \cos^2 \frac{\theta}{3} \) can not be one in reference to the equation \((2.33)\), then we have necessary \( 3|p' \implies p' = 3p'' \) with \( p'' \neq 1 \), if not \( p = 3p'' = 3 \times 3p'' = 9 \) but \( p = A^{2m} + B^{2n} + A^m B^n > 9 \), the hypothesis \( p'' = 1 \) is impossible, then \( p'' > 1 \). Hence:

\[
A^{2m} = \frac{4p' a}{3} = \frac{4 \times 3p'' a}{3} = 4p'' a \quad \text{(3.26)}
\]

\[
B^n C^t = \frac{p}{3} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) = p' \left( 3 - 4 \frac{a}{b} \right) = \frac{3p''(9 - 4a)}{3} = p''.(9 - 4a) \quad \text{(3.27)}
\]

As \( \frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \implies 3 < 4a < 9 \implies a = 2 \) as \( a > 1 \).

\[
\text{As} \quad a = 2, \quad \text{we obtain:}
\]

\[
A^{2m} = \frac{4p' a}{3} = \frac{4 \times 3p'' a}{3} = 4p'' a = 8p'' \quad \text{(3.28)}
\]

\[
B^n C^t = \frac{p}{3} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) = p' \left( 3 - 4 \frac{a}{b} \right) = \frac{3p''(9 - 4a)}{3} = p'' \quad \text{(3.29)}
\]

The two last equations give that \( p'' \) is not prime. Then we use the same methodology described above for the case 3.2.1.3., and we have: \( A, B \) and \( C \) solutions of \((2.1)\) have a common factor.
3.2.1.5. Case $3|p$ and $b = p$ : We have:
\[\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}\]
and:
\[A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4a}{3}\]  
(3.30)
As $A^{2m}$ is an integer, this implies that $3|a$, but $3|p \implies 3|b$. As $a$ and $b$ are co-primes, hence the contradiction. Then the case $3|p$ and $b = p$ is impossible.

3.2.1.6. Case $3|p$ and $b = 4p$ : $3|p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 4p = 12p'$.  
\[A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{a}{3} \implies 3|a\]  
(3.31)
because $A^{2m}$ is an integer. But $3|p \implies 3|(4p) = b$, that is in contradiction with the hypothesis $a, b$ are co-primes. Then the case $b = 4p$ is impossible.

3.2.1.7. Case $3|p$ and $b = 2p$ : $3|p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 2p = 6p'$.  
\[A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{2a}{3} \implies 3|a\]  
(3.32)
because $A^{2m}$ is an integer. But $3|p \implies 3|(2p) \implies 3|b$, that is in contradiction with the hypothesis $a, b$ are co-primes. Then the case $b = 2p$ is impossible.

3.2.1.8. Case $3|p$ and $b \neq 3$ is a divisor of $p$ : We have $b = p' \neq 3$, and $p$ is written as:
\[p = kp' \quad \text{with} \quad 3|k \implies k = 3k'\]  
(3.33)
and:
\[A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4 \times 3.k'p' a}{3p'} = 4ak'\]  
(3.34)
We calculate $B^nC^l$:
\[B^nC^l = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3}\right) = k'(3p' - 4a)\]  
(3.35)

1st Sous-case: $k' \neq 1$, we use the same methodology described for the case 3.1.2.3., and we obtain: $A, B$ and $C$ solutions of (2.1) have a common factor.

2nd sous-case: $k' = 1 \implies p = 3b$  
(3.36)
then we have:
\[A^{2m} = 4a \implies a \quad \text{is even}\]  
(3.37)
and:
\[A^nB^n = 2\sqrt[p]{\cos \frac{\theta}{3}} \sqrt[p]{\left(\sqrt[3]{\sin \frac{\theta}{3} - \cos \frac{\theta}{3}}\right)} = \frac{p\sqrt[3]{3} \sin \frac{\theta}{3}}{3} - 2a\]  
(3.38)
Using the expression of \( A^2m + 2A^mB^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2b\sqrt{3}\sin \frac{2\theta}{3} \) (3.39)

The left member of (3.39) is an integer and \( b \) also, then \( 2\sqrt{3}\sin \frac{2\theta}{3} \) can be written in the form:

\[
2\sqrt{3}\sin \frac{2\theta}{3} = \frac{k_1}{k_2}
\]

where \( k_1, k_2 \) are two co-primes integers and \( k_2|b \implies b = k_2k_3 \).

\( \diamond \) - We suppose \( k_3 \neq 1 \). Hence:

\[
A^{2m} + 2A^mB^n = k_3k_1
\]

Let \( \mu \) is an prime integer such that \( \mu|k_3 \). If \( \mu = 2 \Rightarrow 2|b \) but \( 2|a \) that is contradiction with \( a, b \) co-primes. We suppose \( \mu \neq 2 \) and \( \mu|k_3 \), then \( \mu|A^m(A^m + 2B^n) \implies \mu|A^m \) or \( \mu|(A^m + 2B^n) \).

*A-1* - If \( \mu|A^m \Rightarrow \mu|A^{2m} \Rightarrow \mu|4a \Rightarrow \mu|a \). As \( \mu|k_3 \Rightarrow \mu|b \) and that \( a, b \) are co-primes hence the contradiction.

*A-2* - If \( \mu|(A^m + 2B^n) \Rightarrow \mu \nmid A^m \text{ and } \mu \nmid 2B^n \text{ then } \mu \neq 2 \text{ and } \mu \nmid B^n \). \( \mu|(A^m + 2B^n) \), we can write:

\[
A^m + 2B^n = \mu t' \quad t' \in \mathbb{N}^+
\]

It follows:

\[
A^m + B^n = \mu t' - B^n \Rightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}
\]

Using the expression of \( p \), we obtain:

\[
p = t'^2\mu^2 - 2t' B^n \mu + B^n(B^n - A^n)
\]

As \( p = 3b = 3k_2k_3 \) and \( \mu|k_3 \) hence \( \mu|p \implies p = \mu t' \), so we have :

\[
\mu t' = \mu(\mu t'^2 - 2t' B^n) + B^n(B^n - A^n)
\]

and \( \mu|B^n(B^n - A^n) \Rightarrow \mu|B^n \) or \( \mu|(B^n - A^n) \).

*A-2-1* - If \( \mu|B^n \Rightarrow \mu|B \) which is in contradiction with *A-2*.

*A-2-2* - If \( \mu|(B^n - A^n) \) and using \( \mu|(A^m + 2B^n) \), we obtain:

\[
\mu|3B^n \Rightarrow \begin{cases} 
\mu|B^n \Rightarrow \mu|B \text{ which is impossible} \\
\text{or} \\
\mu = 3
\end{cases}
\]

*A-2-2-1* - If \( \mu = 3 \Rightarrow 3|k_3 \Rightarrow k_3 = 3k_3' \), and we have \( b = k_2k_3 = 3k_2k_3' \), it follows \( p = 3b = 9k_2k_3' \) then \( 9|p \), but \( p = (A^m - B^n)^2 + 3A^mB^n \) then :

\[
9k_2k_3' - 3A^mB^n = (A^m - B^n)^2
\]
we write it as:

\[ 3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \]  

(3.46)

hence \( 3|(3k_2k'_3 - A^mB^n) \Rightarrow 3|A^mB^n \Rightarrow 3|A^m \) or \( 3|B^n \).

*A-2-2-1-1* If \( 3|A^m \Rightarrow 3|A \) and we have also \( 3|A^{2m} \), but \( A^{2m} = 4a \Rightarrow 3|4a \Rightarrow 3|a \). As \( b = 3k_2k'_3 \) then \( 3|b \), but \( a, b \) are co-primes hence the contradiction. Then \( 3 \nmid A \).

*A-2-2-1-2* If \( 3|B^n \Rightarrow 3|B \), but the \( 3|A^{2m} \) gives \( 3|(A^m - B^n)^2 \Rightarrow 3|(A^m - B^n) \Rightarrow 3|A^m \Rightarrow 3|A \). But using the result of the last paragraph *A-2-2-1-1*, we obtain \( 3 \nmid A \). Then the hypothesis \( k_3 \neq 1 \) is impossible.

\( \diamond \) Now we suppose that \( k_3 = 1 \Rightarrow b = k_2 \) and \( p = 3b = 3k_2 \). We have then:

\[ 2\sqrt{\frac{3}{2}} \sin \frac{2\theta}{3} = \frac{k_1}{b} \]  

(3.47)

with \( k_1, b \) co-primes. We write \( 3 \) as:

\[ 4\sqrt{3} \sin \frac{\theta}{3} \cos \frac{\theta}{3} = \frac{k_1}{b} \]

Taking the square of the two members and replacing \( \cos^2 \frac{\theta}{3} \) by \( \frac{a}{b} \), we obtain:

\[ 3 \times 4^2(a-b) = k_1^2 \]  

(3.48)

which implies that:

\( 3|a \quad \text{or} \quad 3|(b-a) \)

*B-1- If \( 3|a, \) as \( A^{2m} = 4a \Rightarrow 3|A^{2m} \Rightarrow 3|A \). But \( p = (A^m - B^n)^2 + 3A^mB^n \) and that \( 3|p \Rightarrow 3|(A^m - B^n)^2 \Rightarrow 3|(A^m - B^n) \). But \( 3|A \) hence \( 3|B^n \Rightarrow 3|B \), it follows \( 3|C \Rightarrow 3|C \).

We obtain: \( A, B \) and \( C \) solutions of \( 2.1 \) have a common factor.

*B-2- Considering now that \( 3|(b-a) \). As \( k_1 = A^m(A^m + 2B^n) \) by the equation \( 3.41 \) and that \( 3|k_1 \Rightarrow 3|A^m(A^m + 2B^n) \Rightarrow 3|A^m \) or \( 3|(A^m + 2B^n) \).

*B-2-1- If \( 3|A^m \Rightarrow 3|A \Rightarrow 3|A^{2m} \) then \( 3|4a \Rightarrow 3|a \). But \( 3|(b-a) \Rightarrow 3|b \) hence the contradiction with \( a, b \) are co-primes.

*B-2-2- If:

\[ 3|(A^m + 2B^n) \Rightarrow 3|(A^m - B^n) \]  

(3.49)

But \( p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n \) then \( p - 3A^mB^n = (A^m - B^n)^2 \Rightarrow 9|(p - 3A^mB^n) \) or \( 9|(3b - 3A^mB^n) \), then \( 3|(b - A^mB^n) \) but \( 3|(b - a) \Rightarrow 3|(a - A^mB^n) \). As \( A^{2m} = 4a = (A^m)^2 \Rightarrow a' \in \mathbb{N}^* \) and \( a = a'^2 \Rightarrow A^m = 2a' \). We arrive to \( 3|(a'^2 - 2a'B^n) \Rightarrow 3|a'(a' - 2B^n) \).

*B-2-2-1- If \( 3|a' \Rightarrow 3|A^m \Rightarrow 3|A, \) but \( 3|(A^m + 2B^n) \Rightarrow 3|2B^n \Rightarrow 3|B^n \Rightarrow 3|B, \) it follows \( 3|C \).
Hence $A, B$ and $C$ solutions of $[2.1]$ have a common factor.

*B-2-2-* Now if $3|(a' - 2B^n) \implies 3|(2a' - 4B^n) \implies 3|(A^m - 4B^n) \implies 3|(A^m - B^n)$, we refine the hypothesis (3.49) above.

The study of the case 3.2.1.8. is finished.

**3.2.1.9 Case $3|p$ and $b|4p$:** As $3|p \Rightarrow p = 3p'$ and $b|4p \Rightarrow \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1 b$.

Let us calculate $A^m B^n$:

$$A^m B^n = 2\sqrt{p} \cos \frac{\theta}{3} \sqrt{\frac{3}{2}} \left( \sqrt{3} \sin \frac{\theta}{3} - \cos \frac{\theta}{3} \right) = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - \frac{a}{2} \quad (3.50)$$

Let:

$$A^m + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2p' \sqrt{3} \sin \frac{2\theta}{3} \quad (3.51)$$

The left member of the equation (3.51) is an integer and also $p'$, then $2\sqrt{3} \sin \frac{2\theta}{3}$ can be written as:

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_2}{k_3} \quad (3.52)$$

where $k_2, k_3$ are two co-primes integers and $k_3 | p' \implies p' = k_3 k_4$.

◊ - We suppose that $k_1 \neq 1$ then:

$$A^m + 2A^m B^n = k_2 k_4 \quad (3.53)$$

Let $\mu$ one prime integer with $\mu | k_4$. Then $\mu | A^m (A^m + 2B^n) \implies \mu | A^m$ or $\mu | (A^m + 2B^n)$.

*A-1-* If $\mu | A^m \implies \mu | A^m \implies \mu | a$. As $\mu | k_4 \implies \mu | p' \implies \mu | (4p = b)$. But $a, b$ are co-primes then the contradiction.

*A-2-* If $\mu | (A^m + 2B^n) \implies \mu | A^m$ and $\mu | 2B^n$ then $\mu \neq 2$ and $\mu | B^n$. $\mu | (A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu t' \quad t' \in \mathbb{N}^* \quad (3.54)$$

It follows:

$$A^m + B^n = \mu t' - B^n \implies A^m = 2^m B^n + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of $p$, we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^n) \quad (3.55)$$
As \( p = 3p' \) and \( \mu|\mu' \Rightarrow \mu|(3p') \Rightarrow \mu|p \), we can write \( \exists \mu' \in \mathbb{N}^* \) and \( p = \mu \mu' \), then we obtain:

\[
\mu' \mu = \mu(\mu \mu'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{3.56}
\]

and \( \mu|B^n(B^n - A^m) \Rightarrow \mu|B^n \) or \( \mu|(B^n - A^m) \).

*A-2-1- If \( \mu|B^n \Rightarrow \mu|B \) which is in contradiction with *A-2.

*A-2-2- If \( \mu|(B^n - A^m) \) and using \( \mu|(A^m + 2B^n) \), we obtain:

\[
\mu|3B^n \Rightarrow \begin{cases} 
\mu|B^n \Rightarrow \mu|B \text{ which is impossible} \\
\mu = 3 
\end{cases}
\tag{3.57}
\]

*A-2-2-1- If \( \mu = 3 \Rightarrow 3|k_4 \Rightarrow k_4 = 3k_4' \), and we obtain \( p' = k_3k_4 = 3k_3k_4' \), it follows \( p = 3p' = 9k_3k_4' \) then \( 9|p \), but \( p = (A^m - B^n)^2 + 3A^m B^n \), then:

\[
9k_4k_4' - 3A^m B^n = (A^m - B^n)^2
\]

that we write:

\[
3(3k_4k_4' - A^m B^n) = (A^m - B^n)^2 \tag{3.58}
\]

then \( 3(3k_4k_4' - A^m B^n) \Rightarrow 3|A^m B^n \Rightarrow 3|A^m \) or \( 3|B^n \).

*A-2-2-1-1- If \( 3|A^m \Rightarrow 3|A^{2m} \Rightarrow 3|a \), but \( 3|p' \Rightarrow 3|4p' \Rightarrow 3|b \), then the contradiction with \( a, b \) co-primes. Then \( 3 \nmid A \).

*A-2-2-1-2- If \( 3|B^n \) but \( A^m = \mu \mu' - 2B^n = 3t' - 2B^n \Rightarrow 3|A^m \), which is in contradiction. Then the hypothesis \( k_4 \neq 1 \) is impossible.

\( \diamond \)- We suppose that \( k_4 = 1 \) \( \Rightarrow p' = k_3k_4 = k_3 \). Then we obtain:

\[
2\sqrt{3}\sin\frac{2\theta}{3} = \frac{k_2}{p'} 
\tag{3.59}
\]

with \( k_2, p' \) co-primes, we write \( \sqrt{3}\sin\frac{2\theta}{3} = \frac{k_2}{p'} \) as:

\[
4\sqrt{3}\sin\frac{\theta}{3}\cos\frac{\theta}{3} = \frac{k_2}{p'}
\]

Taking the square of the two members and replacing \( \cos\frac{\theta}{3} \) by \( \frac{a}{b} \) and \( b = 4p' \), we obtain:

\[
3.a(b - a) = k_2^2 
\tag{3.60}
\]

that implicate:

\( 3|a \) or \( 3|(b - a) \)

*B-1- If \( 3|a \Rightarrow 3|A^{2m} \Rightarrow 3|A, \) as \( p = (A^m - B^n)^2 + 3A^m B^n \) and that \( 3|p \Rightarrow 3|(A^m - B^n)^2 \Rightarrow 3|(A^m - B^n) \). But \( 3|A \), then \( 3|B^n \Rightarrow 3|B \), it follows \( 3|C' \Rightarrow 3|C \).

We obtain : \( A, B \) and \( C \) solutions of (2.1) have a common factor.
3 Proof of the Main Theorem

*B-2- We consider that $3|(b-a)$. As $k_2 = A^{m}(A^{m}+2B^{n})$ given by the equation (3.53) and that $3|k_2 \implies 3|A^{m}(A^{m}+2B^{n}) \implies 3|A^{m}$ or $3|(A^{m}+2B^{n})$.

*B-2-1- If $3|A^{m} \implies 3|A^{2m} \implies 3|a$, but $3|(b-a) \implies 3|b$ then the contradiction with $a,b$ co-primes.

*B-2-2- If:

$$3(A^{m}+2B^{n}) \implies 3(A^{m}-B^{n})$$

(3.61)

but $p = A^{2m} + B^{2n} + A^{m}B^{n} = (A^{m} - B^{n})^{2} + 3A^{m}B^{n}$ then $p - 3A^{m}B^{n} = (A^{m} - B^{n})^{2} \implies 9((p - 3A^{m}B^{n})$ or $9(3p' - 3A^{m}B^{n})$, then $3|p' - A^{m}B^{n}) \implies 3|4(p' - 4A^{m}B^{n}) \implies 3|(b - 4A^{m}B^{n})$ but $3|(b-a) \implies 3|(a-A^{m}B^{n})$. As $3|(A^{2m} - 4A^{m}B^{n}) \implies 3|A^{m}(A^{m} - 4B^{n})$.

*B-2-2-1- If $3|A^{m} \implies 3|A^{2m} \implies 3|a$, but $3|(b-a) \implies 3|b$ then the contradiction with $a,b$ co-primes.

*B-2-2-2- Now if $3|(A^{m} - 4B^{n}) \implies 3|(A^{m} - B^{n})$, we find the hypothesis of the beginning (3.61) above.

**. We suppose $k_1 \neq 3$ and $3|k_1$ $\implies \overline{k_1 = 3k'_{1}}$ with $k'_{1} \neq 1$. we have $4p = 12p' = k_{1}b = 3k'_{1}b \Rightarrow 4p' = k'_{1}b$. $A^{2m}$ can be written as :

$$A^{2m} = \frac{4p}{3} \cos^{2} \theta = \frac{3k'_{1}b}{3} \overline{a} = k'_{1}a$$

(3.62)

and $B^{n}C^{l}$ :

$$B^{n}C^{l} = \frac{p}{3} \left(3 - 4\cos^{2} \theta \right) = \frac{k'_{1}}{4} (3b - 4a)$$

(3.63)

As $B^{n}C^{l}$ is an integer, we must have $4|(3b - 4a)$ or $4|k'_{1}$.

*** We suppose that $4|(3b - 4a) \Rightarrow \frac{3b - 4a}{4} = c \in \mathbb{N}^{*}$, and we obtain:

$$A^{2m} = k'_{1}a$$

$$B^{n}C^{l} = k'_{1}c$$

C-1- If $k'_{1}$ is prime, then $k'_{1}|A^{2m} \Rightarrow k'_{1}|A$ and $k'_{1}|B^{n}C^{l} \Rightarrow k'_{1}|B^{n}$ or $k'_{1}|C^{l}$. If $k'_{1}|B^{n} \Rightarrow k'_{1}|B$, then $k'_{1}|C^{l} \Rightarrow k'_{1}|C$. With the same method if $k'_{1}|C^{l}$, we arrive to $k'_{1}|B$.

We obtain: $A,B$ and $C$ solutions of (2.1) have a common factor.

C-2- $k'_{1}$ not a prime. Let $\mu$ a prime divisor of $k'_{1}$, as described in C-1- above, we obtain : $A,B$ and $C$ solutions of (2.1) have a common factor.

*** We suppose that $4|k'_{1}$.

C-3- $k'_{1}$ = 4 but this case is discussed in the second sous-case of the paragraph (3.2.1.8).
C-4- $k_1' = 4k_1''$ with $k_1'' > 1$. Then, we have:

$$A^{2m} = 4k_1''a$$  \hspace{1cm} (3.64)

$$B^rC^l = k_1''(3b - 4a)$$  \hspace{1cm} (3.65)

C-4-1- If $k_1''$ is prime, then $k_1''|A^{2m} \Rightarrow k_1''|A$ and $k_1''|B^rC^l \Rightarrow k_1''|B^r$ or $k_1''|C^l$.

If $k_1''|B^r \Rightarrow k_1''|B$, then $k_1''|C^l \Rightarrow k_1''|C$. With the same method if $k_1''|C^l$, we arrive to $k_1''|B$.

We obtain: $A, B$ and $C$ solutions of (2.1) have a common factor.

C-4-2- $k_1''$ not a prime. Let $\mu$ a prime divisor of $k_1''$, as described in C-4-1 above, we obtain:

$A, B$ and $C$ solutions of (2.1) have a common factor.

3.2.2 Hypothesis : $\{3|a \text{ and } b|4p\}$

We have:

$$3|a \Rightarrow \exists a' \in \mathbb{N}^* / a = 3a'$$  \hspace{1cm} (3.66)

3.2.2.1. Case $b = 2$ and $3|a$: $A^{2m}$ is written as:

$$A^{2m} = \frac{4p}{3}\cos^2\frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{3} = \frac{4p}{3} \cdot \frac{a}{2} = \frac{2p.a}{3}$$  \hspace{1cm} (3.67)

Using the equation (3.66), $A^{2m}$ becomes:

$$A^{2m} = \frac{2p.3a'}{3} = 2.p.a'$$  \hspace{1cm} (3.68)

But $\cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

3.2.2.2. Case $b = 4$ and $3|a$: $A^{2m}$ is written as:

$$A^{2m} = \frac{4p}{3}\cos^2\frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{3} = \frac{4p}{3} \cdot \frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a'$$

and $\cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Rightarrow a' < 1$  \hspace{1cm} (3.70)

which is impossible.

Then the case $b = 4$ is impossible.

3.2.2.3. Case $b = p$ and $3|a$: Then:

$$\cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p}$$  \hspace{1cm} (3.71)

and:

$$A^{2m} = \frac{4p}{3}\cos^2\frac{\theta}{3} = \frac{4p}{3} \cdot \frac{3a'}{p} = 4a' = (A^{m})^2$$  \hspace{1cm} (3.72)

$$\exists a'' \in \mathbb{N}^* / a' = a''^2$$  \hspace{1cm} (3.73)
3 Proof of the Main Theorem

We calculate $A^mB^n$, hence:

$$A^mB^n = p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a'$$

or

$$A^mB^n + 2a' = p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3}$$

(3.74)

The left member of (3.74) is an integer and $p$ is also, then $2\frac{\sqrt{3}}{3} \sin \frac{2\theta}{3}$ will be written as:

$$2\frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2}$$

(3.75)

where $k_1, k_2$ are two co-primes integers and $k_2|p \implies p = k_2k_3, k_3 \in \mathbb{N}^*$.

◊ - We suppose that $k_3 \neq 1$. We obtain:

$$A^m(A^m + 2B^n) = k_1k_3$$

(3.76)

Let us $\mu$ a prime integer with $\mu|k_3$, then $\mu|b$ and $\mu|A^m(A^m + 2B^n) \implies \mu|A^m$ or $\mu|(A^m + 2B^n)$.

* If $\mu|A^m \implies \mu|A$ and $\mu|A^{2m}$, but $A^{2m} = 4a' \implies \mu|4a' \implies (\mu = 2$ but $2|a')$ or $(\mu|a')$. Then $\mu|a$ hence the contradiction with $a, b$ co-primes.

* If $\mu|(A^m + 2B^n) \implies \mu \notdivides A^m$ and $\mu \notdivides 2B^n$ then $\mu \neq 2$ and $\mu \notdivides B^n$. We write $\mu|(A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu t' \quad t' \in \mathbb{N}^*$$

(3.77)

It follows:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of $p$:

$$p = t'^2\mu^2 - 2t'\mu B^n + B^n(B^n - A^m)$$

(3.78)

Since $p = b = k_2k_3$ and $\mu|k_3$ then $\mu|b \implies 3\mu' \in \mathbb{N}^*$ and $b = \mu\mu'$, so we can write:

$$\mu\mu' = \mu(t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

(3.79)

From the last equation, we get $\mu|B^n(B^n - A^m) \implies \mu|B^n$ or $\mu|(B^n - A^m)$. If $\mu|B^n$ which is contradiction with $\mu \notdivides B^n$. If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, on arrive to:

$$\mu|3B^n \implies \begin{cases} 
\mu|B^n \implies \text{which is contradiction} \\
\mu = 3
\end{cases}$$

(3.80)

Si $\mu = 3$, then $3|b$, but $3|a$ thus the contradiction with $a, b$ co-primes.

◊ - We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^mB^n = k_1$$

(3.81)

$$b = k_2$$

(3.82)

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{b}$$

(3.83)
Taking the square of the last equation, we obtain:
\[
\frac{4}{3} \sin^2 \frac{2\theta}{3} = \frac{k_1^2}{b^2}
\]
\[
\frac{16}{3} \sin^2 \frac{\theta}{3} \cos^2 \frac{\theta}{3} = \frac{k_1^2}{b^2}
\]
\[
\frac{16}{3} \sin^2 \frac{2\theta}{3} \frac{3a'}{b} = \frac{k_1^2}{b^2}
\]
Finally:
\[
4^2 a'(p - a) = k_1^2 
\]  
(3.84)
but \(a' = a^{-2}\) then \(p - a\) is a square. Let us:
\[
\lambda^2 = p - a 
\]  
(3.85)
The equation (3.84) becomes:
\[
4^2 a'^2 \lambda^2 = k_1^2 \implies k_1 = 4a''\lambda 
\]  
(3.86)
taking the positive square root. Using (3.81), we get:
\[
k_1 = 4a''\lambda 
\]  
(3.87)
But \(k_1 = A^n(A^m + 2B^n) = 2a''(A^m + 2B^n)\), it follows:
\[
A^m + 2B^n = 2\lambda 
\]  
(3.88)
Let \(\lambda_1\) prime \(\ne 2\), a divisor of \(\lambda\) (if not \(\lambda_1 = 2|\lambda \implies 2|\lambda^2 \implies 2|(p - a)\) but \(a\) is even, then \(2|p \implies 2|b\) which is contradiction with \(a,b\) co-primes).

We consider \(\lambda_1 \ne 2\) and :
\[
\lambda_1|\lambda \implies \lambda_1|\lambda^2 \quad \text{and} \quad \lambda_1|(A^m + 2B^n) 
\]  
(3.89)
\[
\lambda_1|(A^m + 2B^n) \implies \lambda_1 \nmid A^m \quad \text{if not} \quad \lambda_1|2B^n 
\]  
(3.90)
But \(\lambda_1 \ne 2\) hence \(\lambda_1|B^n \implies \lambda_1|B\), it follows:
\[
\lambda_1|(p = b) \quad \text{and} \quad \lambda_1|A^m \implies \lambda_1|2a^n \implies \lambda_1|a 
\]  
(3.91)
hence the contradiction with \(a,b\) co-primes.

We assume now \(\lambda_1 \nmid A^m\). \(\lambda_1|(A^m + 2B^n) \implies \lambda_1|(A^m + 2B^n)^2\) that is \(\lambda_1|(A^{2m} + 4A^m B^n + 4B^{2n})\), we write it as \(\lambda_1|(p + 3A^m B^n + 3B^{2n}) \implies \lambda_1|(p + 3B^n(A^m + 2B^n) - 2B^{2n})\). But \(\lambda_1|(A^m + 2B^n) \implies \lambda_1|(p - 3B^{2n})\), as \(\lambda_1|(p - a)\) hence by difference, we obtain \(\lambda_1|(a - 3B^{2n})\) or \(\lambda_1|(3a' - 3B^{2n}) \implies \lambda_1|3(a' - B^{2n}) \implies \lambda_1 = 3\) or \(\lambda_1|(a' - B^{2n})\).

* A-1- If \(\lambda_1 = 3\) but \(3|a \implies 3|(p = b)\) hence the contradiction with \(a,b\) co-primes.

* A-2- If \(\lambda_1|(a' - B^{2n}) \implies \lambda_1|(a'^2 - B^{2n}) \implies \lambda_1|(a'' - B^{2n})(a'' + B^{2n}) \implies \lambda_1|(a'' + B^{n})\) or \(\lambda_1|(a' - B^n)\), because \((a'' - B^n) \ne 1\) if not we obtain \(a'^2 - B^{2n} = \)
a^n + B^n \Rightarrow a^{2n} - a^n = B^n - B^{2n}$. The left member is positive and the right member is negative, then the contradiction.

\[ \lambda_1 | (a^n - B^n) \Rightarrow \lambda_1 | 2(a^n - B^n) \Rightarrow \lambda_1 | (A^m - 2B^n) \]

hence $\lambda_1 | 2A^m \Rightarrow \lambda_1 | A^m$, $\lambda_1 \neq 2$, it follows $\lambda_1 | A^m$ hence the contradiction with (3.90).

\[ \lambda_1 | (a^n + B^n) \Rightarrow \lambda_1 | 2(a^n + B^n) \iff \lambda_1 | (A^m + 2B^n) \]

We refine the condition (3.89).

Then the case $k_3 = 1$ is impossible.

3.2.2.4. Case $b | p \Rightarrow p = b.p', p' > 1, b \neq 2, b \neq 4 \text{ and } 3 | a :$

\[ A^{2m} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4bp'.3a'}{3b} = 4p'a' \] (3.92)

We calculate $B^n C^l$:

\[ B^n C^l = \sqrt[p^2]{3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}} = \sqrt[p^2]{3 - 4cos^2\frac{\theta}{3}} \] (3.93)

But $\sqrt[p^2]{\frac{p}{3}} = \frac{3a'}{b}$:

\[ B^n C^l = \sqrt[p^2]{3 - 4cos^2\frac{\theta}{3}} = \frac{p}{3} \left( 3 - 4\frac{a'}{b} \right) = p \left( 1 - 4\frac{a'}{b} \right) = p'(b - 4a') \] (3.94)

As $p = b.p'$, and $p' > 1$, we have then:

\[ B^n C^l = p'(b - 4a') \] (3.95)

\[ \text{and } A^{2m} = 4p'.a' \] (3.96)

A - Let $\lambda$ a prime divisor of $p'$ (we suppose $p'$ not prime). From (3.96), we have:

\[ \lambda | A^{2m} \Rightarrow \lambda | A^m \text{ as } \lambda \text{ is a prime, then } \lambda | A \] (3.97)

From (3.95), as $\lambda | p'$ we have:

\[ \lambda | B^n C^l \Rightarrow \lambda | B^n \text{ or } \lambda | C^l \] (3.98)

If $\lambda | B^n$, $\lambda$ is a prime $\lambda | B$, but $C^l = A^m + B^n$, then we have also :

\[ \lambda | C^l \text{ as } \lambda \text{ is a prime, then } \lambda | C \] (3.99)

By the same way, if $\lambda | C^l$, we obtain $\lambda | B$. then $A, B \text{ and } C$ solutions of (2.1) have a common factor.

B - We suppose now that $p'$ is prime, from the equations (3.95) and (3.96), we obtain then:

\[ p'|A^{2m} \Rightarrow p'|A^m \Rightarrow p'|A \] (3.100)
and:
\[ p'|B^nC^l \Rightarrow p'|B^n \text{ or } p'|C^l \quad (3.101) \]

If \( p'|B^n \Rightarrow p'|B \) \quad (3.102)

As \( C^l = A^m + B^n \) and that \( p'|A, p'|B \Rightarrow p'|A^m, p'|B^n \Rightarrow p'|C^l \Rightarrow p'|C \) \quad (3.103)

By the same way, if \( p'|C^l \), we arrive to \( p'|B \).

Hence: \( A, B \) and \( C \) solutions of (2.1) have a common factor.

3.2.2.5. Case \( b = 2p \text{ and } 3|a \) : We have:
\[ \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \quad \Rightarrow \quad A^{2m} = \frac{4p \cdot a}{3b} = \frac{4p}{3} \cdot \frac{3a'}{2p} = 2a' \quad \Rightarrow \quad 2|A^m \quad \Rightarrow \quad 2|a \quad \Rightarrow \quad 2|a' \]

Then \( 2|a \) and \( 2|b \) which is contradiction with \( a, b \) co-primes.

3.2.2.6. Case \( b = 4p \text{ and } 3|a \) : We have:
\[ \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \quad \Rightarrow \quad A^{2m} = \frac{4p \cdot a}{3b} = \frac{3a'}{4p} = a' \]

Calculate \( A^mB^n \), we obtain:
\[ A^mB^n = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{2p}{3} \cos^2 \frac{\theta}{3} = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{a'}{2} \quad \Rightarrow \quad A^mB^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} \quad (3.104) \]

let:
\[ A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} \quad (3.105) \]

The left member of (3.105) is an integer and \( p \) is an integer, then \( \frac{2\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} \) will be written:
\[ \frac{2\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} = \frac{k_1}{k_2} \quad (3.106) \]

where \( k_1, k_2 \) are two co-primes integers and \( k_2|p \quad \Rightarrow \quad p = k_2k_3 \).

\[ \diamond \quad \text{Firstly, we suppose that } k_3 \neq 1. \text{ Hence:} \]
\[ A^{2m} + 2A^mB^n = k_3k_1 \quad (3.107) \]

Let \( \mu \) a prime integer and \( \mu|k_3 \), then \( \mu|A^{m}(A^m + 2B^n) \quad \Rightarrow \quad \mu|A^m \text{ or } \mu|(A^m + 2B^n) \).

* If \( \mu|A^m \quad \Rightarrow \quad \mu|A \). As \( \mu|k_3 \quad \Rightarrow \quad \mu|p \) and that \( p = A^{2m} + B^{2n} + A^mB^n \quad \Rightarrow \quad \mu|B^{2n} \)

then \( \mu|B \), it follows \( \mu|C^l \), hence \( A, B \) and \( C \) solutions of (2.1) have a common factor.

* If \( \mu|(A^m + 2B^n) \quad \Rightarrow \quad \mu \nmid A^m \) and \( \mu \nmid B^n \) then:
\[ \mu \neq 2 \quad \text{and} \quad \mu \nmid B^n \quad (3.108) \]
\( \mu \mid (A^m + 2B^n) \), we write:

\[
A^m + 2B^n = \mu t \quad t' \in \mathbb{N}^*
\]  
(3.109)

Then:

\[
A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}
\]  
(3.110)

As \( b = 4p = 4k_2k_3 \) and \( \mu \mid k_3 \) then \( \mu \mid b \implies \exists t' \in \mathbb{N}^* \) that \( b = \mu t' \), we obtain:

\[
\mu' \mu = \mu (4\mu t'^2 - 8t' B^n) + 4B^n (B^n - A^m)
\]  
(3.111)

If \( \mu = 3 \), then \( 3 \mid b \), but \( 3 \mid a \) which is contradiction with \( a, b \) co-primes.

\( \Diamond \) - We assume now \( k_3 = 1 \). Hence:

\[
A^{2m} + 2A^mB^n = k_1
\]  
(3.113)

\[
p = k_2
\]  
(3.114)

\[
\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{p}
\]  
(3.115)

Taking the square of the last equation, we obtain:

\[
\frac{4}{3} \sin \frac{2\theta}{3} = \frac{k_1^2}{p^2}
\]

\[
\frac{16}{3} \sin^2 \frac{2\theta}{3} \cos^2 \frac{2\theta}{3} = \frac{k_1^2}{p^2}
\]

\[
\frac{16}{3} \sin^2 \frac{2\theta}{3} \cdot \frac{3a'}{3} = \frac{k_1^2}{p^2}
\]

Finally:

\[
a' (4p - 3a') = k_1^2
\]  
(3.116)

but \( a' = a'^2 \) then \( 4p - 3a' \) is a square. Let us:

\[
\lambda^2 = 4p - 3a' = 4p - a = b - a
\]  
(3.117)

The equation (3.116) becomes:

\[
a'^2 \lambda^2 = k_1^2 \implies k_1 = a'^2 \lambda
\]  
(3.118)

taking the positive square root. Using (3.113), we get:

\[
k_1 = a'^2 \lambda
\]  
(3.119)
But \( k_1 = A^n(A^m + 2B^n) = a^n(A^m + 2B^n) \), it follows:

\[
(A^m + 2B^n) = \lambda \tag{3.120}
\]

Let \( \lambda_1 \) prime \( \neq 2 \), a divisor of \( \lambda \) (if not \( \lambda_1 = 2|\lambda \implies 2|\lambda^2 \). As \( 2|(b = 4p) \implies 2|(a = 3a') \), which is contradiction with \( a, b \) co-primes).

We consider \( \lambda_1 \neq 2 \) and:

\[
\lambda_1|\lambda \implies \lambda_1|(A^m + 2B^n) \tag{3.121}
\]

\[
\implies \lambda_1 \nmid A^m \text{ if not } \lambda_1|2B^n \tag{3.122}
\]

But \( \lambda_1 \neq 2 \) hence \( \lambda_1|B^n \implies \lambda_1|B \), it follows:

\[
\lambda_1|(b = 4p) \text{ and } \lambda_1|A^m \implies \lambda_1|2a^n \implies \lambda_1|a \tag{3.123}
\]

hence the contradiction with \( a, b \) co-primes.

We assume now \( \lambda_1 \nmid A^m \). \( \lambda_1|(A^m + 2B^n) \implies \lambda_1|(A^m + 2B^n)^2 \) that is \( \lambda_1|(A^{2m} + 4A^m B^n + 3B^{2n}) \), we write it as \( \lambda_1|(p+3A^m B^n + 3B^{2n}) \implies \lambda_1|((p+3B^n(A^m + 2B^n) - 3B^{2n}) \). But \( \lambda_1|(A^m + 2B^n) \implies \lambda_1|(p-3B^{2n}) \), as \( \lambda_1|(4p - a) \) hence by difference, we obtain \( \lambda_1|(a - 3(B^{2n} + p)) \) or \( \lambda_1|(3a' - 3(B^{2n} + p)) \implies \lambda_1|3(a' - 3(B^{2n} - p) \implies \lambda_1 = 3 \) or \( \lambda_1|(a' - (B^{2n} + p)) \).

*A-1* If \( \lambda_1 = 3|\lambda \implies 3|\lambda^2 \implies 3|b - a \) but \( 3|a \implies 3|(p = b) \) hence the contradiction with \( a, b \) co-primes.

*A-2* If \( \lambda_1 \neq 3 \) and \( \lambda_1|(a' - B^{2n} - p) \implies \lambda_1|(A^m B^n + B^{2n}) \implies \lambda_1|B^n(A^m + 2B^n) \implies \lambda_1|B^n \) or \( \lambda_1|(A^m + B^n) \). The case \( \lambda_1|B^n \) was studied above.

*A-2-1* If \( \lambda_1|(A^m + 2B^n) \). We re-find the condition \((3.121)\).

Then the case \( k_3 = 1 \) is impossible.

3.2.2.7. Case \( 3|a \) and \( b = 2p' \neq 2 \) with \( p'|p : 3|a \implies a = 3a', b = 2p' \) with \( p/k.p' \), hence:

\[
A^{2m} = \frac{4.p.a}{3'b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a' \tag{3.124}
\]

Calculate \( B^n C^l \):

\[
B^n C^l = \sqrt[n]{p^2} \left(3\sin^2\frac{\theta}{3} - \cos^2\frac{\theta}{3}\right) = \sqrt[n]{p^2} \left(3 - 4\cos^2\frac{\theta}{3}\right) \tag{3.125}
\]

But \( \sqrt[n]{p^2} = \frac{p}{3} \) hence en using \( \cos^2\frac{\theta}{3} = \frac{3.a'}{b} \):

\[
B^n C^l = \sqrt[n]{p^2} \left(3 - 4\cos^2\frac{\theta}{3}\right) = \frac{p}{3} \left(3 - 4\frac{3.a'}{b}\right) = p. \left(1 - \frac{4.a'}{b}\right) = k(p' - 2a') \tag{3.126}
\]

As \( p = b.p' \), and \( p' > 1 \), we have then:

\[
B^n C^l = k(p' - 2a') \tag{3.127}
\]

and \( A^{2m} = 2k.a' \tag{3.128} \)
A - Soit $\lambda$ a prime divisor of $k$ (we suppose $k$ not a prime). From (3.128), we have:

$$\lambda | A^{2m} \Rightarrow \lambda | A^n$$  \hspace{1cm} (3.129)

From (3.127), as $\lambda | k$, we have:

$$\lambda | B^n C^l \Rightarrow \lambda | B^n \quad \text{or} \quad \lambda | C^l$$  \hspace{1cm} (3.130)

If $\lambda | B^n$, $\lambda$ is prime $\lambda | B$, and as $C^l = A^m + B^n$ then we have also:

$$\lambda | C^l \quad \text{as} \quad \lambda \text{ is prime then} \quad \lambda | C$$  \hspace{1cm} (3.131)

By the same way, if $\lambda | C^l$, we obtain $\lambda | B^n$.

Hence: $A$, $B$ and $C$ solutions of (2.1) have a common factor.

B - We suppose now that $k$ is prime, from the equations (3.127) and (3.128), we obtain:

$$k | A^{2m} \Rightarrow k | A^n \Rightarrow k | A$$  \hspace{1cm} (3.132)

and:

$$k | B^n C^l \Rightarrow k | B^n \quad \text{or} \quad k | C^l$$  \hspace{1cm} (3.133)

if $k | B^n \Rightarrow k | B$  \hspace{1cm} (3.134)

as $C^l = A^m + B^n$ and that $k | A, k | B \Rightarrow k | A^n, k | B^n \Rightarrow k | C^l \Rightarrow k | C$  \hspace{1cm} (3.135)

By the same way, if $k | C^l$, we arrive to $k | B$.

Hence: $A$, $B$ and $C$ solutions of (2.1) have a common factor.

3.2.2.8. Case $3 | a$ and $b = 4p'$ $b \neq 2$ with $p'|p$ : 3/|a $\Rightarrow$ a $= 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$ if not $b = 4p$ a case has been studied (paragraph 3.2.2.6), then we have:

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a'$$  \hspace{1cm} (3.136)

Writing $B^n C^l$:

$$B^n C^l = \sqrt[3]{p^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{p^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right)$$  \hspace{1cm} (3.137)

But $\sqrt[3]{p^2} = \frac{p}{3}$, hence en using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{p^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \cdot \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = k(p' - a')$$  \hspace{1cm} (3.138)

As $p = b.p'$, and $p' > 1$, we have:

$$B^n C^l = k(p' - 2a')$$  \hspace{1cm} (3.139)

and $$A^{2m} = 2k.a'$$  \hspace{1cm} (3.140)
Proof of the Main Theorem

A - Let \( \lambda \) a prime divisor of \( k \) (we suppose \( k \) not a prime). From (3.140), we have:
\[
\lambda | A^{2m} \Rightarrow \lambda | A^m \quad \text{as} \quad \lambda \text{ is prime then} \quad \lambda | A
\]  
(3.141)

From (3.139), as \( \lambda | k \) we obtain:
\[
\lambda | B^n C^l \Rightarrow \lambda | B^n \quad \text{or} \quad \lambda | C^l
\]  
(3.142)

If \( \lambda | B^n \), \( \lambda \) is a prime \( \lambda | B \), and as \( C^l = A^m + B^n \), then we have:
\[
\lambda | C^l \quad \text{as} \quad \lambda \text{ is prime, then} \quad \lambda | C
\]  
(3.143)

By the same way if \( \lambda | C^l \), we obtain \( \lambda | B \). Then: \( A, B \) and \( C \) solutions of (2.1) have a common factor.

B - We suppose now that \( k \) is prime, from the equations (3.139) and (3.140), we have:
\[
k | A^{2m} \Rightarrow k | A^m \Rightarrow k | A
\]  
(3.144)

and:
\[
k | B^n C^l \Rightarrow k | B^n \quad \text{or} \quad k | C^l
\]  
(3.145)

if \( k | B^n \Rightarrow k | B \)
\[
as \quad C^l = A^m + B^n \quad \text{and that} \quad k | A, k | B \Rightarrow k | A^m, k | B^n \Rightarrow k | C^l
\]  
(3.146)

\[
\Rightarrow k | C
\]  
(3.147)

By the same way if \( k | C^l \), we arrive to \( k | B \).

Hence: \( A, B \) and \( C \) solutions of (2.1) have a common factor.

3.2.2.9. Case \( 3 | a \) and \( b | 4p \) : \( a = 3a' \) and \( 4p = k_1 b \) with \( k_1 \in \mathbb{N}^* \). As \( A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} - \frac{p}{3} \right) = \frac{p}{3} \left( 3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left( 3 - 4 \frac{3a'}{b} \right) = \frac{k_1}{4} (b - 4a')
\]  
(3.148)

As \( B^n C^l \) is an integer, we must have \( 4 | k_1 \) or \( 4 | (b - 4a') \).

**-1- If \( k_1 = 1 \Rightarrow b = 4p \) : it is the case (3.2.2.6) above.

**-2- If \( k_1 = 4 \Rightarrow p = b \) : it is the case (3.2.2.3) above.

**-3- We suppose that \( 4 | k_1 \) with \( k_1 > 4 \Rightarrow k_1 = 4k_1' \), then we have:
\[
A^{2m} = 4k_1' a'
\]
\[
B^n C^l = k_1'(b - 4a')
\]

By discussing \( k_1' \) is a prime integer or not, we arrive easily to: \( A, B \) and \( C \) solutions of (2.1) have a common factor.

**-4- If \( 4 \nmid (b - 4a') \) and \( 4 \nmid k_1' \) it is impossible. If \( 4 | (b - 4a') \Rightarrow (b - 4a') = 4c \), with \( c \in \mathbb{N}^* \), then we obtain:
\[
A^{2m} = k_1 a'
\]
\[
B^n C^l = k_1 c
\]
4 A Numerical Example

By discussing \( k_1 \) is a prime integer or not, we arrive easily to: \( A, B \) and \( C \) solutions of (2.1) have a common factor.

\[ \Box \]

The main theorem is proved.

4 A Numerical Example

We consider the example:

\[ 6^3 + 3^3 = 3^5 \quad (4.1) \]

with \( A^n = 6^3, B^n = 3^3 \) and \( C^l = 3^5 \). With the notations used in the paper, we obtain:

\[ p = 3^6 \times 73, \quad (4.2) \]
\[ q = 8 \times 3^{11}, \quad (4.3) \]
\[ \Delta = 4 \times 3^{11} (3^6 \times 4^2 - 73^3) < 0, \quad (4.4) \]
\[ \rho = \frac{p\sqrt{p}}{3\sqrt{3}} = \frac{3^8 \times 73\sqrt{73}}{3}, \quad (4.5) \]
\[ \cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \quad (4.6) \]

As \( A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} \Rightarrow \cos^2 \frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \Rightarrow a = 3 \times 2^4, b = 73; \)
then:

\[ \cos \frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}} \quad (4.7) \]
\[ p = 3^6 b \quad (4.8) \]

Let us verify the equation (4.6) using the equation (4.7):

\[ \cos \theta = \cos 3(\theta/3) = 4\cos^3 \frac{\theta}{3} - 3\cos \frac{\theta}{3} = 4 \left( \frac{4\sqrt{3}}{\sqrt{73}} \right)^3 - 3 \frac{4\sqrt{3}}{\sqrt{73}} = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \quad (4.9) \]

That’s OK. For this example, we can use the two conditions of (3.10) as \( 3 \mid p, b \mid 4p \) and \( 3 \mid a \). The cases 3.2.1.3 and 3.2.2.4 are respectively used. We find for both cases that \( A^m, B^n \) and \( C^l \) of the equation (4.1) have a common prime factor which is true.

References