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Ghost-free Formulation of Quantum Gauge Theory on Fractal Spacetime

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Abstract

It is known that quantization of massless spin-1 particles runs into several related complications such as the redundancy of gauge orbits, the presence of extra degrees of freedom and the need to introduce "ghost" fields. The textbook interpretation of quantum gauge theory is that "ghosts" are unphysical objects whose function is to preserve Lorentz covariance and unitarity. In particular, Faddeev-Popov "ghosts" (FPG) violate the spin-statistics theorem and are devoid of measurable properties. FPG are shown to decouple from the spectrum of observable states, yet it remains unclear how their presence in loop diagrams and their interaction with gauge fields is even possible in the absence of any physical attributes. The object of this work is to suggest that, at least in principle, the concept of spacetime endowed with minimal fractality enables a "ghost"-free formulation of quantum gauge theory. Our approach opens the door for a non-perturbative understanding of vacuum polarization in Quantum Electrodynamics (QED).

Key words: Path Integral Quantization, Gauge Theory, Ghost Fields, Faddeev-Popov Method, Gauge Fixing, Minimal Fractal Manifold.

1. Introduction

Spin 1 particles are critical components of QED and non-Abelian field theory. Quantization of both classical electrodynamics and Yang-Mills theories is confronted by several related challenges due to the redundant polarizations carried by vector fields, the over-counting of gauge orbits and the need to reinforce Lorentz covariance and unitarity through fictitious ("ghost") fields [1-3, 6]. In particular, Fadeev-Popov "ghosts" (FPG) evolve within loop diagrams and interact with spin 1 fields, yet they do not contribute to the spectrum of observable states. While the mathematical basis for "ghost" theory is on solid ground, its physical interpretation is at least un-natural. A legitimate question one is compelled to ask is: *How is it possible to evolve and couple fictitious entities to physical fields, the latter being either real or virtual particles?* Building on our previous research [4], here we suggest that the concept of fractal spacetime endowed with minimal deviations from four-dimensionality (the so-called *minimal fractal manifold,* MFM in short) allows for a "ghost-free" formulation of quantum gauge theory.

The paper is organized as follows: Next section surveys the array of challenges involved in the standard quantization of spin 1 fields. Sections 3 to 5 analyze the implications of placing classical electrodynamics on the MFM. Concluding remarks are detailed in the last section. For the sake of clarity and accessibility, the paper is presented in a pedagogical manner that focuses primarily on the physical content and leaves aside excessive mathematical details. Readers familiar with the topic may skip the next two sections. We caution that the framework of ideas developed here is in its infancy. Follow-up research is required to independently confirm, expand or refute our tentative conclusions.

2. Challenges of gauge field quantization

We begin with a brief survey of the main difficulties confronting quantization of abelian and non-abelian fields. The interested reader may consult [1-3, 6] for a deeper analysis and additional technical details.

2.1 Standard quantization of the electromagnetic field

2.1.1) The classical electromagnetic Lagrangian in the absence of external sources is given by

$$L_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

where the field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

Maxwell equations read

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow [\eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu] A^\nu = 0 \tag{3}$$

The Lagrangian (1) is invariant under the group of *local gauge transformations*

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{4}$$

for any function $\Lambda(x)$ satisfying the commutation condition

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \Lambda(x) = [\partial_\mu, \partial_\nu] \Lambda(x) = 0 \tag{6}$$

As a result, the field strength (2) stays unchanged under (4), namely,

$$F_{\mu\nu} \to \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) = F_{\mu\nu} \tag{7}$$

A fundamental difficulty in quantizing the Maxwell theory is that the second-differential operator

$$D_{\mu\nu} = \eta_{\mu\nu}(\partial^{\rho}\partial_{\rho}) - \partial_{\mu}\partial_{\nu} \tag{8}$$

has no inverse as it annihilates any function of the form $\partial_{\mu}\Lambda(x)$. This implies that, for any given initial data, one cannot uniquely find the potential $A_{\mu}(x)$ at later times since there is no way of distinguishing between $A_{\mu}(x)$ and $A_{\mu}(x)+\partial_{\mu}\Lambda(x)$. This defines the *redundancy problem of gauge theory*: the phase space of Maxwell's theory is "foliated" by gauge orbits that are inherently over-counted.

2.1.2) A related difficulty of vector field quantization lies in the number of real components carried by massless spin 1 operators. The electromagnetic potential $A_{\mu}(x)$ has four independent components, yet the photon has only two independent degrees of freedom called *polarization states*. Let us elaborate on this point with additional details. To examine the plane-wave solutions of Maxwell equations (3), it is customary to consider the momentum space representation of $A_{\mu}(x)$

$$A_{\mu}(k) = \frac{1}{(2\pi)^{4}}\int d^{4}x A_{\mu}(x)\exp(-ik\cdot x) \tag{9}$$

Under the gauge transformation, the potential (9) changes as

$$A_{\mu}(k) \to A_{\mu}(k) + \Lambda(k)k_{\mu} \tag{10}$$

Field equations take the form

$$k^{2}A_{\mu}(k) - k_{\mu}k^{\nu}A_{\nu}(k) = 0 \tag{11}$$

and are invariant under (10). One can conveniently resolve $A_\mu(x)$ into four independent vectors, $\varepsilon_\mu(\mathbf{k},\lambda)$, $k_\mu = (\mathbf{k}, k^0)$ and $k^\mu = (\mathbf{k}, -k^0)$, defined by

$$k^\mu \varepsilon_\mu(\mathbf{k},\lambda) = 0 \ , \ \varepsilon_0(\mathbf{k},\lambda) = 0 \qquad (\lambda = 1,2) \tag{12}$$

Hence,

$$A_\mu(k) = a^\lambda(k)\varepsilon_\mu(\mathbf{k},\lambda) + b(k)k_\mu + c(k)k_\mu \tag{13}$$

and the field equations (11) turn into

$$k^2 a^\lambda(k)\varepsilon_\mu(\mathbf{k},\lambda) + b(k)[k^2 - (k\cdot k)k_\mu] = 0 \, , \ (k\cdot k) > 0 \tag{14}$$

which forces the coefficient functions to vanish, namely,

$$k^2 a^\lambda(k) = 0, \quad (\lambda = 1,2) \tag{13a}$$

$$b(k) = 0 \tag{13b}$$

Relations (13) show that the field equations cannot fix the value of the coefficient $c(k)$. This implies that $c(k)$ can be set to zero by means of a suitable gauge transformation, which, in turn, means that $c(k)$ has no physical meaning. One arrives at the conclusion that there are only two independent plane wave solutions on the light cone ($k^2 = 0$) and *two transverse polarization vectors*.

The standard solution to the gauge redundancy problem of Maxwell theory is *gauge fixing*. The method reduces the number of allowed orbits to a smaller set, where all the orbits are related by smaller gauge group symmetry. Since quantum gauge theory is often described using the path-integral (PI) formulation, a generalization of gauge fixing to non-abelian fields is required to ensure internal consistency of the theory. Details on the Fadeev-Popov (FP) gauge fixing method are briefly examined in paragraph 2.2.

2.1.3) Unlike the case of massive fields, the spin of a massless particle cannot be defined relative to its rest frame of reference. As a result, the three-dimensional rotation group is no longer adequate for characterizing the photon spin and it is replaced by the group of two-dimensional rotations around the three-momentum vector $\mathbf{k}$. The existence of only two transverse photon polarizations hints to a violation of Lorentz invariance stemming from the fact that transversality is not preserved by Lorentz transformations. It can be shown, however, that Lorentz symmetry is restored provided that photons couple to *conserved currents* defined through $\partial_\mu J^\mu = 0$. The existence of such currents is a direct consequence of gauge invariance.

2.2 The Fadeev-Popov (FP) method

The FP procedure consists in applying a suitable constraint to the PI description of gauge theory that automatically removes the ambiguity associated with the gauge transformation. Consider the generating functional

$$Z[J] = \int DA_\mu \exp i \int d^4x \,(L + J^\mu A_\mu) \qquad (14)$$

The integral measure $DA_\mu = \Pi_x \, dA_\mu(x)$ spans over all possible vector potentials A_μ and necessarily includes their gauge transforms (4). Explicitly writing (4) as

$$A_\mu = \overline{A_\mu} + \partial_\mu \Lambda(x) \tag{15}$$

factors out the contribution of $\overline{A_\mu}$ and $\Lambda(x)$ in (14), namely,

$$Z[J] = \int D\overline{A_\mu} \exp i \int d^4x \,(L + J^\mu \overline{A_\mu}) \int D\Lambda \tag{16}$$

The presence of the second integral over the arbitrary field $\Lambda(x)$ causes the generating functional to diverge due to the unaccountable many $\Lambda(x)$ contributing to (16). Following the FP method, the generating functional (16) is cast in the equivalent form

$$Z[J] = \int D\Lambda F[A_\mu, \Lambda; J] \Delta[A_\mu] \delta(G[A^\Lambda]) \tag{17}$$

where

$$F[A_\mu, \Lambda; J] = \int dA_\mu \exp i \int d^4x (L + J^\mu A_\mu) \tag{18}$$

and

$$G[A^\Lambda] = \partial^\mu A_\mu = \partial^\mu [\overline{A_\mu} + \partial_\mu \Lambda(x)] \tag{19}$$

The FP determinant is defined as

$$\Delta[A] = \det(\frac{\delta G[A^\Lambda]}{\delta \Lambda}), \quad \Lambda = 0 \tag{20}$$

and leads to the introduction of "ghost" and "anti-ghost" fields. In particular, the "ghost" part of the Lagrangian in Yang-Mills theory is given by

$$L_g = \partial_\mu \overline{c^a}\,\partial^\mu c^a + g f^{abc} (\partial^\mu \overline{c^a}) A_b^\mu c^c \tag{21}$$

Here, "a" is the index of the gauge group, "g" stands for the coupling charge and "f^{abc}" for the structure constants. The first term is the kinetic component of the Lagrangian built up from the contribution of "ghosts" (c^a) and their antiparticles ($\overline{c^a}$), whereas the second term reflects the interaction of "ghosts" with the gauge field. In Yang-Mills theory, "ghosts" violate the spin-statistics theorem in that they are spinless complex scalar fields with fermion statistics.

3. Maxwell fields on the Minimal Fractal Manifold

The "*minimal fractal manifold*" (MFM) is a concept inspired by the Renormalization Group program of Quantum Field Theory (QFT) and it denotes a spacetime model having arbitrarily small but continuous deviations from four-dimensionality ($\varepsilon = 4 - D << 1$). There are reasons to believe that postulating the MFM is the only sensible way of asymptotically matching all consistency requirements mandated by QFT up to the low Terascale sector of probing energies. The underlying motivation, theoretical benefits and implications of the MFM for the development of QFT, in general, and the Standard Model of high-energy physics, in particular, are extensively discussed in [4] and included references.

Consider classical electrodynamics acting on the MFM in the absence of external sources. To keep matters as simple as possible, we adopt below a symbolic "vector-like" convention for the gauge potential in which the Minkowski index is explicitly omitted, that is, $A_\mu(x) \to \mathbf{A}$. In the context of *low-level fractality* [5], the gradient operator of $\mathbf{A}$ may be presented as

$$\partial^{1-\varepsilon(x)}\mathbf{A} = [\partial + \varepsilon(x) D_1^1]\,\mathbf{A} \tag{22}$$

where

$$D_1^1 \mathbf{A} = \mathbf{A}^{(1)}(0)\ln(x) + \gamma \mathbf{A}^{(1)}(x) + \int_0^x \mathbf{A}^{(2)}(x)\ln(x-\sigma)d\sigma \tag{23a}$$

$$\varepsilon(x) = 4 - D(x)\,,\ \varepsilon(x) \ll 1 \tag{23b}$$

in which $D(x)$ stands for the locally defined spacetime dimension, asymptotically reaching the standard $D = 4$ in the continuous limit $\varepsilon(x) \to 0$.

The field strength (2) and Lagrangian (1) assume the *symbolic* expression

$$F_{\mu\nu} \to F_{\mu\nu} \sim \partial\mathbf{A} + \varepsilon(x) D_1^1 \mathbf{A} \tag{24}$$

$$L \to L \sim (\partial\mathbf{A})^2 + \varepsilon(x)(\partial\mathbf{A}\cdot D_1^1\mathbf{A}) + \varepsilon^2(x)(D_1^1\mathbf{A})^2 \tag{25}$$

Electromagnetic fields propagating in free space are plane wave solutions of Maxwell's equations and are given by

$$\mathbf{A} \sim \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)] \tag{26}$$

where the wave vector $\mathbf{k}$ and angular frequency ω are related through the simple dispersion relation

$$kc = \omega \tag{27}$$

To further simplify matters, we assume that the dimensional parameter $\varepsilon(x)$ is reasonably close to an uniform function, that is, $\varepsilon(x) \approx \varepsilon$. On account of (26), the second term in (25) can be reasonably well approximated as

$$\varepsilon\,(\partial \mathbf{A} \cdot D_1^1 \mathbf{A}) \;\sim\; \varepsilon \mathbf{A}^2 \tag{28}$$

which may be directly mapped to a mass term in the Proca Lagrangian. As it is known, the Proca Lagrangian in free space represents the simplest generalization of Maxwell's Lagrangian that explicitly breaks local gauge invariance [1, 6]. It has the form

$$L_P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m_\gamma^2}{2} A_\mu A^\mu \tag{29}$$

where "m_γ" is the non-vanishing photon mass. A quick glance at (28) and (29) reveals that

$$\boxed{m_\gamma^2 = O(\varepsilon)} \tag{30a}$$

Following [4], (23b) represents the background polarization of spacetime induced by low-level fractality and may be understood as the primary source of particle masses, gauge charges and spins. Stated differently, the transition from smooth to low-level fractality near the electroweak scale turns the *passive* four-dimensional spacetime of classical and quantum physics into an *active-like* medium. In particular, the minimal fractal texture of spacetime acts as the primary source of electric charge (e_0) according to [4]

$$\boxed{e_0^2 = O(\varepsilon)} \tag{30b}$$

Drawing from this interpretation, what appears to be an infinitesimal but non-vanishing photon mass stems from the residual energy encoded in the topology of the MFM. Next section elaborates on this result.

4. Implications of arbitrarily small photon mass

Although a non-vanishing photon mass spoils all consistency requirements mandated by QFT, the Proca Lagrangian can be regarded as a gauge-fixed version of the Stückelberg mechanism, which restores gauge invariance, unitarity and renormalizability [7-9]. There are several far-reaching consequences of massive photons, ranging from the variation of the speed of light, charge conservation, photon instability, the Casimir and Bohr-Aharonov effects, to some major implications in astrophysics and cosmology [7-9]. The laboratory upper limit on the photon mass is currently placed around $m_\gamma < 10^{-18} eV$, which draws near the theoretical limit derived from the uncertainty principle and the estimated age of the Universe [7-9].

Previous paragraphs have surveyed the slew of challenges posed by masslessness of the photon in quantum gauge theory. Besides these, there are known theoretical difficulties in infrared QED caused by the continuous exchange of zero-frequency "soft" photons between charged particles. Likewise, the so-called *bremsstrahlung graphs* diverge for zero photon momenta, which complicate the correct estimations of cross sections and decay rates involved in radiation and absorption of "soft" photons [1].

In response to these difficulties, coupling the free Maxwell field to the MFM generates an arbitrarily small *"residual" photon polarization* (30, a-b). The benefit of this scenario is that it automatically removes all shortcomings related to massless photons while staying compatible with photon mass uncertainties derived from experiments.

5. Vacuum polarization from the Minimal Fractal Manifold

The *photon self-energy* (or the vacuum polarization) describes a QED process in which a background Maxwell field creates a virtual electron-positron pair (e^+e^-). The virtual pair is short-lived and it changes the initial distribution of current and charges generated by the Maxwell field. In addition, because the pair is charged, it produces an electric dipole that polarizes the vacuum and contributes to a partial screening effect. As a result of vacuum polarization, at large distances, the "effective" Maxwell field is weaker. Photon self-energy leads to infinities that are typically removed by renormalization.

In QED, the classic Coulomb potential is obtained by Fourier transforming the propagator

$$V(r) = \int \frac{d^3p}{(2\pi)^3} \frac{e^2}{p^2} \exp(i\mathbf{px}) = \frac{e^2}{r} \tag{31}$$

The virtual (e^+e^-) pair adds a loop correction inside the photon line which, in turn, gives a contribution to (31) proportional to e^4. For instance, the *Uehling potential* in spinor QED arises as a radiative effect whose closed-form expression at 1-loop is given by

$$V_U(r) = -\frac{e^4}{24\pi^2 r} \int_0^1 dx \exp(-2mrx) \left(\frac{2x^2+1}{2x^4}\right) \sqrt{x^2-1} \tag{32}$$

in which "m" represents the mass of the charged fermion [1].

It follows these considerations and from (25), (30a-b) that coupling the free Maxwell field to the MFM leads to a term quadratic in ε that mimics a *"residual" photon self–energy* imparted by the fractal texture of spacetime. This self-energy contribution goes smoothly to zero in the continuum limit ($\varepsilon \to 0$).

6. Concluding remarks

We have shown that Maxwell and Yang-Mills Lagrangians acting on the MFM pick up two vanishingly small contributions: a) a mass-like term in the form of "residual" polarization, which is linear in ε and b) a vacuum polarization term in the form a "residual" self-energy, which is quadratic in ε. Both contributions develop from the fractal texture of spacetime near the electroweak scale and automatically bypass the complications associated with quantization of gauge fields in QFT. An added value of this approach is that it opens the door for a non-perturbative understanding of vacuum polarization in QED.

Needless to say, our treatment is entirely preliminary. Follow-up studies on this topic may focus on a deeper understanding of both (30, a-b) and of the implications associated with the residual vacuum polarization contained in the last term of (25). For example, one needs to properly connect the finite cross-section of pair-creation/ annihilation in perturbative QED (γ <=> $e^+ + e^-$) with the nearly vanishing residual vacuum polarization driven by ε^2. To this end, it is necessary to evaluate the closed-form expression of $(D_1^1\mathbf{A})^2$ upon appropriate normalization of x near or above the electroweak scale. It is in this dynamic regime that the MFM is expected to surface [4].

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