Comment on "Exploring the origin of Minkowski spacetime"

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Abstract

In the article Chappell, Hartnett, Iannella, Iqbal, Abbott, "*Exploring the origin of Minkowski spacetime*" [2] authors almost completely revise their article [3], but there is some mathematical and other discrepancies in their articles.

Keywords: *multivector amplitude, Lorentz transformations, proper time, speed of light, geometric algebra, conserved quantities*

Introduction

In articles ([2] and [3]), authors discussed a nice idea of a general transformation of multivectors in Cl_3 (3D real Clifford geometric algebra) that could serve as a new framework for relativity. The author of this article commented the article [3] in [12] and that is known to a leading author of [3].

New theory and old assumptions

On the page 3 of [2], authors discussed the spacetime algebra (STA), Hestenes (1966), and the algebra of the physical space (APS), Baylis (1996), and stated that "*the authors axiomatically assume the Minkowski metric*". This is correct for the STA, but for the algebra of the physical space (APS) is not. In fact, in the APS, an amplitude of a paravector is defined in exactly the same way as in [2] and [3] and with the same motivation. Using the form of a multivector as in [3] $M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb$, we define a *complex number* $z = t + jb$ (belongs to the center of the algebra) and a *complex vector* $\boldsymbol{F} = \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}}$. For a paravector $t + \vec{\boldsymbol{x}}$ we could use a form $t + \|\vec{\boldsymbol{x}}\|\hat{\boldsymbol{h}}$ and notice that we have hyperbolic numbers-like property $\hat{\boldsymbol{h}}^2 = 1$, therefore, a natural choice is to define a paravector amplitude (square of) as in the case of hypercomplex numbers: $(t + \vec{\boldsymbol{x}})(t - \vec{\boldsymbol{x}}) = t^2 - x^2$. But this is just a part of the Clifford conjugation restricted on the real part of the algebra. For a whole multivector, we have the same situation after noticing that $\boldsymbol{F}^2 = (\vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}})^2$ is a complex scalar (belongs to the center of the algebra) and we could write the multivector as $M = z + \boldsymbol{F} = z + z_F \hat{\boldsymbol{F}}, \quad z_F = \sqrt{\boldsymbol{F}^2}, \quad \hat{\boldsymbol{F}}^2 = 1$. Thus, there is the another hyperbolic number-like property and a multivector amplitude based on the Clifford conjugation is a natural choice: $M\bar{M} = (z + \boldsymbol{F})(z - \boldsymbol{F}) = z^2 - \boldsymbol{F}^2$. In [13] it is used this striking similarity with hyperbolic numbers formalism and, relaying on the spectral basis, derived the closed form for many functions of multivector variables. In conclusion, in the paravector model of space-time the Minkowski metric is a natural consequence of hyperbolic properties of vectors and it is not imposed as assumption. The complex and hypercomplex properties are in the heart of Cl_3.

In [2], on the page 5, authors stated the Fundamental multivector involution theorem, but that is not stated clearly. A phrase “commutative amplitude” suggests that $MI(M)=I(M)M$ for the multivector amplitude only. A counter example is for the involution $I(M)=z^*-\boldsymbol{F}=t-\vec{\boldsymbol{x}}-j\vec{\boldsymbol{n}}-jb$, where $z^*=t-jb,$ because

$$\begin{aligned}&(z+\boldsymbol{F})(z^*-\boldsymbol{F})=\\&zz^*+\boldsymbol{F}z^*-z\boldsymbol{F}-\boldsymbol{F}^2=z^*z-\boldsymbol{F}z+z^*\boldsymbol{F}-\boldsymbol{F}^2=\\&(z^*-\boldsymbol{F})(z+\boldsymbol{F}).\end{aligned} \tag{0.1}$$

Consequently, the multivector amplitude defined in the [2] and [3] is not a unique commutative amplitude, however, it is the unique commutative amplitude that in addition produces a complex scalar (so the result of multiplication is in the center of the algebra). Here are theorems from [12]:

Theorem 1. *If* $I(z)\in\mathbb{C}$, *then* $I(M)M=MI(M)$ *iff* $I(\boldsymbol{F})=\pm\boldsymbol{F}$.

The condition $I(M)M=MI(M)$ leads to

$$[z+\boldsymbol{F}][I(z)+I(\boldsymbol{F})]-[I(z)+I(\boldsymbol{F})][z+\boldsymbol{F}]=0\Rightarrow$$

$$\boldsymbol{F}I(\boldsymbol{F})-I(\boldsymbol{F})\boldsymbol{F}=0\Rightarrow I(\boldsymbol{F})=\pm\boldsymbol{F}\ .$$

This condition is met by the Clifford conjugation (up to a sign), but also for (see above) $z^*-\boldsymbol{F}$ etc., however, $(z^*-\boldsymbol{F})(z+\boldsymbol{F})$ is not a complex scalar.

Theorem 2. *Clifford conjugation is the unique involution that meets* $M\bar{M} \in \mathbb{C}$, where $\mathbb{C}$ is the center of $Cl3$.

Proof relays on fact that $\boldsymbol{F}^2$ is a complex scalar, while $\boldsymbol{F}I(\boldsymbol{F})$ generally contains a vector $j\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{n}}$ as a component, orthogonal to vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{n}}$, except for the Clifford conjugation $\boldsymbol{F}I(\boldsymbol{F}) = -\boldsymbol{F}^2 \in \mathbb{C}$. S straightforward proof can be easily obtained by multiplying multivectors

$$\left(t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb\right)\left(s_0 t + s_1\vec{\boldsymbol{x}} + js_2\vec{\boldsymbol{n}} + js_3 b\right), \qquad s_i = \pm 1,$$

where then will appear $j\left(s_1\vec{\boldsymbol{n}}\vec{\boldsymbol{x}} + s_2\vec{\boldsymbol{x}}\vec{\boldsymbol{n}}\right) \Rightarrow s_1 = s_2$ (to have a double inner product in the brackets). Reasoning in [2] is incorrect (page 5, relation 6), because, for example, condition $s_2 = -s_3$ from the last term is superfluous, factors b and $\boldsymbol{n}$ do commute.

Discussing properties of a general transformation that preserve a multivector amplitude in [2], authors settled the condition $|X|^2 = |Y|^2 = 1$, however, there is the possibility $|X|^2 = |Y|^2 = -1$, discussed in [12]. This is corrected in [3]. However, the phase transformation following from condition $|X|^2 = |Y|^2 = -1$ is ignored "*in order to investigate the Lorentz group*". Presented general multivector transformations are more complex than the "*Lorentz transformations*" and main results from the special relativity in the framework of Lorentz transformations are not automatically valid here generally, they should be proven, if possible.

On pages 8 and 9, there is some confusion with formulas 17, 18 and 19. Starting with formula 17: $M' = e^{-j\boldsymbol{w}/2}e^{\boldsymbol{v}/2}Me^{\boldsymbol{v}/2}e^{j\boldsymbol{w}/2}$ authors claim that "*using the multivectors formalism we can now write this as a single operation*" $M' = e^{(\boldsymbol{v}-j\boldsymbol{w})/2}Me^{(\boldsymbol{v}+j\boldsymbol{w})/2}$, but there is no such a "*formalism*", the formulas are just inconsistent, unless $\boldsymbol{v}$ and $\boldsymbol{w}$ are commuting. We could tray, for example, to solve the equation $e^{-j\boldsymbol{w}/2}e^{\boldsymbol{v}/2} = e^C$ to find $C = \left(\boldsymbol{v}' - j\boldsymbol{w}'\right)/2 \neq \left(\boldsymbol{v} - j\boldsymbol{w}\right)/2$. We could reformulate the result in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$, but that's not mentioned. Similar problem arises from the formulas 18 and 19, $\boldsymbol{v}$ and $\boldsymbol{w}$ are not the same in these formulas. If one ignores those problems, the claims in the text are true ones.

On the page 9 starts the discussion on a proper time. It is noticed nicely that a real proper time is necessary to define an action and derive a conserved quantities. The problem here is that conditions for a proper time to be real are not discussed in general, but assuming just restricted Lorentz group, meaning that an "inertial" reference frame is the one with the speed of a particle zero, and that the speed of a particle is necessary less than the speed of light, leading to the transformation rule $h' = \gamma^2 \left(h - \boldsymbol{v} \cdot \boldsymbol{w} \right)$ (formula 23). But on the same page the authors are discussing "*the full set of possible transformations*" and conclude that from $h' = 0$ follows $h = \boldsymbol{v} \cdot \boldsymbol{w}$. However, this is generally true if the formula 23 is correct, which means in the frame of restricted Lorentz transformations, and certainly not in the frame of "*the full set of possible transformations*". The authors also conclude that the relativistic factor γ is real, but their conclusion is invalid even if the speed of particle v is less than the speed of light (consider $\boldsymbol{v} \cdot \boldsymbol{w} = vw$ and $w = v/2$). We cannot assume that the speed of light is necessarily true in "*the full set of possible transformations*", we have to prove it. Nevertheless, later in the text, authors discuss a possibility of a superluminal effects, following from the obtained formulas.

Let us rethink *Cl*3 a little. There is several subspaces in it, for example those based on grades: real scalars, vectors, bivectors and pseudoscalars. A one dimensional motion of a particle is just a possibility, widely explored in the restricted Lorentz group (regarded frequently as the only "physical" in the special relativity). But there are the other subspaces of the algebra, and why not to regard their symmetries as equally possible sources of dynamics? For example, an electron is rather strange object, not the classical one for sure, it possesses a spin, which is not like a cargo in a fast Einstein train or spaceship. Regarding the restricted Lorentz transformations in an electron theory seems to me as a strong restriction. What that it means an “inertial reference frame” when we talk about the electron. If we accept a mechanical point of view regarding “inertial reference frame” then we are taking just one possible symmetry of the 3D space as important, namely a 1D translational symmetry. From this follows conclusion about the maximal speed of particles. However, here we are talking about general transformations, wider than the restricted Lorentz group, and shouldn’t we take all possibilities that such a theory is offering us? In [2], the authors nicely conclude about a conserved quantities, but why the momentum is preferred one (as it is in the special relativity)?

In [12], the author of this article discussed other logical possibilities from demand that proper time was real. Just briefly, from a general expression for multivector amplitude, comparing real and imaginary parts we have

$$t^2 - x^2 + n^2 - b^2 = t'^2 - x'^2 + n'^2 - b'^2 \tag{0.2}$$

$$tb - \vec{\mathbf{x}} \cdot \vec{\boldsymbol{n}} = t'b' - \vec{\mathbf{x}}' \cdot \vec{\boldsymbol{n}}'. \tag{0.3}$$

Defining the differential of the multivector as $dX = dt + d\vec{\mathbf{x}} + jd\vec{\boldsymbol{n}} + jdb$, we have the multivector amplitude of the differential

$$|dX| = dt^2 - dx^2 + dn^2 - db^2 + 2j\left(dbdt - d\vec{\mathbf{x}} \cdot d\vec{\boldsymbol{n}}\right),$$

and hence we can ask which conditions have to be met to define a real proper time τ .

There is a possibility, already discussed here, to define a proper time as $d\tau = |dX'|_{v'=0} \in \mathbb{R}$ (“rest frame”), but then generally remains dependence of the ratio dt'/dt for quantities from different referent frames. One can easily obtain a proper time assuming that all quantities in $|dX'|$, except dt', are equal to zero. Assuming that this is not the case (for example, an electron has a spin in every reference frame) and still regarding a proper time to be real, the imaginary part of the multivector amplitude must be zero in every referent frame:

$$dbdt - d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{n}} = dt^2 \left(d\dot{b} - d\dot{\vec{\boldsymbol{x}}} \cdot d\dot{\vec{\boldsymbol{n}}} \right) = dt^2 \left(h - d\dot{\vec{\boldsymbol{x}}} \cdot d\dot{\vec{\boldsymbol{n}}} \right) = 0 \Rightarrow h = d\dot{\vec{\boldsymbol{x}}} \cdot d\dot{\vec{\boldsymbol{n}}},$$

where we defined $d\dot{b} = h$ and $h' = d\dot{\vec{\boldsymbol{x}}}' \cdot d\dot{\vec{\boldsymbol{n}}}'$. Defining $\dot{\vec{\boldsymbol{n}}} = \vec{\boldsymbol{w}}$ we have $h = \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}}$. A bivector part of a multivector is not transforming like an area [3], therefore it is reasonable to assume for the vector $\vec{\boldsymbol{w}}$ to be proportional to some *angular momentum-like quantity* (AMLQ). Now $\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}}$ may be associated with flow of AMLQ, or helicity. It turns out that this quantity could be associated with a new law of conservation.

One could regard conditions for a real proper time $d\tau$ to be:

i) $d\tau \in \mathbb{R}$

ii) $dt' / dt \equiv \gamma(M, M') = \gamma(M) = dt' / d\tau.$

The condition ii) is natural, relativistic factor γ now depends on quantities from a single reference frame only. From i) and ii) follows that

$$|dX| = |dX'| = d\tau^2 = dt^2 - dx^2 + dn^2 - db^2,$$

$$1 = \frac{dt^2}{d\tau^2} \left(1 - \frac{dx^2}{dt^2} + \frac{dn^2}{dt^2} - \frac{db^2}{dt^2} \right) = \gamma^2 \left(1 - v^2 + w^2 - h^2 \right),$$

$$\gamma = 1 / \sqrt{1 - v^2 + w^2 - (\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}})^2} = 1 / \sqrt{1 - v^2 + w^2 - w^2 v^2 \cos^2 \alpha}. \tag{0.4}$$

Recalling that the factor γ is real (the ratio of two reals) we have the condition

$$1 - v^2 + w^2 - w^2 v^2 \cos^2 \alpha > 0 \Rightarrow v_{\max} < \sqrt{\frac{1 + w^2}{1 + w^2 \cos^2 \alpha}}. \tag{0.5}$$

For $\cos\alpha = \pm 1$ or $w = 0$ follows that $v_{\max} = 1$, but $v_{\max} > 1$ otherwise. Therefore, if the vector $\vec{\boldsymbol{w}}$ has a physical meaning, it follows that the maximum speed varies. A natural assumption is that we do not require $w' = 0$ generally because it could be an internal characteristic of a system (like spin) and could not be reduced to zero by the selection of a suitable reference frame, i.e., there is no a reference frame for an electron ceased to be a fermion.

We have real $|dX'| = dt'^2\left(1 - v'^2 + w'^2 - w'^2 v'^2 \cos^2 \alpha'\right)$, and it would be easiest to conclude that the $w' = 0, \quad v' = 0$, as discussed already. Regarding $v' = 0$, the relativistic factor γ becomes dependent on w', so, remains the possibility $-v'^2 + w'^2 - w'^2 v'^2 \cos^2 \alpha' = 0$, which means

$$v'_\tau = \frac{w'}{\sqrt{1 + w'^2 \cos^2\left(\alpha'\right)}}, \tag{0.6}$$

and we have a real proper time in the referent frame of a moving particle. What could be a physical meaning of that? In relativistic physics, one usually relays on a real scalars and real vectors and defines a proper time regarding $\vec{\boldsymbol{p}} = 0$. But under bilinear transformations, which preserve a multivector amplitude, one could regard $-v'^2 + w'^2 - w'^2 v'^2 \cos^2 \alpha' = 0$, which is equivalent to $\gamma = 1$. This could be possible to justify physically, because after extending the Lorentz transformations and including all the other motions and their symmetries, there is no preferable momentum-zero condition, but rather „center of energy-momentum-AMLQ-flow-zero“ condition, whatever that means. A conclusion on limiting speed 1 is based on preferring the momentum as the main form of motion in space-time. Also, the important motivation for the use of geometry contained in Cl_3 is just equal treatment of all kind of movements (for the author surely). It is interesting that the speed v'_τ generally could be greater than 1, having upper limit $1/\cos\alpha'$ (but there is a question of limiting AMLQ somehow). Instead of the "inertial reference frame" of the special relativity, all we demand for a proper time to be real is the condition $\gamma = 1$.

Having a (really) real proper time, we could define derivative of a multivector by the proper time

$$V=\frac{dX}{d\tau}=\frac{dt}{d\tau}+\frac{d\vec{\mathbf{x}}}{dt}\frac{dt}{d\tau}+j\frac{d\vec{\mathbf{n}}}{dt}\frac{dt}{d\tau}+j\frac{db}{dt}\frac{dt}{d\tau}=\gamma\left(1+\vec{\mathbf{v}}+j\vec{\mathbf{w}}+jh\right), \tag{0.7}$$

$$|V|=1\Rightarrow\frac{d|V|^2}{d\tau}=\frac{dV}{d\tau}\bar{V}+V\frac{d\bar{V}}{d\tau}=0, \tag{0.8}$$

which we could understand as a kind of orthogonality of multivectors (velocity and acceleration). Defining $A=dV/d\tau$ we have $A\bar{V}+V\bar{A}=A\bar{V}+\overline{A\bar{V}}=0$ which means that multivector $A\bar{V}$ (or $V\bar{A}$) is a complex vector. In [2], the condition (0.8) is stated as $dV^2/d\tau=0$, suggesting that $V^2=const$, but this is not true generally.

It is interesting that giving the energy to a particle, we have

$$\gamma=E/m=1/\sqrt{1-v^2+w^2-\left(\vec{\mathbf{w}}\cdot\vec{\mathbf{v}}\right)^2}\Rightarrow$$

$$v=\sqrt{\frac{1+w^2-m^2/E^2}{1+w^2\cos^2\alpha}}=\sqrt{\frac{1+\left(l/E\right)^2-m^2/E^2}{1+\left(l/E\right)^2\cos^2\alpha}}\geq\sqrt{1-m^2/E^2},$$

Consequently, one could expect that after obtaining the energy, a particle with spin should be faster than a particle without spin (but possessing the equal mass). An effect for an electron is rather small and this could be a challenge for experimental physicists.

Conclusion

Starting from the articles [2, 3], it is shown a few consequences of the introduction of a bilinear transformations of multivectors that preserve the multivector amplitude and commented some statements from [2]. There is some interesting possibilities not discussed in [2], but discussed here and in [12]. A particles with a spin, like an electron, should possess properties not contained in the Einstein special relativity. For them, the speed of light, perhaps, is not a limiting speed.

References

1. Baylis, *Geometry of Paravector Space with Applications to Relativistic Physics*, Kluwer Academic Publishers, 2004
2. Chappell, Hartnett, Iannella, Iqbal , Abbott, *Exploring the origin of Minkowski spacetime*, arXiv:1501.04857v3
3. Chappell, Hartnett, Iannella, Abbott, *Deriving time from the geometry of space*, arXiv:1501.04857v2
4. Chappell, Iqbal, Gunn, Abbott, *Functions of multivector variables*, arXiv:1409.6252v1
5. Chappell, Iqbal, Iannella, Abbott, *Revisiting special relativity: A natural algebraic alternative to Minkowski spacetime*, PLoS ONE 7(12)
6. Doran, *Geometric Algebra and its Application to Mathematical Physics*, thesis
7. Doran, Lasenby, Gull, *Gravity as a gauge theory in the spacetime algebra*, Fundamental Theories of Physics, 55, 1993
8. Dorst, Fontijne, Mann, *Geometric Algebra for Computer Science* (Revised Edition), Morgan Kaufmann Publishers, 2007
9. Hestenes, *New Foundations for Classical Mechanics*, Kluwer Academic Publishers, 1999
10. Hestenes, Sobczyk, *Clifford Algebra to Geometric Calculus--A Unified Language for Mathematics and Physics*, Kluwer Academic Publishers, 1993
11. Hitzer, Helmstetter, Ablamowicz, *Square Roots of -1 in Real Clifford Algebras*, arXiv:1204.4576v2
12. Josipović, *Some Remarks on Cl3 and Lorentz Transformations*, http://vixra.org/abs/1507.0045
13. Josipović, Functions of Multivectors in 3D Euclidean Geometric Algebra Via Spectral Decomposition (For Physicists and Engineers), http://vixra.org/abs/1507.0086
14. Mornev, *Idempotents and nilpotents of Clifford algebra* (russian), Гиперкомплексные числа в геометрии и физике, 2(12), том 6, 2009
15. Sobczyk, *New Foundations in Mathematics*: *The Geometric Concept of Number*, Birkhäuser, 2013
16. Sobczyk, *Special relativity in complex vector algebra*, arXiv:0710.0084v1
17. Sobczyk, *Geometric matrix algebra*, Linear Algebra and its Applications 429 (2008) 1163–1173
18. Sobczyk, *Vector Analysis of Spinors*, http://www.garretstar.com
19. Sobczyk, Yarman, Principle of Local Conservation of Energy-Momentum, arXiv:0805.3859v1