SMARANDACHE – R-MODULE AND COMMUTATIVE AND BOUNDED BE-ALGEBRAS

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ABSTRACT

In this paper we introduced Smarandache – 2 – algebraic structure of R-Module namely Smarandache – R-Module. A Smarandache – 2 – algebraic structure on a set N means a weak algebraic structure A₀ on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure A₁, stronger algebraic structure means satisfying more axioms, by proper subset one understands a subset different from the empty set, from the unit element if any, from the whole set. We define Smarandache – R-Module and obtain some of its characterization through Commutative and Bounded BE-Algebras. For basic concepts we refer to Florentin smarandache[2] and Raul Padilla[9].


1. INTRODUCTION

New notions are introduced in algebra to study more about the congruence in number theory by Florentin smarandache[2]. By <proper subset> of a set A, We consider a set P included in A and different from A, different from the empty set, and from the unit element in A – if any they rank the algebraic structures using an order relationship.

The algebraic structures S₁ << S₂ if :both are defined on the same set :: all S₁ laws are also S₂ laws; all axioms of S₁ law are accomplished by the corresponding S₂ law; S₂ law strictly accomplishes more axioms than S₁ laws, or in other words S₂ laws has more laws than S₁.

For example : semi group << monoid << group << ring << field, or Semi group << commutative semi group, ring << unitary ring, etc. they define a General special structure to be a structure SM on a set A, different from a structure SN, such that a proper subset of A is an SN structure, where SM << SN.
2. Prerequisites

**Definition 2.1:** An algebra \((A; *, 1)\) of type \((2, 0)\) is called a BE-algebra if for all \(x, y\) and \(z\) in \(A\),

\begin{align*}
\text{(BE1)} & \quad x * x = 1 \\
\text{(BE2)} & \quad x * 1 = 1 \\
\text{(BE3)} & \quad 1 * x = x \\
\text{(BE4)} & \quad x * (y * z) = y * (x * z).
\end{align*}

In \(A\), a binary relation \(\leq\) is defined by \(x \leq y\) if and only if \(x * y = 1\).

**Definition 2.2:** A BE-algebra \((X; *, 1)\) is said to be self-distributive if \(x * (y * z) = (x * y) * (x * z)\) for all \(x, y\) and \(z\) \(\in A\).

**Definition 2.3:** A dual BCK-algebra is an algebra \((A; *, 1)\) of type \((2,0)\) satisfying \(\text{(BE1)}\) and \(\text{(BE2)}\) and the following axioms for all \(x, y, z \in A\).

\begin{align*}
\text{(dBCK1)} & \quad x * y = y * x = 1 \text{ implies } x = y \\
\text{(dBCK2)} & \quad (x * y) * ((y * z) * (x * z)) = 1 \\
\text{(dBCK3)} & \quad x * ((x * y) * y) = 1.
\end{align*}

**Definition 2.4:** Let \(A\) be a BE-algebra or dual BCK-algebra. \(A\) is said to be commutative if the following identity holds:

\[ x \lor_B y = y \lor_B x \text{ where } x \lor_B y = (y * x) * x \text{ for all } x, y \in A. \]

**Definition 2.5:** Let \(A\) be a BE-algebra. If there exists an element \(0\) satisfying \(0 \leq x\) (or \(0 * x = 1\)) for all \(x \in A\), then the element “0” is called unit of \(A\). A BE-algebra with unit is called a bounded BE-algebra.

**Note:** In a bounded BE-algebra \(x * 0\) denoted by \(xN\).

**Definition 2.6:** In a bounded BE-algebra, the element \(x\) such that \(xNN = x\) is called an involution.

Let \(S(A) = \{x \in A \mid xNN = x\}\) where \(A\) is a bounded BE-algebra. \(S(A)\) is the set of all involutions in \(A\). Moreover, since \(1NN = (1 * 0) * 0 = 0 * 0 = 1\) and \(0NN = (0 * 0) * 0 = 1* 0 = 0\), We have \(0, 1 \in S(A)\) and so \(S(A) \neq \emptyset\).

**Definition 2.7:** Each of the elements \(a\) and \(b\) in a bounded BE-algebra is called the complement of the other if \(a \lor b = 1\) and \(a \land b = 0\).

**Definition 2.8:** Now we have introduced our concept smarandache – R – module: “Let \(R\) be a module, called R-module. If \(R\) is said to be smarandache – R – module. Then there exist a proper subset \(A\) of \(R\) which is an algebra with respect to the same induced operations of \(R\)”
3. Theorem

Theorem 3.1: Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which (BE1) to (BE4) are hold, then the following conditions are satisfied,

(i) \( 1N = 0, 0N = 1 \)

(ii) \( x \leq xNN \)

(iii) \( x \ast yN = y \ast xN \)

(iv) \( 0 \lor x = xNN, x \lor 0 = x \).

Proof. Let \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (BE1) to (BE4) then \( A \) is bounded BE-algebras.

(i) We have \( 1N = 1 \ast 0 = 0 \) and \( 0N = 0 \ast 0 = 1 \). by using (BE1) and (BE3)

(ii) Since \( x \ast xNN = x \ast ((x \ast 0) \ast 0) = (x \ast 0) \ast (x \ast 0) = 1 \)

We get \( x \leq x \) (by (BE1) and (BE4))

(iii) We have \( x \ast yN = x \ast (y \ast 0) \) (by using (BE4))

\[ = y \ast (x \ast 0) \]

\[ = y \ast xN. \]

(iv) By routine operations, we have \( 0 \lor x = (x \ast 0) \ast 0 = xNN \) and \( x \lor 0 = (0 \ast x) \ast x = 1 \ast x = x \).

Theorem 3.2: Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which (BE1) to (BE4) are hold, then the following conditions are satisfied \( x \ast y \leq (y \lor x) \ast y \) for all \( x, y \in A \).

Proof. Let \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (BE1) to (BE4) then \( A \) is bounded BE-algebras.

Since

\[ (x \ast y) \ast ((y \vee x) \ast y) = (y \vee x) \ast ((x \ast y) \ast y) = (y \vee x) \ast (y \vee x) = 1 \]

We have \( x \ast y \leq (y \vee x) \ast y \).

Theorem 3.3: Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which (BE1) to (BE4) are hold, In addition to that satisfy \( x \ast (y \ast z) = (x \ast y) \ast (x \ast z) \) then the following conditions are satisfied \( \text{for all } x, y, z \in A \)

(i) \( x \ast y \leq yN \ast xN \)

(ii) \( x \leq y \text{ implies } yN \leq xN. \)

Proof. Since \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (BE1) to (BE4) then \( A \) is bounded and Self-Distributive BE-algebras.

(i) Since \( (x \ast y) \ast (yN \ast xN) \)

\[ = (x \ast y) \ast ((y \ast 0) \ast (x \ast 0)) \]

\[ = (y \ast 0) \ast ((x \ast y) \ast (x \ast 0)) \text{ (by BE4)} \]
\[ (y * 0) * (x * (y * 0)) \text{ (by distributivity)} \\
= x * ((y * 0) * (y * 0)) \text{ (by BE4)} \\
= x * 1 \text{ (by BE1)} \\
= 1 \text{ (by BE2)} \]

We have \( x * y \leq yN * xN \).

(i) It is trivial by \( x \leq y \), We have \( z * x \leq z * y \)

\[ \text{then } y * z \leq x * z \text{ for all } x, y, z \in A. \]

**Theorem 3.4:** Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which (BE1) to (BE4) are hold, In addition to that satisfy \( x * (y * z) = (x * y) * (x * z) \), then the following conditions are satisfied

(i) \((y \lor x) * y \leq x * y.\)

(ii) \(x * (x * y) = x * y.\)

**Proof.** Since \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (BE1) to (BE4) then \( A \) is a Self-Distributive BE-algebras.

(i) Since
\[
x * (y \lor x) = x * ((x * y) * y) \\
= (x * y) * (x * y) \\
= 1.
\]

We have \( x \leq y \lor x \). By \( z * x \leq z * y \)

We have \( (y \lor x) * y \leq x * y \text{ for all } x, y, z \in A \)

(ii) By using self distributive definition, (BE1) and (BE3), we have
\[
x * (x * y) = (x * x) * (x * y) \\
= 1 * (x * y) \\
= x * y.
\]

**Theorem 3.5:** Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which (BE1) to (BE4) are hold, In addition to that satisfy \( 0 \leq x \) (or \( 0 * x = 1 \)), then the following conditions are satisfied for all \( x, y \in A \)

(i) \( x N N = x \)

(ii) \( x N \land y N = (x \lor y) \)
(iii) \( x \lor y = (x \land y) \)
(iv) \( x \lor y = y \lor x \).

**Proof.** Since \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (BE1) to (BE4) then \( A \) is a bounded and Commutative BE-algebras.

(i) It is obtained that
\[
\begin{align*}
    x \lor 0 &= (x \land 0) \lor 0 \\
    &= (0 \lor x) \land x \\
    &= 1 \land x \\
    &= x.
\end{align*}
\]

(ii) By the definition of “\( \land \)” and (i) we have that
\[
\begin{align*}
    x \land y &= (x \lor y) \land (x \lor y) \\
    &= (x \lor y) \land (x \lor y) \\
    &= x \lor y.
\end{align*}
\]

(iii) By the definition of “\( \land \)” and (i) we have that
\[
\begin{align*}
    (x \land y) \land N &= (x \lor y) \land (x \lor y) \\
    &= x \lor y.
\end{align*}
\]

(iv) We have
\[
\begin{align*}
    x \lor y &= (x \lor 0) \lor (y \lor 0) \\
    &= (x \lor 0) \lor (y \lor 0) \\
    &= x \lor y.
\end{align*}
\]

**Theorem 3.6:** Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which (BE1) to (BE4) are hold, In addition to that, there exists a complement of any element of \( A \) and then it is unique.

**Proof.** Since \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (BE1) to (BE4) then \( A \) is a bounded and Commutative BE-algebras. Let \( x \in A \) and \( a, b \) be two complements of \( x \). Then we know that \( x \land a = x \land b = 0 \) and \( x \lor a = x \lor b = 1 \).

Also since \( x \lor a = (x \lor a) \lor a = 1 \) and \( a \lor (x \lor a) = x \lor (a \lor a) = x \lor 1 = 1 \),

We have \( x \lor a \leq a \) and \( a \leq x \lor a \). So we get \( x \lor a = a \).

Similarly
\[
\begin{align*}
    x \lor b &= b, \\
    a \lor b &= (x \lor a) \lor (x \lor b) = (a \lor x) \lor (b \lor x) \\
    &= b \lor ((a \lor x) \lor (b \lor x)) = b \lor 1 = b. \\
\end{align*}
\]

Hence we obtain \( b \lor a = 1 \).

With similar operations, we have \( b \lor a = 1 \).

Hence we obtain \( a = b \) which gives that the complement of \( x \) is unique.
Theorem 3.7: Let R be a smarandache-R-module, if there exists a proper subset A of R in which (BE1) to (BE4) are hold, In addition to that satisfy \( 0 \leq x \) (or \( 0 \ast x = 1 \)), then the following conditions are equivalent for all \( x, y \in A \):

(i) \( x \land xN = 0 \)
(ii) \( xN \lor x = 1 \)
(iii) \( xN \ast x = x \)
(iv) \( x \ast xN = xN \)
(v) \( x \ast (x \ast y) = x \ast y \).

Proof. Since R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (BE1) to (BE4) then A is a Commutative and bounded BE-algebras.

(i) \( \Rightarrow \) (ii) Let \( x \land xN = 0 \). Then it follows that
\[
xN \lor x = (xN \lor x) \text{ by Theorem 2.5 (i)}
= (xN \land xN) \text{ by Theorem 2.5 (ii)}
= (x \land xN) \text{ by Theorem 2.5 (i)}
= 0N
= 1.
\]

(ii) \( \Rightarrow \) (iii) Let \( xN \lor x = 1 \). Then, since
\[
xN \ast x = x \lor xN = 1 \text{ and}
\]
\[
x \ast (xN \ast x) = xN \ast (x \ast x) = xN \ast 1 = 1
\]
We get \( xN \ast x = x \) by (dBCK1).

(iii) \( \Rightarrow \) (iv) Let \( xN \ast x = x \). Substituting \( xN \) for \( x \) and using Theorem 2.5 (i)
We obtain the result.

(iv) \( \Rightarrow \) (v) Let \( x \ast xN = xN \). Then
We get \( yN \ast (x \ast xN) = yN \ast xN \).
Hence we have \( x \ast (yN \ast xN) = yN \ast xN \). Using Theorem 2.5 (iv)
We obtain \( x \ast (x \ast y) = x \ast y \).

(v) \( \Rightarrow \) (ii) Let \( x \ast (x \ast y) = x \ast y \). Then
We have \( xN \lor x = (x \ast (xN)) \ast xN \)
\[
= (x \ast (x \ast 0)) \ast xN
= (x \ast 0) \ast (x \ast 0)
= 1.
\]

(ii) \( \Rightarrow \) (i) Let \( xN \lor x = 1 \). Then
We obtain \( N \land x = xN \land xNN \)
\[
= (x \lor xN) \text{ by Theorem 2.5 (ii)}
= 1N
= 0.
\]
REFERENCES


