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Group Semirings Using Distributive Lattices As Semirings

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Abstract

In this paper group semirings using distributive lattices as semirings is studied. The condition on the distributive lattices as well as on the groups are given for the group semiring to have zero divisors, idempotents, S-zero divisors, S-anti zero divisors and S-idempotents. This is the first time such analysis has been carried out on these group semirings.

Key words: Chain Lattice, Semirings, distributive lattice, Boolean Algebra, S-zero divisors, S-idempotent, S-anti zero divisors, group semirings.

1 Introduction

In this paper the study of group semirings is made using finite or infinite distributive lattices. Group rings and semigroup rings have been studied by several authors. Both these structures are only studied over rings with unit or a field. In case of group rings several researchers have studied about the zero divisors, idempotents, units etc;[3,4]. Likewise study of semigrouprings that is semigroups over rings or fields have been studied and also special elements like zero divisors, units are analysed by several researchers[3,4]. However[5] have studied semirings and group semirings.

In this paper study of group semigroups where the groups over semirings where semirings are taken as distributive lattices is carried out. This study is very new for group semirings have been studied by researchers very sparingly[5]. In this paper a systematic study of this type is carried out. This paper has four sections. Section one is introductory in nature. Section two studies group semirings where semirings are chain lattices. Section three introduces the new study of group semirings by taking semirings which are distributive lattices as well as Boolean algebras. The conclusions are given in the final section.

2 Group Semirings of Semirings which are chain lattices and their Properties :

Throughout this section C_n will denote a chain lattice of length n and G will denote a group under multiplication. First for the sake of completeness the definition of group semiring is recalled.

Definition 2.1: Let $S = C_n$ be the chain lattice, a semiring. G be any group. The group semiring $C_nG = SG$ of the group G over the semiring C_n (= S) contains all finite formal sums of the form $\left\{ \sum_{i=1}^{n} s_i g_i \,\middle|\, \begin{array}{l} s_i \in S = C_n \\ g_i \in G \end{array} \right\}$ *; on which two binary operations '+' (the union in the lattice C_n) and $\times$ (the $\cap$ on the lattice C_n) are defined on SG as follows.*

Let $\alpha = \sum_{i=1}^{n} s_i\alpha_i$ *and* $\beta = \sum_{i=1}^{m} r_i\alpha_i$ *where* $s_i, r_i \in C_n = S$ *and* $\alpha_i \in G$

i) $\alpha + \beta = \sum_{i=1}^{n} s_i\alpha_i + \sum_{i=1}^{m} r_i\alpha_i$

$= \sum_{i=1}^{m \text{ or } n} (s_i \cup r_i)\alpha_i$ *(which ever m or n is greater)*

$= \sum_{i=1}^{n} m_i \alpha_i$ $(m_i \in C_n = S)$.

ii) $\alpha \times \beta = \left(\sum_{i=1}^{n} s_i \alpha_i \right) \times \left(\sum_{i=1}^{m} r_i \alpha_j \right)$

$= \sum_{k=1}^{t} (s_i \cap r_j)\, \alpha_k = \sum_{k=1}^{t} \gamma_k \alpha_k$

where $\alpha_k = \alpha_i \alpha_j \in G$ *and* $\gamma_k \in S$.

iii) For $e = 1 \in G$ *(the identity of G)*

$1 \times s_i = s_i \times 1 = s_i$ *for all* $s_i \in C_n$ *and* $s_i g = g\alpha_i$ *for all* $g \in G$.

iv) For $1 \in C_n = S$ *we have*

$1 \cdot g_i = g_i \cdot 1 = g_i$ *for all* $g_i \in G$.

v) For $0 \in C_n = S$ *we have*

$0 \cdot g_i = g_i \cdot 0 = 0$ *for all* $g_i \in G$.

vi) Further $1 \cdot G \subseteq SG$ *and* $S.1 \subseteq SG$

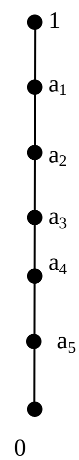
(Here 1 of G and 1 of C_n=S is defined 1).

The identity element 1 of $SG = C_nG$. SG is a semiring.

Some examples of the group semirings is given in the following.

Example 2.1: Let

C_7 =



be the semiring (chain semi lattice) and $G = \langle g \mid g^{20} = 1 \rangle$ be the cyclic group of order 20. C_7G be the group semiring of the group G over the semiring C_7.

Clearly number of elements in C_7G is finite so the group semiring is of finite order. Since G is a commutative group so is the group semiring C_7G.

Let

$\alpha = a_2g^6 + a_3g^2 + a_4g + a_5$ and $\beta = a_3g^7 + a_4g^6 + a_1g + 1 \in C_7G$.

$$\begin{aligned}\alpha + \beta &= (a_2g^6 + a_3g^2 + a_4g + a_5) + (a_3g^7 + a_4g^6 + a_1g + 1)\\ &= a_3g^7 + (a_2 \cup a_4)\, g^6 + a_3g^2 + (a_4 \cup a_1)g + a_5 \cup 1\\ &= a_3g^7 + a_2g^6 + a_3g^2 + a_1g + 1 \in SG.\end{aligned}$$

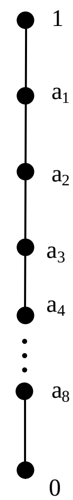
$$\begin{aligned}\alpha \times \beta &= (a_2g^6 + a_3g^2 + a_4g + a_5) \times (a_3g^7 + a_4g^6 + a_1g + 1)\\ &= (a_2 \cap a_3)\, g^6 \times g^7 + (a_3 \cap a_3)\, g^2 \times g^7 + (a_4 \cap a_3)\, g \times g^7\\ &\quad + (a_5 \cap a_3)\, g^7 + (a_2 \cap a_4)\, g^6 \times g^6 + (a_3 \cap a_4)\, g^2 \times g^6\\ &\quad + (a_4 \cap a_4)g \times g^6 + (a_5 \cap a_4)\, g^6 + (a_2 \cap a_1)\, g^6 \times g\\ &\quad + (a_3 \cap a_1)g^2 \times g + (a_4 \cap a_1)\, g \times g + (a_5 \cap a_1)\, g\\ &\quad + (a_2 \cap 1)\, g^6 + (a_3 \cap 1)\, g^2 + (a_4 \cap 1)\, g + a_5 \cap 1\\ &= a_3\, g^{13} + a_3g^9 + a_4\, g^8 + a_5g^7 + a_4g^{12} + a_4g^8 + a_4g^7 + a_5g^6\\ &\quad + a_2g^7 + a_3g^3 + a_4g^2 + a_5g + a_2g^6 + a_3g^2 + a_4g + a_5\\ &= a_3g^{13} + a_4g^{12} + a_3g^9 + a_4g^8 + (a_5g^7 + a_4g^7 + a_2g^7)\\ &\quad + (a_5g^6 + a_2g^6) + a_3\, g^3 + (a_3g^2 + a_3g^2)\\ &\quad + (a_5g + a_4g) + a_5\\ &= a_3g^{13} + a_4g^{12} + a_3g^9 + a_4g^8 + a_2g^7 + a_2g^6 + a_3g^3 + a_3g^2 + a_4g + a_5 \in C_7G\end{aligned}$$

This is the way sum and product are performed of the group semiring using the semiring as the chain lattice C_n.

Example 2.2: Let $G = S_3$ the permutation group of degree three and $S = C_{10}$ be the chain lattice. SG be the group semiring of the group G over the semiring C_{10}. Clearly order of SG is finite but SG is non commutative.

$$G = S_3 = \left\{\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} = e = 1, \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = p_1, \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = p_2,\right.$$
$$\left.\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} = p_3, \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} = p_4, \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = p_5\right\}$$

be the symmetric group of degree three.



Let $\alpha = a_1p_5 + a_3p_3 + a_2p_1 + a_7$ and $\beta = a_8p_4 + a_2p_3 + a_5p_1 + a_6 \in C_nG = SS_3$.

$$\begin{aligned}\alpha+\beta &= (a_1p_5 + a_3p_3 + a_2p_1 + a_7) + (a_8p_4 + a_2p_3 + a_5p_1 + a_6)\\ &= a_1p_5 + a_8p_4 + (a_3p_3 + a_2p_3) + (a_2p_1 + a_5p_1) + (a_7 + a\text{-}_6)\\ &= a_1p_5 + a_8p_4 + (a_3 \cup a_2)\, p_3 + (a_2 \cup a_5)\, p_1 + a_7 \cup a_6\\ &= a_1p_5 + a_8p_4 + a_2p_3 + a_2p_1 + a_6 \in SS_3\end{aligned}$$

Consider $\alpha \times \beta$

$= (a_1p_5 + a_3p_3 + a_2p_1 + a_7) \times (a_8p_4 + a_2p_3 + a_5p_1 + a_6)$

$= (a_1 \cap a_8)\, p_5 \times p_4 + (a_3 \cap a_8)\, p_3 \times p_4 + (a_2 \cap a_8)\, p_1 \times p_4 + (a_7 \cap a_8)\, p_4 + (a_1 \cap a_2)\, p_5 \times p_3 + (a_3 \cap a_2)\, p_3 \times p_3 + (a_2 \cap a_5)\, p_1 \times p_3 + (a_7 \cap a_2)\, p_3 + (a_1 \cap a_5)\, p_5 \times p_1 + (a_3 \cap a_5) p_3 \times p_1 + (a_2 \cap a_5)\, p_1 \times p_1 + (a_3 \cap a_6)\, p_3 + (a_7 \cap a_5)\, p_1 + (a_1 \cap a_6)\, p_5 + (a_2\ \ a_6)\, p_1 + a_7 \cap a_6$

$= a_8 + a_8\, p_2 + a_8\, p_3 + a_8\, p_4 + a_2p_2 + a_3 \cdot 1 + a_5p_4 + a_7p_3 + a_5p_3 + a_5p_5 + a_5 .1 + a_7p_1 + a_6\, p_5 + a_6p_3 + a_6p_1 + a_7$

$= (a_8 \cup a_7 \cup a_5) + (a_8 \cup a_2)\, p_2 + (a_8 \cup a_7 \cup a_5 \cup a_6)p_3 + (a_8 \cup a_5)p_4 + (a_5 \cup a_6)p_5 + (a_6 \cup a_7)\, p_1$

$= a_3 + a_2p_2 + a_5p_3 + a_5p_4 + a_5p_5 + a_6p_1 \in SS_3.$ I

Now find $\beta \times \alpha$

$= (a_8p_4 + a_2p_3 + a_5p_1 + a_6) \times (a_1p_5 + a_3p_3 + a_2p_1 + a_7)$

$= (a_8 \cap a_1)\, p_4p_5 + (a_2 \cap a_1)\, p_3 \times p_5 + (a_5 \cap a_1)\, p_1 \times p_5 + (a_6 \cap a_1)\, p_5 + (a_8 \cap a_3)p_4 \times p_3 + (a_2 \cap a_3)\, p_3 \times p_3 + (a_5 \cap a_3)\, p_1 \times p_3 + (a_6 \cap a_3)\, p_3 + (a_8 \cap a_2)\, p_4 \times p_1 + (a_2 \cap a_2)\, p_3 \times p_1 + (a_5 \cap a_2)\, p_1 \times p_1 + (a_6 \cap a_2)\, p_1 + (a_8 \cap a_7)\, p_4 + (a_2 \cap a_7)\, p_3 + (a_5 \cap a_7)\, p_1 + (a_6 \cap a_7)$

$= a_8 \cdot 1 + a_2p_1 + a_5p_2 + a_6p_5 + a_8p_1 + a_3 \cdot 1 + a_5p_4 + a_6p_3 + a_8p_2 + a_2p_5 + a_5 \cdot 1 + a_6p_1 + a_8p_4 + a_7p_3 + a_7p_1 + a_7$

$= (a_8 \cup a_3 \cup a_5 \cup a_7) + (a_2 \cup a_8 \cup a_6 \cup a_7)p_1 + (a_5 \cup a_8)p_2 + (a_6 \cup a_2)p_5 + (a_5 \cup a_8)\, p_4 + (a_6 \cup a_7)\, p_3$

$= a_3 + a_2p_1 + a_5p_2 + a_2p_5 + a_5p_4 + a_6p_3 \in SS_3.$ II

Clearly I and II are not equal so $\alpha\times\beta \neq \beta\times\alpha$ for this $\alpha, \beta \in SS_3$, hence SS_3 is a non commutative group semiring of finite order.

The following proposition characterizes the group semiring of a group G over the semiring which is a chain lattice.

Proposition 2.1: Let C_n be a finite chain lattice. G be a group and C_nG be the group semiring of the group G over the semiring, C_n.

C_nG is a commutative group semiring if and only if G is a commutative group.

Proof: Given C_n is a chain lattice so C_n is a commutative semiring. Let G be a commutative group clearly the group semiring C_nG is a commutative group semiring.

Suppose let C_nG be a commutative group semiring of the group G over the lattice C_n. To prove G is a commutative group, it enough to prove for every g, h $\in$ G; gh = hg.

Given SG is commutative let g, h $\in$ SG (g, h $\in$ G) then gh = hg as SG is commutative. This is true for every g, h $\in$ G hence G is a commutative group.

Next it is proved C_nG is a finite group semiring.

Proposition 2.2: Let C_n be a semiring (chain lattice of order n) and G a group. C_nG be the group semiring. C_nG is of finite order if and only if G is a finite group.

Proof. Given the group semiring C_nG is of finite order.

Clearly if G is not of finite order, since $G \subseteq C_nG$; C_nG would be of infinite order. Hence G must be a group of finite order.

Suppose G is a group of finite order clearly C_nG the group semiring will be of finite order as C_n is a finite semiring.

Now an example of group ring of infinite order is given.

Example 2.3: Let G = R \ {0} be the group of real numbers under product and C_{15} be the semiring (chain lattice of order 15). C_nG be the group semiring. Clearly C_nG is of infinite order as R \ {0} is an infinite group; so C_nG is of infinite order.

$C_n = 0 < a_{n-2} < a_{n-1} < \ldots < a_1 < 1$ be the chain lattice of order n.

Let

$$\alpha = a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2$$

and

$$\beta = a_5 + a_8\, 0.3 + a_{13}\, 9 \in C_nG.$$

$$\begin{aligned}
\alpha + \beta &= a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_5 + a_8\, 0.3 + a_{13}\, 9 \\
&= (a_1 \cup a_5) + (a_5 \cup a_8)\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_{13}\, 9 \\
&= a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_{13}\, 9 \in C_n G
\end{aligned}$$

$$\begin{aligned}
\alpha \times \beta &= (a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2) \times (a_5 + a_8\, 0.3 + a_{13}\, 9) \\
&= a_1 \cap a_5 + (a_5 \cap a_5)\, 0.3 + (a_9 \cap a_5)\, 120 + (a_{10} \cap a_5)\, 6.2 + (a_1 \cap a_{-8}) 0.3 \\
&\quad + (a_5 \cap a_8)\, (0.3 \times 0.3) + (a_9 \cap a_8)\, (120 \times 0.3) + (a_{10} \cap a_8)\, (6.2 \times 0.3) \\
&\quad + (a_1 \cap a_{13})\, 9 + (a_5 \cap a_{13})\, (0.3 \times 9) + (a_9 \cap a_{13})\, (120 \times 9) + (a_{10} \cap a_{13})\, 6.2 \times 9. \\
&= a_5 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_8\, 0.3 + a_8 \times 0.09 + \\
&\quad a_9\, 36 + a_{10}\, 1.86 + a_{13} 9 + a_{13}\, 2.7 + a_{13}\, 1080 + a_{13}\, 55.8 \\
&= a_5 + (a_5 \cup a_8)\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_{8-}\, 0.09 + a_9\, 3.6 + \\
&\quad a_{10}\, 1.86 + a_{13} 9 + a_{13}\, 1080 + a_{13}\, 2.7 + a_{13}\, 55.8 \in C_n G.
\end{aligned}$$

This is the way product is defined on the infinite group semiring.

There are several group semirings of infinite order.

Just recall all chain lattices are semifield. For more about semifields refer[5].

Theorem 2.1: Let C_n be the semiring which is a semifield G be a commutative group. C_n G the group semiring is a semifield.

Proof: Given C_nG is a commutative group semiring.

Clearly if $\alpha, \beta \in C_nG$; $\alpha+\beta = 0$ is possible only when $\alpha = 0$ and $\beta = 0$.

For in C_n; $a_i + a_j = 0 = a_i \cup a_j$ if only if $a_i = 0 = a_j$ for
$C_n = 0 < a_{n-2} < a_{n-3} < \ldots < a_1 < 1$.

Further $\alpha \times \beta = 0$ is possible in C_nG only if $\alpha = 0$ and $\beta = 0$. For in C_n; $a_ia_{-j} = 0 = a_i \cap a_j$ if and only if $a_i = 0$ or $a_j = 0$.

Thus C_nG is a semifield.

Corollary 2.1: Let C_nG be a group semiring of a group G; if G is a non commutative group then C_nG is a semidivision ring.

Proof: Since C_nG has no zero divisors and for every $\alpha, \beta \in C_nG$; $\alpha + \beta = 0$ implies $\alpha = 0$ and $\beta = 0$. C_nG is a semidivision ring as G is a non commutative group.

Example 2.4: Let $G = < g \mid g^{15} = 1 >$ be the cyclic group of order 15. C_{16} be the chain lattice of order 16. $C_{16}G$ be the group semiring of the group G over the semiring C_{16}. $C_{16}G$ is a semifield of finite order.

Example 2.5: Let $G = < a, b \mid a^2 = b^7 = 1, bab = a >$ be the dihedral group of order 14. C_{27} be the semiring of order 27. $C_{27}G$ is the group semiring of finite order which is a semidivision ring.

This proves that group semiring, C_nG where C_n is a chain lattice has no zero divisors. This is in contrast with group rings for every group ring of a finite group G over any field finite or infinite field has zero divisors.

Next the units and idempotents in C_nG are discussed in the following.

C_n, the chain lattice has only idempotents and has no units.

For any $x, y \in C_n$ we have $x \cap y = 1$ is not possible unless $x = y = 1$.

So C_n has no units. Further every element in C_n is an idempotent as C_n as $a_i \cap a_i = a_i$ for every $a_i \in C_n$ as C_n is a is the chain lattice.

In view of this the following theorems are proved.

Theorem 2.2: Let C_n be the chain lattice and G any group. The group semiring C_nG has no nontrivial zero divisors.

Proof: Follows from the fact C_nG is a semifield.

Theorem 2.3: Let C_nG be the group semiring. The only units of C_nG are $1 \cdot g = g$ for every $g \in G \subseteq C_nG$.

Proof: Follows from the fact $G \subseteq C_nG$ and every $g \in G$ has a unique inverse. However if $\alpha \in C_n \setminus \{1\}$ then it not a unit only an idempotent as C_n is a chain lattice.

If $\alpha = \sum_{i=1}^{n} \alpha_i g_i$ then $\alpha^2 = 1$ is impossible as C_nG is proved to be a semifield so no zero divisors to cancel of or add to 1.

Hence the chain.

However C_nG has idempotents if G is a group of finite order.

Example 2.6: Let G = < g | $g^{12} = 1$ > be the cyclic group of order 12. C_8 be the chain lattice. C_nG be the group semiring. Consider $\alpha = (1 + g^6) \in G$,

$$\begin{aligned}
\alpha^2 &= (1 + g^6) \times (1 + g^6) \\
&= (1 + g^6 + g^6 + g^{12}) \\
&= 1 \cup 1 + (1 \cup 1)\, g^6 \\
&= 1 + g^6 = \alpha.
\end{aligned}$$

Thus α is an idempotent in C_nG. Consider

$$\begin{aligned}
\beta &= 1 + g^3 + g^6 + g^9 \in C_nG \\
\beta^2 &= (1 + g^3 + g^6 + g^9) \times (1 + g^3 + g^6 + g^9) \\
&= 1 + g^3 + g^6 + g^9 + g^3 + g^6 + g^9 \\
&\quad g^{12} + g^6 + g^9 + g^{12} + g^3 + g^9 + g^{12} + g^3 + g^6 \\
&= 1 + g^3 + g^6 + g^9 \quad (\text{as } 1 \cup 1 = 1) \\
&= \beta.
\end{aligned}$$

Thus β is an idempotent. Similarly $\gamma = 1 + g^4 + g^8 \in C_nG$. Clearly $\gamma^2 = \gamma$ is an idempotent in C_nG,

Finally $\delta = 1 + g + g^2 \ldots + g^{11} \in C_nG$ is also an idempotent of C_nG.

Thus apart from all elements C_n as $C_n \subseteq C_nG$ are also idempotents of C_nG as $a_i \times a_i = a_i \cap a_i$ for all $a_i \in C_n$.

In view of this we have the following theorem.

Theorem 2.4: Let C_n be the group semiring of the group G of finite order over the chain lattice C_n.

i) All $\alpha \in C_n$; $C_n \subseteq C_nG$ are idempotents of C_nG.

ii) If $H_i \subseteq G$ is a subgroup of G and $H_i = \{1, h_1, \ldots, h_t\}$ then $\beta = 1 + h_1 + \ldots + h_t \in C_nG$ is an idempotent of C_nG. This is true for every subgroup of G.

iii) If $G = \{1, g_1, \ldots, g_m\}$ then $\gamma = 1 + g_1 + g_2 + \ldots + g_m \in C_nG$ is an idempotent of C_nG.

Proof: For every $\alpha \in C_n$ it is clear $\alpha \times \alpha = \alpha \cap \alpha$ is an idempotent of C_nG as $C_n \subseteq C_nG$,

Further $|G| < \infty$ if $G = \{1, g_1, \ldots, g_m\}$ then $\beta = 1 + g_1 + \ldots + g_m \in C_nG$ is such that $\beta^2 = \beta$. Finally every H_i a subgroup in G is of finite order and if $H_i = \{1, h_1, h_2, \ldots, h_t\}$ then $\gamma = 1 + h_1 + \ldots + h_t \in C_nG$ is such that $\gamma^2 = \gamma$.

Hence the theorem.

Next subsemirings and ideals of the group semiring C_nG are discussed in the following.

Example 2.7: Let C_9 be a chain lattice and $G = \langle g \mid g^{18} = 1 \rangle$ be the cyclic group of order 18. C_9G be the group semiring.

$A_1 = \{0, 1, (1 + g + \ldots + g^{17})\} \subseteq C_9G$ is a subsemiring of order 3.

$A_2 = \{0, 1, (1 + g^2 + g^4 + \ldots + g^{16})\} \subseteq C_9G$ is again a subsemiring of order 3.

$A_3 = \{0, 1, (1 + g^3 + g^6 + g^9 + \ldots + g^{15})\} \subseteq C_9G$ is also a subsemiring of order 3.

$A_4 = \{0, 1, \{1 + g^6 + g^{12}\}\} \subseteq C_9G$ is also a subsemigroup of C_9G.

$A_5 = \{0, 1, \{1 + g^9\}\} \subseteq C_9G$ is a subsemiring of order 3.

Now let $P_1 = \{1, g^2, g^4, \ldots, g^{16}\} \subseteq G$ be a subgroup of G. $C_9\ P_1 \subseteq C_9G$ is a subsemiring of C_9G.

Let $P_2 = \{1, g^9\} \subseteq G$ be a subgroup of G. $C_9P_2 \subseteq C_9G$ is a subsemiring of C_9G.

Let $P_3 = \{1, g^6, g^{12}\} \subseteq G$ be a subgroup of G. $C_9P_3 \subseteq C_9G$ is a subsemiring of C_9G.

Let $P_4 = \{1, g^3, g^6, .., g^{15}\} \subseteq G$ be a subgroup $C_9P_4 \subseteq C_9G$ be a subsemiring of C_9G.

Let $M_1 = \{0, a_5, 1\} \subseteq C_9$ is a sublattice of $C_9 = 0 < a_7 < a_6 < \ldots a_2 < a_1 < 1$.

Now $M_1P_1 \subseteq C_9G$ is a subsemiring of C_9G.

Let $M_2 = \{0, a_6, 1\} \subseteq C_9$ is a sublattice of C_9 and M_2P_1, M_2P_2, M_2P_3 and M_2P_4 are all subsemirings of C_9G.

Thus C_9G has several subsemirings but all of them are not ideals of C_9G only a few of them are ideals.

Further M_2P_1, M_2P_2, M_2P_3 and M_2P_4 are only subsemirings and none of them are ideals of C_9G.

Example 2.8: Let C_2 be the chain lattice. $G = \langle g \mid g^3 = 1 \rangle$ be the cyclic group of degree three. C_2G be the group semiring of the group G over the semiring C_2.

$P = \{0, 1, 1 + g + g^2\} \subseteq C_2G$ is a subsemiring. This is not an ideal of C_2G.

In view of all these the following proposition is proved.

Proposition 2.3: Let C_nG be the group semiring of the group G over the semiring C_n. If M is a subsemiring of C_nG then M is not an ideal of C_nG.

Proof: Proved using an example. In the example 2.7 of this paper there are several subsemirings of the group semiring which is not an ideal.

Next the concept of right and left ideal exist only when C_nG is a non commutative group semiring. Consider the following example.

Example 2.9: Let C_2S_3 be the group semiring of the symmetric group S_3 over the semiring C_2. C_2S_3 has right ideals which are not left ideal. Thus as in case of group rings which are non commutative in case of group semirings which are non commutative has right ideals that are not left ideals and vice versa.

Another interesting feature is in case of a field, field has no ideals other than (0) and F but however semifields which are group semirings of the form C_nG has ideals.

In the next section the study of groupsemirings using distributive lattices which are not chain lattice is carried out.

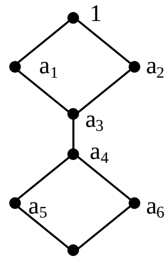
3 Study of group semirings using distributive lattices which are not chain lattices:

In this section a study of group semirings using distributive lattices L which are not chain lattices is carried out. Unlike chain lattices in case of distributive lattices the group semi rings in general are not semifields. However in case of certain lattices the group semiring can be a semifield.

Since the replacing of semiring (chain lattice) by a distributive lattice will not alter the definition of a group semiring so here the definition of group semiring using distributive lattices are not made once again[1-2].

First a few examples of them are given.

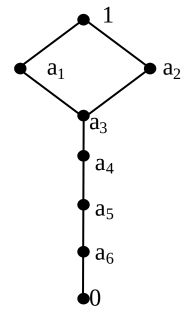
Example 3.1: Let L



be a distributive lattice.

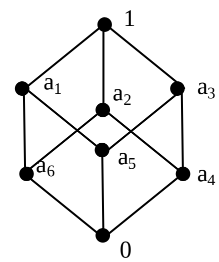
$G = \langle g \mid g^{10} = 1\rangle$ be the cyclic group of order 10. LG be the group semiring of the group G over the semiring L which is a distributive lattice. LG is not a field. In the first place L is a semiring and not a semifield as $a_5 \cap a_6 = 0$. Thus LG has zero divisors so LG is only a semiring.

Example 3.2: Let L



be a distributive lattice. $G = S_4$ be the symmetric group of degree 4. LG the group semiring is a semi division ring. Further L is not a semi field as LG is non commutative semiring. Thus LG has no zero divisors but LG is a non-commutative semi ring.

Example 3.3: Let B



be the Boolean algebra. $G = \langle g \mid g^{12} = 1\rangle$ be the cyclic group of order 12. BG the group semiring. BG has zero divisors, units and idempotents $\alpha = (a_6g^6 + a_5g^2)$ and $\beta = a_4g^2 \in$ BG.

$\alpha\beta = (a_6g^6 + a_5g^2) \times a_4g^2 = 0$. Let $\alpha = 1 + g^3 + g^6 + g^9 \in$ BG.

Clearly $\alpha^2 = \alpha$ so α is an idempotent. Let $g^7 \in$ BG, $g^5 \in$ BG is such that $g^7 \times g^5 = 1$. All elements in G are units of BG as $G \subseteq$ BG.

Theorem 3.1: Let L be a distributive

lattice and G any group. LG the group semiring of the group G over the semiring L.. LG has zero divisors if and only if L is a distributive lattice which is not a semi field.

Proof: If L is not a semi field. That is there exist a_i , $a_j \in L \setminus \{0\}$, $a_i \neq a_j$ such that $a_i \cap a_j = 0$. Take $\alpha = a_i g_1$ and $\beta = a_j g_2 \in LG$; $g_1, g_2 \in G$.

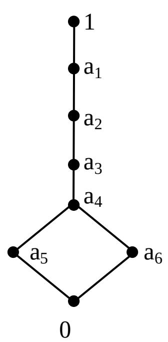
$$\begin{aligned} \alpha \cap \beta &= a_i g_1 \cap a_j g_2 \\ &= (a_1 \cap a_j)(g_1 g_2) \\ &= 0 (g_1 g_2) \quad = \quad 0. \end{aligned}$$

Thus LG has zero divisors.

Corollary 3.1: If L is a semifield then the group semiring LG has no zero divisors.

Proof follows from the fact L is a semifield and no $\alpha, \beta \in LG \setminus \{0\}$ is such that $\alpha \times \beta = (0)$.

***Example 3.4**:* Let L lattice given by following diagram.



$G = S_3$ be the symmetric group of degree three.

LG be the group semiring of the group G over S_3.

Let $R_1 = \{1, p_1\}$ be a subgroup. LR_1 is a subsemiring which is not an ideal.

Let $R_2 = \{1, p_2\} \subseteq S_3$ be the subgroup. LR_2 is again a subsemiring.

LR_1 and LR_2 are isomorphic as subsemirings by mapping p_1 to p_2 and rest of the elements to itself.

The following theorem is interesting which describes a semiring which is not a semifield.

Theorem 3.2: Let a be a finite group. L is a Boolean algebra of order greater than or equal to four. LG the group semiring has zero divisors.

Proof: Follows from the fact all Boolean algebras of order greater than or equal to four has elements $a, b \in L \setminus \{0\}$ with $a \cap b = 0$. This will contribute for zero divisors of the form $\alpha\beta = 0$ when $\alpha = ag_1$ and $\beta = bg_2$ with $\alpha \times \beta = \alpha\beta = ag_1 \times bg_2 = (a \cap b) g_1 g_2 = 0$.

Theorem 3.3: Let G be a group of finite order L be a distributive lattice which is not a chain lattice. LG be the group semiring of the group G over the lattice L. LG has non trivial idempotents.

Proof: Given $| G | = n < \infty$ a finite group. LG the group semiring.

Take $\alpha = (1 + g_1 + \ldots + g_{n-1}) \in LG$. Clearly $\alpha^2 = \alpha$ so α is an idempotent of LG.

Likewise if $H_1, H_2, \ldots, H_t$ are non-trivial subgroups of order $p_1, p_2, \ldots, p_t$ respectively then $\beta_1 = 1 + h_1 + h_2 + \ldots + h_{p_1-1}$

$\in LG$ where $H_1 = \{1, h_1, h_2, \ldots, h_{p_1-1}\} \subseteq G$ is such that $\beta_1^2 = \beta_1$.

Let $\beta_2 = 1 + k_1 + k_2 + \ldots + k_{p_2-1}$ $\in LG$, where $H_2 = \{1, k_1, k_2, \ldots, k_{p_2-1}\} \subseteq G$ is $\beta_2^2 = \beta_2$.

Likewise if $H_t = \{1, m_1, m_2, \ldots, m_{p_t-1}\}$ $\subseteq G$ the subgroup.
$\beta_t = 1 + m_1 + m_2 + \ldots + m_{p_t-1} \in LG$ is such that $\beta_t^2 = \beta_t$.
Hence the theorem.

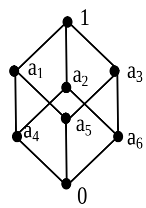
The idempotent in L will be called as trivial idempotents. Likewise zero divisors in L will be defined as trivial zero divisors of LG.

Clearly L has no units and units contributed by the group G will be termed as trivial units of LG.

For the definition and properties of Smarandache zero divisors please refer[5].

Conditions for Smarandache zero divisors to exist in group semirings; BG where B is a Boolean algebra is obtained in the following.

Example 3.5: Let B



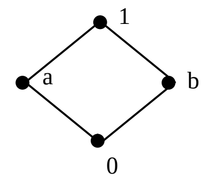
be a Boolean algebra. $G = \langle g \mid g^{16} = 1\rangle$ be the cyclic group of order 16. BG be the group semiring of the group G over the semiring B.

Let $x = a_4 (g^{12} + g^2)$ and $y = a_5 (g^7 + g^5) \in BG$

$x \times y = a_4 (g^{12} + g^2) \times a_5 (g^7 + g^5)$
$= (a_4 \cap a_5) (g^{12} + g^2) (g^7 + g^5)$
$= 0.$

Let $a = a_6 (g^{10} + g)$ and $b = a_2 (g^3 + g^{11} + g^{13}) \in BG$. $x \times a = 0$ and $y \times b = 0$ but $a \times b \neq 0$. Thus $x, y \in BG$ is a Smarandache zero divisor[1-5].

Example 3.6: Let B



Be a Boolean algebra or order four and G be any group. BG be the group semiring of the group G over the semiring B. BG has no S-zero divisors.
In view of this the following theorem is proved.

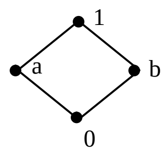
Proposition 3.1: Let G be any group and B a Boolean algebra BG the group semiring.

i. *BG the group semiring has S-zero divisors if $|B| > 4$*
ii. *BG has only zero divisors and no S-zero divisors if $|B| = 4$*
iii. *BG has no zero divisors if $|B| = 2$.*

Proof: Proof of i. follows from the fact if $|B| > 4$ then B has zero divisors as well as S-zero divisors.

Hence BG will have S-zero divisors (refer example 3.5).

Proof of ii. If | B | = 4 then

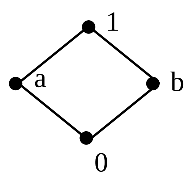
B = 

Clearly $a \times b = 0$ is a zero divisor and cannot find another $y \neq 0$ with $ay = 0$ and a. $x = 0$ with $bx = 0$ and $xy \neq 0$. Hence the claim.

Proof of iii. If | B | = 2 then B is a chain lattice hence BG has no zero divisors.

Next the study about the existence of S-anti zero divisors is discussed.

Example 3.7: Let B =



be the Boolean algebra of order four; $G = S_3$, the symmetric group of degree 3. BS_3 be the group of semiring of the group S_3 over the semiring B.

Let $\alpha = (1 + p_1 + p_2)$, $\beta = (1 + p_4) \in BS_3$. Clearly $\alpha \times \beta \neq (0)$.

Take $x = a\,(p_2 + p_5)$ and $y = b\,(p_5 + p_3 + 1) \in BS_3$.

Consider

$\alpha x = (1 + p_1 + p_2)\, a\,(p_2 + p_5)$

$= 1 \cap a\,[(1 + p_1 + p_2) \times (p_2 + p_5)]$

$= a(p_2 + p_5 + 1 + p_5 + p_2 + p_3)$

$= a\,(p_2 + 1 + p_5 + p_5 + p_2 + p_3)$

$= a\,(1 + p_5 + p_2 + p_3) \neq 0.$

Consider

$\beta y = (1 + p_4) \times b\,(1 + p_3 + p_5)$

$= b \cap 1\,[(1 + p_4) \times (1 + p_3 + p_5)]$

$= b\,(1 + p_4 + p_3 + p_1 + 1 + p_5)$

$= b\,(1 + p_1 + p_3 + p_4 + p_5) \neq 0.$

So $\beta y \neq 0$ and $\alpha x \neq 0$ but

$xy = a\,(p_2 + p_5) \times b\,(p_5 + p_3 + 1)$

$= a \cap b\,[(p_2 + p_5) \times (p_5 + p_3 + 1)]$

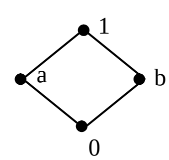
$= 0.$

Thus α is a Smarandache anti-zero divisor of BG.

In view of the following proposition is proved.

Proposition 3.2: Let B be a Boolean algebra of order four. G any group and BG the group semiring of the group G over the semiring B. BG has S-anti zero divisors.

Proof: Let $\alpha = \Sigma\, g_i$ and $\beta = \Sigma\, h_i$, g_i, $h_i \in G$ (all coefficients of g_i in α and h_i in β are 1). Take $x = (\Sigma\, ak_i)$ and $y = \Sigma\, bm_j$ (k_i, $m_j \in$ G). B =



Clearly $\alpha\beta \neq 0$. Further $\alpha x \neq 0$ and $\beta y \neq 0$ but $\alpha\beta = 0$. Thus x is a S-anti zero divisor in BG.

Theorem 3.4: Let BG be the group semiring of the group G over the Boolean algebra of order four. Let $\alpha \in BG$ be a S-anti zero divisor then α need not be a zero divisor.

Proof: Follows from the example 3.7. For α in that example is not a zero divisor in BG.

Proposition 3.3: Let BG be the group semiring of the group G over the Boolean algebra of order greater than 4. BG has both S-zero divisors as well as S-anti zero divisors.

Next we study the Smarandache idempotents in these group semirings. At first it is important to know that the distributive lattices or for that matter any lattice L will not contain any Smarandache idempotent as every element in L is such that $a \times a = a \cap a = a^2 = a$ for all $a \in L$.

However it is an interesting feature to analyse whether the group semiring of a group G over a distributive lattice L have Smarandache idempotents. Let BG be the group semiring of the group $G = S_3$ over the semiring B which is a Boolean algebra.

Take $a = (1 + p_4 + p_5)$ and $b = (p_1 + p_2 + p_3)$ we see $a^2 = a$ and $b^2 = a$; $ab = a$. Thus group semiring has S-idempotents.

Example 3.8: Let $G = \langle g \mid g^8 = 1\rangle$ be the cyclic group of order 8. B be any Boolean algebra or a distributive lattice. BG be the group semiring of the group G over the semiring B.

Let $\alpha = 1 + g^2 + g^4 + g^6$ and $\beta = g + g^3 + g^5 + g^7 \in BG$.

Clearly $\alpha^2 = \alpha$, $\beta^2 = \alpha$ and $\alpha\beta = \beta$.

In view of all these the following interesting theorem for cyclic groups of even order is given.

Theorem 3.5: Let $G = \langle g \mid g^{2n} = 1\rangle$ be the cyclic group of order 2n. B be a distributive lattice or a Boolean algebra. BG the group semiring has a Smarandache idempotent.

Proof: Take $\alpha = (1 + g^2 + g^4 + g^6 + g^8 + \ldots + g^{2n-2}) \in BG$

Let $\beta = (g + g^3 + g^5 + g^7 + \ldots + g^{2n-1}) \in BG$. Clearly $\alpha^2 = \alpha$, $\alpha\beta = \beta$ and $\beta^2 = \alpha$. So α is a Smarandache idempotent.

Example 3.9: Let B be a distributive lattice or a Boolean algebra.

$D = \{a,b \mid a^2 = b^2 = 1, bab = a\}$ be the dihedral group. BD be the group semiring of the group BD over the semiring B.

Take $\alpha = (1 + b + b^2 + \ldots + b^{19})$ and $\beta = (a + ab + ab^2 + \ldots + ab^{19}) \in BD$.

Clearly $\alpha^2 = \alpha$ and $\beta^2 = \alpha$ with $\alpha\beta = \beta$. The α is a Smarandache idempotent in BD.

In view of this the following theorem.

Theorem 3.6: Let L be a distributive lattice or a Boolean algebra. Let $G = D_{2n}$

= {a,b / $a^2 = b^n = 1$, bab = a}; n an even integer say 2m. LG be the group semiring of the group G over the semiring L. LG has S-idempotents.

Proof: Consider $\alpha = (1 + b + b^2 + \ldots + b^{2m-1})$ and $\beta = (a + ab + ab^2 + \ldots + ab^{2m-1}) \in$ LG. Clearly $\alpha^2 = \alpha$ and $\beta^2 = \alpha$ and $\alpha\beta = \beta$. Thus α a Smarandache idempotent of LG.

Example 3.10: Let A_4 be the alternating subgroup of S_4; L be a distributive lattice or a Boolean algebra. LA_4 the group semiring.

Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in LA_4$$

$$\alpha^2 = \alpha.$$

$$\beta^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \alpha$$

$$\alpha\beta = \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}\right] \times \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \beta.$$

Thus α is a Smarandache idempotent of LA_4.

In view of all these the following theorems are proved.

Theorem 3.7: Let G be a group of order n and G has a subgroup of order m (m/n; m an even number). L any distributive lattice or a Boolean algebra. LG the group semiring has S-idempotents.

Proof: Let H be a subgroup of order say m = 2t and let P = $\{1, g_1, \ldots, g_{t-1}\}$ be a sub group of H. Then take $\alpha = 1 + g_1 + \ldots + g_{t-1}$ and $\beta = \sum_{h_i \in H \setminus P} h_i \in LG$.

Clearly $\alpha^2 = \alpha$ and $\beta^2 = \alpha$ and $\alpha\beta = \alpha$. Thus α is the S-idempotent of LG.

Theorem 3.8: Let S_n be the symmetric group of degree n (n even or odd). L any distributive lattice. LS_n the group semiring. LS_n has S-idempotents.

Proof:

Case 1: n is even. S_n has a subgroup of order n which is cyclic. Hence using this subgroup say $G = \{1, g, \ldots, g^{n-1}\}$ contributes to the S-idempotent; $\alpha = 1 + g^2 + \ldots + g^{n-2}$ and $\beta = (g + g^3 + \ldots + g^{n-1}) \in LS_n$ is such that $\alpha^2 = \alpha$; $\beta^2 = \alpha$ and $\alpha\beta = \beta$.

Let n be odd then n – 1 is even. Let H = cyclic group generated by $h \in S_n$ of order n – 1.

Now $(1 + h^2 + h^4 + \ldots + h^{n-3}) = \alpha$ and $\beta = (h + h^3 + \ldots + h^{n-1}) \in LS_n$ are such that $\alpha^2 = \alpha$, $\beta^2 = \alpha$ and $\alpha\beta = \beta$. Thus α is a S-idempotent of LS_n. Hence the theorem.

4. Conclusions

In this chapter group semirings of groups over semirings which are distributive lattices is carried out. Further in case of chain lattices C_n the group semiring C_nG is a semifield in case G is abelian and a semi division ring in case G is a non-commutative group.

Further if the distributive lattice L has zero divisors then only the group semiring LG will have zero divisors.

Finally idempotents which are in LG\L are identified. The concept of Smarandache zero divisors and Smarandache idempotents in group semirings (where semirings are distributive lattices) are carried out and conditions for their existence is also determined in this paper. However in case of group semi rings LG over distributive lattices it is impossible to find units or S-unit in LG\G.

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