There are infinitely many primes of the form n^2+1

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It is trivial to note that n has to be even in order that n^2+1 could be odd.

It is also easy to see that 3 can not be a factor of n^2+1 when n is even. In fact, for n even, n^2+1 is either congruent to 2 modulus 3 when 3 is not a factor of n (as for 5, 17, 65, 101, 197 and so on) or congruent to 1 modulus 3 when 3 is a factor of n (as for 37, 145 and so on). (In fact it happens that the possible even factors of n^2+1 are only primes of the form 4k+1).

On the other side, 5 is instead able to sieve n^2+1 infinitely many times, namely for $n=10k\pm2$, but still leaving infinitely many more n^2+1 candidate to be prime.

Let us focus only on the n=10k: it is sufficient to prove that there are infinitely many primes of the form $100k^2+1$ in order to prove the conjecture in the title.

Obviously they can not be sieved be 5. If we consider the higher primes that could sieve them, we notice that every one, p_i, of them sieves $100k^2+1$ twice every 10 p_i. For instance, for p_1 =13, this is true for $100k^2+1=(10(h13\pm6))^2$, where h goes form 0 to infinity; for p_2 =17, this is true for $100k^2+1=(10(h17\pm3))^2$.

This means that $(p_i -2)/ p_i$ times each of the infinite primes p_i (of the form 4k+1) greater than 5 is not able to sieve $100k^2+1$: 13 is not able to sieve them 11 times out of every 13; 17 is not able to sieve them 15 times out of every 17; and so on.

This also means that both p_i and p_j are jointly not able to sieve them $(p_i -2)(p_j -2)/ p_i p_j$ times: for instance, 13 and 17 are jointly not able to sieve them 11*15/13*17 times, and so on.

Then, all the infinitely many involved primes are jointly not able to sieve them as many times as in the productory of such fractions: $\prod_i (p_i -2)/ p_i$.

When i tends to the infinite, both the numerator and the denominator jointly tend together to the infinite with the same strength, even if the numerator is much smaller, the fraction tending slower (each fraction tends to increase toward 1 while i is increasing) and slower to 0 without reaching it at any finite value of i.

Thus, as also the numerator goes to the infinity with the same strength of the denominator, there will be infinitely many naturals of the form $100k^2+1$ which are not sieved by any of the infinite possible primes, and thus they are themselves primes.

Thus the conjecture mentioned in the title is true.