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# Smarandache-R-Module and Mcrita Context

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**Abstract**: In this paper we introduced Smarandache-2-algebraic structure of $R$-module namely Smarandache-$R$-module. A Smarandache-2-algebraic structure on a set $N$ means a weak algebraic structure $A0$ on $N$ such that there exist a proper subset $M$ of $N$, which is embedded with a stronger algebraic structure $A1$, stronger algebraic structure means satisfying more axioms, by proper subset one understands a subset from the empty set, from the unit element if any, from the whole set. We define Smarandache-$R$-module and obtain some of its characterization through $S$-algebra and Morita context. For basic concept we refer to Raul Padilla.

**Key Words**: $R$-module, Smarandache-$R$-module, $S$-algebras, Morita context and Cauchy modules.

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## §1. Preliminaries

**Definition** 1.1 *Let $S$ be any field. An $S$-algebra $A$ is an $(R, R)$-bimodule together with module morphisms $\mu : A\otimes_R A \to A$ and $\eta : R \to A$ called multiplication and unit linear maps respectively such that*

$$A \otimes_R A \otimes_R A \underset{1_A\otimes\mu}{\overset{\mu\otimes 1_A}{\Longrightarrow}} A \otimes_R A \xrightarrow{\mu} A \text{ with } \mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu) \text{ and}$$

$$R \underset{1_A\otimes\eta}{\overset{\eta\otimes 1_A}{\Longrightarrow}} A \otimes_R A \xrightarrow{\mu} A \text{ with } \mu \circ (\eta \otimes 1_A) = \mu \circ (1_A \otimes \eta).$$

**Definition** 1.2 *Let $A$ and $B$ be $S$-algebras. Then $f : A \to B$ is an $S$-algebra homomorphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.*

**Definition** 1.3 *Let $S$ be a commutative field with $1_R$ and $A$ an $S$-algebra $M$ is said to be a left $A$-module if for a natural map $\pi : A \otimes_R M \to M$, we have $\pi \circ (1_A \otimes \pi) = \pi \circ (\mu \otimes 1_M)$.*

**Definition** 1.4 *Let $S$ be a commutative field. An $S$-coalgebra is an $(R, R)$-bimodule $C$ with $R$-linear maps $\Delta : C \to C\otimes_R C$ and $\varepsilon : C \to R$, called comultiplication and counit respectively such*

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*that* $C \xrightarrow{\Delta} C \otimes_R C \underset{\Delta\otimes 1_C}{\overset{1_C\otimes\Delta}{\rightrightarrows}} C \otimes_R C \otimes_R C$ *with* $(1_C \otimes \Delta) \circ \Delta = (\Delta \otimes 1_C) \circ \Delta$ *and* $C \xrightarrow{\Delta} C \otimes_R C \underset{\varepsilon\otimes 1_C}{\overset{1_C\otimes\varepsilon}{\rightrightarrows}} R$ *with* $(1_C \otimes \varepsilon) \circ \Delta = 1_C = (\varepsilon \otimes 1_C) \circ \Delta$.

**Definition** 1.5 *Let $C$ and $D$ be $S$-coalgebras. A coalgebra morphism $f : C \to D$ is a module morphism if it satisfies $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\varepsilon_D \circ f = \varepsilon_C$.*

**Definition** 1.6 *Let $A$ be an $S$-algebra and $C$ an $S$-coalgebra. Then the convolution product is defined by $f * g = \mu \circ (f \otimes g) \circ \Delta$ with $1Hom_R(C, A) = \eta \circ \in (1_R)$ for all $f, g \in Hom_R(C, A)$.*

**Definition** 1.6 *For a commutative field $S$, an $S$-bialgebra $B$ is an $R$-module which is an algebra $(B, \mu, \eta)$ and a coalgebra $(B, \Delta, \varepsilon)$ such that $\Delta$ and $\varepsilon$ are algebra morphisms or equivalently $\mu$ and $\eta$ are coalgebra morphisms.*

**Definition** 1.7 *Let $R, S$ be fields and $M$ an $(R, S)$-bimodule. Then, $M^* = Hom_R(M, R)$ is an $(S, R)$-bimodule and for every left $R$-module $L$, there is a canonical module morphism $\alpha_L^M : M^* \otimes_R L \to Hom_R(M, L)$ defined by $\alpha_L^M(m^* \otimes l)(m) = m^*(m)l$ for all $m \in M, m^* \in M^*$ and $l \in L$. If $\alpha_L^M$ is an isomorphism for each left $R$-module $L$, then ${}_RM_S$ is called a Cauchy module.*

**Definition** 1.8 *Let $R, S$ be fields with multiplicative identities $M$, an $(S, R)$-bimodule and $N$, an $(R, S)$-bimodule. Then the six-tuple datum $K = [R, S, M, N, \langle,\rangle_R, \langle,\rangle_S]$ is said to be a Morita context if the maps $\langle,\rangle_R : N \otimes_S M \to R$ and $\langle,\rangle_S : M \otimes_R N \to S$ are binmodule morphisms satisfying the following associativity conditions:*

$$m'\langle n, m\rangle_R = \langle m', n\rangle s\ m \text{ and } \langle n, m\rangle Rn' = n\langle m, n'\rangle s$$

*$\langle,\rangle_R$ and $\langle,\rangle_S$ are called the Morita maps.*

## §2. Smarandache-$R$-Modules

**Definition** 2.1 *A Smarandache-R-module is defined to be such an $R$-module that there exists a proper subset $A$ of $R$ which is an $S$-Algebra with respect to the same induced operations of $R$.*

## §3. Results

**Theorem** 3.1 *Let $R$ be a $R$-module. There exists a proper subset $A$ of $R$ which is an $S$-coalgebra iff $A^*$ is an $S$-algebra.*

*Proof* Let us assume $A^*$ is an $S$-algebra. For proving that $A$ is an $S$-coalgebra we check the counit conditions as follows:

$$\varepsilon : A \approxeq A \otimes_R S \xrightarrow{1_A\otimes\mu} A \otimes_R A^* \xrightarrow{\psi_A} R.$$

Next, we check the counit condition as follows:

$$\begin{aligned}\Delta : A \quad \approxeq \quad & A \otimes_R S \otimes_R S \overset{1_A \otimes_\eta End_S(A)}{\longrightarrow} A \otimes_R (A^* \otimes_R A) \otimes_R A^* \\ & \overset{1_A \otimes_A \otimes_A^*}{\longrightarrow} (A \otimes_R A) \otimes_R (A \otimes_R A)^* \overset{1_A \otimes_R A}{\longrightarrow} (A \otimes_R A) \otimes_R (A \otimes_R A)^* \\ & \overset{1_A \otimes_A}{\longrightarrow} (A \otimes_R A) \otimes_R A^* \overset{\approxeq}{\longrightarrow} (A \otimes_R A) \otimes_R A \overset{\approxeq}{\longrightarrow} A \otimes_R A \overset{\approxeq}{\longrightarrow} A.\end{aligned}$$

Thus, $A$ is an $S$-coalgebra.

Conversely, Let us assume $A$ is an $S$-coalgebra. Now to prove that $A^*$ is an $S$-algebra, we check the unit conditions as follows

$$\eta : R \overset{\eta End_S(A)}{\longrightarrow} A \otimes_R A^* \to 1_A \otimes A \overset{\approxeq}{\longrightarrow} A.$$

We check the multiplication conditions as follows $A$ is a Cauchy module. Notice that

$$\begin{aligned}& A \otimes_R A \to R, \\ & A \approxeq A \otimes_R A \otimes_R R \overset{1_A \otimes_\eta End_S(A)}{\longrightarrow} A \otimes_R A \otimes_R A^* \to R \otimes_R A^* \overset{\approxeq}{\longrightarrow} A^*, \\ & \mu : A \otimes_R A \overset{\varepsilon \otimes 1_A}{\longrightarrow} A^* \otimes_R A \overset{\approxeq}{\longrightarrow} R \otimes_R A^* \overset{\approxeq}{\longrightarrow} A^*.\end{aligned}$$

Thus, $A^*$ is an $S$-algebra. By definition, $R$ is a smarandache $R$-module. □

**Theorem** 3.2 *Let $R$ be an $R$-module. Then there exists a proper subset $End_S(M)^*$ of $R$ which is an $S$-algebra.*

*Proof* Let us assume that $R$ be an $R$-module. For proving that $End_S(M)$ is an $S$-coalgebra which satisfies multiplication and unit conditions $\mu : End_S(M) \otimes_R End_S(M) \to End_S(M)$ and $\eta : R \to End_S(M)$, we check the comultiplication condition as follows:

$$\Delta : End_S(M) \approxeq End_S(M) \otimes_R \overset{1_{End(M)} \otimes_n}{\longrightarrow} End_S(M) \otimes_R End_S(M).$$

Next, we check the counit conditions as follows:

$$\begin{aligned}\varepsilon : End_S(M) \quad \approxeq \quad & End_S(M) \otimes_R R \overset{1_{End(M)} \otimes_\eta}{\longrightarrow} End_S(M) \otimes_R End_S(M) \\ & \overset{\approxeq \otimes \approxeq}{\longrightarrow} Hom_R(M, M) \otimes_R Hom_R(M, M) \\ & \overset{\approxeq \otimes \approxeq}{\longrightarrow} (M' \otimes_R M) \otimes_R (M' \otimes_R M) \overset{\psi M \otimes_\psi M}{\longrightarrow} R \otimes_R R \overset{\approxeq}{\longrightarrow} R.\end{aligned}$$

Thus $End_S(M)$ is an $S$-coalgebra. By Theorem 3.1, $End_S(M)^*$ is an $S$-algebra. Hence, $R$ is a Smarandache $R$-module. □

**Theorem** 3.3 *Let $R$ be an $R$-module. Then there exists a proper subset $M \otimes_R M^*$ of $R$ which is an $S$-algebra.*

*Proof* For proving that $M \otimes_R M^*$ is an $S$-algebra, we check the multiplication and unit conditions as follows:

$$\begin{aligned}\mu : (M \otimes_R M^*) \otimes (M \otimes_R M^*) &\xrightarrow{\cong} M \otimes_R (M^* \otimes_R M) \otimes_R M^* \\ &\xrightarrow{1_M \psi_M \otimes 1_M} M \otimes_R R \otimes_R M^* \\ &\xrightarrow{1_M \otimes \psi_M} M \otimes_R M^*.\end{aligned}$$

As $M$ is a Cauchy module, we have

$$\eta : R \to End_S(M) \xrightarrow{\cong} M \otimes_R M^*,$$

which implies that $M \otimes_R M^*$ is an $S$-algebra. Hence, $R$ is a Smarandache $R$-module. □

**Theorem** 3.4 *Let $R$ be an $R$-module. Then there exists a proper subset the datum $[R, M, N, \langle,\rangle_R]$ a morita context $(M \otimes_R N)^*$ of $R$ which is an $S$-algebra.*

*Proof* Let us assume that $R$ be an $R$-module. For proving that $M \otimes_R N$ is an $S$-algebra, we have

$$\begin{aligned}\mu : (M \otimes_R N) \otimes_R (M \otimes_R N) &\rightarrow M \otimes_R (N \otimes_R M) \otimes_R N \\ &\xrightarrow{1_M \otimes \langle,\rangle \otimes 1_N} M \otimes_R R \otimes_R N \xrightarrow{\cong} M \otimes_R N,\end{aligned}$$

which shows that the multiplication condition is satisfied.

Also, since $M$ and $N$ are Cauchy $R$-modules, there exist maps

$$\eta End_R(M) : R \to M^* \otimes_R M \quad \text{and} \quad \eta End_S(N) : R \to N^* \otimes_R N$$

that can be used to prove the unit condition as follows:

$$\begin{aligned}\eta : R \cong R \otimes_R R &\xrightarrow{\eta End_S(M) \otimes \eta End_S(N)} (M^* \otimes_R M) \otimes_R (N^* \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} (M^* \otimes_R N^*) \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} (M \otimes_R N)^* \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} R^* \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong} R \otimes_R (M \otimes_R N) \xrightarrow{\cong} (M \otimes_R N),\end{aligned}$$

which implies that $M \otimes_R N$ is an $S$-algebra. By definition, $R$ is a Smarandache $R$-module.

**Theorem** 3.5 *Let $R$ be an $R$-module. Then there exists a proper subset the datum $[R, M, N, \langle,\rangle R]$ a morita context $M \otimes_R N$ of $R$ which is an $S$-coalgebra.*

*Proof* Let us assume that $R$ be an $R$-module. For proving that $(M \otimes_R N)$ is an $S$-coalgebra, we have

$$\begin{aligned}
\Delta : M \otimes_R N \quad & = \quad (M \otimes_R N) \otimes_R (R \otimes_R) \\
& \overset{1_{M\otimes N}\otimes \eta End_S(M) \otimes \eta End_S(N)}{\longrightarrow} \quad (M^* \otimes_R M) \otimes_R (N^* \otimes_R N) \\
& \overset{1_{M\otimes N}\otimes \cong}{\longrightarrow} \quad (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R (M^* \otimes_R N^*) \\
& \overset{1_{M\otimes N}\otimes \cong}{\longrightarrow} \quad (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R (M \otimes_R N)^* \\
& \overset{1_{M\otimes N}\otimes \langle , \rangle R^*}{\longrightarrow} \quad (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R R \\
& \overset{\cong}{\longrightarrow} \quad (M \otimes_R N) \otimes_R (M \otimes_R N).
\end{aligned}$$

Also, we have the counit condition as follows:

$$\begin{aligned}
\varepsilon : M \otimes_R N \quad & \cong \quad (M \otimes_R N) \otimes_R R \overset{1_{M\otimes N}\otimes \eta End_S(M)}{\longrightarrow} (M \otimes_R N) \otimes_R (M^* \otimes_R M) \\
& \overset{\langle . \rangle_R \otimes 1_{M^* \otimes M}}{\longrightarrow} \quad R \otimes_R M^* \otimes_R M \overset{\cong}{\longrightarrow} M^* \otimes_R M \overset{\psi_M}{\longrightarrow} R,
\end{aligned}$$

which implies that $\Longrightarrow$ $M \otimes_R N$ is an $S$-coalgebra. Hence, $R$ is a Smarandache $R$-module. □

**Theorem** 3.6 *Let $R$ be an $R$-module. Then there exists a proper subset the datum $[R, M, N, \langle , \rangle R]$ a Morita context iff $M \otimes_R N$ is an $S$-bialgebra.*

*Proof* First, if $M \otimes_R N$ is an $S$-bialgebra by Theorem 3.5, we know that $M \otimes_R N$ is an $S$-algebra and $M \otimes_R N$ is an $S$-coalgebra. Hence by definition, $R$ is a Smarandache $R$-module.

If $M \otimes_R N$ is an $S$-bialgebra, we have the map

$$\varepsilon = \langle . \rangle_R : M \otimes_R N \to R.$$

Associativity of the map $\varepsilon = \langle , \rangle_R$ holds because the diagram

$$\begin{array}{ccc}
(M \otimes_R N) \otimes_R M & \overset{\cong}{\longrightarrow} & M \otimes_R (N \otimes_R M) \\
\varepsilon \otimes 1_M \searrow & & \swarrow 1_M \otimes \varepsilon \\
& \mathrm{M} &
\end{array}$$

is commutative. Hence the datum $[R, M, N, \langle , \rangle R]$ is a Morita context. □

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