The Bianchi Identities in Weyl Space

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Abstract

As far as the writer is aware, the Bianchi identities associated with a Weyl space have never been presented. That space was discovered by the noted German mathematical physicist Hermann Weyl in 1918, and represented the geometry underlying a tantalizing theory that appeared to successfully unify the gravitational and electromagnetic fields. One of theory's problems involved one form of the Bianchi identities, which in Riemannian space are used to derive the divergenceless Einstein tensor $G^{\mu\nu}$. Such a derivation is generally not applicable in a non-Riemannian geometry like Weyl's, in which the covariant derivative of the metric tensor is non-zero. But it turns out that such a derivation is not only possible but quite straightforward, with a result that hints at a fundamental relationship between Weyl's geometry and electromagnetism.

1. Preliminaries

In 1918 the German mathematical physicist Hermann Weyl proposed a unification of gravitation and electromagnetism based on the invariance of physics with respect to a conformal (or scale) transformation of the metric tensor $g_{\mu\nu} \to \exp(\pi) g_{\mu\nu}$, where $\pi(x)$ is an arbitrary scalar function. That invariance led Weyl to a formalism involving a non-vanishing covariant derivative of the metric tensor (called the *non-metricity tensor*), which he determined to be

$$g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha \tag{1.1}$$

where

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha}$$

Here the double-bar and single-bar subscripts represent covariant differentiation and ordinary partial differentiation, respectively, and the $\Gamma^\lambda_{\mu\nu}$ quantities are the Weyl connection coefficients (not to be confused with the Christoffel symbols). This formalism represents a geometry that is today known as a *Weyl space*. The quantity ϕ_α represents a new field which Weyl subsequently identified as the electromagnetic 4-potential.

Using Weyl's definition for $g_{\mu\nu||\alpha}$ in (1.1) is it an easy matter to show that

$$g^{\mu\nu}{}_{||\alpha} = -2g^{\mu\nu}\phi_\alpha, \quad \left(\sqrt{-g}\right)_{||\alpha} = 4\sqrt{-g}\,\phi_\alpha \tag{1.2}$$

where $\sqrt{-g}$ is the determinant of the metric tensor in four dimensions. We thus have the identity

$$\left(\sqrt{-g}\,g^{\mu\nu}g^{\alpha\beta}\right)_{||\lambda} = 0 \tag{1.3}$$

which will be of use later on.

It is important to note that the traditional Bianchi identities of ordinary Riemannian geometry, given by

$$R^\lambda{}_{\mu\nu\alpha} + R^\lambda{}_{\alpha\mu\nu} + R^\lambda{}_{\nu\alpha\mu} = 0, \tag{1.4}$$

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0, \tag{1.5}$$

$$R^\lambda{}_{\mu\nu\alpha||\beta} + R^\lambda{}_{\mu\beta\nu||\alpha} + R^\lambda{}_{\mu\alpha\beta||\nu} = 0 \tag{1.6}$$

are also valid in a Weyl space, where $R^\lambda{}_{\mu\nu\alpha}$ is the Riemann-Christoffel curvature tensor given by

$$R^\lambda{}_{\mu\nu\alpha} = \Gamma^\lambda_{\mu\nu|\alpha} - \Gamma^\lambda_{\mu\alpha|\nu} + \Gamma^\lambda_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\lambda_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{1.7}$$

In Weyl space this tensor is a complicated mixture of the Christoffel symbols, the metric tensor and the Weyl field ϕ_μ. Fortunately, we will not be needing it.

2. Derivation of the Einstein Tensor from the Bianchi Identities

A common exercise for students is to use the set of Bianchi identities in (1.6) to show that the covariant divergence of the Einstein tensor $G^{\alpha\beta} = R^{\alpha\beta} - 1/2 g^{\alpha\beta} R$ vanishes. In Riemannian geometry $g_{\mu\nu||\alpha}$ is zero, allowing us to pull the metric tensor inside the covariant differentiation process for index raising and lowering purposes. We also have the indispensible identity

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} \tag{2.1}$$

Using this identity, setting $\beta = \lambda$ in (1.5) and multiplying by $g^{\mu\nu}$, it is a simple matter to show that

$$\left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right)_{||\beta} = 0 \tag{2.2}$$

from which the traditional Einstein field equations for free space

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = 0 \tag{2.3}$$

are obtained.

3. Consequences of Non-Metricity

The Riemann-Christoffel tensor is most conveniently derived via the identity

$$\xi_{\mu||\alpha||\beta} - \xi_{\mu||\beta||\alpha} = -\xi_\lambda R^\lambda{}_{\mu\alpha\beta}$$

where ξ_λ is an arbitrary rank 1 tensor. This can be extended to tensors of higher rank; for the metric tensor itself, this can be written as

$$g_{\mu\nu||\alpha||\beta} - g_{\mu\nu||\beta||\alpha} = -g_{\mu\lambda} R^\lambda{}_{\nu\alpha\beta} - g_{\lambda\nu} R^\lambda{}_{\nu\alpha\beta} \tag{3.1}$$

or

$$g_{\mu\nu||\alpha||\beta} - g_{\mu\nu||\beta||\alpha} = -R_{\mu\nu\alpha\beta} - R_{\nu\mu\alpha\beta} \tag{3.2}$$

Thus, in a space in which the non-metricity tensor does not vanish the identity in (2.1) is no longer valid, and a straightforward derivation of (2.2) cannot be performed. It's also complicated by the fact that the metric tensor cannot be pulled into the covariant differentiation process without having to deal with terms like $g^{\mu\nu}{}_{||\alpha}$.

4. The Bianchi Identities in Weyl Space

Fortunately, Weyl's definitions for the non-metricity tensor and its determinant are simple enough that we can replicate most of the steps used in the derivation of (2.2) without difficulty. To begin we note that, for a Weyl space, (3.2) reduces to

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} - 2g_{\mu\nu} F_{\alpha\beta} \tag{4.1}$$

where we define $F_{\alpha\beta} = \phi_{\alpha||\beta} - \phi_{\beta||\alpha}$. We now use (1.6) with the contraction $\beta = \lambda$, which we write as

$$\left(g^{\lambda\kappa} R_{\kappa\mu\nu\alpha}\right)_{||\lambda} + R_{\mu\nu||\alpha} - R_{\mu\alpha||\nu} = 0 \tag{4.2}$$

where we have used the identity $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = -R^\lambda{}_{\mu\nu\lambda}$ by virtue of the contraction properties of the curvature tensor. We now multiply (4.2) by the quantity $\sqrt{-g}\, g^{\mu\nu} g^{\alpha\beta}$ and use Weyl's definitions in (1.1) and (1.2) to pull these quantities into (4.2). Equation (4.1) now allows us to raise the indices of $R_{\mu\kappa\nu\alpha}$, which will result in various terms involving the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar $R = g^{\alpha\beta} R_{\alpha\beta}$. The rest is basically just a lot of algebra, which is simplified considerably using (1.3). After some straightforward simplification, index relabeling and reduction of the remaining terms, it is easy to show that

$$\left[\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2} g^{\mu\nu} R\right)\right]_{||\alpha} = \left(\sqrt{-g} F^{\beta\alpha}\right)_{||\alpha} \tag{4.3}$$

Note now that the term on the right-hand side is identical to the source density of the electromagnetic field, or $\sqrt{-g}S^\beta = \left(\sqrt{-g}F^{\beta\alpha}\right)_{||\alpha}$, leaving us with the tensor density divergence expression

$$\left[\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\right]_{||\alpha} = \sqrt{-g}S^\beta \tag{4.4}$$

which reduces to (2.2) in the absence of the Weyl field ϕ_α.

(The reader will note that the simplicity of this expression is the result of bringing the metric determinant into the analysis.) This is a most interesting result, considering its simplicity and the apparent connection of Weyl's formalism to electrodynamics, but what does it mean? Ordinarily, one associates the Einstein tensor with the mass-energy tensor $T^{\alpha\beta}$ which, for the electromagnetic field, is given by

$$T^{\alpha\beta} = F^{\alpha\lambda}F_\lambda^\beta - \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}$$

The covariant divergence of both the Einstein and mass-energy tensors is assumed to vanish, which conventionally is interpreted as a conservation law for these quantities. Here the notion of conservation seems to be turned on its head. True, the covariant divergence of the source vector density vanishes (implying conservation of electric charge), but how can the same be true of the left-hand side of (4.4)?

5. Discussion

Although Einstein showed Weyl's 1918 theory to be non-physical, the theory frequently appears in the literature today, usually in the context of dark matter and dark energy theories, while any connection it may have to electromagnetism is generally ignored. Nevertheless, it is fascinating that the theory continues to intimate a fundamental relationship between geometry and the electromagnetic field. Indeed, in Weyl's original theory he was able to derive a connection between the Ricci scalar and the electromagnetic source vector from a conformally invariant action principle, which resulted in the identification

$$S^\beta = kg^{\alpha\beta}\left(R\phi_\alpha + \frac{1}{2}R_{|\alpha}\right)$$

with k a constant.

As is well known, a consistent interpretation of gravitational energy conservation via the vanishing divergence of the Einstein tensor is problematic. This is due at least in part to the fact that a gravitational field contains energy, which in turn generates an additional gravitational field that can act on itself, a conclusion that is apparent from the highly nonlinear nature of the Einstein field equations themselves. It should therefore come as no surprise that theories of gravitational radiation, which should at least superficially resemble Maxwell's equations, have historically relied on weak-field approximations of the field equations. Exactly how electrodynamics influences gravity (and vice versa) will likely not be known until a consistent and workable quantum gravity theory presents itself.

Strangely, it appears that Weyl himself never explored the derivation of the Einstein tensor from the Bianchi identities using his theory. If he had done so, perhaps he would have been much amused.

Reference

R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*. McGraw-Hill, 2nd Edition, 1975.