THE SIGNIFICANCE OF THE NON-TRIVIAL ZEROS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT

This paper expounds the role of the non-trivial zeros of the Riemann zeta function ζ and supplements the author's earlier papers on the Riemann hypothesis. There is a lot of mystery surrounding the non-trivial zeros.

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To understand what Riemann wanted to achieve with the non-trivial zeros, we need to understand the part played by the complex plane.

First, the terms in the Riemann zeta function ζ:-

$$\zeta(s) = \sum_{n=1} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots$$

where s is the complex number $1/2 + bi$

For the term $1/2^{1/2 + bi}$ above, e.g., whether it would be positive or negative in value would depend on which part of the complex plane this term $1/2^{1/2 + bi}$ would be found in, which depends on $2(n)$ and b (it does not depend on 1/2 – 1/2 and $2(n)$ only determine how far the term is from zero in the complex plane). This term could be in the positive half (wherein the term would have a positive value) or the negative half (wherein the term would have a negative value) of the complex plane. Thus, some of the terms in the Riemann zeta function ζ would have positive values while the rest have negative values (depending on the values of n and b). The sum of the series in the Riemann zeta function ζ is calculated with a formula, e.g., the Riemann-Siegel formula, or, the Euler-Maclaurin summation formula.

Riemann evidently anticipated that there would be an equal, or, almost equal number of primes among the terms in the positive half and the negative half of the complex plane when there is a zero. In other words, he thought that the distribution of the primes would be statistically fair, the more terms are added to the Riemann zeta function ζ, the fairer or "more equal" would be the distribution of the primes in the positive half and the negative half of the complex plane when there is a zero. (Compare: The tossing of a coin wherein the more tosses there are the "more equal" would be the number of heads and the number of tails.) That is, in the longer term, with more and more terms added to the Riemann zeta function ζ, more or less 50% of the primes should be found in the positive half of the complex plane and the balance 50% should be found in the negative half of the complex plane, the more terms there are the fairer or "more equal" would be this distribution, when there is a zero.

A non-trivial zero indicates the point in the Riemann zeta function ζ wherein the total value of the positive terms equals the total value of the negative terms. There is an infinitude of such points, i.e., non-trivial zeros. Riemann evidently thought that for the case of a zero the number of primes found among the positive terms would be more or less equal to the number of primes found among the negative terms, which represents statistical fairness. It is evident that through a zero the order or pattern of the distribution of the primes could be observed.

Next, the error term in the following J function for calculating the number of primes less than a given quantity:-

$$J(n) = Li(n) - \sum_p Li(n^p) - \log 2 + \int_n^{\infty} dt/(t(t^2 - 1) \log t)$$

where the 1st. term $Li(n)$ is generally referred to as the "principal term" and the 2nd. term $\sum_p Li(n^p)$ had been called the "periodic terms" by Riemann, Li being the logarithmic integral

$\sum_p Li(n^p)$, the secondary term of the function, the error term, represents the sum taken over all the non-trivial zeros of the Riemann zeta function ζ. n here is a real number raised to the power of p, which is in this instance a complex number of the form $1/2 + bi$, for some real number b, $n^{1/2}$ being $\sqrt{n}$. If the Riemann hypothesis is true, for a given number n, when computing the values of n^p for a number of different zeta zeros p, the numbers we obtain are scattered round the circumference of a circle of radius $\sqrt{n}$ in the complex plane, centred on zero, and are either in the positive half or negative half of the complex plane.

To evaluate $\sum_p Li(n^p)$ each zeta zero has to be paired with its mirror image, i.e., complex conjugate, in the south half of the argument plane. These pairs have to be taken in ascending order of the positive imaginary parts as follows:-

zeta zero: $1/2 + 14.134725i$ & its complex conjugate: $1/2 - 14.134725i$
zeta zero: $1/2 + 21.022040i$ & its complex conjugate: $1/2 - 21.022040i$
zeta zero: $1/2 + 25.010858i$ & its complex conjugate: $1/2 - 25.010858i$

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(Note: The complex conjugates are all also zeros.)

If, e.g., we let $n = 100$, then the error term for $n = 100$ would be $\sum_p Li(100^p)$. To calculate this error term, we have to first raise 100 to the power of a long list zeta zeros in ascending order of the positive imaginary parts (the 1st. 3 zeta zeros are shown above), the longer the list of zeta zeros the better, e.g., 100,000 zeta zeros, in order to achieve the highest possible accuracy in the error term. Then we take the logarithmic integrals of the above powers (100,000 pairs of zeta zeros & their complex conjugates) and add them up, which is as follows:-

$$100^{1/2 + 14.134725i} + 100^{1/2 - 14.134725i}$$

$$+ 100^{1/2 + 21.022040i} + 100^{1/2 - 21.022040i}$$
$$+ 100^{1/2 + 25.010858i} + 100^{1/2 - 25.010858i}$$

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The imaginary parts of the zeta zeros would cancel out the imaginary parts of their complex conjugates, leaving behind their respective real parts. For example, for the 1st. zeta zero 1/2 + 14.134725*i*, its imaginary part +14.134725*i* would cancel out the imaginary part -14.134725*i* in its complex conjugate 1/2 - 14.134725*i*, leaving behind only the real parts $100^{1/2}$ for each of them. That is, for $100^{1/2 + 14.134725i} + 100^{1/2 - 14.134725i}$, we only have to add together the logarithmic integral of $100^{1/2}$(from $100^{1/2 + 14.134725i}$) and the logarithmic integral of $100^{1/2}$(from $100^{1/2 - 14.134725i}$) to get the 1st. term. The same is to be carried out for the next 99,999 powers in ascending order of the positive imaginary parts, giving altogether a total of 200,000 logarithmic integrals (of both the zeta zeros & their complex conjugates) to be added together to give the 100,000 terms. These terms have either positive or negative values, an equal or almost equal number of positive and negative values, which depend on whether they are in the positive or negative half of the complex plane, as is described above. The positive values and the negative values of these 100,000 terms are added together and should cancel out each other, slowly converging. The difference between the positive values and the negative values of these 100,000 terms constitutes the error term. (Note that the Riemann hypothesis asserts that the difference between the true number of primes $p(n)$ and the estimated number of primes $q(n)$ would be not much larger than $\sqrt{n}$ – not much larger than $\sqrt{100}$ ($\sqrt{100}$ is also expressed as $100^{1/2}$) in the above case. Like the case of tossing a coin wherein the statistical probability is that in the long run the number of heads would practically equal the number of tails, there should be equal or almost equal quantities of positive terms and negative terms, i.e., 50,000 or thereabout positive terms and 50,000 or thereabout negative terms, which would be statistically fair, the discrepancy if any being the error.)

All this is evidently a laborious process, though the ingenuity of the ideas behind the Riemann hypothesis should be acknowledged.

Someone might ask the following question: Why go through all this trouble to find the exact number of primes less than a given quantity when the sieve of Eratosthenes could actually provide the exact answer without any error at all?

REFERENCES

[1] H. M. Edwards, 2001, Riemann's Zeta Function, Dover Publications, Inc.

[2] G. H. Hardy and E. M. Wright, 1979, An Introduction To Theory Of Numbers, Oxford, England: Clarendon Press

[3] B. B. Mandelbrot, 1977, The Fractal Geometry Of Nature, W. H. Freeman

[4] K. Mathew, 2012, Logic: Its Proper Use [How To Think Logically] Volume 1 & Volume 2, www.Amazon.com

[5] B. Riemann, 1859, On The Number Of Prime Numbers Less Than A Given Quantity, Berlin Academy of Sciences