A kappa deformed Clifford Algebra, Hopf Algebras and Quantum Gravity

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Abstract

Explicit deformations of the Lorentz (Conformal) algebra are performed by recurring to Clifford algebras. In particular, deformations of the boosts generators are possible which still retain the form of the Lorentz algebra. In this case there is an invariant value of the energy that is set to be equal to the Planck energy. A discussion of Clifford-Hopf κ-deformed quantum Poincare algebra follows. To finalize we provide further deformations of the Clifford geometric product based on Moyal star products associated with noncommutative spacetime coordinates.

Keywords: Clifford Algebras, Relativity, κ-deformed Poincare algebra.

Sometime ago Magueijo and Smolin [5] proposed a modification of special relativity in which a physical energy, which may be the Planck energy, joins the speed of light as an invariant. This was accomplished by a *nonlinear* modification of the action of the Lorentz group on momentum space, generated by adding a dilatation to each boost in such a way that the Planck energy remains invariant. The associated algebra has unmodified structure constants, and they highlighted the similarities between the group action found and a transformation previously proposed by Fock [6].

In this work we shall take a different approach and construct deformations of the Lorentz (Conformal) algebra by recurring solely to Clifford algebras and leading to an invariant value of the energy. A discussion of Clifford-Hopf κ-deformed quantum Poincare algebra follows based on the work by [2].

We begin by reviewing [7] how the conformal algebra in four dimensions admits a Clifford algebra realization; i.e. the generators of the conformal algebra can be expressed in terms of the Clifford algebra basis generators. The conformal algebra in four dimensions $so(4,2)$ is isomorphic to the $su(2,2)$ algebra.

Let $\eta_{ab} = (-,+,+,+)$ be the Minkowski spacetime (flat) metric in $D = 3+1$-dimenisons. The epsilon tensors are defined as $\epsilon_{0123} = -\epsilon^{0123} = 1$, The real Clifford $Cl(3,1,R)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by the anticommutators

$$\{ \Gamma_a, \ \Gamma_b \} \ \equiv \ \Gamma_a \ \Gamma_b \ + \ \Gamma_b \ \Gamma_a \ = \ \ = \ 2 \ \eta_{ab} \tag{1.1a}$$

such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab}, \quad \Gamma_5 \ = \ - \ i \ \Gamma_0 \ \Gamma_1 \ \Gamma_2 \ \Gamma_3, \quad (\Gamma_5)^2 \ = \ 1; \quad \{\Gamma_5, \Gamma_a\} \ = \ 0; \tag{1.1b}$$

$$\Gamma_{abcd} \ = \ \epsilon_{abcd} \ \Gamma_5; \quad \Gamma_{ab} = \frac{1}{2}(\Gamma_a \Gamma_b - \Gamma_b \Gamma_a). \tag{1.2a}$$

$$\Gamma_{abc} \ = \ \epsilon_{abcd} \ \Gamma_5 \ \Gamma^d; \quad \Gamma_{abcd} = \epsilon_{abcd} \ \Gamma_5. \tag{1.2b}$$

$$\Gamma_a \ \Gamma_b \ = \ \Gamma_{ab} + \eta_{ab}, \quad \Gamma_{ab} \ \Gamma_5 \ = \ \frac{1}{2} \epsilon_{abcd} \ \Gamma^{cd}, \tag{1.2c}$$

$$\Gamma_{ab} \ \Gamma_c \ = \ \eta_{bc} \ \Gamma_a - \eta_{ac} \ \Gamma_b + \epsilon_{abcd} \ \Gamma_5 \ \Gamma^d \tag{1.2d}$$

$$\Gamma_c \ \Gamma_{ab} \ = \ \eta_{ac} \ \Gamma_b - \eta_{bc} \ \Gamma_a + \epsilon_{abcd} \ \Gamma_5 \ \Gamma^d \tag{1.2e}$$

$$\Gamma_a \ \Gamma_b \ \Gamma_c \ = \ \eta_{ab} \ \Gamma_c + \eta_{bc} \ \Gamma_a - \eta_{ac} \Gamma_b + \epsilon_{abcd} \ \Gamma_5 \ \Gamma^d \tag{1.2f}$$

$$\Gamma^{ab} \ \Gamma_{cd} \ = \ \epsilon^{ab}{}_{cd} \ \Gamma_5 \ - 4\delta^{[a}_{[c} \ \Gamma^{b]}{}_{d]} \ - 2\delta^{ab}_{cd}. \tag{1.2g}$$

$$\delta^{ab}_{cd} \ = \ \frac{1}{2} \ (\delta^a_c \ \delta^b_d \ - \ \delta^a_d \ \delta^b_c \ ). \tag{1.2.h}$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas.

At this stage we may provide the relation among the $Cl(3,1)$ algebra generators and the the conformal algebra $so(4,2) \sim su(2,2)$ in $4D$. It is well known to the experts that the operators of the Conformal algebra can be written in terms of the Clifford algebra generators as

$$P_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ - \ \Gamma_5); \quad K_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ + \ \Gamma_5); \quad D \ = \ - \ \frac{1}{2} \ \Gamma_5, \quad L_{ab} \ = \ \frac{1}{2} \ \Gamma_{ab}. \tag{1.3}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. In order to match the physical dimensions of momentum in (1.3) a mass parameter should be introduced. For convenience it is set to unity as well as $c = 1$. The total number of generators is respectively $4+4+1+6 = 15$. From the above realization of the conformal algebra generators (1.3), the explicit evaluation of the commutators yields

$$[P_a, \ D] \ = \ P_a; \quad [K_a, \ D] \ = \ - \ K_a; \quad [P_a, \ K_b] \ = \ - \ 2g_{ab} \ D \ + \ 2 \ L_{ab}$$

$$[P_a,\ P_b]\ =\ 0;\quad [K_a, K_b]\ =\ 0$$

$$[L_{ab},\ L_{cd}]\ =\ g_{bc}\ L_{ad}\ -\ g_{ac}\ L_{bd}\ +\ g_{ad}\ L_{bc}\ -\ g_{bd}\ L_{ac},\ \ldots \tag{1.4}$$

which is consistent with the $su(2,2) \sim so(4,2)$ commutation relations. We should notice that the K_a, P_a generators in (1.3) are both comprised of Hermitian Γ_a *and* anti-Hermitian $\pm\Gamma_a\Gamma_5$ generators, respectively. The dilation D operator is Hermitian, while the Lorentz generator L_{ab} is anti-Hermitian. The fact that Hermitian and anti-Hermitian generators are required is consistent with the fact that $U(2,2)$ is a pseudo-unitary group

If one wishes to deform the Clifford generators Γ_A one may choose for deformation generator the following

$$F = \frac{P_0}{2\kappa}\ \Gamma_5 \tag{1.5}$$

such that upon exponentiation it yields

$$\tilde{\Gamma}_A\ =\ e^F\ \Gamma_A\ e^{-F}\ =\ e^{[F,\ ]}\ \Gamma_A\ =$$

$$\Gamma_A\ +\ [\ F,\ \Gamma_A\ ]\ +\ \frac{1}{2!}\ [\ F,\ [\ F,\ \Gamma_A\ ]\ ] + \frac{1}{3!}\ [\ F,\ [\ F,\ [\ F,\ \Gamma_A\ ]\ ]\ ] + \ldots \tag{1.6}$$

The first order deformations of the Lorentz generators $\Gamma_{\mu\nu}$ are

$$\tilde{\Gamma}_{\mu\nu}\ =\ \Gamma_{\mu\nu}\ +\ [\frac{P_0}{2\kappa}\ \Gamma_5, \Gamma_{\mu\nu}]\ =\ \Gamma_{\mu\nu}\ +\ \frac{1}{2\kappa}\ (\ \eta_{\mu 0}P_\nu\ -\ \eta_{\nu 0}P_\mu\ )\Gamma_5 \tag{1.7}$$

From eq-(1.7) one can infer that $\tilde{\Gamma}_{ij} = \Gamma_{ij}$ so that the rotation generators remain undeformed to first oder. Due to the vanishing commutator $[P_0\ \Gamma_5,\ \Gamma_{ij}] = 0$ the higher order contributions remain zero and the rotation generators remain undeformed to *all* orders $\Gamma_{ij} = \tilde{\Gamma}_{ij}$. The second order contributions to the deformed boosts are

$$\frac{1}{4\kappa^2}\ [\ P_0\ \Gamma_5\ ,\ [\ P_0\ \Gamma_5,\ \Gamma_{0i}\ ]\ ] = \frac{1}{4\kappa^2}[\ P_0\ \Gamma_5,\ P_i\ \Gamma_5\ ]\ =\ 0 \tag{1.8}$$

and similar findings occur with the higher order nested commutators. Therefore, the higher order contributions to the deformed boost generators are *zero* and one has then that the deformed boosts are given by

$$\tilde{\Gamma}_{0i}\ =\ \Gamma_{0i}\ -\ \frac{1}{2\kappa}\ P_i\ \Gamma_5 \tag{1.9}$$

and the following commutator becomes

$$[\tilde{\Gamma}_{0i},\ P_0]\ =\ P_i\ (1 - \frac{P_0}{\kappa}) \tag{1.10}$$

so that $P_0 = \kappa$ is an *invariant* energy under deformed boosts because the commutator (1.10) *vanishes* when $P_0 = \kappa$. The deformed boosts (1.9) can also be rewritten in terms of the dilatation generator $D = -\frac{1}{2}\Gamma_5$ of eq-(1.3) as $\tilde{\Gamma}_{0i} = \Gamma_{0i}\ +\ \frac{1}{\kappa}\ P_i\ D$ in agreement with the results of [5].

From eqs-(1.6, 1.7,1.9) one learns that

$$[\tilde{\Gamma}_{0i},\ \tilde{\Gamma}_{0j}] \ = \ \Gamma_{ij} = \tilde{\Gamma}_{ij} \tag{1.11}$$

and

$$[\tilde{\Gamma}_{ij},\ \tilde{\Gamma}_{0k}] \ = \ [\Gamma_{ij},\ \tilde{\Gamma}_{0k}] \ = \ - \ \eta_{ik}\ \Gamma_{0j} \ + \ \eta_{jk}\ \Gamma_{0i} \ + \frac{1}{2\kappa}\ \eta_{ik} P_j \Gamma_5 \ - \ \frac{1}{2\kappa}\ \eta_{jk} P_i \Gamma_5 =$$

$$\eta_{jk}\ \tilde{\Gamma}_{0i} \ - \ \eta_{ik}\ \tilde{\Gamma}_{0j} \tag{1.12}$$

such that the Lorentz algebra (1.11, 1.12) remains *unmodified* despite having deformed the boost generators in eq-(1.9).

The procedure in this work is based entirely on Clifford algebras and differs from the approach made by [5]. Many other deformations of the boosts/rotation generators are possible. Some of these deformations will not affect the Lorentz (Conformal) algebra while others will also deform the Lorentz (Conformal) algebra. Let us provide examples where the Lorentz (Conformal) algebra is also modified. Choosing for instance the following exponential operator

$$e^F \ = \ e^{P_0 \Gamma_{03}/\kappa} \tag{1.13}$$

it leads to

$$[\ \tilde{\Gamma}_{03},\ P_3\ ] \ = \ [\ \Gamma_{03} \ + \ \frac{P_3}{\kappa}\ \Gamma_{30},\ P_3\ ] \ = \ P_0\ (1 - \frac{P_3}{\kappa}) \tag{1.14}$$

Let us study the *higher order* contributions to the *deformed* boost generators $\tilde{\Gamma}_{0i}, i = 1,2,3$. Given

$$(\Gamma_0)^2 = -1,\ (\Gamma_1)^2 = (\Gamma_2)^2 = (\Gamma_3)^2 = 1,\ \ (\Gamma_5)^2 \ = \ 1,\ \ \{\Gamma_5,\ \Gamma_a\} \ = \ 0 \tag{1.15}$$

one can show that

$$(P_0)^2 = \frac{1}{4}\ (\Gamma_0 - \Gamma_0\Gamma_5)\ (\Gamma_0 - \Gamma_0\Gamma_5) \ = \ 0 \tag{1.16}$$

and similarly

$$(P_1)^2 \ = \ (P_2)^2 \ = \ (P_3)^2 \ = \ 0 \tag{1.17}$$

consequently the Clifford algebraic realization of the conformal algebra generators (1.3) yields *nilpotent* momentum generators. As a result of the nilpotent conditions of eqs-(1.16,1.17) the higher order contributions to the deformed boost generators are *zero*

$$\frac{1}{2\kappa^2}\ [\ P_0\ \Gamma_{03}\ ,\ [\ P_0\ \Gamma_{03},\ \Gamma_{03}\ ]\ ] \ = \ 0 \tag{1.18}$$

and similar findings occur with the higher order nested commutators. Hence the deformed boost generator is given by $\tilde{\Gamma}_{03} = \Gamma_{03} - \frac{P_3}{\kappa}\Gamma_{03}$, and the deformed commutator to all orders becomes

$$[\ \tilde{\Gamma}_{03},\ P_3\ ]\ =\ [\ \Gamma_{03}\ -\ \frac{P_3}{\kappa}\ \Gamma_{03},\ P_3\ ]\ =\ P_0\ (1-\frac{P_3}{\kappa}) \tag{1.19}$$

such that $P_3 = \kappa$ is now an *invariant* momentum value under the *deformed* boost generators $\tilde{\Gamma}_{03}$ to all orders since the commutator (1.19) vanishes when $P_3 = \kappa$. Despite that there is an invariant momentum, there is now no condition on the energy because

$$[\ \tilde{\Gamma}_{03},\ P_0\ ]\ =\ P_3 \tag{1.20}$$

as it occurs under ordinary Lorentz boost transformations.

Conversely, one can find a deformed boost generator such that there is an invariant energy given by κ, with *no* no condition on the momentum. For example, let us choose in this case $F = e^{P_3\Gamma_{03}/\kappa}$ instead of $F = e^{P_0\Gamma_{03}/\kappa}$. Repeating the same process with the new value of F one gets

$$[\ \tilde{\Gamma}_{03},\ P_0\ ]\ =\ P_3\ (1-\frac{P_0}{\kappa}) \tag{1.21}$$

such that the commutator (1.21) vanishes when $P_0 = \kappa$ leading to an *invariant* energy $P_0 = \kappa$, with *no* condition on the momentum since

$$[\ \tilde{\Gamma}_{03},\ P_3\ ]\ =\ P_0 \tag{1.22}$$

as it occurs under ordinary Lorentz boost transformations. The *key* difference among these three examples is that only in the case of deformed boost generators given by eq-(1.5, 1.9) the Lorentz algebra (1.11, 1.12) still remains *unmodified*.

Next we will discuss why the procedure in this work (based entirely on Clifford algebras) differs from the work [5]. Besides the nilpotent momentum generator conditions given by eqs-(1.16, 1.17) one can also show that

$$P_a\ P_b\ =\ \frac{1}{4}\ (\Gamma_a - \Gamma_a\Gamma_5)\ (\Gamma_b - \Gamma_b\Gamma_5)\ =\ 0 \tag{1.23}$$

after using eq-(1.15). Consequently the commutator in eq-(1.10) will then *reduce* to P_i, as in ordinary boost transformations, since the term $P_iP_0 = 0$ in eq-(1.23) vanishes. The latter result is also compatible with the condition $P_i\Gamma_5 = -P_i$ resulting from the momentum operators realization in eq-(1.3) and leading to a deformed boost $\tilde{\Gamma}_{0i} = \Gamma_{0i} - \frac{1}{2\kappa}P_i\Gamma_5 = \Gamma_{0i} + \frac{1}{2\kappa}P_i$ given by a *linear* combination of boosts and momentum generators.

Hence one must distinguish the momentum *operator* realization of P_a in terms of Clifford algebra generators and the approach taken by [5] ; i.e. the operators P_0, P_1, P_2, P_3 are realized in terms 4×4 matrices associated with the Clifford algebra generators instead of being the four momentum components corresponding to the momentum four-vector $p_a = (p_o, p_1, p_2, p_3)$ in momentum space. Upper case letters P_a belong to the momentum operators realized in terms of Clifford generators (matrices) while lower case letters p_a belong to the momentum four-vector.

The latter approach by [5] recurs to a *nonlinear* modification of the action of the Lorentz group on momentum space, generated by adding a dilatation to each boost in such a way that the Planck energy remains invariant. The deformed boost (nonlinear in the momentum) is given in terms of the dilatation generator in momentum space $D = p^j \frac{\partial}{\partial p_j}$ as $\tilde{L}_{0i} = L_{0i} + \frac{1}{\kappa} p_i D$. This should be contrasted with our Clifford algebraic realization $\tilde{\Gamma}_{0i} = \Gamma_{0i} - \frac{1}{2\kappa} P_i \Gamma_5 = \Gamma_{0i} + \frac{1}{\kappa} P_i D = \Gamma_{0i} + \frac{1}{2\kappa} P_i$. A recent study of Lorentz invariant deformations of momentum space can be found in [13].

Many approaches have been taken in order to formulate a consistent theory of quantum gravity. At very high energies the gravitational effects can no longer be neglected and spacetime is no longer a smooth manifold, but a fuzzy, noncommutative space [9]. Physical theories on such noncommutative manifolds require a new framework provided by noncommutative geometry [10]. In this framework, the search for generalized (quantum) symmetries that leave the physical action invariant leads to deformations of the Poincare symmetry, with κ-Poincare symmetry being among the most extensively studied [3].

Let us discuss briefly these deformations of the Lorentz (Poincare algebra) from the Clifford algebras perspective. The authors [2] formulated the κ-Poincare algebra as a quantum Clifford-Hopf algebra, using the Wick isomorphism that relates quantum Clifford algebras to their respective standard Clifford algebras. The Minkowski spacetime quantum Clifford algebra structure associated with the conformal group and the Clifford-Hopf alternative κ-deformed quantum Poincare algebra was investigated by [2]. The resulting algebra $\mathcal{U}_q(SO(2,3))$ turned out to be equivalent to the deformed anti-de Sitter algebra.

Quantum deformations of the Poincare algebra were introduced by [3] and followed by the doubly special relativity (DSR), which contains two observer-independent parameters : the velocity of light and the Planck length [4]. The DSR framework coincides with the algebraic structure of the Poincare algebra κ-deformation, where the deformation parameter κ is related to the Planck mass.

Quantum Clifford algebras are denoted as $Cl(V, B)$, where B is an arbitrary bilinear *not* necessarily symmetric form, and have been investigated in [8]. Let $B = g + A$, where $A = \frac{1}{2}(B - B^T)$ is the antisymmetric piece of the bilinear form. The B-dependent Clifford product is defined as

$$\gamma_\mu \, \gamma_\nu \; = \; (\; g_{\mu\nu} \; + \; A_{\mu\nu} \;) \; \mathbf{1} \; + \; \gamma_\mu \wedge \gamma_\nu \tag{1.24}$$

resulting in a *deformed* exterior product $\tilde{\wedge}$ given by

$$\gamma_\mu \; \tilde{\wedge} \; \gamma_\nu \; = \; A_{\mu\nu} \; \mathbf{1} \; + \; \gamma_\mu \wedge \gamma_\nu, \quad A_{\mu\nu} = -A_{\nu\mu} \tag{1.25}$$

For example, the $B = g + A$ dependent Clifford product $\gamma_\mu \gamma_\nu \gamma_\rho$ will now have the extra terms $A_{\mu\nu}\gamma_\rho, A_{\mu\rho}\gamma_\nu - A_{\nu\rho}\gamma_\mu$ in addition to the $g_{\mu\nu}\gamma_\rho, g_{\mu\rho}\gamma_\nu - g_{\nu\rho}\gamma_\mu$ and $\gamma_{\mu\nu\rho}$ terms. Hence the algebra $Cl(V, B)$ clearly differs from $Cl(V, g)$

It is possible to express every antisymmetric bilinear form $A(u, v)$ as the contraction of $u \wedge v$ with F where F is an appropriately chosen grade-two element of the exterior algebra $\Lambda^2(V)$. The Wick isomorphism between the Quantum Clifford algebra $Cl(V, B)$ and the ordinary Clifford algebra $Cl(V, g)$ given by [2]

$$Cl(V, B) \; = \; e_\wedge^{-F} \; \wedge \; Cl(V, g) \; \wedge \; e_\wedge^F \tag{1.26}$$

where the outer exponential is defined as

$$e^F_\wedge \;=\; 1 \;+F \;+\frac{1}{2}\; F\wedge F \;+\frac{1}{3!}\; F\wedge F\wedge F \;+\; ..... \tag{1.27}$$

and which is a *finite* series when dim V is finite. Full details of the construction of κ-Poincare algebra as a quantum Clifford-Hopf algebra, using the Wick isomorphism (1.26) can be found in [2]. Quantum Clifford algebras are essential tools which allow us to construct quantum deformations of the (Conformal) Poincare algebra which has been postulated as a fundamental symmetry in the theory of quantum gravity. Other examples of quantum Clifford-Hopf algebras were found by [11] as the hidden quantum group symmetry of the eight vertex free fermion model.

To finalize we shall provide further deformations of the Clifford geometric product based on Moyal star products associated with noncommutative spacetime coordinates. The associative and noncommutative Moyal star product when the (inverse) symplectic form $\Omega^{\mu\nu} = -\Omega^{\mu\nu}$ does *not* have an X-dependence is defined as

$$(\;A_1 \;*\; A_2\;)(Z) \;=\; exp\left(\;\frac{1}{2}\;\Omega^{\mu\nu}\;\partial_{X^\mu}\;\partial_{Y^\nu}\;\right)\;A_1(X)\;A_2(Y)|_{X=Y=Z} \;=$$

$$\sum_{n=0}^{\infty}\;\frac{(\frac{1}{2})^n}{n!}\;\Omega^{\mu_1\nu_1}\;\Omega^{\mu_2\nu_2}\;..........\;\Omega^{\mu_n\nu_n}\;(\partial^n_{\mu_1\mu_2......\mu_n}\;A_1)\;(\partial^n_{\nu_1\nu_2......\nu_n}\;A_2) \tag{1.28}$$

$$\partial^n_{\mu_1\mu_2......\mu_n}\;A_1(Z)\;\equiv\;\partial_{\mu_1}\;\partial_{\mu_2}\;......\;\partial_{\mu_n}\;A_1(Z). \tag{1.29a}$$

$$\partial^n_{\nu_1\nu_2......\nu_n}\;A_2(Z)\;\equiv\;\partial_{\nu_1}\;\partial_{\nu_2}\;......\;\partial_{\nu_n}\;A_2(Z). \tag{1.29b}$$

For simplicity we shall take the very special case of canonical noncommutativity of the spacetime coordinates $[x^\mu, x^\nu]_* = i\Theta^{\mu\nu} = \Omega^{\mu\nu} = constants$, such that the star product is the standard Moyal one. In the particular case when $\Theta^{01} = L_P^2 = \kappa^{-2} \neq 0$, and the rest of the $\Theta^{\mu\nu}$ components are set to zero, one can construct a star deformed Clifford product by replacing the derivatives in eq-(1.28) by momentum operators. Due to the (nilpotent) conditions in eqs-(1.16, 1.17, 1.23) one has in this particular case

$$\gamma_a \;*\; \gamma_b \;=\; \gamma_a\;\gamma_b \;+\frac{i}{2\kappa^2}\;[\;(P_0\;\gamma_a)\;(P_1\;\gamma_b)\;-\;(P_1\;\gamma_a)\;(P_0\;\gamma_b)\;] \tag{1.30}$$

where $P_0 = \frac{1}{2}\gamma_0(1-\gamma_5)$; $P_1 = \frac{1}{2}\gamma_1(1-\gamma_5)$. The deformed commutator is $[\gamma_a, \gamma_b]_* = \gamma_a \;*\; \gamma_b - \gamma_b \;*\; \gamma_a$. In the case of $D = 2$ the product $\gamma_a \;*\; \gamma_b$ becomes $\gamma_a\;\gamma_b$ and the deformation is trivially zero. It remains to verify whether or not the deformed commutator satisfies the Jacobi identities in higher dimensions than two.

Other star deformed Clifford products can be constructed using ordinary derivatives. In curved spacetime backgrounds one writes $\gamma_\mu = e^a_\mu(x)\gamma_a$ in terms of the vielbeins $e^a_\mu(x)$ and the tangent space Clifford algebra generators γ_a. The star deformed Clifford product is given by

$$\gamma_\mu \;*\; \gamma_\nu \;=\; (e^a_\mu(x) * e^b_\nu(x))\;\gamma_a\;\gamma_b \;=\; \gamma_\mu\;\gamma_\nu \;+\frac{i}{2\kappa^2}\;\Theta^{\alpha\beta}\;(\partial_\alpha e^a_\mu(x))\;(\partial_\beta e^b_\nu(x)) \;+\; \ldots \tag{1.31}$$

In flat spacetimes the product (1.31) reduces to the ordinary one, whereas the product (1.30) does have first order corrections in powers of $1/\kappa^2$ in contrast to powers of $1/\kappa$ in κ-deformed Poincare algebras. Moyal Deformations of Clifford Gauge Theories of Gravity based on the Seiberg-Witten map can be found in [12].

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