Lectures on Affine, Hyperbolic and Quantum Algebras

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Abstract

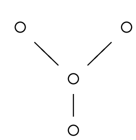
Many suspect that exceptional structures play a pivotal role in color physics and in gravity, along with the nonassociative octonions. On the other hand, nonassociativity belongs to the axioms of higher dimensional categories with a tensor product. This nonassociativity disappears in low dimensional field theories, for which the special modular tensor categories are strict [1]. A non trivial braiding exists for a generic representation category. Starting with the classification of complex Lie algebras, we see how the indefinite metrics of spacetime are associated to exceptional structure. The octonions are introduced in the context of affine algebras.

1 Overview

It has long been considered that color $SU(3)$ and quark confinement depend on the nonassociative octonions $\mathbb{O}$ [2][3][4][5][6][7][8]. The complex exceptional Lie algebras are all related to the octonions $\mathbb{O}$. For instance, in [9][10][11] one obtains a real form of the group E_6 as the group $SL(3,\mathbb{O})$, the 3×3 matrices of determinant 1 over $\mathbb{O}$. Such matrices act on the 3×3 Hermitian elements $\mathbf{p}$ of the exceptional Jordan algebra $\mathcal{H}_3(\mathbb{O})$. For the three octonion off diagonal elements of $\mathbf{p}$, say $(\overline{a},\overline{b},c)$, this action [9] is

$$(a,b,c)\mapsto(\overline{q}a\overline{q},bq,qc) \tag{1}$$

for a unit octonion q. These are exactly the one vector and two spinor 8 dimensional representations of $SO(8)$, called respectively V, S^+ and S^- [6]. The special S_3 action on the Dynkin diagram



for $SO(8)$ is known as *triality*. For infinite dimensional affine algebras, there is no such Dynkin diagram. Rather, the Dynkin triangle gives the affine form of $SU(3)$, and there is also an S_3 symmetry for affine $\mathbf{E}_6{}^{(1)}$. The affine polygons for $SU(N)$ are required to label the braid group generators for B_N.

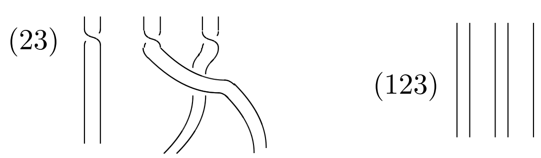
The restriction to the 2×2 matrices in $SL(2, \mathbb{O})$ defines $SO(9,1)$, the Lorentz group in $D = 10$ dimensions [10][12]. This critical dimension appears in infinite dimensional algebras, as described below. The $9+1$ dimensions are reduced to $3+1$ when $SL(2,\mathbb{O})$ is reduced to the double cover of the Lorentz group by selecting a complex subalgebra. This is the usual action of $SL(2,\mathbb{C})$ on the Hermitian elements

$$P = \begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix} \tag{2}$$

of Minkowski spacetime $\mathbb{R}^{1,3}$. Observe that we would require a pure imaginary X, Y and Z in order to interpret P as a quaternion in $\mathbb{H}$. Classical spacetime points are just inherently non multiplicative, generating a translation group $\mathbf{T}$ using only addition. The Poincare group, whose representations characterise rest mass and spin, is the semidirect product of $SL(2,\mathbb{C})$ and $\mathbf{T}$.

The X, Y and Z directions are naturally associated to the positive definite metrics of ordinary Lie algebras. In a quantum setting, one wants to work with the category of representations of an algebra $\mathfrak{g}$. Such a category is properly constructed using the associative universal enveloping algebra $U(\mathfrak{g})$, which is a Hopf algebra. In a true quantum representation category, a permutation action on N objects must be deformed to an action of the braid group B_N. Elements of S_3 now look like the ribbon graphs

(3)



where the half twist appears because the *permutation representation* of S_3 really lives in S_6. This is a finite analogue of an adjoint representation. A cycle (231) can contain a full ribbon twist on each strand, if the ribbon twists match the braiding in B_3. The braid group B_3, associated to $SU(3)$, covers the modular group $SL(2,\mathbb{Z})$. This modular group is canonically represented in a quantum representation category.

We start with the Dynkin classification of Lie algebras of ADE type. First, let the single node for $SU(2)$ (actually $sl(2)$) represent a set of spinors $\mathbb{CP}^1$. The Dynkin triangle for affine $SU(3)$ gives three copies of this $\mathbb{CP}^1$, each pair intersecting in a line. This is a basis for $\mathbb{R}^3$, with the triangle representing the origin. Morally, space begins with affine Dynkin diagrams.

The duality between points and planes underlies the diagrams of a *modular tensor category* [13][14], which is a category with a braiding and ribbon twists. The ambient plane stands for a single zero dimensional object. A braid strand is a one dimensional object, going from the left hand side of the plane to the right [15]. A generic map between two such strands is a pointlike node that divides them, from top to bottom.

For classical Hopf algebras, such as the universal enveloping algebra of a finite dimensional Lie algebra, the braiding in the representation category turns out to be *symmetric*, returning braids in B_n to their underlying permutations. Another example of a classical Hopf algebra is the group algebra $\mathbb{R}G$ of a finite group G. Even the octonions $\mathbb{O}$ can be made to look associative, as a ribbon vertex inside the symmetric monoidal category for $\mathbb{R}(\mathbb{F}_2^3)$ [16][6]. In fact, this parity cube $\mathbb{F}_2^3$ is instrumental in *defining* the associator laws of a symmetric monoidal category. However, there is no reason to assume symmetry when dealing with fermions, and the usual categorical axioms are easily broken in higher dimensions.

One would like to assign twisted ribbon graphs to the massless charged leptons, with one full twist on three ribbon strands specifying a unit of electromagnetic charge [9][17], and the cycles (231) and (213) determining chirality for electrons and positrons. These fermions emerge from the axiomatic structure of quantum representation categories, without resorting to a Lagrangian formalism. For affine algebras, one considers deformation parameters $t \neq 0, \pm 1$ for $U_t(\mathfrak{g})$ [18][19][20], the quantum version of the universal enveloping algebra $U(\mathfrak{g})$. In a nice modular tensor category, the value of t is entirely determined by finiteness conditions on the representations.

Beyond low dimensional field theories, braid strands are themselves nonassociative, as if representing a single octonion rather than an eight dimensional representation $\mathbb{O}$. To see how closely octonions are related to Lie algebras, we must study the fundamental root lattices of an ADE algebra, given in lecture 3. Lecture 2 gives the classification of ADE algebras, not necessarily of finite dimension. After some categorical preliminaries we look at affine algebras in lecture 5 and finally quantum algebras in lecture 6.

Hyperbolic algebras are, by definition, associated to metrics of signature $(1, n)$. They are neither finite nor affine. The octonionic form of (2) hides the $(1, 9)$ metric of spacetime, which is related to the hyperbolic $\mathbf{E}_{10}$ diagram (see Figure 1). As discussed in [21], the integer elements of $SL(2, \mathbb{O})$ form a lattice $\mathcal{E}_{10}$, which is a Lorentzian extension of the $\mathbf{E}_8$ root lattice. This is the $\mathbf{E}_{10}$ lattice. The 24 dimensional Leech lattice comes from the integral elements of the Jordan algebra $\mathcal{H}_3(\mathbb{O})$ [22][21]. This may be extended to Lorentzian lattices in dimensions 26 and 27.

2 Dynkin Diagrams of ADE Type

For algebras of finite type, a primitive *root system* R is a finite subset of the Euclidean space $\mathbb{R}^n$ with inner product. For any non zero $\alpha \in R$, the *coroot* $\alpha^\vee$ is the vector

$$\alpha^\vee = \frac{2\alpha}{\alpha \cdot \alpha}. \tag{4}$$

We begin with the ADE case, where all basis roots are normalised to length $\alpha \cdot \alpha = 2$. A root α is *simple* if it cannot be written as an integer combination $\sum x_i \alpha_i$ of other primitive roots, and there is a basis of simple roots.

Now let $\langle \alpha, \beta \rangle$ stand for $\alpha \cdot \beta^\vee$. Then a reflection orthogonal to α is the map

$$\rho \mapsto \rho - \langle \rho, \alpha \rangle \alpha. \tag{5}$$

for any root ρ. These reflections fix the root system R, so that $\pm\alpha$ are the only primitive roots lying on a line. For α and β both in R, the number $\langle \alpha, \beta \rangle$ is an integer. Hence

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \tag{6}$$

is an integer, given by $4\cos^2\theta$, where θ is the angle between α and β. Observe that the angle between two simple roots cannot be acute, so that $\cos\theta$ is negative. By inspection,

Lemma 2.1. *The positive integers p with $p \geq 2$ for which*

$$4\cos^2(2\pi/(p+1)) \in \{0, 1, 2, 3\}$$

are $p \in \{2, 3, 5, 7, 11\}$.

These are the Galois primes, associated to roots of unity ω_{p+1}. Roots also occur for the relative angles $3\pi/4$ and $5\pi/6$.

Example 2.2 The $SU(2)$ root system has one simple root, $\sqrt{2}$. The $SU(3)$ root system in $\mathbb{R}^2$ has two simple roots, $\alpha = (\sqrt{2}, 0)$ and $\beta = (-\sqrt{2}^{-1}, \sqrt{3}\sqrt{2}^{-1})$.

Definition 2.3 The *Cartan matrix A* associated to an ADE root system is a symmetric matrix defined by $A_{ij} = \langle \alpha_i, \alpha_j \rangle$ for a set of simple roots α_i.

By normalisation, the diagonal entries of A are 2. In the case of a finite type algebra, all principal minors must be positive, so that the off diagonal entries in

$$\begin{pmatrix} 2 & A_{12} \\ A_{21} & 2 \end{pmatrix} \tag{7}$$

must obey $A_{ij} A_{ji} = {A_{ij}}^2 \leq 3$. But this forces $A_{ij} \in \{0, -1\}$, since simple roots are not relatively acute.

We will now classify all possible Cartan matrices A using the associated Dynkin diagrams. This includes non finite algebras. For an $n \times n$ Cartan matrix, there are n nodes in the diagram. Two distinct nodes i and j are joined by a line only if the entry $A_{ij} = -1$.

Consider first a diagram with at most one ternary node. For example, the $SO(8)$ Dynkin diagram in the introduction has three lines of length 2 coming into the central node. We call this $SO(8)$ graph a $(p, q, r) = (2, 2, 2)$ graph. There is a ternary graph of type (p, q, r) for any $p, q, r \geq 2$, where we assume that $p \geq q \geq r$. Allowing also length 1 lines, the ternary graphs include the $SU(N)$ graphs of type $(N - 1, 1, 1)$. The single node is the $(1, 1, 1)$ graph. A useful number is [23]

$$D(p, q, r) \equiv \frac{1}{p} + \frac{1}{q} + \frac{1}{r}. \tag{8}$$

Lemma 2.4. *For a ternary graph, the determinant of A equals*

$$pqr(D(p, q, r) - 1).$$

Proof: For the $SU(N)$ matrices, the determinant $N = p + q$ follows by induction on N. By the symmetry of ternary graphs, the general determinant must be a combination

$$a(p + q + r) + b(pq + qr + pr) + cpqr$$

for some integers a, b and c. The case $r = 1$ forces $c = -b$ and $a = 1 - b$, and we fix $b = 1$ using $(2, 2, 2)$.

A connected Dynkin graph is of finite type only if it contains no triangles, since

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{9}$$

has zero determinant. Similarly, any polygon defines a matrix A whose rank is less than maximal, and hence of zero determinant. Altogether, there are three basic types [23] of ADE diagram:

1. Finite: When ternary, $D(p, q, r) > 1$.

2. Affine: When ternary, $D(p, q, r) = 1$. In general, A has zero determinant. An affine graph extends a finite type graph by only one node.

3. Hyperbolic: Every proper subgraph is either of finite or affine type. In the ternary case, $D(p, q, r) < 1$. A hyperbolic $n \times n$ matrix A has signature $(1, n - 1)$.

Theorem 2.5. *All possible connected Dynkin diagrams of finite, affine or hyperbolic ADE type are listed in Figure 1.*

Proof: The series for $SU(N)$ is obvious. Consider other finite ternary graphs. When $q = r = 2$, we obtain the $SO(2N)$ graphs. Otherwise $p, q \geq 3$, so that $p^{-1} + q^{-1} \leq 2/3$. From $D(p,q,r) > 1$ it then follows that

$$1 + \frac{2}{3}r > r,$$

forcing $r = 2$. If $q \geq 4$, it is not possible to satisfy $D(p,q,r) > 1$, so $q = 3$. Then $p < 6$. The three solutions, $(3,3,2)$, $(4,3,2)$ and $(5,3,2)$, are the exceptional graphs of type $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ respectively. There are only two possible quartic graphs, one for the affine $SO(8)^{(1)}$ and a similar hyperbolic one. Any further extension would be of infinite type. A polygon is clearly affine. Consider now ternary affine graphs. The exceptional solutions to $D(p,q,r) = 1$ are $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$. The affine $SO(2N)$ series for $N \geq 5$ has two distinct ternary nodes, and this is the only new way to extend the finite $SO(2N)$ graphs. Finally, consider hyperbolic graphs. Ternary solutions include $(4,3,3)$, $(5,4,2)$ and $(7,3,2)$. An extended octagon is disallowed since it contains a $(5,4,2)$ subgraph. The extended affine $SO(2N)$ graphs stop at ten nodes, since this contains the affine $\mathbf{E}_8^{(1)}$ graph. The quintic graph on six nodes is allowed, but higher valencies would have hyperbolic subgraphs.

Affine and hyperbolic $SU(2)$ are optional extras. The full Cartan classification for finite dimensional Lie algebras also includes some asymmetric matrices A, when some of the A_{ij} equal -2 or -3. This includes the series $SO(2N+1)$, a series of quaternion algebras, and two exceptional diagrams, known as G_2 and F_4, which are closely related to the octonions and the exceptional Jordan algebra.

Lemma 2.6. *There are only two affine Cartan matrices such that $A_{ij} = -3$ for some $i \neq j$. These are known as affine G_2 and ${D_4}^{(3)}$.*

Proof: The matrices

$$\begin{pmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -3 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

have negative determinant, and are therefore hyperbolic.

Definition 2.7 Given a primitive root system of ADE type with simple roots $\{\alpha_i\}$, the *root lattice* Q is the set of all multiples $\sum_i k_i \alpha_i$ with integer k_i. Q is a sublattice of the *weight lattice* P, which is the set of all β such that

$$\langle \beta, \alpha \rangle$$

is an integer, for all roots α.

Example 2.8 The dual lattice for $SU(3)$, with simple roots α and β as in (2), is given by integer combinations of

$$\nu = \frac{\alpha - \beta}{3}$$

and either β or α.

Next we look at how to build the root lattice for $\mathbf{E}_8$ using four copies of the $SU(3)$ lattice.

3 The Arithmetic of Lattices

The divisor function $\sigma_p(n)$ on positive integers is defined by

$$\sigma_p(n) = \sum_{d|n} d^p \tag{10}$$

where the sum is over all divisors d of n including 1. For any odd $p \geq 2$ these numbers are collected into an *Eisenstein series* E_{p+1}. In particular,

$$E_4 = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + \cdots \tag{11}$$

$$E_6 = -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + 3126q^5 + \cdots$$

$$E_{12} = \frac{691}{130 \cdot 504} + q + 2049q^2 + 177148q^3 + 4196353q^4 + 48828126 \cdots$$

E_4 and E_6 form a basis for modular forms for the modular group $PSL(2, \mathbb{Z})$ [24], which is generated by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

The first three terms in $\equiv 240E_4$ count the number of root vectors of length 0, 2 and 4 in the root lattice $\mathcal{E}_8$ for $\mathbf{E}_8$, remembering that simple roots are normalised to length 2. Actually, modular forms are completely determined by their first few terms. This is not obvious. Define also the *modular discriminant*

$$\Delta \equiv \sum_n \tau(n) q^n = \frac{240^3 {E_4}^3 - 504^2 {E_6}^2}{12^3} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots \tag{13}$$

Then

$$\theta_{12} \equiv (240E_4)^3 - 720\Delta = 1 + 196560q^2 + 16773120q^3 + \cdots \tag{14}$$

counts the root vectors in the Leech lattice [25][26]. In this case, the minimal vectors have norm 4. Using the complex form of the $SU(3)$ lattice, we will think of $\mathcal{E}_8$ as a lattice in $\mathbb{C}^4$ and the Leech lattice as a lattice in $\mathbb{C}^{12}$.

Definition 3.1 An *integral lattice* $\mathcal{L}$ of rank c is a free Abelian group embedded in $\mathbb{R}^c$ with a bilinear form $\langle , \rangle$ which is $\mathbb{Z}$ valued on $\mathcal{L}$. $\mathcal{L}$ is *even* when $\langle x, x\rangle$ is an even integer for every x in a basis set for $\mathcal{L}$. Given any basis $x_1, \cdots, x_c$ for $\mathcal{L}$, we say that $\mathcal{L}$ is *unimodular* if the determinant of the matrix $\langle x_i, x_j\rangle$ equals 1.

The only even, unimodular lattice of rank 8 is the $\mathcal{E}_8$ lattice. The $SO(16)$ lattice is the even sublattice of the simple square lattice in $\mathbb{R}^c$, but its determinant equals 4. Let us now build the $\mathcal{E}_8$ lattice [27][28]. Let α and β be the simple roots of the $SU(3)$ lattice in $\mathbb{C}$, as in (2), and ν the dual basis vector

$$\nu \equiv \frac{\alpha - \beta}{\sqrt{3}} = \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}} i \tag{15}$$

of length 2/3. There are four $SU(3)$ directions in $\mathbb{C}^4$, and a basis for the $\mathcal{E}_8$ lattice is given using two types of vector: those like $(0, \nu, \nu, \nu)$, and $(0, \beta, 0, 0)$ and its permutations. The trick is to assign signs to the coordinates using the non zero *tetracode* words in ${\mathbb{F}_3}^4$, namely

$$\begin{array}{cc} 0,1,1,1 & 0,-1,-1,-1 \\ 1,1,-1,0 & -1,-1,1,0 \\ 1,-1,0,1 & -1,1,0,-1 \\ 1,0,1,-1 & -1,0,-1,1. \end{array} \tag{16}$$

Observe that for any two codewords with zeroes in different places, there is a sign mismatch in the two remaining places. Therefore,

$$\langle (+\nu, +\nu, -\nu, 0), (0, -\nu, -\nu, -\nu)\rangle = \nu\nu - \nu\nu = 0, \tag{17}$$

and similarly for all other pairs of codewords. The mostly positive words give the four basis vectors $(0, \nu, \nu, \nu)$, $(\nu, \nu, -\nu, 0)$, $(\nu, 0, \nu, -\nu)$ and $(\nu, -\nu, 0, \nu)$. Since $\langle \nu, \beta\rangle = -1$, the matrix $\langle a_i, a_j\rangle$ on the 8 (not simple) basis vectors is

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 2 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}, \tag{18}$$

which has determinant 1. Each of ν, $\nu + \beta$ and $\nu - \alpha$ has norm 2/3. Then

$$\langle \nu, \nu + \beta\rangle = \langle \nu, \nu - \alpha\rangle = -\frac{1}{3}. \tag{19}$$

Thus all vectors u, w in $\mathbb{C}^4$ with three non zero coordinates taking the values ν, $\nu+\beta$ and $\nu-\alpha$ satisfy $\langle u, w\rangle \in \mathbb{Z}$. For instance,

$$\langle(\nu,\nu,\nu,0),(\nu,\nu,\nu+\beta,0)\rangle = \frac{2}{3}+\frac{2}{3}-\frac{1}{3} = 1. \tag{20}$$

There is then a total of $216+24=240$ root vectors of norm 2: three coordinates in three non zero places gives 27 vectors, times 8 for the tetracode signs, and then the 24 roots in the 4 copies of $SU(3)$. This is $\mathcal{E}_8$.

The standard definition of $\mathcal{E}_8$ uses octonion coordinates, considered as 240 basis vectors in $\mathbb{R}^8$ of norm 2. For octonion units e_i with $i=0,1,\cdots,7$, there are 112 vectors of type $\pm(e_i+e_j)$ for distinct i and j. The remaining 128 vectors are of the form

$$(\pm e_0 \pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7)/2$$

with an even number of minus signs, so that all dot products are in $\mathbb{Z}$.

It is easier to define E_8 using the corresponding affine lattice. In an affine root system, there is one additional basis root. The *dual Coxeter* number $h^\vee$ is given by the sum of vertex labels m_i on the Dynkin diagram, as follows. Each vertex on an E type graph is associated to a translation vector a_i, $i=0,\cdots,c$, whose coordinates in $\mathbb{R}^{c+1}$ sum to zero. In this hyperplane, the finite $\mathcal{E}_8$ lattice is recovered. Each multiplicity m_i is a positive integer. The set of m_i satisfies

$$\sum_i m_i a_i = 0 \tag{21}$$

where $m_0=1$. For $\mathcal{E}_9$, the lattice for $\mathbf{E}_8{}^{(1)}$, the Dynkin diagram is

$$\begin{array}{ccccccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 4 & & 2 \\
\circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\
 & & & & & & & & & & | & & & & \\
 & & & & & & & & & & \circ & & & & \\
 & & & & & & & & & & 3 & & & &
\end{array}$$

and $h^\vee = 30$. The corresponding set of a_i are given by

$$a_0 = (1,-1,0,0,0,0,0,0,0), \tag{22}$$

$$a_1 = (0,1,-1,0,0,0,0,0,0), \quad \cdots \quad a_7 = (0,0,0,0,0,0,0,1,-1),$$

$$a_8 = (-1,-1,-1,-1,-1,-1,2,2,2)/3.$$

All vectors are of norm 2 and a_8 sits at the node marked 3. To obtain the affine lattice, the additional vector a_0 is adjusted by a constant vector that is orthogonal to the vectors in $\mathbb{R}^8$,

$$a_0 + (1,1,\cdots,1) = (2,0,1,1,1,1,1,1,1). \tag{23}$$

The vector $a_5 = (0,0,0,0,0,1,-1,0,0)$ sits at the ternary node. A basic triality works on the $\mathbb{R}^3$ vectors $(1,-1,0,0)$, $(0,1,-1,0)$, $(0,0,1,-1)$ and $(-1,-1,1,1)$, with respective multiplicities 1, 2, 1, 1.

Hyperbolic lattices use indefinite metrics. The lattice for $\mathbf{E}_{10}$ lives in $\mathbb{R}^{9,1}$ with the Lorentzian metric [29]. Define this lattice $\mathcal{L}_{9,1}$ to be the set of all $x \in \mathbb{R}^{9,1}$ with either all $\mathbb{Z}$ coordinates or all half integral $\mathbb{Z}+1/2$ coordinates, such that

$$x \cdot u \equiv x \cdot (1,1,\cdots,1)/2 \tag{24}$$

is an integer. From [29],

Theorem 3.2. *The lattice $\mathcal{L}_{9,1}$ has exactly 10 fundamental roots, which are the vectors v such that $v \cdot v = 2$ and $v \cdot w = -1$, where*

$$w \equiv (0,1,2,3,4,5,6,7,8,38).$$

These ten roots are the vectors u and

$$b_0 = (1,-1,0,0,0,0,0,0,0,0), \quad \cdots \quad b_7 = (0,0,0,0,0,0,0,1,-1,0), \tag{25}$$

$$b_9 = (-1,-1,0,0,0,0,0,0,0,0,0),$$

where the last coordinate is the time direction [21]. Take the eight vectors $b_9 + b_0, b_1, b_2, \cdots, b_6, u$. Turn them into vectors in $\mathbb{R}^8$ by lopping off the first and last coordinates. Exercise: prove that these vectors generate the lattice for $\mathcal{E}_8$.

The Lorentzian Leech lattice in $\mathbb{R}^{25,1}$ is defined similarly [29] using the remarkable vector

$$w = (0,1,2,3,4,5,6,7,8,9,10,11,\cdots,24,70) \tag{26}$$

of norm zero. It has an infinite number of fundamental roots. The ordinary Leech lattice $\mathcal{L}_L$ [25][26] in $\mathbb{R}^{24}$ may be defined using octonion triplets [22]. As in (14), the vectors of norm $2n$ in $\mathcal{L}_L$ [27] are counted by

$$\theta_{12} = 240(33600 \cdot {E_4}^3 + 441 \cdot {E_6}^2). \tag{27}$$

The form E_{12} is not independent of E_4 and E_6. For instance, using [30]

$$3\Delta = \frac{65}{252} E_{12} - 691 \cdot {E_6}^2, \tag{28}$$

we have

$$E_{12} = \frac{252 \cdot 50}{13}(96 \cdot {E_4}^3 + {E_6}^2) \tag{29}$$

and

$$\theta_{12} = 240(\frac{1}{3} E_{12} + 84 \cdot {E_6}^2). \tag{30}$$

4 Noncommutativity and Nonassociativity

Let ω be a complex number. In category theory, one talks about turning an asymmetric product

$$ij = \omega ji \tag{31}$$

on special elements of an algebra $\mathcal{A}$ into a symmetric product $i \circ j$. Introduce a complex valued function $\phi(a, b)$ on $\mathcal{A} \times \mathcal{A}$ so that

$$i \circ j = \phi(i, j)ij = \phi(j, i)ji = j \circ i. \tag{32}$$

In particular, distinct anticommuting units satisfy $ij = -ji$, and

$$\phi(i, j) + \phi(j, i) = 0. \tag{33}$$

In the case of non associativity, anticommuting objects reduce the 12 possible length 3 terms down to the three products

$$(ij)k, \qquad (jk)i, \qquad (ki)j \tag{34}$$

for distinct i, j and k. For instance, let i and j be two distinct imaginary units in $\mathbb{O}$ [16][6], so that $i^2 = j^2 = -1$ and $ij = -ji$.

Theorem 4.1. *Let i, j, k be three imaginary units in $\mathbb{O}$ such that all six units*

$$e \in U = \{i, j, k, ij, jk, ki\}$$

are distinct, where $e^2 = -1$. Then the seven units $U \cup -(ij)k$ generate $\mathbb{O}$, and

$$(ij)k = (jk)i = (ki)j.$$

When U has only three distinct elements, the i, j, k define a quaternion subalgebra of $\mathbb{O}$ with $(ij)k = -1$. When U has two elements, the subalgebra is $\mathbb{C}$.

For such anticommuting units, an associator map $(ij)k \to i(jk)$ is just a minus sign. The products in (34) are easier to remember than the Fano plane for $\mathbb{O}$ [6]. The simplest associator rule for $\phi(a, b)$ is

$$\phi(ij, k) + \phi(jk, i) + \phi(ki, j) = 0. \tag{35}$$

The rules (33) and (35) make the function ϕ into a *cocycle* for the algebra $\mathcal{A}$. These cohomological functions are crucial to the definition of an affine Lie algebra.

Let us now consider some basic category theory. Recall that a one dimensional *category* [1] is a collection of objects X, $Y \cdots$ and arrows $f : X \to Y$ such that

1. there exists an identity $1_X : X \to X$ for each object,

2. any pair of arrows $f : X \to Y$ and $g : Y \to Z$ has a composition fg from X to Z,

3. composition is associative.

A *monoid* is a category with only one object, so composition behaves as a product on arrows, and a *group* is a monoid whose arrows all have inverses. A group representation is then a *functor* from the group category to some linear category. By definition, a functor preserves identities and satisfies $F(fg) = F(f)F(g)$. A category with a tensor product $\otimes$ on objects is lifted by one dimension: the object $X \otimes Y$ acts as a one dimensional arrow $X \to Y$, and the arrows become two dimensional arrows f, g, h etc. A $\otimes$ category is not automatically associative, and requires associator arrows

$$a_{XYZ} : X(YZ) \to (XY)Z \tag{36}$$

for every triplet of objects X, Y and Z. Up to dimension three, the associators obey the Mac Lane pentagon [1]

$$\begin{aligned}(1_X \otimes a_{YZT})(a_{X(YZ)T})(a_{XYZ} \otimes 1_T) \\ = (a_{XY(ZT)})(a_{(XY)ZT}).\end{aligned} \tag{37}$$

A *braided monoidal* category is a three dimensional category with only one object and only one 1-arrow, so that the 2-arrows and 3-arrows form a $\otimes$ category, and there exist braiding arrows

$$\gamma_{XY} : X \otimes Y \to Y \otimes X \tag{38}$$

for every pair of 2-arrows X and Y. A braiding generalises anticommutativity. These γ_{XY} obey the hexagon axioms [31],

$$\begin{aligned}a_{XYZ} \cdot \gamma_{X(YZ)} \cdot a_{YZX} \\ = (\gamma_{XY} \otimes 1_Z) \cdot a_{YXZ} \cdot (1_Y \otimes \gamma_{XZ}),\end{aligned} \tag{39}$$

and similarly for a^{-1}. We ignore the additional left and right unit maps [31]. Homework: draw these axioms with objects. A braided $\otimes$ category is *symmetric* if $\gamma_{XY}\gamma_{YX} = 1$ for all X, Y. A general arrow

$$(ij)k \to (jk)i \tag{40}$$

combines an associator and a braiding [15]. When multiplication is an arrow

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \tag{41}$$

associativity is an arrow $m(1 \otimes m) \to (m \otimes 1)m$, often drawn as a square. The Mac Lane pentagon now covers five sides of a cube. This is the correct axiom for a representation category whenever a group has higher dimensional structure, particularly a semidirect product.

5 Affine Algebras

We give a concrete construction of affine algebras, beginning with the traditional definition of the Virasoro algebra. This comes with a *central charge* c. This is the infinite dimensional algebra on a set of generators L_n for $n \in \mathbb{Z}$ with commutators

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{42}$$

The first term on the right hand side follows from defining L_n on the space of formal Laurent polynomials $\mathbb{C}[z, z^{-1}]$. For now, think of L_n as $-z^{n+1} \cdot d/dz$ for $n \in \mathbb{Z}$. This Virasoro algebra occurs naturally alongside the affine version of a simple finite dimensional Lie algebra $\mathfrak{g}$, as follows [23].

The central charge term comes from a certain 2-cocycle ϕ. By definition, this is a bilinear odd function of m, which must satisfy the cocycle condition

$$\phi([f,g],h) + \phi([g,h],f) + \phi([h,f],g) = 0. \tag{43}$$

Since it is only non zero for $m = -n$, for some $a \neq 0$ we get

$$\phi([a, m-a], -m) + \phi([m-a], -m) + \phi([-m, a], m-a) = 0 \tag{44}$$

$$= (2a-m)\phi(m,-m) + \phi(2m-a)\phi(-a,a) - (m+a)\phi(a-m, m-a).$$

These terms only cancel out for ϕ a multiple of either m or m^3. We choose the combination $m^3 - m$ in (42) so that (i) the algebra works nicely in a representation category and (ii) for $c \in \{2,4,6,8,24\}$, this term is always an integer. The choice $c = 8$ for $D = 10$ corresponds to the Lie algebra $\mathbf{E}_8$, while $c = 6$ for $D = 12$ appears with $\mathbf{E}_6$.

Now let A and B be two Laurent polynomials in $\mathbb{C}[z, z^{-1}]$. The *residue* $r(A)$ [23] of a polynomial A is the coefficient of z^{-1}. On the *loop algebra* $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$ for some complex Lie algebra $\mathfrak{g}$, there is a bracket

$$[A \otimes x, B \otimes y] \equiv AB \otimes [x,y]. \tag{45}$$

A 2-cocycle ψ for this algebra uses the bilinear form $(x|y)$ on $\mathfrak{g}$, and is given by [23]

$$\psi(A \otimes x, B \otimes y) = (x|y) \cdot r(\frac{dA}{dt}B). \tag{46}$$

This cocycle extension of the loop algebra introduces another generator, called K. The bracket now looks like

$$[A \otimes x + \lambda_1 K, B \otimes y + \lambda_2 K] = [A \otimes x, B \otimes y] + \psi(A \otimes x, B \otimes y)K, \tag{47}$$

for any $\lambda_i \in \mathbb{C}$. This can be tidied up a lot by putting in the L_0 generator, so that $L_0 K = 0$. On the monomial terms, the final bracket is

$$\begin{aligned} &[z^m \otimes x + \lambda_1 K - \mu_1 L_0, z^n \otimes y + \lambda_2 K - \mu_2 L_0] \\ &= z^{m+n} \otimes [x,y] + \mu_1 n z^n \otimes y - \mu_2 m z^m \otimes x \\ &\qquad + m\delta_{m,-n}(x|y)K, \end{aligned} \tag{48}$$

where $\mu_i \in \mathbb{C}$. Altogether, we have the affine algebra

$$\mathfrak{g}^{(1)} \equiv \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}(-L_0).$$

One combines $\mathfrak{g}^{(1)}$ with the Virasoro algebra, in a semidirect product.

Everything is only well defined in the context of universal enveloping algebras. The usual enveloping algebra for a Lie algebra $\mathfrak{g}$ is *classical*, as a Hopf algebra involving a symmetric braiding law. In the next lecture, we define quantum Hopf algebras.

Choose a finite dimensional ADE Lie algebra $\mathfrak{g}$ of rank c. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of the maximal torus defined by the root lattice. For an affine algebra, $\mathfrak{h}$ defines a $c+2$ dimensional (over $\mathbb{C}$) subalgebra

$$(1 \otimes \mathfrak{h}) \oplus \mathbb{C}K \oplus \mathbb{C}(-L_0) \tag{49}$$

in $\mathfrak{g}^{(1)}$. The value c is chosen here by the formula [23][32][33]

$$c(k) = \frac{\dim \mathfrak{g} \cdot k}{k + h^\vee} \tag{50}$$

for the standard representation category, where we only consider $k = 1$. For $\mathbf{E}_8{}^{(1)}$, $h^\vee = 30$ and $c = 8$. For $\mathbf{E}_6^{(1)}$, $h^\vee = 12$ and $c = 6$. In the ADE case, the dual Coxeter number is just the Coxeter number.

In the ADE affine lattice, the basic roots α are supplemented by an object L_0, giving a full root system for $\mathfrak{g}^{(1)}$ [23][34],

$$\{nL_0 + \alpha \mid \alpha \in R, n \in \mathbb{Z}\} \cup \{mL_0 \mid m \in \mathbb{Z}, m \neq 0\}. \tag{51}$$

In order to obtain a nice modular tensor category [13][14] of representations for an ADE algebra, $c(1)$ in (50) must also define the deformation parameter associated to the braiding. That is, let $\mu = c^{-1} \cdot \dim \mathfrak{g}$. Then $\mu = h^\vee + 1$ and the root of unity

$$t = \exp(\frac{\pi i}{\mu}) \equiv \omega_{2\mu} \tag{52}$$

is selected as a deformation parameter.

6 Affine and Quantum Algebras

Take any deformation parameter $t \in \mathbb{C}$, $t \neq 0, \pm 1$, such as in (52). For the classical universal enveloping algebra $U(\mathfrak{g})$, there is a set of standard generators $\{E_i, F_i, H_\alpha\}$ for $i = 1, \cdots, c$ and α in the root lattice. $U(\mathfrak{g})$ is generated by monomials in the $\{E_i, F_i, H_\alpha\}$. A basis for the affine $\mathfrak{g}^{(1)}$ is given by the elements (for $k \in \mathbb{Z}$) $z^k \otimes E_i$, $z^k \otimes F_i$, $z^k \otimes H_\alpha$, H_α, K and L_0.

For both finite and affine Lie algebras, the deformed universal enveloping algebra $U_t(\mathfrak{g})$ [18][35] (for $t \neq 0, \pm 1$) has generators $\{E_i, F_i, K_\alpha\}$, for $i \in$

$1, 2, \cdots, c$. Let α_i be the root vector associated to the index i. Given the entries c_{ij} of the ADE Cartan matrix, the defining relations for $U_t(\mathfrak{g})$ are

$$K_\alpha K_\beta = K_{\alpha+\beta} = K_\beta K_\alpha, \qquad K_0 = 1, \tag{53}$$
$$K_\alpha E_j K_\alpha^{-1} = t^{\langle\alpha,\alpha_j\rangle} E_j, \qquad K_\alpha F_j = t^{-\langle\alpha,\alpha_j\rangle} F_j K_\alpha,$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_j}}{t - t^{-1}},$$
$$E_i{}^2 E_j - (t + t^{-1}) E_i E_j E_i + E_j E_i{}^2 = 0 \text{ for } c_{ij} = -1, \tag{54}$$
$$F_i{}^2 F_j - (t + t^{-1}) F_i F_j F_i + F_j F_i{}^2 = 0 \text{ for } c_{ij} = -1,$$
$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i \text{ for } c_{ij} = 0. \tag{55}$$

This is an algebra over $\mathbb{Q}(t)$. The relations (54) are known as the Serre relations. The classical case is recovered from Drinfeld's version of $U_t(\mathfrak{g})$, given by

$$t \mapsto e^{\hbar}, \;\; K_{\alpha_i} \mapsto e^{hK_{\alpha_i}} \tag{56}$$

for a parameter $\hbar$. $\mathcal{A} = U_t(\mathfrak{g})$ has a *comultiplication* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ given by

$$\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \;\; \Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i, \tag{57}$$
$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha.$$

A *bialgebra* $\mathcal{A}$ has both a multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a comultiplication Δ.

Definition 6.1 A bialgebra is *coassociative* if

$$\Delta(1 \otimes \Delta) = (\Delta \otimes 1)\Delta.$$

A *Hopf algebra* over $\mathbb{F}$ is a bialgebra with an additional three maps: a *unit* $\eta : \mathbb{F} \to \mathcal{A}$, a *counit* $\epsilon : \mathcal{A} \to \mathbb{F}$, which sends K_α to 1 and E_i, F_i to 0, and an *antipode* $S : \mathcal{A} \to \mathcal{A}$, with

$$S(E_i) = -K_{-\alpha_i} E_i, \;\; S(F_i) = -F_i K_{\alpha_i}, \tag{58}$$
$$S(K_\alpha) = K_{-\alpha}.$$

Theorem 6.2. *The coassociative algebra* $U_t(\mathfrak{g})$ *obeys the Hopf algebra counit rule, so that* $\Delta(1 \otimes \epsilon)$ *and* $\Delta(\epsilon \otimes 1)$ *are isomorphisms, and the antipode rule*

$$\Delta(S \otimes 1)m = \epsilon \cdot \eta = \Delta(1 \otimes S)m.$$

The braid group action [18] on $U_t(\mathfrak{g})$ is given by the following maps σ_i and τ_i for $i = 1, \cdots c$.

$$\sigma_i(E_i) = S(F_i), \quad \sigma_i(F_i) = S(E_i), \quad \sigma_i(K_{\alpha_j}) = K_{\alpha_j} K_{\alpha_i}{}^{-c_{ij}}, \tag{59}$$
$$\sigma_i(E_j) = -E_i E_j + t^{-1} E_j E_i, \qquad c_{ij} = -1,$$

$$\sigma_i(F_j) = tF_iF_j - F_jF_i, \qquad c_{ij} = -1,$$

$$\sigma_i(E_j) = E_j, \qquad \sigma_i(F_j) = F_j, \qquad c_{ij} = 0.$$

The τ_i are the same as σ_i except for

$$\tau_i(E_i) = -K_{-\alpha_i}F_i, \quad \tau_i(F_i) = -E_iK_{\alpha_i}. \tag{60}$$

Theorem 6.3. *The maps σ_i and τ_i obey the braid rules*

$$\sigma_i\tau_i = \tau_i\sigma_i = 1,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \;\; \tau_i\tau_j = \tau_j\tau_i, \quad \textit{for } c_{ij} = 0,$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \;\; \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j, \quad \textit{for } c_{ij} = -1.$$

Example 6.4 For the rank 2 lattice $SU(3)$, there are generators σ_1, σ_2, $\tau_1 = {\sigma_1}^{-1}$ and $\tau_2 = {\sigma_2}^{-1}$ for the braid group B_3. For any affine $SU(N)$ lattice, with $N \geq 2$, one obtains the braid group B_N on N strands. This covers the permutation Weyl group S_N.

A representation category is given by a *finite* number of simple objects X_i, so that all reasonable representations are words in these letters. The category is a braided $\otimes$ category. An object is a ribbon strand, drawn vertically on the page. For axioms that depend on the the existence of dual representations (ie. *rigidity*), see [13][14]. The representation category also requires *twist* arrows $\tau_X : X \to X$ for each object X. These are the ribbon twists. Finally, the *balancing axiom* [31]

$$\tau_{X\otimes Y} = \gamma_{Y\otimes X}\gamma_{X\otimes Y}(\tau_X \otimes \tau_Y) \tag{61}$$

for ribbon categories defines the twist on tensor products.

When t takes the special value (52), there is a finitely generated ribbon category of representations [13][14]. Links and twists on the finite basis define a representation of the modular group $SL(2,\mathbb{Z})$. However, the braid group B_3 is more fundamental than the modular group. It may be represented by 2×2 *reduced Burau* matrices [36] using a parameter $t \in \mathbb{C}$.

Beyond these low dimensional structures, triality works on a triplet of octonion spaces. Three copies of the $\mathbf{E}_8$ lattice are closely related to the Leech lattice in $\mathbb{R}^{24}$, given by integral elements of $SL(3,\mathbb{O})$. This is like three copies of $\mathbf{E}_{10}$, glued along three hyperbolic planes, for a lattice in $\mathbb{R}^{25,1}$. Thus we expect that the nonassociativity of $\mathbb{O}$ is closely linked to the nonassociativity of general braids, that may be cyclic in the plane.

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With the blossoming of the internet in 2005, I started receiving enthusiastic emails from a certain Mike Rios, aka the archangel Metatron. Mike worked with octonions and the exceptional Jordan algebra. Unfortunately, circumstances have been difficult. But then one day, in April 2015, I met a real octonion theorist who was thinking about gravity, namely Rob Wilson. I am very grateful to Rob for sharing his insights into triality and gravity. Thanks also to Graham Dungworth, who once wrote at *Galaxy Zoo*. The University of Auckland has provided resources.

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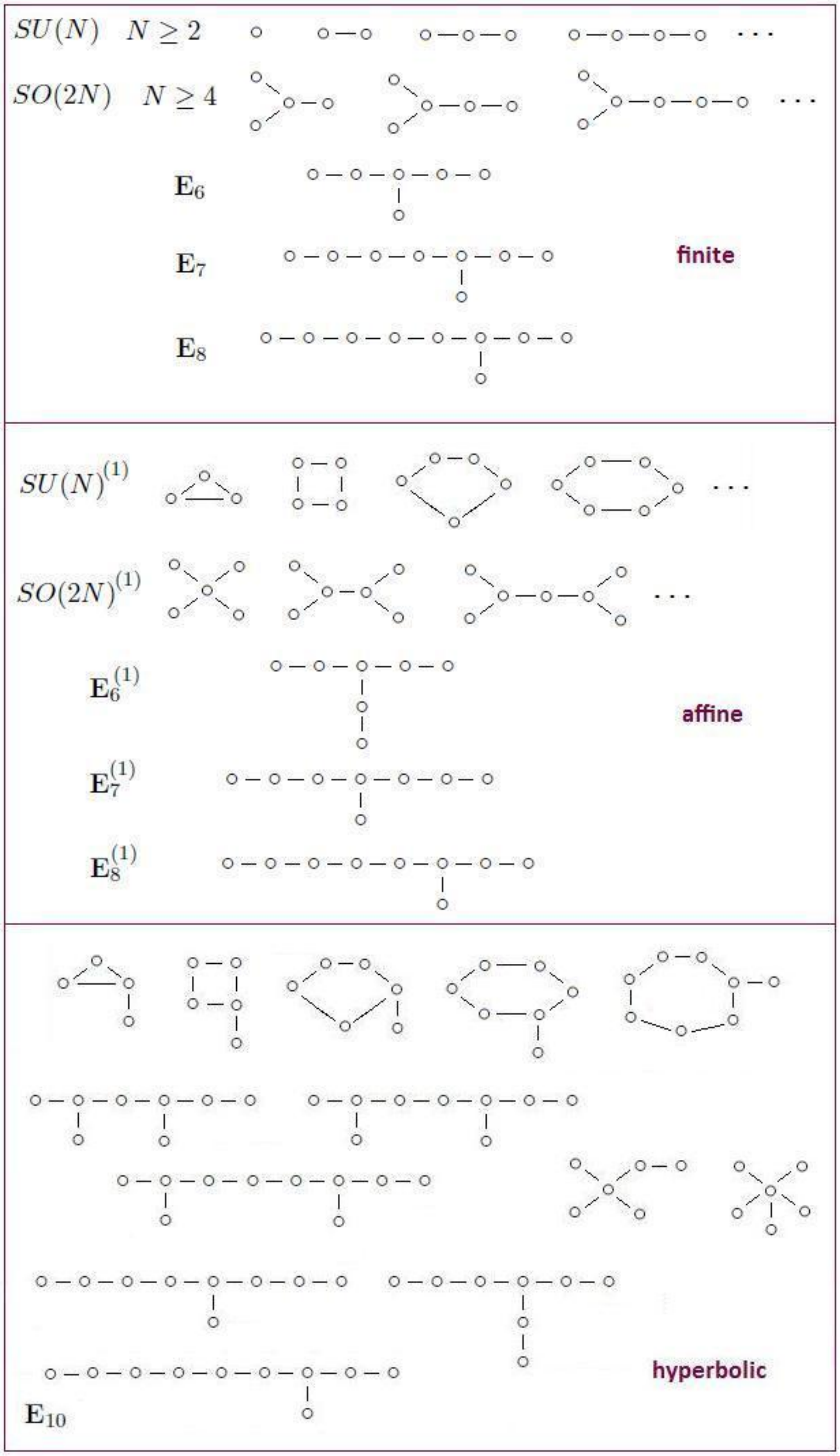

Figure 1: ADE Classification