The “Time since the Singularity” of Expanding Uniform Dust

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Abstract

The calculations of Oppenheimer and Snyder revealed discrepancies in the behavior of a dust ball described using “standard” in place of “comoving” coordinates: periodic metric and energy density singularities that occur in “comoving” coordinates are completely eliminated by the Oppenheimer-Snyder transformation to “standard” coordinates. In addition, the definition of “comoving time” worryingly involves the clocks of an infinite number of different observers, removing it from the realm of what is physically observable. Nonetheless, the Oppenheimer-Snyder example of second-guessing “comoving” dust ball results by transforming them to physically less questionable coordinates has since rarely been emulated. We here treat a dust ball whose energy density always decreases; its “comoving” result has a familiar singularity at a sufficiently early time, which has prompted “age of the universe” terminology. But upon Oppenheimer-Snyder type transformation to “standard” coordinates, the singularity is eliminated, pulling the rug out from under “age” talk. Furthermore, the transformed dust ball was at no stage as small as its Schwarzschild radius. But in a time-reversed variant of well-known Oppenheimer-Snyder abrupt dust contraction, the transformed dust ball likely underwent an inflationary epoch. In addition to the above extension of the Oppenheimer-Snyder transformation, we point out why singular metrics are inconsistent with General Relativity fundamentals, and how “comoving coordinates” also contravene GR fundamentals.

Introduction

In “comoving coordinates” the definition of “time” requires the clocks *of an infinite number of different observers* [1], which obviously renders “comoving time” *physically unobservable*.

Furthermore, *all* “comoving-coordinate” metric tensors, such as, for example, the spherically-symmetric general form [2],

$$ds^2 = (cdt)^2 - U(r,t)dr^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2), \tag{1}$$

are required to adhere to the condition [3],

$$g_{00} = 1,$$

which can *contradict* two general physical properties of g_{00}, namely that in the static weak-field limit $(g_{00} - 1)/2$ is the Newtonian gravitational potential ϕ [4], and that in the static limit $(g_{00})^{-\frac{1}{2}}$ is the gravitational time-dilation factor [5]. In fact “comoving coordinates” are *able* to adhere to the unphysical restriction $g_{00} = 1$ *only* by dint of *physically unobservable* “comoving time”, *which requires the clocks of an infinite number of different observers*.

It is clear from the foregoing two paragraphs that “results” presented exclusively in terms of “comoving coordinates” cannot be assumed to be physically interpretable. For example, an Oppenheimer-Snyder finite-radius ball of dust that is treated in “comoving coordinates” and starts out with a *momentarily static* uniform finite energy density, proceeds to periodically transit through a state whose metric tensor is *singular* and whose energy density is *infinite* [6], but that dust-ball system manifests no trace of metric or configuration singularity *after* the application of Oppenheimer and Snyder’s analytic space-time transformation of its “comoving coordinate” metric solution to “standard” coordinates [7].

Similarly, a finite-radius ball of dust that is treated in “comoving coordinates” and starts with a *decreasing* uniform finite energy density will *invariably* have transited *at an earlier time* through a state whose metric tensor is singular and whose energy density is infinite [8]. Regrettably, despite the caveat of the preceding paragraph that “results” presented exclusively in terms of “comoving coordinates” cannot be assumed to be physically interpretable, *precautionary* space-time transformation of the “comoving-coordinate” results for dust balls with initially decreasing uniform finite energy density to some other coordinate system which doesn’t have g_{00} rigidly fixed to unity *is almost never carried out*. As a consequence of this incautious omission, the “earlier-time” *singularity* in the “comoving” results for dust with initially decreasing uniform finite energy density has been routinely presumed to be very important physics [8]; to cite one typical example, Steven Weinberg declares that [9], “. . . the time elapsed since this singularity . . . may justly be called the age of the universe.”

Such a bold pronouncement is of course hardly in keeping with the fact that the closely related “comoving coordinate” result for dust which starts with an initially momentarily static uniform finite energy

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density *has all of its periodic singularities eliminated by the Oppenheimer-Snyder space-time transformation to "standard" coordinates* [7]. Nor is it at all expected that *mathematical singularities* play a role in *classical* theoretical physics; indeed the usual *informal* proscription of mathematical singularities in classical theoretical physics turns out to be a *formally* open and shut matter for the metric tensors of General Relativity in light of the Principle of Equivalence and the tensor contraction theorem [10].

The proof of the tensor contraction theorem for a given space-time transformation $\bar{x}^{\alpha}(x^{\mu})$ and its inverse transformation $x^{\nu}(\bar{x}^{\alpha})$ hinges on the validity of the relationship [11],

$$(\partial\bar{x}^{\alpha}/\partial x^{\mu})(\partial x^{\nu}/\partial\bar{x}^{\alpha}) = \delta^{\nu}_{\mu}, \tag{2}$$

which, if each component of the Jacobian matrix $\partial\bar{x}^{\alpha}/\partial x^{\mu}$ is well-defined in terms of the finite real numbers at a given space-time point x^{μ}, and *also* each component of its inverse matrix is thus well-defined in terms of the finite real numbers, follows at that space-time point from the chain rule of the calculus. However, because the right-hand side δ^{ν}_{μ} of Eq. (2) is *always* well-defined in terms of the finite real numbers, Eq. (2) *becomes self-inconsistent* at any space-time point x^{μ} where any component of the Jacobian matrix $\partial\bar{x}^{\alpha}/\partial x^{\mu}$ or any component of its inverse matrix *fails to be a well-defined finite real number*. Because the tensor contraction theorem *is indispensable to GR* (e.g., the Einstein tensor is constructed from *contractions* of the Riemann curvature tensor), a space-time transformation can be considered *physical* in the context of GR *only* at space-time points where *each* component of its Jacobian matrix *is a well-defined finite real number* and the *same* is true of *each* component of the *inverse* of that Jacobian matrix.

The Principle of Equivalence implies that a metric tensor is at each space-time point the congruence transformation of the Minkowski metric tensor with the Jacobian matrix of some space-time transformation [12]. Therefore in light of the foregoing discussion of physical space-time transformations in the context of GR, a metric tensor can *only* be considered *physical* in the context of GR at those space-time points where both it and its inverse consist solely of components which are well-defined finite real numbers and its signature is equal to the $\{+,-,-,-\}$ signature of the Minkowski metric tensor [13].

We therefore see that mathematical singularities have absolutely no place in physically legitmate GR metrics, but they *can* occur in "comoving coordinate" metrics as a consequence of the unphysical $g_{00} = 1$ restriction that is imposed on those metrics; that restriction is enforced by reference to the physically unobservable set of clocks of an infinite number of observers.

The next section of this article is concerned with carrying out the almost universally neglected space-time transformation to "standard" coordinates of the "comoving-coordinate" results for a certain class of dust balls with *initially decreasing* uniform finite energy density. This of course *parallels* the Oppenheimer-Snyder space-time transformation to "standard" coordinates of the "comoving-coordinate" results for dust balls with *initially momentarily static* uniform finite energy density, which transformation, as is well-known, eliminates the periodic metric and energy-density singularities that are an artifact of the use of the unphysical "comoving coordinates" [14]. Not surprisingly, we find that our space-time transformation to "standard" coordinates is technically very similar indeed to that of Oppenheimer and Snyder, and that it analogously eliminates the "earlier-time" metric singularity which of course is an artifact of the use of the unphysical "comoving coordinates". Therefore Weinberg's "time elapsed since this singularity" is devoid of physical interpretability or meaning, precisely as is the periodic time interval which separates the physically nonexistent "metric singularities" that are features of the unphysical "comoving coordinate" version of the Oppenheimer-Snyder dust-ball system *only before* its space-time transformation to "standard" coordinates [14].

We find that the "age" of an expanding dust-ball "universe" is in fact infinite, precisely as the "the time of contraction" of an Oppenheimer-Snyder dust-ball system is infinite [7], and that the "size" of an expanding dust-ball "universe" has never in the past been as small as its Schwarzschild radius, just as an Oppenheimer-Snyder dust-ball system never will contract to being as small as its Schwarzschild radius [7]. Interestingly, however, just as there exists a certain time interval during which an Oppenheimer-Snyder dust-ball system typically *contracts especially rapidly* [15], an expanding dust-ball "universe" would correspondingly be expected to experience an epoch of expecially rapid expansion, i.e., "inflation" would seem to be a natural attribute of such a model.

Finally, although dust models themselves of course cannot encompass any quantum effects, we mention that a rule of thumb for a quantum dynamical system is that its inherent uncertainty energy varies inversely with its size. Therefore an *expanding* physical system might be expected to experience a nonclassical extra kinetic "push" from the diminishing of its store of uncertainty energy.

Uniform density dust balls in "comoving" and "standard" coordinates

In "comoving coordinates", the $0 \le r \le a$ interior region of a uniform energy-density dust ball of radius a is described by the spherically-symmetric metric of Eq. (1) with [16, 14],

$$V(r,t) = R^2(t)r^2, \tag{3a}$$

and,

$$U(r,t) = R^2(t)/(1 + \Omega(r^2/c^2)), \tag{3b}$$

where the dimensionless function $R(t)$ satisfies the initial condition,

$$R(t_0) = 1, \tag{3c}$$

and the first-order in time equation of motion,

$$(\dot{R}(t))^2 = (\omega^2/R(t)) + \Omega, \tag{3d}$$

where the constant ω^2 reflects the universal gravitational constant G and the dust ball interior's initial uniform energy density $\rho(t_0)$,

$$\omega^2 \stackrel{\text{def}}{=} (8/3)\pi G\rho(t_0)/c^2, \tag{3e}$$

and the constant Ω, which occurs in *both* of Eqs. (3d) and (3b), is readily evaluated from Eq. (3d) in terms of ω^2 and the initial value of the time derivative of $R(t)$, namely,

$$\Omega = (\dot{R}(t_0))^2 - \omega^2. \tag{3f}$$

In *addition* to the relations given by Eqs. (3a) through (3f), which determine the "comoving" metric of Eq. (1) in the $0 \le r \le a$ *interior* region of the dust ball, we *also* have that in "comoving coordinates" the energy density $\rho(r,t)$ of the dust ball is given *everywhere* in space-time by,

$$\rho(r,t) = \begin{cases} \rho(t_0)/(R(t))^3 & \text{if } 0 \le r \le a, \\ 0 & \text{if } r > a. \end{cases} \tag{3g}$$

The fact that the region $r > a$ *is empty space* permits us to make use of the Birkhoff theorem *on that region's* $r = a$ *boundary* to *aid* us in working out the space-time transformation of the "comoving-coordinate" metric of Eqs. (1) and (3) to "standard" coordinates [17, 14].

In the Oppenheimer-Snyder case, we have that the energy density $\rho(r,t)$ *is initially momentarily static*, i.e.,

$$\dot{\rho}(r,t_0) = 0, \tag{4a}$$

From Eq. (3g) we see that Eq. (4a) is assured if,

$$\dot{R}(t_0) = 0. \tag{4b}$$

Eqs. (4b) and (3f) imply that,

$$\Omega = -\omega^2, \tag{4c}$$

which *specializes* Eq. (3d) to the following *time-cycloidal* form [18],

$$(\dot{R}(t))^2 = \omega^2[(1/R(t)) - 1]. \tag{4d}$$

The nonnegative *continuous* (although not continuously differentiable) time-cycloidal $R(t)$ which satisfies Eq. (4d) and Eq. (3c) (namely $R(t_0) = 1$) is periodic with period π/ω, and *vanishes* at $t = t_0 + (n+1/2)(\pi/\omega)$, $n = 0, \pm 1, \pm 2, \ldots$ [14], with the consequence that the "comoving" metric described by Eqs. (1), (3a), (3b), (3c), (3e), (4c) and (4d) is periodically *singular* at those times, as is the energy density $\rho(r,t)$ of Eq. (3g).

We would on the grounds of the inherent character of General Relativity (and *even* on the grounds of the inherent character of classical theoretical physics in general) of course expect that these periodic *singularities* are *artifacts* of the unphysical "comoving coordinate system" in which the metric described by Eqs. (1), (3a), (3b), (3c), (3e), (4c) and (4d) is expressed; *indeed these singularities are completely absent after that metric has been space-time transformed from "comoving" to "standard" coordinates* [14].

Instead of the initially momentarily static energy density of Oppenheimer and Snyder, we here are interested in having the energy density $\rho(r,t)$ monotonically *decrease* from the initial time t_0 onward. We see from Eq. (3g) that such decrease is assured if $R(t)$ monotonically *increases* from the initial time t_0 onward. The doubtless most *convenient* way to ensure such increase in $R(t)$ is to set $\dot{R}(t_0)$ equal to ω, which causes Eq. (3f) to become simply,

$$\Omega = 0, \tag{5a}$$

and which furthermore, in conjunction with the initial condition $R(t_0) = 1$ of Eq. (3c), *specializes* Eq. (3d) to,

$$\dot{R}(t) = \omega/(R(t))^{\frac{1}{2}}. \tag{5b}$$

The solution of Eq. (5b) which satisfies the $R(t_0) = 1$ initial condition is,

$$R(t) = (1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}, \tag{5c}$$

which clearly monotonically increases from the initial time t_0 onward.

The $R(t)$ of Eq. (5c), however, *vanishes* at one particular time t_s, and t_s is *earlier* than the *initial* time t_0 for which $\dot{R}(t_0) = \omega$, namely,

$$t_s = t_0 - \tfrac{2}{3}\omega^{-1}. \tag{5d}$$

Therefore the "comoving" metric of Eqs. (1), (3a), (3b), (3e), (5a) and (5c) is *singular* at that *earlier* time t_s, as is the energy density $\rho(r,t)$ of Eq. (3g).

We would on the grounds of the inherent character of General Relativity (and *even* on the grounds of the inherent character of classical theoretical physics in general) of course expect this metric *singularity* at the earlier time t_s to be an *artifact* of the unphysical "comoving" coordinates in which the metric is expressed. To *check* that expectation, we now launch into the intricate and lengthy procedure needed to work out the space-time transformation of the metric of Eqs. (1), (3a), (3b), (3e), (5a) and (5c) from "comoving" to "standard" coordinates.

We need to transform the "comoving coordinates" (t, r, θ, ϕ), in terms of which the invariant line element ds^2 is given by Eq. (1), into "standard coordinates" [19] $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$, in terms of which the *same* invariant line element ds^2 is *also* given by,

$$\begin{aligned} ds^2 &= B(\bar{r},\bar{t})(cd\bar{t})^2 - A(\bar{r},\bar{t})(d\bar{r})^2 - \bar{r}^2((d\bar{\theta})^2 + (\sin\bar{\theta}d\bar{\phi})^2) \\ &= (cdt)^2 - U(r,t)(dr)^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2). \end{aligned} \tag{6a}$$

Inspection in Eq. (6a) *of the rightmost two terms* of the line element ds^2 in *both* its "standard" and its "comoving" form immediately reveals three very convenient transformation choices,

$$\bar{\theta} = \theta,\ \bar{\phi} = \phi \text{ and } \bar{r} = (V(r,t))^{\frac{1}{2}} = r(1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}, \tag{6b}$$

where we have evaluated $(V(r,t))^{\frac{1}{2}}$ by using Eqs. (3a) and (5c).

Next we would like to obtain $\bar{t}$ as a function of r and t, as has been done in Eq. (6b) for $\bar{r}$. Inspection of Eq. (6a), however, shows that that task is completely enmeshed with the determination of B and A as functions of r and t; moreover $\bar{t}(r,t)$ *itself* cannot be extracted from Eq. (6a), *only certain combinations of its partial derivatives* $(c(\partial\bar{t}/\partial r))$ *and* $(\partial\bar{t}/\partial t)$ *can.* We thus must solve *both* simultaneous algebraic *and* first-order partial differential equations to obtain $\bar{t}(r,t)$.

We now present in greater detail *that part of* Eq. (6a) *which isn't rendered redundant by the three transformation choices of* Eq. (6b), namely,

$$B[(\partial\bar{t}/\partial t)(cdt) + c(\partial\bar{t}/\partial r)dr]^2 - A[(1/c)(\partial\bar{r}/\partial t)(cdt) + (\partial\bar{r}/\partial r)dr]^2 = (cdt)^2 - U(r,t)(dr)^2. \tag{6c}$$

Since the three bilinear differential forms $(cdt)^2$, $(2c\,dt\,dr)$ and $(dr)^2$ are linearly independent, Eq. (6c) produces *the three simultaneous equations*,

$$B(\partial\bar{t}/\partial t)^2 - A((1/c)(\partial\bar{r}/\partial t))^2 = 1, \tag{7a}$$

$$B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r) = 0, \tag{7b}$$

$$B(c(\partial\bar{t}/\partial r))^2 - A(\partial\bar{r}/\partial r)^2 = -U. \tag{7c}$$

We now eliminate A and B from Eqs. (7) in order to obtain the partial differential equation for $\bar{t}$. Solving Eq. (7b) for A yields,

$$A = \frac{B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}. \tag{8a}$$

We now insert this value of A into each one of Eqs. (7a) and (7c) and follow that by solving each one for $(1/B)$,

$$(1/B) = \frac{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{t}/\partial t)[(\partial\bar{r}/\partial r)(\partial\bar{t}/\partial t) - ((1/c)(\partial\bar{r}/pat))(c(\partial\bar{t}/\partial r))]}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}, \tag{8b}$$

$$(1/B) = \frac{(1/U)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r))[(\partial\bar{r}/\partial r)(\partial\bar{t}/\partial t) - ((1/c)(\partial\bar{r}/pat))(c(\partial\bar{t}/\partial r))]}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}. \tag{8c}$$

If the common rightmost factor which occurs in the numerators of the right-hand sides of both Eq. (8b) and Eq. (8c) *vanished*, so would $(1/B)$. Since we do *not* want B to be singular, *we shall assume that this common factor of the right-hand sides of* Eqs. (8b) *and* (8c) *is nonzero*. With that assumption, equating the right-hand side of Eq. (8b) to the right-hand side of Eq. (8c) produces the relation,

$$((1/c)(\partial\bar{r}/\partial t))(\partial\bar{t}/\partial t) = (1/U)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r)). \tag{9a}$$

From Eqs. (3b), (5a) and (5c), we obtain that,

$$U(r,t) = (1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{4}{3}}, \tag{9b}$$

and from Eq. (6b) we obtain both that,

$$(\partial\bar{r}/\partial r) = (1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}, \tag{9c}$$

and that,

$$((1/c)(\partial\bar{r}/\partial t)) = (\omega r/c)/(1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{1}{3}}. \tag{9d}$$

Putting Eqs. (9b), (9c) and (9d) into Eq. (9a) yields the following first-order linear partial differential equation for $\bar{t}(r,t)$,

$$(1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{1}{3}}(\partial\bar{t}/\partial t) = (c^2/(\omega r))(\partial\bar{t}/\partial r), \tag{9e}$$

which is separable in r and t. Putting,

$$\bar{t}(r,t) = \omega^{-1}\alpha(r)\beta(t), \tag{10a}$$

into Eq. (9e) yields,

$$(1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{1}{3}}(\dot{\beta}(t)/\beta(t)) = (c^2/(\omega r))(\alpha'(r)/\alpha(r)) = -\omega p, \tag{10b}$$

where p is an arbitrary dimensionless constant. Eq. (10b) is satisfied by,

$$\beta(t) = b_0(p)\exp(-p[(1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}]), \tag{10c}$$

and,

$$\alpha(r) = a_0(p)\exp(-p[\tfrac{1}{2}(\omega r/c)^2]), \tag{10d}$$

where $b_0(p)$ and $a_0(p)$ are arbitrary dimensionless constants that can vary with p.

Because the Eq. (9e) partial differential equation is *linear*, any linear combination of its solutions will be a solution as well. Combining this linear superposition property of the solutions of Eq. (9e) with Eqs. (10a), (10c) and (10d), we see that the general solution of Eq. (9e) will be of the form,

$$\bar{t}(r,t) = \omega^{-1}\int dp\, a_0(p)b_0(p)\exp(-p[\tfrac{1}{2}(\omega r/c)^2 + (1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}]). \tag{10e}$$

Eq. (10e) tells us that $\bar{t}(r,t)$ is equal to ω^{-1} times the Laplace representation of a general dimensionless function of the dimensionless variable $(\frac{1}{2}(\omega r/c)^2 + (1 + \frac{3}{2}\omega(t - t_0))^{\frac{2}{3}})$.

Therefore, given *any* differentiable dimensionless function $\phi(u)$ of the dimensionless variable u, the function $\bar{t}(r,t)$ which is given by,

$$\bar{t}(r,t) = \omega^{-1}\phi(\tfrac{1}{2}(\omega r/c)^2 + (1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}), \tag{10f}$$

satisfies the Eq. (9e) partial differential equation, and also has the dimensions of time. Note that it is straightforward to verify by direct substitution that the general form of $\bar{t}(r,t)$ which is specifically described by Eq. (10f) satisfies the Eq. (9e) partial differential equation.

With the Eq. (10f) form of $\bar{t}(r,t)$ in hand, we are now almost in a position to obtain the two "standard" metric functions B and A in terms of r and t from Eqs. (8), but first we need to calculate the two partial derivatives $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ from Eq. (10f),

$$(\partial\bar{t}/\partial t) = \phi'(\tfrac{1}{2}(\omega r/c)^2 + (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}})/(1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{1}{3}}, \tag{11a}$$

$$(c(\partial\bar{t}/\partial r)) = (\omega r/c)\phi'(\tfrac{1}{2}(\omega r/c)^2 + (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}). \tag{11b}$$

We can now substitute Eqs. (11a) and (11b), along with Eqs. (9c) and (9d), into Eq. (8a), with the result,

$$A(r,t) = \frac{B(r,t)[\phi'(\frac{1}{2}(\omega r/c)^2 + (1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})]^2}{(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}}. \tag{12a}$$

We can make the same four substitutions into Eq. (8b) (or, with exactly the same effect, make those same four substitutions together with the substitution of Eq. (9b), into Eq. (8c)), with the result,

$$B(r,t) = \frac{(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}}{[\phi'(\frac{1}{2}(\omega r/c)^2+(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})]^2[1-(\omega r/c)^2/(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}]}. \tag{12b}$$

The substitution of Eq. (12b) into Eq. (12a) then yields the "standard" metric function $A(r,t)$,

$$A(r,t) = \frac{1}{1-(\omega r/c)^2/(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}}. \tag{12c}$$

We note that while in Eq. (12b) $B(r,t)$ is expressed in terms of a function $\phi'(\frac{1}{2}(\omega r/c)^2 + (1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})$ which has not yet been determined, the expression in Eq. (12c) for $A(r,t)$ is the finished product. This expression is valid *only* for $0 \le r \le a$, where dust of uniform energy density $\rho(t) = \rho(t_0)/(R(t))^3$ *is present* (and where, of course, $R(t) = (1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}$). However, we *also* know that the region $r > a$ *is empty space*, so we *in addition* expect the Birkhoff theorem to apply on the $r = a$ *boundary* of that empty-space region. In order to *check* whether that is in fact the case for the $A(r,t)$ of Eq. (12c), we first eliminate the "comoving time coordinate" t from $A(r,t)$ in favor of the "standard radial coordinate" $\bar{r} = r(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}$ (see Eq. (6b)), then we *fix* r to its *boundary value* a, and then we *compare* A as a function of $\bar{r}$ with the static Schwarzschild "standard form" of this metric component for a system of effective mass $M = a^3(4/3)\pi\rho(t_0)/c^2$). For the $A(r,t)$ of Eq. (12c), this prescription yields,

$$A(r=a,\bar{r}) = \frac{1}{1-((a^3\omega^2)/(c^2\bar{r}))}, \tag{12d}$$

and from Eq. (3e),

$$\omega^2 = (8/3)\pi G\rho(t_0)/c^2 = 2GM/a^3, \tag{12e}$$

so that Eq. (12d) reads,

$$A(r=a,\bar{r}) = \frac{1}{1-((2GM)/(c^2\bar{r}))}, \tag{12f}$$

which is indeed the Schwarzschild "standard form" of this metric component.

The *same* approach permits one *to pin down* the not yet determined function $\phi'(u)$ which appears in the Eq. (12b) expression for the metric component $B(r,t)$. On one hand we of course require $B(r=a,\bar{r})$ to have its static Schwarzschild "standard form", namely,

$$B(r=a,\bar{r}) = 1 - ((2GM)/(c^2\bar{r})) = 1 - ((a^3\omega^2)/(c^2\bar{r})) = 1 - ((\omega a/c)^2/(\bar{r}/a)), \tag{13a}$$

where we have used Eq. (12e). On the other hand, applying the relation,

$$(1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}} = \bar{r}/r, \tag{13b}$$

which follows from Eq. (6b), to the Eq. (12b) expression for $B(r,t)$ permits one to obtain the result that,

$$B(r=a,\bar{r}) = \frac{(\bar{r}/a)}{[\phi'(\frac{1}{2}(\omega a/c)^2+(\bar{r}/a))]^2[1-((\omega a/c)^2/(\bar{r}/a))]}. \tag{13c}$$

Equating the right-hand side of Eq. (13c) to the expression which appears after the last equal sign on the right-hand side of Eq. (13a) then permits one to solve for $\phi'(\frac{1}{2}(\omega a/c)^2 + (\bar{r}/a))$,

$$\phi'(\tfrac{1}{2}(\omega a/c)^2 + (\bar{r}/a)) = \frac{(\bar{r}/a)^{\frac{3}{2}}}{(\bar{r}/a) - (\omega a/c)^2}. \tag{13d}$$

We now obtain $\phi'(u)$ by setting $(\bar{r}/a)$ to $u - \frac{1}{2}(\omega a/c)^2$ in both sides of Eq. (13d),

$$\phi'(u) = \frac{(u - \frac{1}{2}(\omega a/c)^2)^{\frac{3}{2}}}{u - \frac{3}{2}(\omega a/c)^2} \tag{13e}$$

In order to obtain $\bar{t}(r,t) = \omega^{-1}\phi(\frac{1}{2}(\omega r/c)^2 + (1 + \frac{3}{2}\omega(t - t_0))^{\frac{2}{3}})$ as per Eq. (10f), we must *integrate* the specific $\phi'(u)$ which we have obtained in Eq. (13e),

$$\phi(u) = \phi(u_0) + \int_{u_0}^{u} du' \, \frac{(u' - \frac{1}{2}(\omega a/c)^2)^{\frac{3}{2}}}{u' - \frac{3}{2}(\omega a/c)^2} = \phi(u_0) + \int_{u_0 - \frac{1}{2}(\omega a/c)^2}^{u - \frac{1}{2}(\omega a/c)^2} ds \, \frac{s^{\frac{3}{2}}}{s - (\omega a/c)^2}. \tag{14}$$

To conveniently have the lower limit of the integration over the variable s in Eq. (14) be equal to unity, we choose u_0 equal to $1 + \frac{1}{2}(\omega a/c)^2$. With that choice of u_0 we use the $\phi(u)$ of Eq. (14) to implement Eq. (10f) for $\bar{t}(r,t)$,

$$\bar{t}(r,t) = \omega^{-1}\phi(\tfrac{1}{2}(\omega r/c)^2 + (1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}) = \bar{t}(r = a, t = t_0) + \omega^{-1}\int_{1}^{S(r,t)} ds \, \frac{s^{\frac{3}{2}}}{s - (\omega a/c)^2}, \tag{15}$$

where $S(r,t) \stackrel{\text{def}}{=} \frac{1}{2}(\omega/c)^2(r^2 - a^2) + (1 + \frac{3}{2}\omega(t - t_0))^{\frac{2}{3}}$. (Note that $\omega^{-1}\phi(1 + \frac{1}{2}(\omega a/c)^2) = \bar{t}(r = a, t = t_0)$ because $S(r = a, t = t_0) = 1$.)

The Eq. (15) expression for $\bar{t}(r,t)$ is, of course, only valid for $0 \le r \le a$, i.e., within the dust ball. The *most crucial feature* of $\bar{t}(r,t)$ is that it *diverges* when $0 \le r \le a$ and $S(r,t) \le (\omega a/c)^2 = (2GM)/(c^2 a)$, where the last equality follows from Eq. (12e).

In particular, when $t = t_s = t_0 - \frac{2}{3}\omega^{-1}$, we see that for $0 \le r \le a$, $S(r,t_s) = \frac{1}{2}(\omega/c)^2(r^2 - a^2) \le 0$, and therefore $\bar{t}(r,t_s)$ *diverges* for $0 \le r \le a$. The "comoving coordinate singularity" at $t = t_s = t_0 - \frac{2}{3}\omega^{-1}$ is therefore *simply never reached* anywhere within the dust ball, as it would require *the infinite time* $\bar{t}(r,t_s)$ in "standard" coordinates to do so. In terms of the colorful language invoked by Steven Weinberg [9], the "age of the universe" *is infinite*; the dust-ball metric singularity at the "comoving time" $t_s = t_0 - \frac{2}{3}\omega^{-1}$ *is only an artifact* of the use of the *unphysical* "comoving coordinates", *in exactly the same way* that the Oppenheimer-Snyder periodic dust-ball metric singularities are all *likewise* only *artifacts* of the use of the *unphysical* "comoving coordinates" [14]. Of course *the physical nonexistence of metric singularities* is a readily demonstrated *general property* of General Relativity, as we have showed in detail in the Introduction. It would in any case overstretch physical credibility for *mathematical singularities* to actually play a *role* in classical theoretical physics.

It is as well of interest to look into what the requirement $S(r,t) > (\omega a/c)^2 = (2GM)/(c^2 a)$ implies for the $r = a$ outer surface of the dust ball in the "standard" coordinate system (in the unphysical "comoving" coordinate system every dust particle has zero three-velocity [20, 14], so the radius a of the dust ball's outer surface *never changes* in unphysical "comoving" coordinates; *only* the magnitude of the dust ball's uniform *energy density* can change with "comoving time" in unphysical "comoving coordinates"). If we express $S(r,t)$ in terms of the "standard" radial coordinate $\bar{r} = r(1 + \frac{3}{2}\omega(t - t_0))^{\frac{2}{3}}$ in place of the "comoving" time t, we obtain $S(r,\bar{r}) = \frac{1}{2}(\omega/c)^2(r^2 - a^2) + \bar{r}/r$. Specializing to the dust ball's outer surface $r = a$, we obtain $S(r = a, \bar{r}) = \bar{r}/a > (2GM)/(c^2 a)$, which implies that $\bar{r} > (2GM)/c^2$, namely the radius in "standard" coordinates of the dust ball's outer surface *must always exceed* that dust ball's Schwarzschild radius $(2GM)/c^2$—the dust ball's outer surface *actually attaining* its Schwarzschild radius is *impossible* in "standard" coordinates because that would require an *infinite* "standard" time, namely $\bar{t}(r = a, \bar{r})$ *diverges* when $\bar{r}/a = S(r = a, \bar{r}) \le (\omega a/c)^2 = (2GM)/(c^2 a)$. The result here that the radius in "standard" coordinates of the dust ball's outer surface *must always exceed* its Schwarzschild radius *is identical* to what is found in the Oppenheimer-Snyder case [14, 7], and it is *also*, of course, merely *another* instance of *the physical nonexistence of metric singularities* in General Relativity, which is proved in detail in the Introduction (not that it could even be physically credible for mathematical singularities to actually play a role in *any* branch of classical theoretical physics). In terms of the colorful language invoked by Steven Weinberg [9], *not only* is the "age of the universe" *infinite*, also the "size of the universe" could *never* in the past *have been as small as its Schwarzschild radius*.

Finally, just as in the Oppenheimer-Snyder case, analytic evaluation of the integral expression for $\bar{t}(r,t)$ (given here by Eq. (15)) can be carried out *in the region where it doesn't diverge*, namely for $S(r,t) > (\omega a/c)^2 = (2GM)/(c^2 a)$, and it is furthermore only valid when $0 \le r \le a$, i.e., within the dust ball. One must take care *not* to inadvertently *mentally analytically continue* that analytic result into the region $S(r,t) \le (\omega a/c)^2 = (2GM)/(c^2 a)$ *where* $\bar{t}(r,t)$ *diverges*.

To simplify notation during and after evaluation of the integral, we reexpress Eq. (15) in the streamlined form,

$$\omega(\bar{t}_\alpha(S) - \bar{t}_\alpha(1)) = \int_1^S ds\, s^{\frac{3}{2}}/(s-\alpha), \tag{16a}$$

where,

$$S \stackrel{\text{def}}{=} S(r,t) = \tfrac{1}{2}(\omega/c)^2(r^2 - a^2) + (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}, \tag{16b}$$

and,

$$\alpha \stackrel{\text{def}}{=} (\omega a/c)^2 = (2GM)/(c^2 a). \tag{16c}$$

Note that we consider *only* the case where $S > \alpha$; the integral on the right-hand side of Eq. (16a) *diverges* when $S \le \alpha$. Also, Eqs. (16a) through (16c) only apply when $0 \le r \le a$, namely within the dust ball. It may be useful to note that the ω which appears on the left-hand side of Eq. (16a), and in the expression for $S(r,t)$ in Eq. (16b), is related to α via $\omega = (c/a)\alpha^{\frac{1}{2}}$, and it may also be useful to note that $S(r=a, t=t_0) = 1$.

Evaluation of the integral on the right-hand side of Eq. (16a) requires only the simple change of variable $s = v^2$, followed by some mildly tedious algebra,

$$\int_1^S ds\, s^{\frac{3}{2}}/(s-\alpha) = 2\int_1^{S^{\frac{1}{2}}} dv v^4/(v^2-\alpha). \tag{16d}$$

Next we note that,

$$2v^4/(v^2-\alpha) = 2[(v^2-\alpha)+\alpha]^2/(v^2-\alpha) = 2v^2 + 2\alpha + \alpha^{\frac{3}{2}}[(1/(v-\alpha^{\frac{1}{2}})) - (1/(v+\alpha^{\frac{1}{2}}))].$$

We now need only carry out four elementary integrations to obtain the result,

$$\bar{t}_\alpha(S) = \bar{t}_\alpha(1) + \omega^{-1}\Big\{\tfrac{2}{3}(S^{\frac{3}{2}} - 1) + 2\alpha(S^{\frac{1}{2}} - 1) + \alpha^{\frac{3}{2}}\ln\Big[\tfrac{(S^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(1+\alpha^{\frac{1}{2}})}{(1-\alpha^{\frac{1}{2}})(S^{\frac{1}{2}}+\alpha^{\frac{1}{2}})}\Big]\Big\}, \tag{16e}$$

and we reiterate that Eq. (16e) is valid *only* when $S > \alpha$, and also that $\bar{t}_\alpha(S)$ *diverges* when $S \le \alpha$.

Although the result that we have obtained here for $\bar{t}(r,t)$ is analytically simpler than that which is obtained in the Oppenheimer-Snyder case, it also appears to have strong similarities, in particular in a vicinity near its region of divergence [14]. Certainly the technical manner in which the radius of the dust ball's outer surface is prevented from attaining the Schwarzschild value in "standard" coordinates appears to be highly analogous in the two cases.

A notable feature of the Oppenheimer-Snyder result is the existence of certain time interval during which the rate of contraction of the dust sphere in "standard" coordinates is enormously greater than it is at times which lie outside of that particular time interval [15]. Given the apparent strong similarity of our result to that of Oppenheimer and Snyder in space-time regions where critical phenomena occur, it stands to reason that there would exist an epoch in time when our dust ball expands vastly more rapidly in "standard" coordinates than it does at times which lie outside of that particular epoch. The existence of such an "inflationary" epoch could be viewed as the time-reversed version of the well-known Oppenheimer-Snyder "contractionary" epoch [15].

References

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