On the existence of at least one prime number between $5n$ and $6n$

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Abstract

One of the still unsolved conjectures related to prime numbers states that for all integers $n \geq k > 1$ there exists at least one prime number in the interval $[kn, (k+1)n]$. The case $k = 1$ is called Bertrand’s postulate, which was proved Chebyshev in the year 1850. M. El Bachraoui proved the case $k = 2$ in 2006, and the case $k = 2$ was proved by Andy Loo in 2011. This paper gives the proof for the case when $k = 5$.

1. Notations

Throughout the whole paper, we assume that:

- $n \in \mathbb{N}$ and $p \in \mathbb{P}$;
- $E_p(n)$ denotes the exponent of $p$ in the factorization of $n$;
- $\pi(n)$ denotes the prime counting function, i.e. the number of all primes less or equal to $n$;

2. Lemmas

Lemma 1.1. For every $n \geq 8$:

$$\pi(n) \leq \frac{n}{2} \quad (1)$$

Proof. This is trivial since 1, 9 and all nonnegative even numbers are not prime. \hfill \Box

Lemma 1.2. Define two functions $f(x)$ and $g(x)$ such that:

$$f(x) = \sqrt{2\pi x^{x+\frac{1}{2}}} e^{-x} e^{\frac{1}{12x+1}}$$

$$g(x) = \sqrt{2\pi x^{x+\frac{1}{2}}} e^{-x} e^{\frac{1}{12x+1}} \quad (2)$$

Then for every $n \geq 1$:

$$g(n) < n! < f(n) \quad (3)$$
Proof. See [1]. □

**Lemma 1.3.** For every \( x \in \mathbb{R} \) and \( x \geq 3 \):
\[
\prod_{p \leq x} p < 2^{2x-3}
\] (4)

Proof. See [3, Lemma 4: An upper bound on the primorial]. □

3. Main results

Consider the prime factorization of \( \binom{6n}{5n} \), assuming for the contrary that there exists \( n \) such that there is no prime between \( 5n \) and \( 6n \). Divide the prime factors into two groups \( P_1 \) and \( P_2 \) such that:
\[
\binom{6n}{5n} = P_1 P_2 P_3 = \prod_{1<p \leq \sqrt{6n}} P_1 p^{E_p(\binom{6n}{5n})} \prod_{\sqrt{6n}<p \leq 5n} P_2 p^{E_p(\binom{6n}{5n})}
\] (5)

Observe that:
\[
P_1 < (6n)^{\pi(\sqrt{6n})} \leq (6n)^{\frac{\sqrt{6n}}{2}}
\] (6)
as none of the prime powers in the interval \((1, \sqrt{6n}]\) exceed \( 6n \). Obviously, we can only apply Lemma 1.1 when \( \sqrt{6n} \) is greater or equal to \( 8 \) – thus \( n \) must be greater or equal to \( 10^2 \frac{3}{2} \), but since we have assumed that \( n \in \mathbb{N} \), then \( n \) must be greater or equal to \( 11 \). It is easy to see that if \( p > \sqrt{6n} \), then \( E_p(\binom{6n}{5n}) < 2 \) (if \( p > \sqrt{6n} \), then \( p^2, p^3, \ldots, p^{1+k} > 6n \), which is impossible on the strength of Lemma 1.1). It is easy to see that if \( 3n < p < 5p \leq 5n < 6n < 2p \). We apply Lemma 1.3 to obtain:
\[
P_2 \leq \prod_{\sqrt{6n}<p \leq 3n} p \leq \prod_{1<p \leq 3n} p \leq 2^{6n-3}
\] (7)

Now, using Lemma 1.2, we get the approximation of \( \binom{6n}{5n} \):
\[
\binom{6n}{5n} = \frac{(6n)!}{(5n)![(n)!]} > \frac{g(6n)}{f(5n)f(n)} =
\]
\[
= \frac{\sqrt{2\pi}(6n)^{6n+1+1} e^{-6n} e^{\frac{1}{12n+1}}}{(\sqrt{2\pi}(5n)^{5n+1+1} e^{-5n} e^{\frac{1}{12n+1}})(\sqrt{2\pi}n^{n+1} e^{-n} e^{\frac{1}{12n}})}
\] (8)

After simplification, we get:
\[
\binom{6n}{5n} > \frac{\sqrt{3}}{\sqrt{5\pi n}} \left( \frac{46656}{3125} \right)^n e^{\frac{1}{12n+1}} - \frac{1}{6n} - \frac{1}{12n}
\] (9)
Combining inequalities (6), (7) and (9) we obtain:

\[
\frac{\sqrt{3}}{\sqrt{5\pi n}} \left( \frac{46656}{3125} \right)^n e^{\frac{1}{72n+1} - \frac{1}{60n} - \frac{1}{12n}} \leq (6n)^{\sqrt{\frac{2\pi n}{e}}} 2^{6n-3}
\] (10)

Which does not hold true for sufficiently large \(n\), so we conclude that for sufficiently large \(n\) there is always at least one prime number between 5\(n\) and 6\(n\). Quod erat demonstrandum.

References
