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SOME MATHEMATICAL BASES FOR NON-COMMYTATIVE FIELD THEORIES[1]

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Misconceptions have recently been found in the definition of a partial derivative (in the case of the presence of both explicit and implicit dependencies of the function subjected to differentiation) in the classical analysis. We investigate the possible influence of this discovery on quantum mechanics and the classical/quantum field theory. Surprisingly, some commutators of operators of space-time 4-coordinates do *not* equal to zero. Thus, we provide the bases for new-fashioned noncommutative field theory.

To the best of my knowledge, the assumption that the operators of coordinates do *not* commute $[\hat{x}_\mu, \hat{x}_\nu]_- \neq 0$ has been made by H. Snyder [1]. The Lorentz symmetry thus may be broken. Much interest has recently been attracted to this idea [2, 3] in the context of "brane theories".

Moreover, the famous Feynman-Dyson proof of Maxwell equations [4] contains intrinsically the non-commutativity of velocities $[\dot{x}_i(t), \dot{x}_j(t)]_- \neq 0$ that also may be considered as a contradiction with the well-accepted theories.

On the other hand, it was recently discovered that the concept of partial derivative is *not* well defined in the case of both explicit and implicit dependence of the corresponding function, which the derivatives act upon [5, 7] (see also the discussion in [6]). The well-known example of such a situation is the field of an accelerated charge [8].[3]

Let us study the case when we deal with explicite and implicite dependencies $f(\mathbf{p}, E(\mathbf{p}))$. It is well known that the energy in the relativism is connected with the 3-momentum as $E = \pm\sqrt{\mathbf{p}^2 + m^2}$; the unit system $c = \hbar = 1$ is used. In other words, we must choose the 3-dimensional hyperboloid from the entire Minkowski space and the energy is *not* an independent quantity anymore. Let us calculate the commutator of the whole derivative $\hat{\partial}/\hat{\partial}E$ and $\hat{\partial}/\hat{\partial}p_i$.[4] In the general case one has

$$\frac{\hat{\partial} f(\mathbf{p}, E(\mathbf{p}))}{\hat{\partial} p_i} \equiv \frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial p_i} + \frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial E}\frac{\partial E}{\partial p_i}\,. \quad (1)$$

Applying this rule, we surprisingly find

$$\begin{aligned} &[\frac{\hat{\partial}}{\hat{\partial} p_i}, \frac{\hat{\partial}}{\hat{\partial} E}]_- f(\mathbf{p}, E(\mathbf{p})) = \\ &= \frac{\hat{\partial}}{\hat{\partial} p_i}\frac{\partial f}{\partial E} - \frac{\partial}{\partial E}(\frac{\partial f}{\partial p_i} + \frac{\partial f}{\partial E}\frac{\partial E}{\partial p_i}) = \\ &= \frac{\partial^2 f}{\partial E \partial p_i} + \frac{\partial^2 f}{\partial E^2}\frac{\partial E}{\partial p_i} - \frac{\partial^2 f}{\partial p_i \partial E} - \\ &- \frac{\partial^2 f}{\partial E^2}\frac{\partial E}{\partial p_i} - \frac{\partial f}{\partial E}\frac{\partial}{\partial E}(\frac{\partial E}{\partial p_i})\,. \end{aligned} \quad (2)$$

So, if $E = \pm\sqrt{m^2 + \mathbf{p}^2}$ and one uses the generally-accepted representation form of $\partial E/\partial p_i = p^i/E$, one has that the expression (2) appears to be equal to $(p^i/E^2)\frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial E}$. Within the choice of the normalization the coefficient is the longitudinal electric field in the helicity basis (the electric/magnetic fields can

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[3] First, Landau and Lifshitz wrote that the functions depended on t' and only through $t' + R(t')/c = t$ they depended implicitly on x, y, z, t. However, later (in calculating the formula (63.7)) they used the explicit dependence of R on the space coordinates of the observation point too. Chubykalo and Vlayev claimed that the time derivative and curl do *not* commute in their case. Jackson, in fact, disagreed with their claim on the basis of the definitions ("the equations representing Faraday's law and the absence of magnetic charges ... are satisfied automatically"; see his Introduction in [6b]). But, he agrees with [8] that one should find "a contribution to the spatial partial derivative for fixed time t from explicit spatial coordinate dependence (of the observation point)." Škovrlj and Ivezić [6c] calls this partial derivative as '*complete* partial derivative'; Chubykalo and Vlayev [6a], as '*total* derivative with respect to a given variable'; the terminology suggested by Brownstein [7] is 'the *whole*-partial derivative'.

[4] In order to make distinction between differentiating the explicit function and that which contains both explicit and implicit dependencies, the 'whole partial derivative' may be denoted as $\hat{\partial}$.

be derived from the 4-potentials which have been presented in [9]). On the other hand, the commutator

$$[\frac{\hat{\partial}}{\hat{\partial} p_i}, \frac{\hat{\partial}}{\hat{\partial} p_j}]_- f(\mathbf{p}, E(\mathbf{p})) =$$

$$= \frac{1}{|E|^3} \frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial E} [p_i, p_j]_- \,. \qquad (3)$$

This may be considered to be zero unless we would trust to the genious Feynman. He postulated that the velocity (or, of course, the 3-momentum) commutator is equal to $[p_i, p_j] \sim i\hbar\epsilon_{ijk} B^k$, i.e., to the magnetic field.[5]

Furthermore, since the energy derivative corresponds to the operator of time and the i-component momentum derivative, to $\hat{x}_i$, we put forward the following anzatz in the momentum representation:

$$[\hat{x}^\mu, \hat{x}^\nu]_- = \omega(\mathbf{p}, E(\mathbf{p}))\, F_{||}^{\mu\nu} \frac{\partial}{\partial E} \,, \qquad (4)$$

with some weight function ω being different for different choices of the antisymmetric tensor spin basis.

In the modern literature, the idea of the broken Lorentz invariance by this method concurrs with the idea of the *fundamental length*, first introduced by V. G. Kadyshevsky [10] on the basis of old papers by M. Markov. Both ideas and corresponding theories are extensively discussed, e.g. [11]. In my opinion, the main question is: what is the space scale, when the relativity theory becomes incorrect.

Conclusions

We found that the commutator of two derivatives may be *not* equal to zero. As a consequence, for instance, the question arises, if the derivative $\hat{\partial}^2 f/\hat{\partial} p^\nu \hat{\partial} p^\mu$ is equal to the derivative $\hat{\partial}^2 f/\hat{\partial} p^\mu \hat{\partial} p^\nu$ in all cases?[6] The presented consideration permits us to provide some bases for non-commutative field theories and induces us to look for further development of the classical analysis in order to provide a rigorous mathematical basis for operations with functions which have both explicit and implicit dependencies.

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[5]In fact, if we put in the corespondence to the momenta their quantum-mechanical operators (of course, with the appropriate clarification $\partial \to \hat{\partial}$), we obtain again that, in general, the derivatives do *not* commute $[\frac{\hat{\partial}}{\partial x_\mu}, \frac{\hat{\partial}}{\partial x_\nu}]_- \neq 0$.

[6]The same question can be put forward when we have differentiation with respect to the coordinates too, that may have impact on the correct calculations of the problem of accelerated charge in classical electrodynamics.

References

[1] H. Snyder, Phys. Rev. **71**, 38 (1947); ibid. **72**, 68 (1947).

[2] S. Doplicher *et al.*, Phys. Lett. B**331**, 39 (1994); S. Majid and H. Ruegg, Phys. Lett. B**334**, 348 (1994); J. Lukierski, H. Ruegg and W. J. Zakrzewski, Ann. Phys. **243**, 90 (1995); N. Seiberg and E. Witten, JHEP **9909**, 032 (1999), hep-th/9908142; I. F. Riad and M. M. Sheikh-Jabbari, JHEP **08**, 045 (2000), hep-th/0008132; M. M. Sheikh-Jabbari, Phys. Rev. Lett. **84**, 5265 (2000), hep-th/0001167; J. Madore *et al.*, Eur. Phys. J. C**16**, 161 (2000).

[3] S. I. Kruglov, hep-th/0110059 [To be published in a special issue of "Ann. Fond. Broglie" on Contemporary Electrodynamics]; B. G. Sidharth, physics/0110040 [To be published in a special issue of "Ann. Fond. Broglie" on Contemporary Electrodynamics].

[4] F. Dyson, Am. J. Phys. **58**, 209 (1990). S. Tanimura, Ann. Phys. **220**, 229 (1992); M. Land, N. Shnerb and L. Horwitz, hep-th/9308003.

[5] A. E. Chubykalo and R. Smirnov-Rueda, Mod. Phys. Lett. A**12**, 1 (1997), hep-th/9608038; A. E. Chubykalo and R. Alvarado, Hadronic J. [To be published].

[6] A. E. Chubykalo and S. Vlayev, Int. J. Mod. Phys. A**14**, 3789 (1999); J. D. Jackson, hep-ph/0203076, L. Škovrlj and T. Ivezić, hep-ph/0203116.

[7] K. R. Brownstein, Am. J. Phys. **67**, 639 (1999).

[8] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields.* 4th revised English ed. [Translated from the 6th Russian edition] (Pergamon Press, 1979). 402p., §63

[9] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959); H. M. Ruck and W. Greiner, J. Phys. G: Nucl. Phys. **3**, 657 (1977).

[10] V. G. Kadyshevsky, Nucl. Phys. B**141**, 477 (1978); V. G. Kadyshevsky, M. D. Mateev, R. M. Mir-Kasimov and I. P. Volobuev, Theor. Math. Phys. **40**, 800 (1979) [Teor. Mat. Fiz. **40**, 363 (1979)]; V. G. Kadyshevsky and M. D. Mateev, Phys. Lett. B**106**, 139 (1981); V. G. Kadyshevsky and M. D. Mateev, Nuovo Cim. A**87**, 324 (1985); A. D. Donkov, R. M. Ibadov, V. G. Kadyshevsky, M. D. Mateev and M. V. Chizhov, ibid. **87**, 350 (1985); ibid. **87**, 373 (1985). See also the old work: V. G. Kadyshevsky, R. M. Mir-Kasimov and N. B. Skachkov, Yad. Fiz. **9**, 212 (1969). See also references to the old works of M. Markov in the above-cited papers.

[11] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D**52**, 1108 (1995); G. Amelino-Camelia, Nature **408**, 661 (2000); gr-qc/0012051; hep-th/0012238; gr-qc/0106004; J. Kowalski-Glikman, hep-th/0102098; G. Amelino-Camelia and M. Arzano, hep-th/0105120; N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, hep-th/0107039.