Does a single spin-1/2 pure quantum state have a counterpart in physical reality?(accepted version)

Koji Nagata[1] and Tadao Nakamura[2]

[1] *Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea*
E-mail: ko_mi_na@yahoo.co.jp
[2] *Department of Information and Computer Science, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan*
E-mail: nakamura@pipelining.jp

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We discuss the fact that a single spin observable σ_x in a quantum state does not have a counterpart in physical reality. We consider whether a single spin-1/2 pure state has a counterpart in physical reality. It is an eigenvector of Pauli observable σ_z or an eigenvector of Pauli observable σ_x. We assume a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We assume also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. The value of transition probability $|\langle +_z|+_x\rangle|^2$ is $1/2$. We consider the following physical situation. If we detect $|+_z\rangle$, then we assign measurement outcome as $+1$. If we detect $|+_x\rangle$, then we assign measurement outcome as -1. The existence of a single classical probability space for the transition probability within the formalism of the measurement outcome does not coexist with the value of the transition probability $|\langle +_z|+_x\rangle|^2 = 1/2$. We have to give up the existence of such a classical probability space for the state $|+_z\rangle$ or for the state $|+_x\rangle$, as they define the transition probability. It turns out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality, in general. We investigate whether the Stern-Gerlach experiment accepts hidden-variables theories. We discuss that the existence of the two spin-1/2 pure states $|\uparrow\rangle$ and $|\downarrow\rangle$ rules out the existence of probability space of specific quantum measurement. If we detect $|\uparrow\rangle$, then we assign measurement outcome as $+1$. If we detect $|\downarrow\rangle$, then we assign measurement outcome as -1. This hidden-variables theory does not accept the transition probability $|\langle\uparrow|\downarrow\rangle|^2 = 0$. Therefore we have to give up the hidden-variables theory. This implies the Stern-Gerlach experiment cannot accept the hidden-variables theory. A single spin-1/2 pure state (e.g., $|\uparrow\rangle\langle\uparrow|$) is a single one-dimensional projector. In other word, a single one-dimensional projector does not have a counterpart in such physical reality, in general. The one-dimensional projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ are commuting with each other. Our discussion shows that we cannot assign the specific definite values ($+1$ and -1) to the two commuting operators, simultaneously. We study whether quantum phase factor accepts a hidden-variables theory. We discuss that the existence of two spin-1/2 pure states $|0\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $|\theta\rangle = (|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)/\sqrt{2}$ rules out the existence of probability space of a hidden-variables theory. If we detect $|0\rangle$, then we assign measurement outcome as $+1$. If we detect $|\theta\rangle$, then we assign measurement outcome as -1. The hidden-variables theory does not accept the transition probability $|\langle 0|\theta\rangle|^2 = \cos^2(\theta/2)$. Therefore we have to give up the hidden-variables theory for quantum phase factor. We explore phase factor is indeed a quantum effect, not classical. Our research gives a new insight to the quantum information processing which relies on quantum phase factor, such as Deutsch's algorithm.

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I. INTRODUCTION

The quantum theory (cf. [1–5]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

On the other hand, from the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [6], a hidden-variable interpretation of the quantum theory has been an attractive topic of research [2, 3]. There are two main approaches to study the hidden-variable interpretation of the quantum theory. One is the Bell-EPR theorem [7]. This theorem says that the quantum predictions violate the inequality following from the EPR-locality condition. The EPR-locality condition tells that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system.

The other is the no-hidden-variables theorem of Kochen and Specker (KS theorem) [8]. The original KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators. The quantum theory does not accept the KS type of hidden-variable theory. The proof of the original KS theorem relies on intricate geometric argument. Greenberger, Horne, and Zeilinger discover [9, 10] the so-called GHZ theorem for four-partite GHZ state. And, the KS theorem becomes very simple form (see also Refs. [11–15]).

Mermin considers the Bell-EPR theorem in a multipartite state. He derives multipartite Bell inequality [16]. The quantum predictions by n-partite GHZ state violate the Bell-Mermin inequality by an amount that grows exponentially with n. And, several multipartite Bell inequalities are reported [17–25]. They also say that the quantum predictions violate local hidden-variable theories by an amount that grows exponentially with n.

As for the KS theorem, it is begun to research the validity of the KS theorem by using inequalities (see Refs. [26–29]). To find such inequalities to test the validity of the KS theorem is particularly useful for experimental investigation [30]. The KS theorem is related to the algebraic structure of a set of quantum operators. The KS theorem is independent of a quantum state under study. One of authors derives an inequality [29] as tests for the validity of the KS theorem. The quantum predictions violate the inequality when the system is in an uncorrelated state. An uncorrelated state is defined in Ref. [31]. The quantum predictions by n-partite uncorrelated state violate the inequality by an amount that grows exponentially with n.

Leggett-type nonlocal hidden-variable theory [32] is experimentally investigated [33–35]. The experiments report that the quantum theory does not accept Leggett-type nonlocal hidden-variable theory. These experiments are done in four-dimensional space (two parties) in order to study nonlocality of hidden-variable theories.

Recently, it is shown that the two expected values of a spin-1/2 pure state $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ rule out the existence of the actually measured results of von Neumann's projective measurement [36, 37]. More recently, it is also shown that the expected value of a spin-1/2 pure state $\langle\sigma_x\rangle$ rules out the existence of the actually measured results of von Neumann's projective measurement [38, 39].

Many researches address non-classicality of observables. And non-classicality of quantum state itself is not investigated very much (however see [40]). Here we ask: Does a single spin-1/2 pure quantum state have a counterpart in physical reality? We see that two spin-1/2 pure states do not have such a counterpart in physical reality, simultaneously.

We see a single spin-1/2 pure state is used in quantum computation, quantum cryptography and so on. As for quantum computation, we are inputting non-classical information into quantum computer. As for quantum cryptography, we are exchanging non-classical information. Further, in various quantum information processing, we control quantum state by means of Pauli observables, which are non-classical. This manuscript gives new and important insight to quantum information theory, which can be implemented only by non-classical devices.

The double-slit experiment is an illustration of wave-particle duality. In it, a beam of particles (such as photons) travels through a barrier with two slits removed. If one puts a detector screen on the other side, the pattern of detected particles shows interference fringes characteristic of waves; however, the detector screen responds to particles. The system exhibits behaviour of both waves (interference patterns) and particles (dots on the screen).

If we modify this experiment so that one slit is closed, no interference pattern is observed. Thus, the state of both slits affects the final results. We can also arrange to have a minimally invasive detector at one of the slits to detect which slit the particle went through. When we do that, the interference pattern disappears. An analysis of a two-atom double-slit experiment based on environment-induced measurements is reported [41].

We consider the Stern-Gerlach experiment. The Stern-Gerlach experiment, named after German physicists Otto Stern and Walther Gerlach, is an important experiment in quantum mechanics on the deflection of particles. This experiment, performed in 1922, is often used to illustrate basic principles of quantum mechanics. It can be used to demonstrate that electrons and atoms have intrinsically quantum properties, and how measurement in quantum mechanics affects the system being measured.

On the other hand, in quantum mechanics, a phase factor is a complex coefficient $e^{i\theta}$ that multiplies a ket $|\psi\rangle$ or bra $\langle\phi|$. It does not, in itself, have any physical meaning, since the introduction of a phase factor does not change the expectation values of a Hermitian operator. That is, the values of $\langle\phi|A|\phi\rangle$ and $\langle\phi|e^{-i\theta}Ae^{i\theta}|\phi\rangle$ are the same. However, differences in phase factors between two interacting quantum states can sometimes be measurable (such as in the Berry phase) and this can have important consequences. In optics, the phase factor is an important quantity in the treatment of interference.

In this paper, we discuss the fact that a single spin observable σ_x in a quantum state does not have a counterpart in physical reality. We consider whether a single spin-1/2 pure state has a counterpart in physical reality. It is an eigenvector of Pauli observable σ_z or an eigenvector of Pauli observable σ_x. We assume a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We assume also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. The value of transition probability $|\langle +_z|+_x\rangle|^2$ is 1/2. We consider the following physical situation. If we detect $|+_z\rangle$, then we assign measurement outcome as +1. If we detect $|+_x\rangle$, then we assign measurement outcome as -1. The existence of a single classical probability space for the transition probability within the formalism of the measurement outcome does not coexist with the value of the transition probability $|\langle +_z|+_x\rangle|^2 = 1/2$. We have to give up the existence of such a classical probability space for the state $|+_z\rangle$ or for the state $|+_x\rangle$, as they define the transition probability. It turns out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality, in general. We investigate whether the Stern-Gerlach experiment accepts hidden-variables theories. We discuss that the existence of the two spin-1/2 pure states $|\uparrow\rangle$ and $|\downarrow\rangle$ rules out the existence of probability space of specific quantum measurement. If we detect $|\uparrow\rangle$, then we assign measurement outcome as +1. If we detect $|\downarrow\rangle$, then we assign measurement outcome as -1. This hidden-variables theory does not accept the transition probability $|\langle\uparrow|\downarrow\rangle|^2 = 0$. Therefore we have to give up the hidden-variables theory. This implies the Stern-Gerlach experiment cannot accept the hidden-variables theory. A single spin-1/2 pure state (e.g., $|\uparrow\rangle\langle\uparrow|$) is a single one-dimensional projector. In other word, a single one-dimensional projector does not have a counterpart in such physical reality, in general. The one-dimensional projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ are commuting with each other. Our discussion shows that we cannot assign the specific definite values (+1 and -1) to the two commuting operators, simultaneously. We study whether quantum phase factor accepts a hidden-variables theory. We discuss that the existence of two spin-1/2 pure states $|0\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $|\theta\rangle = (|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)/\sqrt{2}$ rules out the existence of probability space of a hidden-variables theory. If we detect $|0\rangle$, then we assign measurement outcome as +1. If we detect $|\theta\rangle$, then we assign measurement outcome as -1. The hidden-variables theory does not accept the transition probability $|\langle 0|\theta\rangle|^2 = \cos^2(\theta/2)$. Therefore we have to give up the hidden-variables theory for quantum phase factor. We explore phase factor is indeed a quantum effect, not classical. Our research gives a new insight to the quantum information processing which relies on quantum phase factor, such as Deutsch's algorithm.

Our paper is organized as follows.

In Sec. II, we discuss the fact that a Pauli observable in a quantum state does not have a counterpart in physical reality.

In Sec. III, we consider whether a single spin-1/2 pure quantum state has a counterpart in physical reality.

In Sec. IV, we investigate whether the Stern-Gerlach experiment accepts hidden-variables theories.

In Sec. V, we study whether a quantum phase factor accepts hidden-variables theories.

In Sec. VI, we give simple discussion about physical meaning.

Section VII concludes this paper.

II. DOES PAULI OBSERVABLE IN A QUANTUM STATE HAVE A COUNTERPART IN PHYSICAL REALITY?

We assume an implementation of the double-slit experiment. There is a detector just after each slit. Interference figure does not appear, and we do not consider such a pattern. Let (σ_z, σ_x) be Pauli vector. We assume that a source of spin-carrying particles emits them in a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z.

We consider a quantum expected value $\langle\sigma_x\rangle$ as

$$\langle\sigma_x\rangle = \langle +_z|\sigma_x|+_z\rangle = 0. \tag{1}$$

We introduce a hidden variables theory for the quantum expected value of Pauli observable σ_x. Then, the quantum expected value given in (1) can be

$$\langle\sigma_x\rangle = \int d\lambda \rho(\lambda) f(\lambda). \tag{2}$$

where λ is some hidden variable, $\rho(\lambda)$ is a probabilistic distribution, and $f(\lambda)$ is the predetermined "hidden" result of the measurement of the dichotomic observable σ_x. The possible values of $f(\lambda)$ are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is +1. If a particle passes another slit, then the value of the result of measurement is -1.

In what follows, we discuss that we cannot assign the truth value "1" for the proposition (2). Assume the proposition (2) is true. We have same proposition

$$\langle\sigma_x\rangle = \int d\lambda' \rho(\lambda') f(\lambda'). \tag{3}$$

An important note here is that the value of the right-hand-side of (2) is equal to the value of the right-hand-side of (3) because we only change the label.

We derive a necessary condition for the quantum expected value given in (2). We derive the maximum value of the product $\langle\sigma_x\rangle^2$ of the quantum expected value. The quantum expected value is $\langle\sigma_x\rangle$ given in (2). We have

$$\begin{aligned}
&\langle\sigma_x\rangle^2 \\
&= \int d\lambda \rho(\lambda) f(\lambda) \times \int d\lambda' \rho(\lambda') f(\lambda') \\
&= \int d\lambda \rho(\lambda) \int d\lambda' \rho(\lambda') f(\lambda) f(\lambda') \\
&= \int d\lambda \rho(\lambda) \int d\lambda' \rho(\lambda') |f(\lambda) f(\lambda')| \\
&\leq \int d\lambda \rho(\lambda) \int d\lambda' \rho(\lambda') = 1.
\end{aligned} \tag{4}$$

Here we use the fact

$$|f(\lambda) f(\lambda')| = 1 \tag{5}$$

since the possible values of $f(\lambda)$ are ± 1. The inequality (4) can be saturated because we have

$$\begin{aligned}
&\{\lambda | f(\lambda) = 1\} = \{\lambda' | f(\lambda') = 1\} \\
&\{\lambda | f(\lambda) = -1\} = \{\lambda' | f(\lambda') = -1\}.
\end{aligned} \tag{6}$$

Hence we derive the following proposition if we assign the truth value "1" for a hidden variables theory for Pauli observable σ_x

$$(\langle\sigma_x\rangle^2)_{\max} = 1. \tag{7}$$

We derive a necessary condition for the quantum expected value for the system in a pure spin-1/2 state $|+_z\rangle$ given in (1). We derive the possible value of the product

$$\langle\sigma_x\rangle \times \langle\sigma_x\rangle = \langle\sigma_x\rangle^2. \tag{8}$$

$\langle\sigma_x\rangle$ is the quantum expected value given in (1). We have the following proposition since $\langle\sigma_x\rangle = 0$

$$\langle\sigma_x\rangle^2 = 0. \tag{9}$$

We have

$$\langle\sigma_x\rangle^2 \leq 0. \tag{10}$$

We have the following proposition concerning quantum mechanics

$$(\langle\sigma_x\rangle^2)_{\max} = 0. \tag{11}$$

We do not assign the truth value "1" for two propositions (7) and (11), simultaneously. We are in the contradiction. We have to give up a hidden variables theory for the expected value of Pauli observable σ_x. The measured observable σ_x in the state does not have a counterpart in physical reality.

III. DOES A SINGLE SPIN-1/2 PURE QUANTUM STATE HAVE A COUNTERPART IN PHYSICAL REALITY?

Let (σ_z, σ_x) be Pauli vector. We assume a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We assume also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. We consider a quantum expected value (transition probability) $|\langle +_z | +_x \rangle|^2$ as

$$|\langle +_z | +_x \rangle|^2 = 1/2. \tag{12}$$

We introduce a hidden variables theory for the quantum state $|+_z\rangle$ and for the quantum state $|+_x\rangle$. Then, the quantum expected value given in (12) can be

$$|\langle +_z|+_x\rangle|^2 = \int d\lambda \rho(\lambda) f(\lambda). \tag{13}$$

The possible values of $f(\lambda)$ are ± 1 (in $\hbar/2$ unit). If we detect $|+_z\rangle$, then we assign measurement outcome as $+1$. If we detect $|+_x\rangle$, then we assign measurement outcome as -1.

In what follows, we discuss that we cannot assign the truth value "1" for the proposition (13). Assume the proposition (13) is true. We have same proposition

$$|\langle +_z|+_x\rangle|^2 = \int d\lambda' \rho(\lambda') f(\lambda'). \tag{14}$$

An important note here is that the value of the right-hand-side of (13) is equal to the value of the right-hand-side of (14) because we only change the label.

We derive a necessary condition for the quantum expected value given in (13). We derive the maximum value of the product $|\langle +_z|+_x\rangle|^4$ of the quantum expected value. The quantum expected value is $|\langle +_z|+_x\rangle|^2$ given in (13). We have

$$\begin{aligned}
&|\langle +_z|+_x\rangle|^4 \\
&= \int d\lambda \rho(\lambda) f(\lambda) \times \int d\lambda' \rho(\lambda') f(\lambda') \\
&= \int d\lambda \rho(\lambda) \int d\lambda' \rho(\lambda') f(\lambda) f(\lambda') \\
&= \int d\lambda \rho(\lambda) \int d\lambda' \rho(\lambda') |f(\lambda) f(\lambda')| \\
&\leq \int d\lambda \rho(\lambda) \int d\lambda' \rho(\lambda') = 1.
\end{aligned} \tag{15}$$

Here we use the fact

$$|f(\lambda) f(\lambda')| = 1 \tag{16}$$

since the possible values of $f(\lambda)$ are ± 1. The inequality (15) can be saturated because we have

$$\begin{aligned}
&\{\lambda | f(\lambda) = 1\} = \{\lambda' | f(\lambda') = 1\} \\
&\{\lambda | f(\lambda) = -1\} = \{\lambda' | f(\lambda') = -1\}.
\end{aligned} \tag{17}$$

Hence we derive the following proposition if we assign the truth value "1" for a hidden variables theory for the quantum state $|+_z\rangle$ and for the quantum state $|+_x\rangle$

$$(|\langle +_z|+_x\rangle|^4)_{\max} = 1. \tag{18}$$

We derive a necessary condition for the quantum expected value given in (12). We derive the possible values of the product

$$|\langle +_z|+_x\rangle|^2 \times |\langle +_z|+_x\rangle|^2 = |\langle +_z|+_x\rangle|^4. \tag{19}$$

$|\langle +_z|+_x\rangle|^2$ is the quantum expected value given in (12). We have the following proposition since $|\langle +_z|+_x\rangle| = 1/2$

$$|\langle +_z|+_x\rangle|^4 = 1/4. \tag{20}$$

We have

$$|\langle +_z|+_x\rangle|^4 \leq 1/4. \tag{21}$$

We have the following proposition concerning quantum mechanics

$$(|\langle +_z|+_x\rangle|^4)_{\max} = 1/4. \tag{22}$$

We do not assign the truth value "1" for two propositions (18) and (22), simultaneously. We are in the contradiction. We have to give up a hidden variables theory for the quantum state $|+_z\rangle$ or for the quantum state $|+_x\rangle$. It turns out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality.

In short, we give up the following situation

$$\overbrace{|+_z\rangle\langle +_z|}^{\text{observable}} \rightarrow \overbrace{+1}^{\text{physical reality}} \quad \text{and} \quad \overbrace{|+_x\rangle\langle +_x|}^{\text{observable}} \rightarrow \overbrace{-1}^{\text{physical reality}}. \tag{23}$$

IV. THE STERN-GERLACH EXPERIMENT AND HIDDEN-VARIABLES THEORIES

Let σ_z be Pauli observable of z-axis. We consider two quantum states $|\uparrow\rangle$ and $|\downarrow\rangle$, which can be described as an eigenvector of Pauli observable σ_z.

We consider a quantum expected value (the transition probability) as

$$|\langle\uparrow|\downarrow\rangle|^2 = 0. \tag{24}$$

We introduce a hidden-variables theory for the quantum expected value of the transition probability. Then, the quantum expected value given in (24) can be

$$|\langle\uparrow|\downarrow\rangle|^2 = \int d\lambda\rho(\lambda)f(\lambda). \tag{25}$$

where λ is some hidden variable, $\rho(\lambda)$ is a probabilistic distribution, and $f(\lambda)$ is the predetermined "hidden" result of measurement. The possible values of $f(\lambda)$ are ± 1 (in $\hbar/2$ unit). If we detect $|\uparrow\rangle$, then we assign measurement outcome as $+1$. If we detect $|\downarrow\rangle$, then we assign measurement outcome as -1.

In what follows, we discuss that we cannot assign the truth value "1" for the proposition (25). Assume the proposition (25) is true. We have the same quantum expected value

$$|\langle\uparrow|\downarrow\rangle|^2 = \int d\lambda'\rho(\lambda')f(\lambda'). \tag{26}$$

An important note here is that the value of the right-hand-side of (25) is equal to the value of the right-hand-side of (26) because we only change the label.

We derive a necessary condition for the quantum expected value given in (25). We derive the maximum value of the product $|\langle\uparrow|\downarrow\rangle|^4$ of the quantum expected value. The quantum expected value is $|\langle\uparrow|\downarrow\rangle|^2$ given in (25). We have

$$\begin{aligned}
&|\langle\uparrow|\downarrow\rangle|^4 \\
&= \int d\lambda\rho(\lambda)f(\lambda) \times \int d\lambda'\rho(\lambda')f(\lambda') \\
&= \int d\lambda\rho(\lambda) \int d\lambda'\rho(\lambda')f(\lambda)f(\lambda') \\
&\leq \int d\lambda\rho(\lambda) \int d\lambda'\rho(\lambda')|f(\lambda)f(\lambda')| \\
&= \int d\lambda\rho(\lambda) \int d\lambda'\rho(\lambda') = 1.
\end{aligned} \tag{27}$$

Here we use the fact

$$|f(\lambda)f(\lambda')| = 1 \tag{28}$$

since the possible values of $f(\lambda)$ are ± 1. The inequality (27) can be saturated because we have

$$\begin{aligned}
&\{\lambda|f(\lambda) = 1\} = \{\lambda'|f(\lambda') = 1\} \\
&\{\lambda|f(\lambda) = -1\} = \{\lambda'|f(\lambda') = -1\}.
\end{aligned} \tag{29}$$

Hence we derive the following proposition if we assign the truth value "1" for a hidden variables theory for the transition probability

$$(|\langle\uparrow|\downarrow\rangle|^4)_{\max} = 1. \tag{30}$$

We derive a necessary condition for the quantum expected value given in (24). We derive the possible value of the product

$$|\langle\uparrow|\downarrow\rangle|^2 \times |\langle\uparrow|\downarrow\rangle|^2 = |\langle\uparrow|\downarrow\rangle|^4. \tag{31}$$

$|\langle\uparrow|\downarrow\rangle|^2$ is the quantum expected value given in (24). We have the following proposition since $|\langle\uparrow|\downarrow\rangle|^2 = 0$

$$|\langle\uparrow|\downarrow\rangle|^4 = 0. \tag{32}$$

We have

$$|\langle\uparrow|\downarrow\rangle|^4 \leq 0. \tag{33}$$

Thus,

$$(|\langle\uparrow|\downarrow\rangle|^4)_{\max}=0. \tag{34}$$

We do not assign the truth value "1" for two propositions (30) and (34), simultaneously. We are in the contradiction. We give up assigning the truth value "1" for the proposition (25).

Assume we give up proposition (25). We cannot assign the specific definite values (+1 and −1) for the quantum state $|\uparrow\rangle$ and for the quantum state $|\downarrow\rangle$ simultaneously. It turns out that the single spin-1/2 pure state $|\uparrow\rangle$ or the single spin-1/2 pure state $|\downarrow\rangle$ does not have counterparts in such physical reality simultaneously. A single spin-1/2 pure state (e.g., $|\uparrow\rangle\langle\uparrow|$) is a single one-dimensional projector. In other word, a single one-dimensional projector does not have a counterpart in such physical reality, in general. The one-dimensional projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ are commuting with each other. Our discussion shows that we cannot assign the specific definite values (+1 and −1) to the two commuting operators, simultaneously.

In short, we give up the following situation

$$\overbrace{|\uparrow\rangle\langle\uparrow|}^{\text{observable}} \rightarrow \overbrace{+1}^{\text{physical reality}} \quad \text{and} \quad \overbrace{|\downarrow\rangle\langle\downarrow|}^{\text{observable}} \rightarrow \overbrace{-1}^{\text{physical reality}}. \tag{35}$$

V. QUANTUM PHASE FACTOR AND HIDDEN-VARIABLES THEORIES

Let σ_z be Pauli observable of z-axis. We consider two quantum states $|\uparrow\rangle$ and $|\downarrow\rangle$, which can be described as an eigenvector of Pauli observable σ_z.

We study whether quantum phase factor accepts a hidden-variables theory. We use the transition probability for two spin-1/2 pure states

$$\begin{aligned}|0\rangle &= (|\uparrow\rangle+|\downarrow\rangle)/\sqrt{2},\\ |\theta\rangle &= (|\uparrow\rangle+e^{i\theta}|\downarrow\rangle)/\sqrt{2}.\end{aligned} \tag{36}$$

We consider the following transition probability

$$|\langle 0|\theta\rangle|^2=\cos^2(\theta/2). \tag{37}$$

We introduce a hidden-variables theory in order to explain the value of the transition probability. If we detect $|0\rangle$, then we assign measurement outcome as +1. If we detect $|\theta\rangle$, then we assign measurement outcome as −1. In this case, the transition probability given in (37) can be depictured as follows:

$$|\langle 0|\theta\rangle|^2=\int d\lambda\rho(\lambda)f(\lambda). \tag{38}$$

The possible values of $f(\lambda)$ are ±1 (in $\hbar/2$ unit).

Assume the proposition (38) is true. We have the same value of the transition probability

$$|\langle 0|\theta\rangle|^2=\int d\lambda'\rho(\lambda')f(\lambda'). \tag{39}$$

An important note here is that the value of the right-hand-side of (38) is equal to the value of the right-hand-side of (39) because we only change the label.

We derive a necessary condition for the transition probability given in (38). We derive the maximum value of the product

$$|\langle 0|\theta\rangle|^2\times|\langle 0|\theta\rangle|^2. \tag{40}$$

We have

$$\begin{aligned}&|\langle 0|\theta\rangle|^4\\ &=\int d\lambda\rho(\lambda)f(\lambda)\times\int d\lambda'\rho(\lambda')f(\lambda')\\ &=\int d\lambda\rho(\lambda)\int d\lambda'\rho(\lambda')f(\lambda)f(\lambda')\\ &=\int d\lambda\rho(\lambda)\int d\lambda'\rho(\lambda')|f(\lambda)f(\lambda')|\\ &\le\int d\lambda\rho(\lambda)\int d\lambda'\rho(\lambda')=1.\end{aligned} \tag{41}$$

Here we use the fact

$$|f(\lambda)f(\lambda')| = 1 \tag{42}$$

since the possible value of $f(\lambda)$ is ± 1. The inequality (41) can be saturated because we have

$$\begin{aligned} &\{\lambda|f(\lambda) = 1\} = \{\lambda'|f(\lambda') = 1\} \\ &\{\lambda|f(\lambda) = -1\} = \{\lambda'|f(\lambda') = -1\}. \end{aligned} \tag{43}$$

Hence we derive the following proposition concerning a hidden-variables theory.

$$(|\langle 0|\theta\rangle|^4)_{\max} = 1. \tag{44}$$

And we derive a necessary condition for the transition probability given in (37). We derive the possible value of the product

$$|\langle 0|\theta\rangle|^2 \times |\langle 0|\theta\rangle|^2. \tag{45}$$

We have the following proposition since the transition probability is $\cos^2(\theta/2)$

$$|\langle 0|\theta\rangle|^4 = \cos^4(\theta/2). \tag{46}$$

We have

$$|\langle 0|\theta\rangle|^4 \le \cos^4(\theta/2). \tag{47}$$

Thus,

$$(|\langle 0|\theta\rangle|^4)_{\max} = \cos^4(\theta/2). \tag{48}$$

We do not assign the truth value "1" for two propositions (44) and (48), simultaneously. We are in the contradiction. We give up proposition (38) and we accept quantum theory. We have to give up the hidden-variables theory in order to explain the value of the transition probability $\cos^2(\theta/2)$. Thus, the quantum phase factor does not accept the hidden-variables theory.

In the case that

$$\theta = \pi/2 \tag{49}$$

we get the same result of the section III.

In the case that

$$\theta = \pi \tag{50}$$

we get the same result of the section IV.

In short, we give up the following general situation

$$\overbrace{|0\rangle\langle 0|}^{\text{observable}} \rightarrow \overbrace{+1}^{\text{physical reality}} \quad \text{and} \quad \overbrace{|\theta\rangle\langle\theta|}^{\text{observable}} \rightarrow \overbrace{-1}^{\text{physical reality}}. \tag{51}$$

VI. SIMPLE DISCUSSION ABOUT PHYSICAL MEANING

This seems a very complicated analysis of a rather simple situation. It would be much easier to understand if we used the language of probability theory (random variables, expectation values etc). Instead of writing out integrals we can just write $E(X)$ and so on. In order to understand this paper we needed to translate it into this simple language.

Let's consider spins (of a spin 1/2 particle) in two directions x and z and suppose we create a particle in the state spin up in direction z. Now the question is what do we mean by transition probabilities? We want that if the particle is initially in state spin up in z direction, then, if it is measured in the direction x the result is equally likely to be up or down. We know that after this, if it is measured again in the z direction, it is equally likely to be up or down again. So any sensible hidden variables model for the spins of one particle has to take account of the measurement changing the spin. We seem to want the spin to be unchanged by the measurement. This is clearly impossible.

VII. CONCLUSIONS

In conclusion, we have discussed the fact that a single spin observable σ_x in a quantum state does not have a counterpart in physical reality. We have considered whether a single spin-1/2 pure state has a counterpart in physical reality. It has been an eigenvector of Pauli observable σ_z or an eigenvector of Pauli observable σ_x. We have assumed a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We have assumed also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. The value of transition probability $|\langle +_z|+_x\rangle|^2$ has been 1/2. We have considered the following physical situation. If we have detected $|+_z\rangle$, then we assign measurement outcome as +1. If we have detected $|+_x\rangle$, then we assign measurement outcome as −1. The existence of a single classical probability space for the transition probability within the formalism of the measurement outcome does not have coexisted with the value of the transition probability $|\langle +_z|+_x\rangle|^2 = 1/2$. We have had to give up the existence of such a classical probability space for the state $|+_z\rangle$ or for the state $|+_x\rangle$, as they define the transition probability. It has turned out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality, in general. We have investigated whether the Stern-Gerlach experiment accepts hidden-variables theories. We have discussed that the existence of the two spin-1/2 pure states $|\uparrow\rangle$ and $|\downarrow\rangle$ rules out the existence of probability space of specific quantum measurement. If we have detected $|\uparrow\rangle$, then we assign measurement outcome as +1. If we have detected $|\downarrow\rangle$, then we assign measurement outcome as −1. This hidden-variables theory does not have accepted the transition probability $|\langle\uparrow|\downarrow\rangle|^2 = 0$. Therefore we have had to give up the hidden-variables theory. This implies the Stern-Gerlach experiment cannot have accepted the hidden-variables theory. A single spin-1/2 pure state (e.g., $|\uparrow\rangle\langle\uparrow|$) has been a single one-dimensional projector. In other word, a single one-dimensional projector does not have had a counterpart in such physical reality, in general. The one-dimensional projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ have been commuting with each other. Our discussion has shown that we cannot assign the specific definite values (+1 and −1) to the two commuting operators, simultaneously. We have studied whether quantum phase factor accepts a hidden-variables theory. We have discussed that the existence of two spin-1/2 pure states $|0\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $|\theta\rangle = (|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)/\sqrt{2}$ rules out the existence of probability space of a hidden-variables theory. If we have detected $|0\rangle$, then we assign measurement outcome as +1. If we have detected $|\theta\rangle$, then we assign measurement outcome as −1. The hidden-variables theory does not have accepted the transition probability $|\langle 0|\theta\rangle|^2 = \cos^2(\theta/2)$. Therefore we have had to give up the hidden-variables theory for quantum phase factor. We have explored phase factor is indeed a quantum effect, not classical. Our research has given a new insight to the quantum information processing which relies on quantum phase factor, such as Deutsch's algorithm.

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