Special Relativity for Beginners
Part I
(Derivation of the Classical Kinetic Energy without the use of the Binomial Expansion)

In this paper I derive the classical formula for the kinetic energy of a particle from Einstein's relativistic kinetic energy formula without the use of the binomial expansion. This method is suitable to be taught in secondary schools' physics senior courses.

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1. Introduction

The traditional method of deriving the classical formula for the kinetic energy of a particle from Einstein's relativistic kinetic energy is based on the following mathematical expression known as the binomial expansion or binomial theorem:

$$(x+y)^n = x^n + \frac{n}{1!}x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \ldots \qquad (1.1)$$

for $x^2 > y^2$. Formula (1.1) is, sometimes, written as

$$(a+x)^n = a^n + \frac{n}{1!}a^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots \qquad (1.2)$$

where *a* is a constant and *x* a variable. The following section shows an alternative method of obtaining the classical kinetic energy expression without using the above relation. The nomenclature of symbols are given in **Appendix 1**.

2. The Method

I shall begin the derivation from Einstein's total relativistic energy formula:

$$E^2 = p^2 c^2 + \left(m_0 c^2\right)^2 \tag{2.1}$$

Subtracting $\left(m_0 c^2\right)^2$ to both sides

$$E^2 - \left(m_0 c^2\right)^2 = p^2 c^2 \tag{2.2}$$

Taking into account that the square difference, $E^2 - \left(m_0 c^2\right)^2$, equals the binomial multiplication, $\left(E + m_0 c^2\right)\left(E - m_0 c^2\right)$, we can rewrite equation (2.2) as follows

$$\left(E + m_0 c^2\right)\left(E - m_0 c^2\right) = p^2 c^2 \tag{2.3}$$

Considering that the definition of Einstein's relativistic kinetic energy is

$$K = E - m_0 c^2 \tag{2.4}$$

we can substitute the second factor on the first side of equation (2.3) with *K.* Thus we get

$$\left(E + m_0 c^2\right) K = p^2 c^2 \tag{2.5}$$

Solving this equation for *K* we obtain

$$K = \frac{p^2 c^2}{E + m_0 c^2} \tag{2.6}$$

Considering that the momentum, *p*, of a particle of mass, *m* is given by

$$p = m v \tag{2.7}$$

we can substitute *p* in equation (2.6) with the second side of equation (2.7). This gives

$$K = \frac{\left(m v c\right)^2}{E + m_0 c^2} \tag{2.8}$$

And according to Einstein's most famous equation

$$E = m c^2 \tag{2.9}$$

Substituting *E* in equation (2.8) with the second side of equation (2.9) we get

$$K = \frac{\left(m v c\right)^2}{m c^2 + m_0 c^2} \tag{2.10}$$

Working algebraically yields

$$K = \frac{m v^2}{1 + \frac{m_0}{m}} \tag{2.11}$$

Now we consider Einstein's relativistic mass law

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.12}$$

We assume that the following condition is satisfied

$$v \ll c \tag{2.13}$$

In other words, if the velocity, *v*, of the body is much lower than the speed of light, *c,* then the relativistic mass, *m*, of the body is approximately equal to its rest mass, m_0 , (because $(v/c)^2 \ll 1$). Mathematically

$$m \approx m_0 \tag{2.14}$$

In virtue of the approximate equation (2.14), equation (2.11) can be rewritten as

$$K \approx \frac{m_0 v^2}{1 + \frac{m_0}{m_0}} \tag{2.15}$$

which is

$$K \approx \frac{1}{2} m_0 v^2 \tag{2.16}$$

The last expression is the classical kinetic energy of a body of rest mass m_0 . Because in classical physics it is customary to denote the mass of a body by m , (and not by m_0) we can finally write

$$K \approx K_{classical} = \frac{1}{2} m v^2 \tag{2.17}$$

for $v \ll c$ (classical physics' equations apply)

3. Conclusions

In summary, this paper shows an alternative method of deriving the classical formula for the kinetic energy of a particle from Einstein's relativistic kinetic energy. Because the method presented here does not use the binomial expansion, is suitable to be taught in secondary schools' physics senior courses (preliminary and/or HSC courses). This paper also shows that less sophisticated mathematical tools can produce identical results as those produced by more advance tools.

Appendix 1
Nomenclature

The following are the symbols used in this paper

c = speed of light in vacuum
v = speed of a body or particle of mass m
m = relativistic mass of a body or particle
m_0 = rest mass of a body or particle
p = momentum of a body or particle
E = total relativistic energy (or simply relativistic energy) of a body or particle
K = relativistic kinetic energy of a body or particle
$K_{classical}$ = classical kinetic energy of a body or particle
SR = Special Relativity (Einstein's theory of special relativity)