Theoretical study of the no-cloning theorem

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(Dated: June 19, 2015)

We review the no-cloning theorem that relies on the properties of the quantum theory. Usually, the no-cloning theorem implies that two quantum states are identical or orthogonal if we allow a cloning to be on the two quantum states. Here, we rely on the maximum value of the square of an expected value. We may result in the fact that the two quantum states under consideration could not be orthogonal if we consider the maximum value of the square of the expected value. The no-cloning theorem may imply that the two quantum states under consideration may be identical if we consider the maximum value of the square of the expected value. The no-cloning theorem itself has this character.

PACS numbers: 03.67.-a (Quantum information theory), 03.65.Ta (Quantum measurement theory), 03.67.Lx (Quantum computer)

I. INTRODUCTION

Quantum mechanics gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

The no-cloning theorem is a result of quantum mechanics that forbids the creation of identical copies of an arbitrary unknown quantum state. It was stated by Wootters and Zurek [1] and Dieks [2] in 1982, and has profound implications in quantum computing and related fields.

The state of one system can be entangled with the state of another system. For instance, one can use the Controlled NOT gate and the Walsh-Hadamard gate to entangle two qubits. This is not cloning. No well-defined state can be attributed to a subsystem of an entangled state. Cloning is a process whose result is a separable state with identical factors. According to Asher Peres and David Kaiser, the publication of the no-cloning theorem was prompted by a proposal of Nick Herbert [3] for a superluminal communication device using quantum entanglement.

A literature concerning quantum cloning topic can be seen in Ref. [4].

Our discussion provides the good security of quantum cryptography. The no-cloning theorem in this discussion implies that the two quantum states under consideration are identical even though an eavesdropper allows a cloning to be on the two quantum states. A probability that the eavesdropper selects unknown and identical quantum state is very small.

In this paper, we reconsider the no-cloning theorem that relies on the properties of the quantum theory. Usually, the no-cloning theorem implies that two quantum states are identical or orthogonal if we allow a cloning to be on the two quantum states.

We review the no-cloning theorem as follows:

$$U|\phi\rangle_A|e\rangle_B = |\phi\rangle_A|\phi\rangle_B. \tag{1}$$

U is the time evolution operator. Alice has a quantum state $|\phi\rangle_A$. Bob has a quantum state $|e\rangle_B$. Bob's state changes into $|\phi\rangle_B$ by using the time evolution operator. Thereby Alice's state is cloned into Bob's state. Let us consider the inner product of them. The inner product is explained as follows: A generalization of the scalar product. Any product $\langle u, v\rangle$ of vectors which satisfies the following conditions. It must be distributive over addition, be reflexive, $\langle au, v\rangle$ must equal $a\langle u, v\rangle$ and $\langle v, v\rangle = 0 \Rightarrow v = 0$ [5].

Then we have

$$\langle e|_B\langle\phi|_A|\psi\rangle_A|e\rangle_B = \langle e|_B\langle\phi|_A U^\dagger U|\psi\rangle_A|e\rangle_B = \langle\phi|_B\langle\phi|_A|\psi\rangle_A|\psi\rangle_B. \tag{2}$$

Thus,

$$\langle\phi|\psi\rangle_A = \langle\phi|\psi\rangle_A\langle\phi|\psi\rangle_B. \tag{3}$$

By omitting subscripts A and B, we have

$$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2. \tag{4}$$

We derive the following proposition:

$$\langle\phi|\psi\rangle^2 = 0 \vee \langle\phi|\psi\rangle^2 = 1. \tag{5}$$

Therefore the no-cloning theorem implies that two quantum states are identical or orthogonal if we allow a cloning to be on the two quantum states. By squaring each propositions, we have

$$\langle\phi|\psi\rangle^4 = 0 \vee \langle\phi|\psi\rangle^4 = 1. \tag{6}$$

We would assume that the two propositions (5) and (6) would be always true. We may not assume the two quantum states are orthogonal:

$$\langle\phi|\psi\rangle^4 = 0 \tag{7}$$

when we consider the possible maximum value of the square of the expected value $\langle\phi|\psi\rangle^4$ which is one. This may mean that we would not assume $\langle\phi|\psi\rangle = 0$. And we may assume $\langle\phi|\psi\rangle = 1$. The no-cloning theorem may imply that the two quantum states under consideration may be identical when we consider the maximum value of the square of the expected value in the discussion below.

II. THE NO-CLONING THEOREM BASED ON THE MAXIMUM VALUE OF THE SQUARE OF AN EXPECTED VALUE

A. Orthogonal case

We consider a quantum expected value as

$$\langle \phi|\psi\rangle^2 = 0. \tag{8}$$

The above quantum expected value is zero if the two quantum states under consideration $|\phi\rangle$ and $|\psi\rangle$ are orthogonal.

We derive a necessary condition for the quantum expected value given in (8). By squaring the proposition (8), we derive the following proposition

$$\langle \phi|\psi\rangle^4 = 0. \tag{9}$$

B. Whether the orthogonal case can be possible

On the other hand, a mean value E satisfies a probability interpretation of quantum measurement theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\langle \phi|\psi\rangle^2)}{m} \tag{10}$$

where l denotes a label and r is the result of quantum measurements. The notation $r_l(\langle \phi|\psi\rangle^2)$ implies that the lth outcome of quantum measurements when we would measure the expected value $\langle \phi|\psi\rangle^2$ in a thought experiment. We can assume the value of r is ± 1.

In what follows, we would not assume the two quantum states are orthogonal, that is, $\langle \phi|\psi\rangle^4 = 0$. And we consider the possible maximum value of the square of the expected value $\langle \phi|\psi\rangle^4$ which is one.

Assume the quantum mean value given in (10) admits a probability interpretation of quantum measurement theory. One has the following proposition concerning a probability interpretation of quantum measurement theory

$$\langle \phi|\psi\rangle^2(m) = \frac{\sum_{l=1}^{m} r_l(\langle \phi|\psi\rangle^2)}{m}. \tag{11}$$

We can assume as follows by Strong Law of Large Numbers [6],

$$\langle \phi|\psi\rangle^2(+\infty) = \langle \phi|\psi\rangle^2. \tag{12}$$

Assume the proposition (11) would be true. By changing the label l into l', we have the same quantum mean value as follows

$$\langle \phi|\psi\rangle^2(m) = \frac{\sum_{l'=1}^{m} r_{l'}(\langle \phi|\psi\rangle^2)}{m}. \tag{13}$$

An important note here is that the value of the right-hand-side of (11) is equal to the value of the right-hand-side of (13) because we only change the label l into l'.

We are very interested in the maximum value of the square of an expected value in a probability interpretation of quantum measurement theory. Therefore we focus on each measurement results providing a probability. In fact, we can easily solve the problem when we may use the Kronecker delta because minus one multiplied by minus one is plus one and plus one multiplied by plus one is plus one, and then all values are plus one. And we obtain the maximum value when we take the summation of them. In short, we can multiply a measuremenet result by the same measurement result. Therefore, we may introduce the Kronecker delta in the discussion below.

The Kronecker delta is explained as follows: The two variable function $\delta_{ll'}$ that takes the value 1 when $l = l'$ and the value 0 otherwise. If the elements of a square matrix are defined by the delta function, the matrix produced will be the identity matrix [5].

We have

$$
\begin{aligned}
&\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m) \times \frac{\delta_{ll'}}{\delta_{ll'}} \\
&= \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\langle\phi|\psi\rangle^2)}{m} \times \frac{\delta_{ll'}}{\delta_{ll'}} \\
&= \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{\sum_{l=1}^{m}} \times \frac{\sum_{l'=1}^{m} r_{l'}(\langle\phi|\psi\rangle^2)}{\sum_{l'=1}^{m}} \times \frac{\delta_{ll'}}{\delta_{ll'}} \\
&= \frac{\sum_{l=1}^{m}}{m} \cdot (r_l(\langle\phi|\psi\rangle^2))^2 \\
&= \frac{\sum_{l=1}^{m}}{m} = 1.
\end{aligned} \tag{14}
$$

Here $\delta_{ll'}$ is the Kronecker delta. We use the following fact

$$(r_l(\langle\phi|\psi\rangle^2))^2 = 1. \tag{15}$$

Thus we may derive a proposition concerning the maximum value of the square of the mean value, that is,

$$\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m) \times \frac{\delta_{ll'}}{\delta_{ll'}} = 1. \tag{16}$$

From Strong Law of Large Numbers, we may have

$$\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2 \times \frac{\delta_{ll'}}{\delta_{ll'}} = 1. \tag{17}$$

Hence we may derive the following proposition concerning the maximum value of the square of the expected value

$$\langle\phi|\psi\rangle^4 \times \frac{\delta_{ll'}}{\delta_{ll'}} = 1. \tag{18}$$

Thus, when $l = l'$, we may have

$$\langle\phi|\psi\rangle^4 = 1. \tag{19}$$

On the other hand, when $l < l'$, we may have

$$
\begin{aligned}
&\frac{\sum_{l=1}^{l'}}{\sum_{l=1}^{l'}} \langle\phi|\psi\rangle^4 \times \frac{\delta_{ll'}}{\delta_{ll'}} = \frac{\sum_{l=1}^{l'}}{\sum_{l=1}^{l'}} 1. \\
&\Rightarrow \langle\phi|\psi\rangle^4 \frac{\sum_{l=1}^{l'}}{\sum_{l=1}^{l'}} \frac{\delta_{ll'}}{\delta_{ll'}} = \frac{l'}{l'}. \\
&\Rightarrow \langle\phi|\psi\rangle^4 = 1.
\end{aligned} \tag{20}
$$

Finally, when $l' < l$, we may have

$$
\begin{aligned}
&\frac{\sum_{l'=1}^{l}}{\sum_{l'=1}^{l}} \langle\phi|\psi\rangle^4 \times \frac{\delta_{ll'}}{\delta_{ll'}} = \frac{\sum_{l'=1}^{l}}{\sum_{l'=1}^{l}} 1. \\
&\Rightarrow \langle\phi|\psi\rangle^4 \frac{\sum_{l'=1}^{l}}{\sum_{l'=1}^{l}} \frac{\delta_{ll'}}{\delta_{ll'}} = \frac{l}{l}. \\
&\Rightarrow \langle\phi|\psi\rangle^4 = 1.
\end{aligned} \tag{21}
$$

Hence we may have

$$\langle\phi|\psi\rangle^4 = 1. \tag{22}$$

Therefore, we may not assume the two quantum states under consideration could be orthogonal:

$$\langle\phi|\psi\rangle^4 = 0 \tag{23}$$

and we may assume that the two quantum states under consideration could be identical:

$$\langle\phi|\psi\rangle = 1 \wedge \langle\phi|\psi\rangle^4 = 1. \tag{24}$$

Hence we may assume the following case

$$|\phi\rangle = |\psi\rangle. \tag{25}$$

The no-cloning theorem may imply that the two quantum states under consideration could be identical if we consider the maximum value of the square of the expected value.

Why do our discussions claim that the expected value takes 1 and it does not take 0? Maybe a probability 0 is not physical, we think. The problem is open.

III. CONCLUSIONS

In conclusion, we have reviewed the no-cloning theorem that relies on the properties of the quantum theory. Usually, the no-cloning theorem has implied that two quantum states are identical or orthogonal if we allow a cloning to be on the two quantum states. Here, we have relied on the maximum value of the square of an expected value. We may have resulted in the fact that the two quantum states under consideration could not be orthogonal if we consider the maximum value of the square of the expected value. The no-cloning theorem may have implied that the two quantum states under consideration may be identical if we consider the maximum value of the square of the expected value. The no-cloning theorem itself has had this character.

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[5] C. Clapham and J. Nicholson, *The Concise Oxford Dictionary of Mathematics* (Oxford University Press, 2014), Fifth ed.
[6] In probability theory, the law of large numbers is a theorem that describes the result of performing the same experiment a large number of times. According to the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed. The strong law of large numbers states that the sample average converges almost surely to the expected value.