Quaternions and Clifford Geometric Algebras

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September 14, 2017 (v2)

1 Preface

1.1 Revision, September 14, 2017 (v2)

Warning: *In the First Draft Edition, the discussion of the EGPB (in sections 2.38 and 2.39) has very serious errors, and the method of EGPB computation as given there is generally wrong, making those sections unsuitable for reading! The faulty method of EGPB calculation seemed to be correct for the particular examples given.*

In this revision, sections 2.38 (The inner product of blades) and 2.39 (Expansion of the geometric product of blades, EGPB) have been almost completely rewritten to correct errors throughout those sections that had rendered those sections completely unsuitable for reading in the First Draft Edition. I am confident in the corrections made in this revision, but no proofs are given and other errors are always possible. For more detailed discussions and proofs of EGPB, consider reading reference [8].

Not much else has been changed in this revision, and other errors may remain to be fixed in a possible future revision. I still do not consider this revision to be a completely reliable and accurate reference. I *do not* recommend citing it as a formal reference. *This book is intended to be a very informal introduction to the subject.* That said, I do have some confidence that most of the content in this book is correct and suitable for reading. Serious students and researchers should read published books in the subject, and perhaps read this book only as a supplementary text to be carefully compared with published books.

I would like to thank Guntram Kunz for the very valuable e-mail discussions that revealed the errors that are present in sections 2.38 and 2.39 of the First Draft Edition.

- Robert Benjamin Easter (`reaster2015@gmail.com`).

1.2 First Draft Edition, June 19, 2015 (v1)

As a first rough draft that has been put together very quickly, this book is likely to contain errata and disorganization. The references list and inline citations are very incompete, so the reader should search around for more references.

I do not claim to be the inventor of any of the mathematics found here. However, some parts of this book may be considered new in some sense and were in small parts my own original research.

Much of the contents was originally written by me as contributions to a web encyclopedia project just for fun, but for various reasons was inappropriate in an encyclopedic volume. I did not originally intend to write this book. This is not a dissertation, nor did its development receive any funding or proper peer review.

I offer this free book to the public, such as it is, in the hope it could be helpful to an interested reader.

June 19, 2015 - Robert B. Easter. (v1)
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Quaternion Algebra

1.1 The Quaternion Formula

In the 1870's, William Kingdon Clifford sought to extend and unify Hamilton's *quaternions* with Hermann Grassmann's *extensive quantities* into a single algebra that Clifford called *geometric algebra*. Clifford's geometric algebra has also been named after him in his honor as *Clifford Algebra*.

Hamilton's quaternions have many representations in geometric algebra, and they have many very important uses. Grassmann's extensive quantities are represented in geometric algebra as the outer or exterior products of vectors.

Quaternions were introduced in 1853 by Sir William Rowan Hamilton in his book *Lectures on Quaternions*[4]. Hermann Grassmann published his book on extensive quantities, called *Die Lineale Ausdehnungslehre*, in 1844.

Hamilton sought to extend the planar, 2D (w, x) complex numbers

$$c = w1 + xi$$

into 3D (w, x, y). His solution was the 4D (w, x, y, z) *quaternion* numbers

$$q = w1 + xi + yj + zk.$$

It took Hamilton a few years to conclude that it required a 4D, four-component complex number, or a *quaternion of numbers*, to extend 2D complex numbers into 3D. For a while, Hamilton studied and wrote papers on number triplets (x, y, z) but was unable to achieve a satisfactory extension of complex numbers using only number triplets.

The *real* part of a quaternion

$$w = \mathrm{Re}(q)$$

is also called the *scalar* or *metric* part, and is usually interpretted as the fourth dimension of the quaternion. The *pure quaternion* or *imaginary* part

$$xi + yj + zk = \mathrm{Im}(q)$$

is interpretted as a *geometric* 3D (x, y, z) space. The three complex planes

$$\begin{gathered} w1 + xi \\ w1 + yj \\ w1 + zk \end{gathered}$$

share the real metric axis 1. In the shared real metric axis 1, the three complex geometric axes i, j, k can be related to each other by an arbitrarily defined formula or *rule*. Hamilton's breakthrough, which culminated in his invention or formulation of quaternions, was the specific formula he found, which very-usefully relates $1, i, j$, k to each other, metrically and geometrically.

The numbers i, j, k are called *complex units*. Any complex unit u has the complex unit *rule* or metric *signature*

$$u^2 = -1$$

which is an impossible square for any *real* number. A complex unit u is also called an *imaginary*. A complex number c can be written in exponential form using the formula of LEONHARD EULER as

$$\begin{aligned} c &= \sqrt{cc^\dagger}e^{i\arg(c)} \\ &= |c|e^{i\theta_c} \\ &= |c|(\cos(\theta_c) + \sin(\theta_c)i). \end{aligned}$$

The *conjugate* of c is

$$c^\dagger = w - xi.$$

In most texts on complex numbers, the *complex conjugate* of c is written

$$\begin{aligned} \bar{c} = c^* &= \mathrm{Re}(c) - \mathrm{Im}(c) \\ &= w - xi \end{aligned}$$

which is the *real* component minus the *imaginary* component of c. These are three different notations for the same conjugate. The notation c^* has a different meaning in Clifford algebra where c^* denotes the *dual* of c, and $\bar{c}$ is often the adjoint of a linear function $\underline{c}$.

The *argument* function $\arg(x + yi)$ has a standard piece-wise function definition, but symbolically it can be defined as

$$\arg(c) = -\ln\left(\frac{c}{|c|}\right)i$$

and gives the radians angle θ_c of c. Using this well-known exponential form of complex number, the rotation of a complex number in the complex c-plane has the very nice formula

$$\begin{aligned} c' &= Rc \\ &= e^{\theta i}c \\ &= e^{\theta i}|c|e^{i\theta_c} \\ &= |c|e^{(\theta+\theta_c)i} \\ &= cR. \end{aligned}$$

The complex number $R = e^{\theta i}$ is called a *rotor* or *rot*ation operat*or*, and it rotates c into c' though an angle θ in the complex c-plane. The complex number rotor is limited to rotating points in a complex plane.

With quaternions, Hamilton succeeded in extending the rotor R to rotating points in a 3D space around arbitrary axes and in arbitrary planes. To achieve this extension of rotors for rotations in 3D, Hamilton derived his famous formula or specific rule for the quaternion complex units i, j, k

$$i^2 = j^2 = k^2 = ijk \ = \ -1.$$

Complex numbers $c = w + xi$ were *not* originally interpretted geometrically as points on a complex c-plane. Instead, complex numbers were first treated as abstract quantities or as just numbers having real $\mathrm{Re}(c) = w$ and imaginary $\mathrm{Im}(c) = xi$ parts. With quaternions, Hamilton originally gave a geometrical interpretation of the three complex units i, j, k. The three complex units i, j, k represent a *orthonormal* frame or basis for 3D *geometric space* as three mutually perpendicular geometrical axes of space that each have *unit length* or length 1. The four numbers w, i, j, k are also in a more-abstract 4D (w, i, j, k) *complex space*. These two different spaces are two interpretations of the units i, j, k, one as geometric *unit vectors* in a 3D geometric space, and the second as *complex units* of a 4D complex space that includes w as the real part.

Hamilton invented the word *vector* to describe the geometrical and physical significance of the units i, j, k. Vectors represent linear motions or displacements in space that can be added and subtracted to move from point to point in space. These motions were originally described as *vections* or *conveyances*. A vector also represents a point of space relative to an origin.

Ever since the 1901 book *Vector Analysis*[2], it has been common to use boldface type for the name of a vector or geometrical quantity. Interpretting the quaternion *complex units* i, j, k as vectors, as Hamilton himself did, then Hamilton's famous formula or rule is written

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} \ = \ -1.$$

The formula is a *definition* for how the complex unit *vectors* $\mathbf{i}, \mathbf{j}, \mathbf{k}$ relate to each other, and it also defines their squares or metric signatures as -1.

In geometric algebra, the sign of the square of a vector is called the *signature* of the vector. The quaternion complex unit vectors are vectors with signature -1. Vectors with signature $+1$ are called *Euclidean* or *hyperbolic* vectors. Vectors can also be defined or constructed that have signature 0, and these vectors are called *null* vectors. In a geometric algebra, vectors of both signatures $+1$ and -1 are often defined and used together in the geometric algebra. The conformal and quadric geometric algebras use vectors of all three signatures.

The pure quaternion *vectors*

$$\mathbf{v} \ = \ x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

have been used extensively in vector analysis, which is taught in courses on *Calculus*. Many popular books on calculus teach vector analysis without ever mentioning quaternions. In vector analysis, the *complex* nature of the vector units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is hidden by the *algebra of vector analysis*, which does not use quaternions and does not make any direct usage of quaternion multiplication. By using only the dot and cross products, detailed in the next sections, vector analysis treats the complex vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as if they were *Euclidean* unit vectors of a Euclidean 3D space. The actual 3D space of the pure quaternion unit vectors $\mathbf{v}$ is a 3D *complex* vector space, where the quaternion vectors are sometimes called *imaginary directed quantities*. In the $\mathcal{G}_3$ geometric algebra of Euclidean 3D space of three vector units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with signature $+1$, the pure quaternion vectors $\mathbf{v}$ with signature -1 have a representation as *pseudovectors*, which is all explained later in this book.

It is interesting to note that, Euclidean vectors often represent real, material, or mass entities having real geometric and physical substance; complex numbers and quaternions often represent imaginary, non-material, or energy entities. Among the first important applications of quaternions is the electromagnetic theory of JAMES CLERK MAXWELL. Euclidean vectors represent *real directed quantities*, such a physical lengths. The square of a Euclidean vector gives a positive squared value, as expected for the squared measurement of a real, material, geometrical object.

A quaternion rotor has the form

$$R = e^{\theta \mathbf{n}}$$

and rotates vectors in the plane through the origin perpendicular to unit vector $\mathbf{n}$ by the angle θ.

The multiplication of complex numbers in a complex c-plane is *commutative*, and $Rc = cR$ for two coplanar complex numbers c and R. However, the multiplication of two quaternions is generally *not* commutative, and generally $Rq \neq qR$ for two *quaternions* q and R.

In geometric algebra, a *grade-k multivector* A is the product of k vectors $\mathbf{a}_i$

$$A = \prod_{i=1}^{k} \mathbf{a}_i = \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_k.$$

The product of two multivectors A and B is generally *non-commutative*, $AB \neq BA$.

Holding the unit vector $\mathbf{n}$ constant and allowing scalars α and β to vary, then quaternions in the form of coplanar complex numbers $\alpha + \beta \mathbf{n}$ have commutative multiplication like standard complex numbers, but otherwise the multiplication of quaternions is non-commutative.

The *conjugate* product of two quaternions is

$$(pq)^\dagger = q^\dagger p^\dagger.$$

In geometric algebra, where p and q are two grade-2 multivectors, the *reverse* product

$$\widetilde{pq} = \tilde{q}\tilde{p}$$

has a meaning that is very similar to the conjugate, and they are often, but not always, equivalent. The grade-2 multivectors are usually a scalar w, plus a *pseudovector* or *bivector* $\mathbf{n}^*$

$$w + x\mathbf{n}^* \;=\; w + x(\mathbf{b} \wedge \mathbf{a}).$$

The *dual* of the unit vector $\mathbf{n}$ is the bivector $\mathbf{n}^* = \mathbf{b} \wedge \mathbf{a}$, which is the *outer product* of the two perpendicular unit vectors $\mathbf{a}$ and $\mathbf{b}$ which are mutually perpendicular to $\mathbf{n}$. The unit vectors $\mathbf{n}, \mathbf{a}, \mathbf{b}$ are a frame of orthonormal unit vectors. The dual $\mathbf{n}^*$ is *not* the complex conjugate of $\mathbf{n}$. From here forward in this book, the notation $\mathbf{n}^*$ denotes the spatial *dual* of $\mathbf{n}$ in a geometric algebra.

Two *perpendicular* pure quaternions or vectors multiply *anti-commutatively* as

$$\mathbf{pq} \;=\; -\mathbf{qp}$$

or as

$$\mathbf{p} \times \mathbf{q} \;=\; -\mathbf{q} \times \mathbf{p}$$

which may be familiar from calculus.

1.2 The Scalar and Vector Parts

A quaternion q can be interpretted as the sum of a real number w and a vector $\mathbf{v}$

$$\begin{aligned} q &= w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= w + \mathbf{v}. \end{aligned}$$

The real number component w is called the *scalar part* of the quaternion. The *vector part* $\mathbf{v}$ is also called the *pure quaternion* part.

The *scalar part* operator S takes the scalar part of a quaternion q as

$$w \;=\; \mathrm{S}q = \mathrm{S}.q = \mathrm{S}(q).$$

The *vector part* operator V takes the vector part of a quaternion q as

$$\mathbf{v} \;=\; \mathrm{V}q = \mathrm{V}.q = \mathrm{V}(q).$$

There are three notations for taking a part. The first is the operator notation, the second is the product or dot notation, and the third is the function notation. The dot or function notation is used as needed to disambiguate the argument of the operator. The operator notation takes the operation on the variable that follows it, and the dot notation takes the operation on the entire expression that follows the dot, or until another dot is encountered. A dot can also be used to indicate where the argument of the operator ends when it helps avoid ambiguity. Some examples are

$$\begin{aligned} \mathrm{S}qr &= \mathrm{S}q.r = \mathrm{S}(q)r \\ \mathrm{S}.qr &= \mathrm{S}(qr) \\ \mathrm{S}.qr.s &= \mathrm{S}(qr)s \\ \mathrm{S}.qr.\mathrm{V}s &= \mathrm{S}(qr)\mathrm{V}s \\ \mathrm{S}.qr\mathrm{V}s &= \mathrm{S}(qr\mathrm{V}s). \end{aligned}$$

The quaternion q can be written as the sum

$$q = \mathrm{S}q + \mathrm{V}q.$$

1.3 The Quaternion Product

Two quaternions q and p

$$\begin{aligned} q &= q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k} = q_w + \mathbf{q} = \mathrm{S}q + \mathrm{V}q \\ p &= p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} = p_w + \mathbf{p} = \mathrm{S}p + \mathrm{V}p \end{aligned}$$

can be multiplied together by the usual algebraic method of multiplying polynomials

$$\begin{aligned} pq &= (p_w + \mathbf{p})(q_w + \mathbf{q}) \\ &= p_w q_w + p_w \mathbf{q} + q_w \mathbf{p} + \mathbf{pq} \\ &= p_w q_w + p_w \mathbf{q} + q_w \mathbf{p} + \frac{1}{2}(\mathbf{pq} + \mathbf{qp}) + \frac{1}{2}(\mathbf{pq} - \mathbf{qp}) \\ &= p_w q_w + p_w \mathbf{q} + q_w \mathbf{p} - \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \times \mathbf{q} \\ &= (p_w q_w - \mathbf{p} \cdot \mathbf{q}) + (p_w \mathbf{q} + q_w \mathbf{p} + \mathbf{p} \times \mathbf{q}). \end{aligned}$$

This multiplication is called the quaternion or *Hamilton product*. It contains components equal to the *dot product* $\mathbf{p} \cdot \mathbf{q}$ and *cross product* $\mathbf{p} \times \mathbf{q}$, which are familiar from calculus.

1.4 The Dot Product

The *dot product* of two pure quaternions $\mathbf{p}$ and $\mathbf{q}$ is their *symmetric product*

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= -\frac{1}{2}(\mathbf{pq} + \mathbf{qp}) \\ &= p_x q_x + p_y q_y + p_z q_z \\ &= |\mathbf{p}||\mathbf{q}|\cos(\theta). \end{aligned}$$

The angle θ is the angle between the vectors. The negative sign results from unit vectors squaring -1. This product is also sometimes called the *metric product* or *anti-commutator product*.

The product is symmetric in the sense that

$$\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$$

such that the dot product of two vectors is commutative.

1.5 The Cross Product

The *cross product* of two pure quaternions $\mathbf{p}$ and $\mathbf{q}$ is their *anti-symmetric product*

$$\begin{aligned}\mathbf{p}\times\mathbf{q} &= \frac{1}{2}(\mathbf{pq}-\mathbf{qp})\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ p_x & p_y & p_z\\ q_x & q_y & q_z\end{vmatrix}\\ &= (p_yq_z-p_zq_y)\mathbf{i}+(p_zq_x-p_xq_z)\mathbf{j}+(p_xq_y-p_yq_x)\mathbf{k}\\ &= |\mathbf{p}||\mathbf{q}|\sin(\theta)\mathbf{n}.\end{aligned}$$

The angle θ is the angle between the vectors. The unit vector $\mathbf{n}$ is perpendicular to the plane of $\mathbf{p}$ and $\mathbf{q}$, and it has the orientation of the quadrantal versor of the plane that rotates from $\mathbf{p}$ toward $\mathbf{q}$. The orientaton of $\mathbf{n}$ is usually determined by the *right-hand rule* on a right-handed system of axes, or by *left-hand rule* on a left-handed system of axes. The magnitude $|\mathbf{p}\times\mathbf{q}|$ is the area of the parallelogram framed by the two vectors. This product is also sometimes called the *vector product* or *commutator product*.

The cross product is anti-symmetric in the sense that

$$\mathbf{p}\times\mathbf{q} = -\mathbf{q}\times\mathbf{p}$$

such that the cross product is anti-commutative.

The quaternion product of two vectors $\mathbf{p}$ and $\mathbf{q}$

$$\begin{aligned}\mathbf{pq} &= -\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\times\mathbf{q}\\ &= |\mathbf{p}||\mathbf{q}|(-\cos(\theta)+\sin(\theta)\mathbf{n})\\ &= |\mathbf{p}||\mathbf{q}|(\cos(\pi-\theta)+\sin(\pi-\theta)\mathbf{n})\end{aligned}$$

has only the dot and cross product terms. If the vectors are both unit vectors, then their Hamilton product, simply called their product or geometric product, is a versor that rotates by the angle $\pi-\theta$ in the $\mathbf{pq}$-plane in the direction of $\mathbf{p}$ to $\mathbf{q}$. The conjugate

$$\begin{aligned}\mathrm{K}.\mathbf{pq} &= -\mathbf{p}\cdot\mathbf{q}-\mathbf{p}\times\mathbf{q} = -\mathbf{q}\cdot\mathbf{p}+\mathbf{q}\times\mathbf{p} = \mathbf{qp}\\ &= |\mathbf{p}||\mathbf{q}|(-\cos(\theta)-\sin(\theta)\mathbf{n})\\ &= |\mathbf{p}||\mathbf{q}|(\cos(\theta-\pi)+\sin(\theta-\pi)\mathbf{n})\end{aligned}$$

rotates by the negative angle $\theta-\pi$. In geometric algebra, $\widetilde{\mathbf{pq}} = \mathbf{qp}$ is the *reverse* of $\mathbf{pq}$. For versors, the *reverse* is usually the same as the *conjugate*, and also the same as the *inverse* since

$$\begin{aligned}\mathbf{pqqp} &= \mathbf{p}(-1)\mathbf{p} = -\mathbf{pp} = 1.\\ (\mathbf{pq})^{-1} &= \mathbf{qp}.\end{aligned}$$

Scalars, or any expressions that result in scalars, can always commute with any expression and can be moved to the left or right-side of the product in which they occur. Commuting scalar-valued expressions, and inserting identities equal to the scalar 1, are useful algebraics when manipulating quaternion expressions.

Two identities that are often useful, but maybe not obvious at first, are

$$\begin{aligned}\mathbf{pq}+\mathbf{qp} &= -2\mathbf{p}\cdot\mathbf{q}\\ \mathbf{pq}-\mathbf{qp} &= 2\mathbf{p}\times\mathbf{q}.\end{aligned}$$

In standard calculus or vector analysis, only the dot and cross products are employed. The dot product is always a scalar, and the cross product is always a vector, and therefore quaternions are always avoided in vector analysis.

The subject of rotations, where quaternions are most useful, is often missing in a course on calculus. However, using only the dot and cross products, the rotation formula of OLINDE RODRIGUES

$$\mathbf{v}_\theta = \cos(\theta)\mathbf{v}+\sin(\theta)(\mathbf{n}\times\mathbf{v})+(1-\cos(\theta))(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$$

provides the way to rotate a vector without using quaternions. The formula rotates the vector $\mathbf{v}$ directly, or *conically*, around the line of unit vector $\mathbf{n}$ by an angle of θ counter-clockwise by *right-hand rule* in a right-handed system of axes, or clockwise by *left-hand rule* in a left-handed system of axes. The Rodrigues rotation formula will be discussed again later in this book. Rotations are usually treated in a course on *linear algebra*, where the *trigonometry* and *matrix algebra* formulas are derived for rotations of vectors.

1.6 Conjugates

For a quaternion

$$\begin{aligned}q &= w+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\\ &= w+\mathbf{v}\\ &= \mathrm{S}q+\mathrm{V}q\end{aligned}$$

its *conjugate* is

$$\begin{aligned}\mathrm{K}q=q^\dagger &= \mathrm{S}q-\mathrm{V}q\\ &= w-\mathbf{v}\\ &= w-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}.\end{aligned}$$

The conjugate $q^\dagger$ has the scalar part w unchanged, but any imaginary, pure quaternion, or vector part is multiplied by -1, or is *inverted* or *reflected* in the origin.

There are two notations, the operator notation $\mathrm{K}q$, and the superscript dagger notation $q^\dagger$. The superscript dagger notation denotes the *conjugate* which extends the complex conjugate to multivectors in Clifford algebra. In this book, the other notations for complex conjugate, $\bar{q}$ and q^*, are *not* used so that notational conflicts are avoided.

The conjugate $\mathbf{A}^\dagger$ of a k-blade $\mathbf{A}$ (the outer product of k vectors $\mathbf{a}_i$)

$$\mathbf{A} = \bigwedge_{i=1}^{k}\mathbf{a}_i=\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_{k-1}\wedge\mathbf{a}_k$$

is defined as

$$\begin{aligned}\mathbf{A}^\dagger &= (-1)^{k_-}(-1)^{k(k-1)/2}\mathbf{A}\\ &= (-1)^{k_-}\tilde{\mathbf{A}}\end{aligned}$$

where

$$\begin{aligned}k &= \mathrm{gr}(\mathbf{A})\\ &= \mathrm{gr}_+(\mathbf{A}) + \mathrm{gr}_-(\mathbf{A}) + \mathrm{gr}_\varnothing(\mathbf{A})\\ &= k_+ + k_- + k_\varnothing\end{aligned}$$

is the grade of $\mathbf{A}$,

- $k_- = \mathrm{gr}_-(\mathbf{A})$ is the grade of $\mathbf{A}$ with signature -1,
- $k_+ = \mathrm{gr}_+(\mathbf{A})$ is the grade of $\mathbf{A}$ with signature $+1$,
- $k_\varnothing = \mathrm{gr}_\varnothing(\mathbf{A})$ is the grade of $\mathbf{A}$ with signature 0,

and $\tilde{\mathbf{A}}$ is the *reverse* of $\mathbf{A}$

$$\begin{aligned}\tilde{\mathbf{A}} = \mathbf{A}^\sim &= \bigwedge_{i=1}^{k} \mathbf{a}_{k-i+1} = \mathbf{a}_k \wedge \mathbf{a}_{k-1} \wedge \cdots \wedge \mathbf{a}_1\\ &= (-1)^{k(k-1)/2}\mathbf{A}.\end{aligned}$$

If all the vectors $\mathbf{a}_i$ have signature $+1$, then $\mathbf{A}^\dagger = \mathbf{A}^\sim$. A null vector with signature 0 is usually the sum of two vectors, one with $+1$ and the other with -1 signature, but it is also possible to define null vector elements $(\mathbf{e}_{\varnothing_i})^2 = 0$ of the algebra.

More generally, the conjugate of a grade-k multivector A is

$$A^\dagger = \sum_{i=0}^{k} \langle A \rangle_i^\dagger$$

which is the sum of the conjugates of each grade-i part. Each grade-i part may be an i-vector, in which case the conjugate of each grade-i component of the i-vector is summed. The conjugate $w^\dagger$ of a grade-0 scalar w is $w^\dagger = w$.

The quaternion versor conjugate $\mathrm{U}q^\dagger$ is similar, or isomorphic, to the *reverse*

$$\tilde{R} = e^{-\frac{1}{2}\theta(\mathbf{n}^*)}$$

of a $\mathcal{G}_3$ rotor

$$R = e^{\frac{1}{2}\theta(\mathbf{n}^*)}$$

for a rotor operation that rotates around Euclidean 3D unit vector $\mathbf{n}$ by angle θ as

$$\mathbf{v}' = R\mathbf{v}\tilde{R}.$$

The bivector or pseudovector $\mathbf{n}^*$ is the *dual* of vector $\mathbf{n}$.

A quaternion versor or rotor R and its *conjugate* $R^\dagger$ rotate in opposite directions around the same unit vector axis $\mathbf{n}$

$$\begin{aligned}R &= e^{\frac{1}{2}\theta\mathbf{n}}\\ \mathrm{K}.R = R^\dagger &= e^{-\frac{1}{2}\theta\mathbf{n}}.\end{aligned}$$

The nature of the anti-commutative multiplication of quaternions leads to the quaternion *rotor operation*, or *versor sandwich*

$$\mathbf{v}' \;=\; R\mathbf{v}R^{\dagger}$$

that performs the same conical rotation by angle θ as the example above does using the geometric algebra rotor.

1.7 Tensor or Magnitude

The magnitude $|q|$ of a quaternion $q = w + \mathbf{v}$, also called the *quaternion tensor* $\mathrm{T}q = |q|$, is given by

$$\begin{aligned} |q| = \mathrm{T}q &= \sqrt{qq^{\dagger}} = \sqrt{(w+\mathbf{v})(w-\mathbf{v})} \\ &= \sqrt{w^2 - \mathbf{v}^2} = \sqrt{w^2 - |\mathbf{v}|^2\hat{\mathbf{v}}^2} \\ &= \sqrt{w^2 + |\mathbf{v}|^2} = \sqrt{w^2 + x^2 + y^2 + z^2}. \end{aligned}$$

This is the quaternion extension of the formula for the magnitude of a complex number.

The vector $\hat{\mathbf{v}}$ denotes the unit directional vector of $\mathbf{v}$, a notation that is commonly used in vector analysis. In quaternions, another notation is $\hat{\mathbf{v}} = \mathrm{U}\mathbf{v}$, which is called the unit or *versor* of $\mathbf{v}$. All quaternion unit vectors $\hat{\mathbf{v}}$ have the square $\hat{\mathbf{v}} = -1$. The magnitude $|\mathbf{v}|$ of vector $\mathbf{v}$ is well-known as the *Euclidean norm*

$$|\mathbf{v}| \;=\; \sqrt{x^2 + y^2 + z^2}.$$

The quaternion tensor is also called the *quaternion norm*.

1.8 Versors

Hamilton gave the original definition of a *versor* as a quaternion with unit tensor. In Hamilton's original terminology, a *versor* is a *vers*ion operat*or*.

Given the quaternion $q = w + \mathbf{v}$, the unit or *versor* of q is

$$\begin{aligned} \mathrm{U}q &= \frac{q}{\mathrm{T}q} = \frac{q}{|q|} = \frac{w}{\mathrm{T}q} + \frac{\mathbf{v}}{\mathrm{T}q} \\ &= \frac{w}{\mathrm{T}q} + \frac{\mathrm{T}\mathbf{v}\mathrm{U}\mathbf{v}}{\mathrm{T}q} = \frac{w}{|q|} + \frac{|\mathbf{v}|\hat{\mathbf{v}}}{|q|} \\ &= \frac{w}{\sqrt{w^2+x^2+y^2+z^2}} + \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{w^2+x^2+y^2+z^2}}\hat{\mathbf{v}} \\ &= \cos(\theta) + \sin(\theta)\hat{\mathbf{v}} \\ &= e^{\theta\hat{\mathbf{v}}}. \end{aligned}$$

This versor is the rotor on a *complex plane* of complex numbers $x + y\hat{\mathbf{v}}$, where i has been replaced with $\hat{\mathbf{v}}$. If the angle $\theta = \frac{\pi}{2}$, then $\mathrm{U}q = \hat{\mathbf{v}}$ and is called a *quadrantal versor*. All unit vectors $\hat{\mathbf{v}}$ are quadrantal versors that rotate complex numbers $x + y\hat{\mathbf{v}}$ in their complex plane by angle $\frac{\pi}{2}$. Holding $\hat{\mathbf{v}}$ constant, then the multiplication of complex numbers of the form $p = x + y\hat{\mathbf{v}}$ is *commutative*, the same as for regular complex numbers $x + yi$. These complex numbers p can be rotated by an angle θ in their plane using the versor of their complex plane as

$$\begin{aligned} p' &= \mathrm{U}q.p = e^{\theta\hat{\mathbf{v}}}p = pe^{\theta\hat{\mathbf{v}}} \\ &= e^{\hat{\mathbf{v}}\theta}|p|e^{\hat{\mathbf{v}}\theta_p} = |p|e^{(\theta+\theta_p)\hat{\mathbf{v}}}. \end{aligned}$$

In Hamilton's terminology, p' is the version of p resulting from the versor operation of the factor $\mathrm{U}q$ into the faciend p.

By the properties of Hamilton's rule

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

we can write

$$\begin{aligned} \mathbf{i} &= \mathbf{k}/\mathbf{j} = e^{\frac{\pi}{2}\mathbf{i}} \\ \mathbf{j} &= \mathbf{i}/\mathbf{k} = e^{\frac{\pi}{2}\mathbf{j}} \\ \mathbf{k} &= \mathbf{j}/\mathbf{i} = e^{\frac{\pi}{2}\mathbf{k}} \end{aligned}$$

which shows that the quaternion vector units are each a certain quadrantal versor. For example, $\mathbf{ij} = \mathbf{k}$ such that $\mathbf{i}$ is the quadrantal versor or rotor that rotates $\mathbf{j}$ into $\mathbf{k}$ through a $\frac{\pi}{2}$ rotation in the geometric $\mathbf{jk}$-plane. For any unit vector or versor such as $\hat{\mathbf{v}}$, a similar truth holds that $\hat{\mathbf{v}}$ is also the quadrantal rotor or versor of a geometric plane perpendicular to $\hat{\mathbf{v}}$. Therefore, the versor $e^{\theta\hat{\mathbf{v}}}$ is also a rotor on *vectors* that lie in the geometric plane through the origin perpendicular to $\hat{\mathbf{v}}$. A quaternion versor is the rotor of a complex plane of complex numbers and also the rotor of a geometric plane of geometric vectors. In most applications, the versor is used as a rotor on vectors, not on the complex numbers. For any quaternion $w + \mathbf{v}$, the vector part $\mathbf{v}$ is the rotational axis of the quaternion. Hamilton's aim to extend complex numbers and their rotor into 3D is achieved by quaternions.

However, a single rotor operation $\mathbf{a}' = e^{\theta\hat{\mathbf{v}}}\mathbf{a}$ is limited to rotating a vector $\mathbf{a}$ that should lie in the plane through the origin perpendicular to $\hat{\mathbf{v}}$. A solution that allows rotating in arbitrary planes involves a rotor and its conjugate in a special *conical rotation operation* often called a *versor sandwich*

$$\mathbf{a}' = e^{\theta\hat{\mathbf{v}}}(\mathbf{a})e^{-\theta\hat{\mathbf{v}}}$$

that rotates any vector $\mathbf{a}$ around the line of the unit vector or quadrantal versor $\hat{\mathbf{v}}$ by an angle 2θ. This so-called versor sandwich will be derived and discussed in later sections.

The versor sandwich is also a very important operation in geometric algebra. In the *Conformal Geometric Algebra* and *Quadric Geometric Algebra*, this form of versor sandwich operation is used for rotations, translations, and dilations. Dilations are a method of scaling.

1.9 Biradials

Hamilton introduced the concept of a biradial in his 1853 book. The biradial concept is useful as a starting point for better understanding the rotor operator or versor sandwich. For simplicity, let vectors **a** and **b** be unit vectors so that magnitudes can be ignored. A quaternion *biradial* is the quotient or product

$$\begin{aligned} b_\theta &= \mathbf{b}/\mathbf{a} = -\mathbf{b}\mathbf{a} \\ &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b} \\ &= \cos(\theta) + \sin(\theta)\mathbf{n} \\ &= e^{\theta\mathbf{n}} \\ &= \mathrm{K}(\mathbf{a}/\mathbf{b}) \end{aligned}$$

$$\begin{aligned} b_{-\theta} &= \mathbf{a}/\mathbf{b} = -\mathbf{a}\mathbf{b} \\ &= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\times\mathbf{b} = \\ &= \cos(-\theta) + \sin(-\theta)\mathbf{n} \\ &= e^{-\theta\mathbf{n}} \\ &= \mathrm{K}(\mathbf{b}/\mathbf{a}) \end{aligned}$$

$$\begin{aligned} b_{\pi-\theta} &= \mathbf{a}\mathbf{b} \\ &= -\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b} \\ &= \cos(\pi-\theta) + \sin(\pi-\theta)\mathbf{n} \\ &= e^{(\pi-\theta)\mathbf{n}} \\ &= \mathrm{K}(\mathbf{b}\mathbf{a}) \end{aligned}$$

$$\begin{aligned} b_{\theta-\pi} &= \mathbf{b}\mathbf{a} \\ &= -\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\times\mathbf{b} \\ &= \cos(\theta-\pi) + \sin(\theta-\pi)\mathbf{n} \\ &= e^{(\theta-\pi)\mathbf{n}} \\ &= \mathrm{K}(\mathbf{a}\mathbf{b}) \end{aligned}$$

of two vectors **a** and **b**. If **a** and **b** are not unit vectors, then their magnitudes T**a** and T**b** commute onto a side of these expressions and multiply or divide into them, but the expressions are essentially the same. Vectors **a** and **b** that are used to define a quaternion biradial or rotor are sometimes called *rays*, and their magnitudes are unimportant and can be arbitrary since the magnitudes of rotors cancel out in a rotor operation. For two rays like **a** and **b**, only their directions are important to the four possible biradials of the two rays.

As Hamilton originally explained, the quaternion biradial $b = \mathbf{b}/\mathbf{a}$ (read "b by a") is an operator that *turns* **a** into **b** as an operator on **a**

$$\mathbf{b} = b\mathbf{a} = (\mathbf{b}/\mathbf{a})\mathbf{a} = \mathbf{b}\mathbf{a}^{-1}\mathbf{a} = \mathbf{b}(-\mathbf{a})\mathbf{a} = -\mathbf{b}\mathbf{a}\mathbf{a} = -\mathbf{b}(-1).$$

The meaning of the biradial operation b is very literally that, $\mathbf{b}$ is obtained by $\mathbf{a}$ as the operand of the operator b. It is $\mathbf{b}$ by $\mathbf{a}$! The simplicity of this is quite important. The quaternion algebra is a division algebra on vectors. Vector quotients or fractions having a vector as the dividend and divisor are a powerful feature of the quaternion algebra.

The rule for division is that it is equivalent to multiplication by the inverse placed on the right-hand side (RHS) as

$$\begin{aligned} \mathbf{b}/\mathbf{a} &= \mathbf{b}\mathbf{a}^{-1} = -\mathbf{b}\mathbf{a} = \text{-K}.\mathbf{ab} \\ &\neq \mathbf{a}^{-1}\mathbf{b} = -\mathbf{ab} = \text{-K}.\mathbf{ba}. \end{aligned}$$

Without fully deriving it yet, we have made use of the result that the inverse of a unit vector is

$$\begin{aligned} \mathbf{aa} &= \mathbf{a}^2 = -1 \\ \mathbf{aa}^{-1} &= 1 = -\mathbf{a}^2 = -\mathbf{aa} = \mathbf{a}(-\mathbf{a}) \\ \mathbf{a}^{-1} &= -\mathbf{a} = \text{K}\mathbf{a} \end{aligned}$$

its conjugate or negative. The inverse of a vector in general is

$$\mathbf{v}^{-1} = \frac{-\mathbf{v}}{|\mathbf{v}|^2} = \frac{\text{K}\mathbf{v}}{(\text{T}\mathbf{v})^2} = \frac{\text{K}\mathbf{v}}{-(\text{T}\mathbf{v}\text{U}\mathbf{v})^2} = \frac{\mathbf{v}}{\mathbf{v}^2} = \frac{1}{\mathbf{v}}.$$

The biradial operator b is a composition of scaling and rotation that scales $\mathbf{a}$ by $\frac{|\mathbf{b}|}{|\mathbf{a}|}$ and then rotates it into the line of $\mathbf{b}$.

1.10 Quaternion Identities

This section serves as summary or quick reference to some useful quaternion identities. For simplicity, let $|\mathbf{a}| = |\mathbf{b}| = 1$ so that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors.

Definition of quaternion units and products:

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= \mathbf{ijk} = -1 \\ q &= q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k} = q_w + \mathbf{q} = \text{S}q + \text{V}q \\ p &= p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} = p_w + \mathbf{p} = \text{S}p + \text{V}p \\ pq &= (p_w q_w - \mathbf{p}\cdot\mathbf{q}) + (p_w\mathbf{q} + q_w\mathbf{p} + \mathbf{p}\times\mathbf{q}). \end{aligned}$$

Unit vectors:

$$\begin{aligned} \mathbf{u} &= u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} = \sin(\phi)[\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}] + \cos(\phi)\mathbf{k} \\ \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{u}^2 &= u_x^2\mathbf{i}^2 + u_y^2\mathbf{j}^2 + u_z^2\mathbf{k}^2 + \\ &\quad u_x u_y(\mathbf{ij} + \mathbf{ji}) + u_x u_z(\mathbf{ik} + \mathbf{ki}) + u_y u_z(\mathbf{jk} + \mathbf{kj}) \\ &= -u_x^2 - u_y^2 - u_z^2 = -|\mathbf{u}|^2 = -1. \end{aligned}$$

Conjugate, norm, and versor operations:

$$\begin{aligned}
\mathrm{K}q &= q^\dagger = \mathrm{K}(\mathrm{S}q + \mathrm{V}q) = \mathrm{S}q - \mathrm{V}q = q_w - \mathbf{q} \\
\mathrm{T}q &= |q| = \sqrt{q\,\mathrm{K}q} = \sqrt{q_w^2 - \mathbf{q}^2} = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} \geq 0 \\
\mathrm{U}q &= \frac{q}{|q|} = \frac{q}{\mathrm{T}q} = q(\mathrm{T}q)^{-1} = q\,\mathrm{T}(\,q^{-1}) = q\,\mathrm{T}q^{-1} \\
q &= \mathrm{T}q\,\mathrm{U}q \\
\mathrm{K}q &= q^{-1}\mathrm{T}q^2 = \frac{\mathrm{T}q}{\mathrm{U}q} = \mathrm{T}q(\mathrm{U}q)^{-1}.
\end{aligned}$$

Multiplicative inverse of a quaternion and a vector:

$$\begin{aligned}
q^{-1} &= \frac{\mathrm{K}q}{q\,\mathrm{K}q} = \frac{\mathrm{K}q}{\mathrm{T}q^2} = \frac{\mathrm{KU}q}{\mathrm{T}q} \\
(\mathrm{U}q)^{-1} &= \frac{\mathrm{KUU}q}{\mathrm{TU}q} = \frac{\mathrm{KU}q}{1} = \mathrm{KU}q = \mathrm{SU}q - \mathrm{VU}q \\
\mathrm{K}q &= \mathrm{T}q\mathrm{KU}q = \mathrm{KT}q\mathrm{U}q \\
\mathbf{a}^{-1} &= \frac{\mathrm{K}\mathbf{a}}{\mathbf{a}\mathrm{K}\mathbf{a}} = \frac{-\mathbf{a}}{|\mathbf{a}|^2} = -\mathbf{a} \\
\mathbf{a}^{-1}\mathbf{a} &= 1.
\end{aligned}$$

Vector products:

$$\begin{aligned}
\mathbf{ab} &= -\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}) + \frac{1}{2}(\mathbf{ab}-\mathbf{ba}) \\
\mathbf{a}\cdot\mathbf{b} &= -\frac{1}{2}(\mathbf{ab}+\mathbf{ba}) = |\mathbf{a}||\mathbf{b}|\cos(\theta) = \cos(\theta) \\
&= a_x b_x + a_y b_y + a_z b_z \\
&= \frac{|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2}{4} \\
\mathbf{a}\times\mathbf{b} &= \frac{1}{2}(\mathbf{ab}-\mathbf{ba}) = |\mathbf{a}||\mathbf{b}|\sin(\theta)\mathbf{n} = \sin(\theta)\mathbf{n} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
-\mathbf{a}\times\mathbf{b} &= \mathbf{b}\times\mathbf{a} \\
\mathbf{ab}+\mathbf{ba} &= -2(\mathbf{a}\cdot\mathbf{b}) = 2\mathrm{S}(\mathbf{ab}) \\
\mathbf{ab}-\mathbf{ba} &= 2(\mathbf{a}\times\mathbf{b}) = 2\mathrm{V}(\mathbf{ab}) \\
\cos(-\theta) &= \cos(\theta) \\
\sin(-\theta) &= -\sin(\theta) \\
\mathrm{K}(\mathbf{ab}) &= -\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\times\mathbf{b} = -\mathbf{b}\cdot\mathbf{a} + \mathbf{b}\times\mathbf{a} = \mathbf{ba} \\
\mathrm{K}(\mathbf{b}/\mathbf{a}) &= \mathrm{K}(-\mathbf{ba}) = -\mathbf{ab} = \mathbf{a}/\mathbf{b}
\end{aligned}$$

The operations $\mathrm{S}q$, $\mathrm{V}q$, $\mathrm{K}q$, $\mathrm{T}q$, $\mathrm{U}q$ are the *scalar* part, *vector* part, *conjugate*, *tensor*, and *versor* of quaternion q in Hamilton's notation and terminology.

Since scalars commute with all elements in the quaternion algebra, so do scalars such as $\mathrm{T}q$, $\mathbf{a}^2$, $\mathbf{b}^2$, and $\mathbf{a}^{-1}\mathbf{a}$.

For *perpendicular vectors* $\mathbf{u}\perp\mathbf{v}$,

$$\begin{aligned}\mathbf{uv} &= \mathbf{u}\times\mathbf{v}\\ \mathbf{vu} &= -\mathbf{uv};\end{aligned}$$

and,

$$\begin{aligned}\mathbf{ij} &= -\mathbf{ji}\\ \mathbf{ik} &= -\mathbf{ki}\\ \mathbf{jk} &= -\mathbf{kj}.\end{aligned}$$

For any scalar t and value i where $i^2=-1$, the Taylor series for e^x gives the Euler formula

$$\sum_{n=0}^{n=\infty}\frac{(ti)^n}{n!}=e^{ti} = \cos(t)+\sin(t)i.$$

1.11 The Biradial b/a

The biradials were introduced in §1.9. In this section, the components of the biradial $\mathbf{b}/\mathbf{a}$ are examined in detail, and some notation is introduced.

The notation

$$\mathbf{b}^{||\mathbf{a}} = \mathrm{proj}_{\mathbf{a}}\mathbf{b}$$

denotes the component of vector $\mathbf{b}$ that is parallel to vector $\mathbf{a}$, and is probably familiar as the *projection* of $\mathbf{b}$ on $\mathbf{a}$. The notation

$$\mathbf{b}^{\perp\mathbf{a}} = \mathrm{rej}_{\mathbf{a}}\mathbf{b}=\mathbf{b}-\mathrm{proj}_{\mathbf{a}}\mathbf{b}$$

denotes the component of vector $\mathbf{b}$ that is perpendicular to vector $\mathbf{a}$, and may be familiar as the *rejection* of $\mathbf{b}$ from $\mathbf{a}$. Sometimes the abridged notations $\mathbf{b}^{||}$ and $\mathbf{b}^{\perp}$ are used when the vector $\mathbf{a}$ is understood in context. Using these notations, the vector $\mathbf{b}$ can be written in the component form

$$\mathbf{b} = \mathbf{b}^{||\mathbf{a}}+\mathbf{b}^{\perp\mathbf{a}}$$

relative to the vector $\mathbf{a}$.

The biradial $b=\mathbf{b}/\mathbf{a}$ can be written in terms of components as

$$\begin{aligned}b &= \mathbf{ba}^{-1}=(\mathbf{b}^{||\mathbf{a}}+\mathbf{b}^{\perp\mathbf{a}})\mathbf{a}^{-1}=-(\mathbf{b}^{||\mathbf{a}}+\mathbf{b}^{\perp\mathbf{a}})\mathbf{a}\\ &= -\left(|\mathbf{b}^{||\mathbf{a}}|\mathbf{a}+|\mathbf{b}^{\perp\mathbf{a}}|\frac{\mathbf{b}^{\perp}}{|\mathbf{b}^{\perp}|}\right)\mathbf{a}=\cos(\theta)+\sin(\theta)\mathbf{a}\frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|}\\ &= \cos(\theta)+\sin(\theta)\mathbf{n}=e^{\theta\mathbf{n}}\\ &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\times\mathbf{b}=-\mathbf{ba}\end{aligned}$$

where θ is the angle from $\mathbf{a}$ to $\mathbf{b}$, and $\mathbf{n}=\mathbf{a}\times\frac{\mathbf{b}^{\perp}}{|\mathbf{b}^{\perp}|}$ is the unit vector normal to the $\mathbf{ab}$-plane.

1.12 Quadrantal Versors and Handedness

When $\theta = \pi/2$ then $b = \mathbf{n}$, showing that $\mathbf{n}$ is the *quadrantal versor* of the $\mathbf{ab}$-plane perpendicular to $\mathbf{n}$. We can also write

$$\begin{aligned}\mathbf{n} &= e^{\frac{\pi}{2}\mathbf{n}}\\ b &= \mathbf{n}^{\frac{2\theta}{\pi}} = \mathbf{n}^t\end{aligned}$$

where t is the number of *quadrants* (multiples of $\frac{\pi}{2}$ radians) of rotation round $\mathbf{n}$ as the axis of rotation following a right-hand rule on a right-handed axes model or following a left-hand rule on a left-handed axes model.

If $\mathbf{b}$ is at a positive angle in the plane from $\mathbf{a}$, this means that on right-handed axes, $b\mathbf{a}$ rotates $\mathbf{a}$ by right-hand rule counter-clockwise from $\mathbf{a}$ into $\mathbf{b}$, while on left-handed axes $b\mathbf{a}$ rotates $\mathbf{a}$ by left-hand rule clockwise from $\mathbf{a}$ into $\mathbf{b}$. Angles are positive counter-clockwise on right-handed axes, and are positive clockwise on left-handed axes.

The choice of right-handed or left-handed axes is a modeling choice outside the algebra that affects the geometric interpretation of the algebra but not the algebra itself.

The relation between the unit vectors $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ as defined by Hamilton, also defines $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as the quadrantal versors or biradials

$$\begin{aligned}\mathbf{i} &= \mathbf{k}/\mathbf{j} = e^{\frac{\pi}{2}\mathbf{i}}\\ \mathbf{j} &= \mathbf{i}/\mathbf{k} = e^{\frac{\pi}{2}\mathbf{j}}\\ \mathbf{k} &= \mathbf{j}/\mathbf{i} = e^{\frac{\pi}{2}\mathbf{k}}\end{aligned}$$

following a right-hand (counter-clockwise) rule on right-handed axes or a left-hand (clockwise) rule on left-handed axes, where $\mathbf{i}^t, \mathbf{j}^t, \mathbf{k}^t$ each rotate by $t\frac{\pi}{2}$ radians in their perpendicular plane according to the matching handedness rule and axes model being used.

1.13 Quaternions and the Left-hand Rule on Left-handed Axes

In a sense, a left-hand rule and axes are a better fit to Hamilton's defined relation since then a versor used as left-side multiplier to operate on a right-side multiplicand performs left-hand clockwise rotation for positive angles, and the same versor used as a right-side multiplier to operate on a left-side multiplicand performs right-hand counter-clockwise rotation by the same angle (the reverse rotation). For example, imagine working on left-handed axes and interpret the geometry represented by the following quadrantal versor rotations

$$\begin{aligned}e^{\frac{\pi}{2}\mathbf{k}}(x\mathbf{i} + y\mathbf{j}) = \mathbf{k}(x\mathbf{i} + y\mathbf{j}) &= x\mathbf{j}(-\mathbf{i})\mathbf{i} + y\mathbf{j}(-\mathbf{i})\mathbf{j} = x\mathbf{j} - y\mathbf{i}\\ (x\mathbf{j} - y\mathbf{i})e^{\frac{\pi}{2}\mathbf{k}} = (x\mathbf{j} - y\mathbf{i})\mathbf{k} &= x\mathbf{jj}(-\mathbf{i}) - y\mathbf{ij}(-\mathbf{i}) = x\mathbf{i} + y\mathbf{j}\\ (x\mathbf{j} - y\mathbf{i})e^{-\frac{\pi}{2}\mathbf{k}} = (x\mathbf{j} - y\mathbf{i})\mathbf{k}^{-1} &= (x\mathbf{j} - y\mathbf{i})(-\mathbf{k}) = -x\mathbf{i} - y\mathbf{j} = e^{\pi\mathbf{k}}(x\mathbf{i} + y\mathbf{j})\\ e^{\frac{\pi}{2}\mathbf{k}}(x\mathbf{i} + y\mathbf{j})e^{-\frac{\pi}{2}\mathbf{k}} &= e^{2\frac{\pi}{2}\mathbf{k}}(x\mathbf{i} + y\mathbf{j}) = -(x\mathbf{i} + y\mathbf{j})\end{aligned}$$

which show that operating with $\mathbf{k}$ on the left-side performs left-hand clockwise rotation, operating with $\mathbf{k}$ on the right-side performs a right-hand counter-clockwise (reverse) rotation, and operating with $\mathbf{k}^{-1}$ on the right-side performs another clockwise rotation.

On left-handed axes, there is arguably more consistency of handedness as if the quaternion relation $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$ was defined for a left-handed system of axes. It is also possible for the relation to have been defined $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{kji}=-1$, inverting the quaternion space and making right-handed axes arguably match the geometric interpretation of the algebra better. In Hamilton's book *Lectures on Quaternions* he often uses left-handed axes with left-hand rule clockwise rotations, and it seems that was often the preferred orientation used by the inventor of quaternions.

1.14 Rotation Formulas

These results so far will be shown to generalize for any quadrantal versor $\mathbf{u}$ rotating a vector $\mathbf{v}$ by $2\theta=2t$ *quadrants* or by $2\theta=2t\frac{\pi}{2}=t\pi$ *radians* in the plane perpendicular to $\mathbf{u}$ as

$$\begin{aligned}\mathbf{v}_{2t} &= \mathbf{u}^t\mathbf{v}\mathbf{u}^{-t}=e^{\frac{t\pi}{2}\mathbf{u}}\mathbf{v}e^{-\frac{t\pi}{2}\mathbf{u}}\\ \mathbf{v}_{2\theta} &= \mathbf{u}^{\frac{2\theta}{\pi}}\mathbf{v}\mathbf{u}^{-\frac{2\theta}{\pi}}=e^{\theta\mathbf{u}}\mathbf{v}e^{-\theta\mathbf{u}}\end{aligned}$$

where for $t=1$ quadrant the result is the reflection of $\mathbf{v}$ in $\mathbf{u}$, or $\mathbf{v}$ conically rotated π round $\mathbf{u}$, or is $\mathbf{v}$ conjugated by $\mathbf{u}$. Note that we need to convert angles in quadrants to radians since the ordinary trigonometric functions $\sin(\theta)$, $\cos(\theta)$ are defined for θ in radians.

1.15 The Scalar Power of a Vector

Any versor can be constructed or expressed using its rotational angle t in *quadrants* and its unit vector or quadrantal versor rotational axis $\mathbf{u}$ as $\mathbf{u}^t$. This expression seems to be seldomly used but is important to understand since it gives a clear geometric meaning to taking the scalar power of a vector.

For a general vector $\mathbf{v}=|\mathbf{v}|\mathbf{u}_v$, its scalar power of t can be expressed as

$$\mathbf{v}^t = |\mathbf{v}|^t\mathbf{u}_v^t$$

and still represents a *versor*

$$(\mathrm{U}\mathbf{v})^t = \mathbf{u}_v^t$$

that rotates around $\mathbf{u}_v$ by t quadrants or $t\frac{\pi}{2}$ radians, and also a scaling or *tensor* $\mathrm{T}\mathbf{v}=|\mathbf{v}|$ that is applied t times.

1.16 Rotation of Vector Components

When $\mathbf{v}$ is a vector in the plane perpendicular to $\mathbf{u}$, then $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and obey the anti-commutative multiplication property

$$\mathbf{vu} = \mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{v} = -\mathbf{uv}$$

leading to

$$\begin{aligned}\mathbf{v}_{2t} &= \mathbf{u}^t\mathbf{v}\mathbf{u}^{-t} = \mathbf{u}^t\,\mathrm{K}(\mathbf{u}^{-t})\,\mathbf{v}\\ &= \mathbf{u}^t\mathbf{u}^t\,\mathbf{v} = \mathbf{u}^{2t}\mathbf{v}\\ &= e^{\frac{t\pi}{2}\mathbf{u}}e^{\frac{t\pi}{2}\mathbf{u}}\mathbf{v} = e^{t\pi\mathbf{u}}\mathbf{v} = e^{2\theta\mathbf{u}}\mathbf{v}\end{aligned}$$

which demonstrates that the conjugate versor $\mathbf{u}^{-t}$ applied as right-side operator on $\mathbf{v}$ rotates the same (in the plane perpendicular to the versor axis $\mathbf{u}$) as the versor $\mathbf{u}^t$ itself applied as left-side operator on $\mathbf{v}$, and vice versa. The reverse rotation can be written

$$\begin{aligned}\mathbf{v} &= \mathrm{K}(\mathbf{u}^t)\mathbf{v}_{2t}\mathrm{K}(\mathbf{u}^{-t}) = \mathbf{u}^{-t}\mathbf{v}_{2t}\mathbf{u}^t\\ &= \mathbf{u}^{-t}(\mathbf{u}^t\mathbf{v}\mathbf{u}^{-t})\mathbf{u}^t\end{aligned}$$

which is also seen as two successive rotations with the second outer rotation undoing or reversing the first inner rotation.

When $\mathbf{v}$ is a vector parallel to $\mathbf{u}$, then their multiplication $\mathbf{vu} = \mathbf{v}\cdot\mathbf{u} = \mathbf{uv}$ is commutative, leading to

$$\mathbf{v}_{2\theta} = e^{\theta\mathbf{u}}\mathbf{v}e^{-\theta\mathbf{u}} = e^{\theta\mathbf{u}}e^{-\theta\mathbf{u}}\mathbf{v} = \mathbf{v}$$

where $\mathbf{v}$ is unaffected by the rotation.

In general, a vector $\mathbf{v}$ will be neither completely parallel nor perpendicular to another vector $\mathbf{u}$. Similar to how $\mathbf{b}$ was written above in the expressions for $\mathbf{b}/\mathbf{a}$, a vector $\mathbf{v}$ can be written $\mathbf{v} = \mathbf{v}^{||\mathbf{u}} + \mathbf{v}^{\perp\mathbf{u}}$ since it is always possible for any vector $\mathbf{v}$ to be written as a sum of components parallel and perpendicular to any other vector $\mathbf{u}$. The parallel part

$$\begin{aligned}\mathbf{v}^{||\mathbf{u}} &= \mathcal{P}_{\mathbf{u}}(\mathbf{v}) = -(\mathbf{v}\cdot\mathbf{u})/\mathbf{u} = \mathbf{v} - \mathbf{v}^{\perp\mathbf{u}}\\ &= (\mathbf{v}\cdot\mathbf{u})\frac{\mathbf{u}}{|\mathbf{u}|^2} = \frac{(\mathbf{u}\cdot\mathbf{v})}{(\mathbf{u}\cdot\mathbf{u})}\mathbf{u}\end{aligned}$$

is the *projection of* $\mathbf{v}$ *on* $\mathbf{u}$. The perpendicular part

$$\begin{aligned}\mathbf{v}^{\perp\mathbf{u}} &= \mathcal{P}^{\perp}_{\mathbf{u}}(\mathbf{v}) = (\mathbf{v}\times\mathbf{u})/\mathbf{u} = \mathbf{v} - \mathbf{v}^{||\mathbf{u}}\\ &= (\mathbf{v}\times\mathbf{u})\times\frac{-\mathbf{u}}{|\mathbf{u}|^2} = \frac{\mathbf{u}}{|\mathbf{u}|^2}\times(\mathbf{v}\times\mathbf{u})\\ &= \frac{\mathbf{u}}{|\mathbf{u}|^2}(\mathbf{v}\times\mathbf{u}) = \frac{(\mathbf{u}\times\mathbf{v})}{(\mathbf{u}\cdot\mathbf{u})}\times\mathbf{u}\end{aligned}$$

is the *rejection of* $\mathbf{v}$ *by* $\mathbf{u}$. We can also write the orthogonalization of $\mathbf{v}$ by $\mathbf{u}$ as

$$\mathbf{v} = \mathbf{vu}/\mathbf{u} \;=\; (-\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\times\mathbf{u})/\mathbf{u} = \mathcal{P}_{\mathbf{u}}(\mathbf{v}) + \mathcal{P}^{\perp}_{\mathbf{u}}(\mathbf{v}).$$

Combining some results from above we get

$$\begin{aligned}
\mathbf{v}_{2\theta} &= e^{\theta\mathbf{u}}\mathbf{v}e^{-\theta\mathbf{u}} = e^{\theta\mathbf{u}}(\mathbf{v}^{||\mathbf{u}} + \mathbf{v}^{\perp\mathbf{u}})e^{-\theta\mathbf{u}}\\
&= e^{\theta\mathbf{u}}\mathbf{v}^{||\mathbf{u}}e^{-\theta\mathbf{u}} + e^{\theta\mathbf{u}}\mathbf{v}^{\perp\mathbf{u}}e^{-\theta\mathbf{u}}\\
&= e^{\theta\mathbf{u}}e^{-\theta\mathbf{u}}\mathbf{v}^{||\mathbf{u}} + e^{\theta\mathbf{u}}e^{\theta\mathbf{u}}\mathbf{v}^{\perp\mathbf{u}}\\
&= \mathbf{v}^{||\mathbf{u}} + e^{2\theta\mathbf{u}}\mathbf{v}^{\perp\mathbf{u}}
\end{aligned}$$

which represents the component $\mathbf{v}^{||\mathbf{u}}$ unaffected by a null (or any) rotation round $\mathbf{u}$, while the component $\mathbf{v}^{\perp\mathbf{u}}$ undergoes a planar rotation by 2θ in the plane perpendicular to $\mathbf{u}$. The sum of these vectors is the rotated *version* $\mathbf{v}_{2\theta}$ of $\mathbf{v}$. This is an important formula for the type of rotation known as *conical rotation* by 2θ round the line of $\mathbf{u}$ relative to the origin. The cone apex is the origin, the cone base is in the plane perpendicular to and intersecting $\mathbf{v}^{||\mathbf{u}}$ at distance $|\mathbf{v}^{||\mathbf{u}}|$ from the origin, and the sides of the cone are swept by the line of $\mathbf{v}$ as it is rotated round axis $\mathbf{u}$ for a complete 2π rotation.

A versor $\mathbf{u}^t$ and its conjugate $\mathbf{u}^{-t}$ have the same angle but inverse axes

$$\begin{aligned}
\angle(\mathbf{u}^t) = \angle(\mathbf{u}^{-t}) &= t\pi/2 = \theta\\
\mathrm{Ax}(\mathbf{u}^t) = -\mathrm{Ax}(\mathbf{u}^{-t}) &= \mathbf{u}
\end{aligned}$$

or they have the same axis but reverse angles

$$\begin{aligned}
\angle(\mathbf{u}^t) = -\angle(\mathbf{u}^{-t}) &= t\pi/2 = \theta\\
\mathrm{Ax}(\mathbf{u}) = \mathrm{Ax}(\mathbf{u}^{-t}) &= \mathbf{u}
\end{aligned}$$

however, there is often a preference to take positive angles and inverse axes. The conjugate or inverse versor $\mathbf{u}^{-t}$ is sometimes also called the *reversor* of $\mathbf{u}^t$.

We've looked at the biradial $\mathbf{b}/\mathbf{a}$ already, so let's now look at the other products.

1.17 The Product ba

The product $\mathbf{ba}$ gives

$$\begin{aligned}
\mathbf{ba} &= \mathbf{ba}^{-1}\mathbf{a}^2 = \mathbf{a}^2\mathbf{ba}^{-1} = -\mathbf{ba}^{-1}\\
&= e^{\pi\mathbf{n}}e^{\theta\mathbf{n}} = e^{(\pi+\theta)\mathbf{n}}\\
&= e^{(\pi+\theta-2\pi)\mathbf{n}} = e^{(\theta-\pi)\mathbf{n}} = e^{(\pi-\theta)(-\mathbf{n})}\\
&= -[\cos(\theta) + \sin(\theta)\mathbf{n}] = -\mathbf{b}\cdot\mathbf{a} + \mathbf{b}\times\mathbf{a} = \mathrm{K}(\mathbf{ab})
\end{aligned}$$

showing that, compared to the quotient $\mathbf{b}/\mathbf{a}$, the product $\mathbf{ba}$ rotates by an additional angle π or an angle $\pi+\theta$ round the same axis $\mathbf{n}$, or rotates by the supplement angle $\pi - \theta$ in the opposite direction around the inverse axis $-\mathbf{n}$. Notice how multiplying by -1 is the same as rotating by an additional $\pm\pi$ round the same axis.

1.18 The Biradial a/b

The biradial $\mathbf{a}/\mathbf{b}$ gives

$$\begin{aligned}
\mathbf{a}\mathbf{b}^{-1} &= -\mathbf{a}\mathbf{b} = -\mathbf{a}(\mathbf{b}^{||\mathbf{a}} + \mathbf{b}^{\perp\mathbf{a}}) \\
&= -\mathbf{a}\left(|\mathbf{b}^{||\mathbf{a}}|\mathbf{a} + |\mathbf{b}^{\perp\mathbf{a}}|\frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|}\right) = \cos(\theta) - \sin(\theta)\mathbf{a}\frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|} \\
&= \cos(\theta) - \sin(\theta)\mathbf{n} = e^{-\theta\mathbf{n}} \\
&= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\times\mathbf{b} = -\mathbf{a}\mathbf{b} = \mathrm{K}(\mathbf{b}/\mathbf{a})
\end{aligned}$$

which is the inverse of $\mathbf{b}/\mathbf{a}$, as expected, that rotates by $-\theta$ round the same axis $\mathbf{n}$.

1.19 The Product ab

The product $\mathbf{ab}$ gives

$$\begin{aligned}
\mathbf{a}\mathbf{b} &= \mathbf{a}\mathbf{b}^{-1}\mathbf{b}^2 = -\mathbf{a}\mathbf{b}^{-1} \\
&= e^{\pi\mathbf{n}}e^{-\theta\mathbf{n}} = e^{(\pi-\theta)\mathbf{n}} \\
&= -[\cos(-\theta) + \sin(-\theta)\mathbf{n}] = -\cos(\theta) + \sin(\theta)\mathbf{n} \\
&= -\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b} = \mathrm{K}(\mathbf{b}\mathbf{a})
\end{aligned}$$

showing that, compared to the quotient $\mathbf{b}/\mathbf{a}$, the product $\mathbf{ab}$ rotates by the supplement angle $\pi-\theta$ round the same axis $\mathbf{n}$.

1.20 Pairs of Conjugate Versors

The quotients and products of $\mathbf{a}$ and $\mathbf{b}$ are two pairs

$$\begin{aligned}
\angle(\mathbf{b}/\mathbf{a}) &= \theta \\
\angle(\mathbf{a}/\mathbf{b}) &= -\theta
\end{aligned}$$

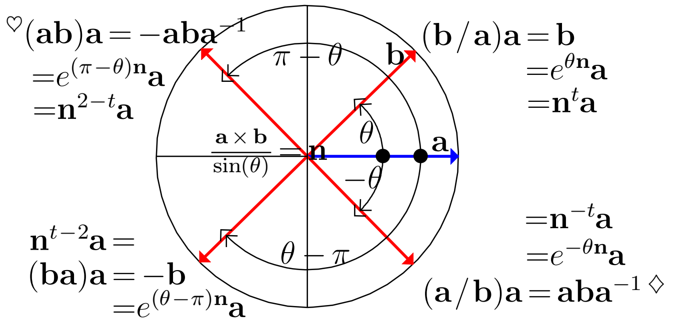
and

$$\begin{aligned}
\angle(\mathbf{ab}) &= \pi-\theta \\
\angle(\mathbf{ba}) &= \theta-\pi
\end{aligned}$$

of conjugate versors.

Conjugates have the same angle of rotation but around inverse axes, and therefore rotate in opposite directions. The conjugate of a versor is also its inverse, so they are both pairs of inverses.

Rotation of $\mathbf{a}$ by conjugate versors $\mathbf{b}/\mathbf{a}$,$\mathbf{a}/\mathbf{b}$ & $\mathbf{ab}$,$\mathbf{ba}$.



Unit vectors $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{n}^2 = -1$. $t = 2\theta/\pi$ quadrants.
♢ Reflection of $\mathbf{b}$ in $\mathbf{a}$ or conical rotation round $\mathbf{a}$ by π.
♡ Reflection of $\mathbf{b}$ in plane perpendicular to $\mathbf{a}$.

Figure 1.1. Conjugate versors.

1.21 The Rotation Formula (b/a)v(a/b)

For the $\mathbf{ab}$-plane of the biradial $\mathbf{b}/\mathbf{a}$, the unit vectors $\mathbf{a}$ and $\mathbf{b}$ span the plane and are a basis for vectors in the plane. Any vector in the plane can be written in the form $\alpha\mathbf{a} + \beta\mathbf{b}$.

The unit vector

$$\mathbf{n} \;=\; \mathbf{a} \times \frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|}$$

spans the third linearly independent spatial dimension perpendicular to the $\mathbf{ab}$-plane.

An arbitrary vector $\mathbf{v}$ of the pure quaternion vector space can be written

$$\begin{aligned} \mathbf{v} &= (\alpha\mathbf{a} + \beta\mathbf{b}) + \gamma\mathbf{n} \\ &= \mathbf{v}^{||} + \mathbf{v}^{\perp} \end{aligned}$$

where $\mathbf{v}^{||}$ is the component in the $\mathbf{ab}$-plane, and $\mathbf{v}^{\perp}$ is the component perpendicular to the $\mathbf{ab}$-plane.

The biradial $\mathbf{b}/\mathbf{a}$ operating on an arbitrary vector $\mathbf{v}$ gives

$$\begin{aligned} \mathbf{ba}^{-1}\mathbf{v} &= e^{\theta\mathbf{n}}(\mathbf{v}^{||} + \mathbf{v}^{\perp}) = e^{\theta\mathbf{n}}((\alpha\mathbf{a} + \beta\mathbf{b}) + \gamma\mathbf{n}) \\ &= \alpha e^{\theta\mathbf{n}}\mathbf{a} + \beta e^{\theta\mathbf{n}}\mathbf{b} + \gamma e^{\theta\mathbf{n}}\mathbf{n} \\ &= \alpha\mathbf{b} + \beta\mathbf{ba}^{-1}\mathbf{b} + \gamma e^{\theta\mathbf{n}} e^{\frac{\pi}{2}\mathbf{n}} \\ &= \alpha\mathbf{b} + \beta\mathbf{ba}^{-1}\mathbf{ba}^{-1}\mathbf{a} + \gamma e^{(\theta + \frac{\pi}{2})\mathbf{n}} \\ &= \alpha\mathbf{b} + \beta e^{\theta\mathbf{n}} e^{\theta\mathbf{n}}\mathbf{a} + \gamma e^{(\theta + \frac{\pi}{2})\mathbf{n}} \\ &= \alpha\mathbf{b} + \beta e^{2\theta\mathbf{n}}\mathbf{a} + \gamma e^{(\theta + \frac{\pi}{2})\mathbf{n}} \end{aligned}$$

where

$$e^{\theta\mathbf{n}}\mathbf{v}^{||} \;=\; \alpha\mathbf{b} + \beta e^{2\theta\mathbf{n}}\mathbf{a}$$

is the component of $\mathbf{v}$ in the $\mathbf{ab}$-plane, rotated by θ in the plane. The other value $\gamma e^{(\theta+\frac{\pi}{2})\mathbf{n}}$ is another quaternion that rotates by $\theta+\frac{\pi}{2}$ and scales by γ, and by itself it is not a useful part of the result; this quaternion value can be eliminated as follows

$$\begin{aligned} \gamma e^{(\theta+\frac{\pi}{2})\mathbf{n}} e^{-\theta\mathbf{n}} &= \gamma e^{\frac{\pi}{2}\mathbf{n}} \\ &= \gamma\mathbf{n} \end{aligned}$$

where we recognize

$$\begin{aligned} e^{-\theta\mathbf{n}} &= \mathbf{a}/\mathbf{b} \\ &= (\mathbf{b}/\mathbf{a})^{-1}. \end{aligned}$$

This suggests trying the following

$$\begin{aligned} (\mathbf{b}/\mathbf{a})\mathbf{v}(\mathbf{a}/\mathbf{b}) &= \left(\alpha\mathbf{b}+\beta e^{2\theta\mathbf{n}}\mathbf{a}+\gamma e^{(\theta+\frac{\pi}{2})\mathbf{n}}\right)e^{-\theta\mathbf{n}} \\ &= \alpha\mathbf{bab}^{-1}+\beta e^{2\theta\mathbf{n}}\mathbf{aab}^{-1}+\gamma\mathbf{n} \\ &= \alpha\mathbf{ba}^{-1}\mathbf{ba}^{-1}\mathbf{a}+\beta e^{2\theta\mathbf{n}}\mathbf{b}+\gamma\mathbf{n} \\ &= \alpha e^{2\theta\mathbf{n}}\mathbf{a}+\beta e^{2\theta\mathbf{n}}\mathbf{b}+\gamma\mathbf{n} \\ &= e^{2\theta\mathbf{n}}(\alpha\mathbf{a}+\beta\mathbf{b})+\gamma\mathbf{n} \\ &= e^{2\theta\mathbf{n}}\mathbf{v}^{||}+\mathbf{v}^{\perp} \end{aligned}$$

and finding that, in general, $b\mathbf{c}b^{-1}$ rotates the component $\mathbf{v}^{||}$ of $\mathbf{v}$ in the plane of the biradial b by *twice* the angle θ of the biradial, and it leaves the component $\mathbf{v}^{\perp}$ of $\mathbf{v}$ perpendicular to the plane unaffected. This type of rotation is called *conical rotation* due to how the line $\mathbf{v}$ turns on a cone as it is rotated through an entire 2π around $\mathbf{n}$ as the axis of the cone.

1.22 The Rays a and b

Although $\mathbf{a}$ and $\mathbf{b}$ were treated as unit vectors for simplicity, the result

$$b\mathbf{v}b^{-1} = \mathbf{ba}^{-1}\mathbf{vab}^{-1} = \frac{|\mathbf{b}|\mathbf{u}_b}{|\mathbf{a}|\mathbf{u}_a}\mathbf{v}\frac{|\mathbf{a}|\mathbf{u}_a}{|\mathbf{b}|\mathbf{u}_b} = \frac{\mathbf{u}_b}{\mathbf{u}_a}\mathbf{v}\frac{\mathbf{u}_a}{\mathbf{u}_b}$$

shows that $\mathbf{a}$ and $\mathbf{b}$ need not be unit vectors. Since the lengths of $\mathbf{a}$ and $\mathbf{b}$ are inconsequential in the rotation, they are sometimes called *rays* of the biradial instead of vectors, as mentioned at the beginning.

1.23 Reflection and Conjugation

In the work above, we encountered

$$\mathbf{bab}^{-1} = e^{2\theta\mathbf{n}}\mathbf{a}$$

as a planar rotation. This can also be viewed as a conical rotation by $2\frac{\pi}{2}=\pi$ around $\mathbf{b}$ as the cone axis, or viewed as a *reflection* of $\mathbf{a}$ in $\mathbf{b}$

$$\begin{aligned}
\mathbf{bab}^{-1} &= e^{\frac{\pi}{2}\mathbf{b}}\mathbf{a}e^{-\frac{\pi}{2}\mathbf{b}} = e^{\frac{\pi}{2}\mathbf{b}}(\mathbf{a}^{||\mathbf{b}}+\mathbf{a}^{\perp\mathbf{b}})e^{-\frac{\pi}{2}\mathbf{b}} \\
&= e^{\frac{\pi}{2}\mathbf{b}}\left(te^{\frac{\pi}{2}\mathbf{b}}+|\mathbf{a}^{\perp\mathbf{b}}|\frac{\mathbf{a}^{\perp\mathbf{b}}}{|\mathbf{a}^{\perp\mathbf{b}}|}\right)e^{-\frac{\pi}{2}\mathbf{b}} = te^{\frac{\pi}{2}\mathbf{b}} - |\mathbf{a}^{\perp\mathbf{b}}|e^{\frac{\pi}{2}\mathbf{b}}e^{-\frac{\pi}{2}\mathbf{b}}\frac{\mathbf{a}^{\perp\mathbf{b}}}{|\mathbf{a}^{\perp\mathbf{b}}|} \\
&= te^{\frac{\pi}{2}\mathbf{b}} + |\mathbf{a}^{\perp\mathbf{b}}|e^{\pi\mathbf{b}}\frac{\mathbf{a}^{\perp\mathbf{b}}}{|\mathbf{a}^{\perp\mathbf{b}}|} = \mathbf{a}^{||\mathbf{b}} - \mathbf{a}^{\perp\mathbf{b}} \\
\mathbf{bab}^{-1} &= -\mathbf{b}(\mathbf{a}^{||\mathbf{b}}+\mathbf{a}^{\perp\mathbf{b}})\mathbf{b} = -\mathbf{b}\left(t\mathbf{b}+|\mathbf{a}^{\perp\mathbf{b}}|\frac{\mathbf{a}^{\perp\mathbf{b}}}{|\mathbf{a}^{\perp\mathbf{b}}|}\right)\mathbf{b} \\
&= \left(t+|\mathbf{a}^{\perp\mathbf{b}}|\frac{\mathbf{a}^{\perp\mathbf{b}}}{|\mathbf{a}^{\perp\mathbf{b}}|}\mathbf{b}\right)\mathbf{b} = t\mathbf{b} - \mathbf{a}^{\perp\mathbf{b}} = \mathbf{a}^{||\mathbf{b}} - \mathbf{a}^{\perp\mathbf{b}}.
\end{aligned}$$

This result

$$\mathbf{b}(\mathbf{a}^{||\mathbf{b}}+\mathbf{a}^{\perp\mathbf{b}})\mathbf{b}^{-1} \;=\; \mathbf{a}^{||\mathbf{b}} - \mathbf{a}^{\perp\mathbf{b}}$$

is also called the *conjugation of* $\mathbf{a}$ *by* $\mathbf{b}$.

For quadrantal versor (unit vector) $\mathbf{u}$ perpendicular to vector $\mathbf{v}$,

$$\mathbf{uvu}^{-1} \;=\; -\mathbf{uvu} = \mathbf{uuv} = \mathbf{u}^2\mathbf{v} = e^{2\frac{\pi}{2}\mathbf{u}}\mathbf{v} = -\mathbf{v}$$

which is the *conjugate* $\mathbf{v}^\dagger = \mathrm{K}\mathbf{v}$ of $\mathbf{v}$, and it is also called the *inversion* of $\mathbf{v}$.

1.24 Rotations as Successive Reflections in Lines

The conical rotation $b\mathbf{v}b^{-1}$ of $\mathbf{v}$ by angle 2θ in the $\mathbf{ab}$-plane of the biradial b with angle θ can also be seen as two successive reflections

$$\begin{aligned}
b\mathbf{v}b^{-1} &= \mathbf{ba}^{-1}\mathbf{vab}^{-1} = \mathbf{b}(\mathbf{ava}^{-1})\mathbf{b}^{-1} = \mathbf{v}'' \\
\mathbf{v}' &= \mathbf{ava}^{-1} = \mathbf{a}(\mathbf{v}^{||\mathbf{a}}+\mathbf{v}^{\perp\mathbf{a}})\mathbf{a}^{-1} = \mathbf{av}^{||\mathbf{a}}\mathbf{a}^{-1} + \mathbf{av}^{\perp\mathbf{a}}\mathbf{a}^{-1} \\
&= \mathbf{a}(t\mathbf{a})\mathbf{a}^{-1} + e^{\pi\mathbf{a}}\mathbf{v}^{\perp\mathbf{a}} = \mathbf{v}^{||\mathbf{a}} - \mathbf{v}^{\perp\mathbf{a}} \\
\mathbf{v}'' &= \mathbf{v}'^{||\mathbf{b}} - \mathbf{v}'^{\perp\mathbf{b}}
\end{aligned}$$

where $\mathbf{v}'$ is the first reflection of $\mathbf{v}$ in $\mathbf{a}$, and $\mathbf{v}''$ is the second reflection of $\mathbf{v}'$ in $\mathbf{b}$.

1.25 Equivalent Versors

Every pair of unit vectors $\mathbf{b}_i, \mathbf{a}_i$ in the same plane and having the same angle θ from $\mathbf{a}_i$ to $\mathbf{b}_i$ form the same biradial

$$\begin{aligned}
b &= \mathbf{b}_i/\mathbf{a}_i \\
&= e^{\theta\mathbf{n}}
\end{aligned}$$

with the same angle $\theta = \cos^{-1}(\mathbf{a}\cdot\mathbf{b})$, and the same normal

$$\mathbf{n} = \frac{\mathbf{a}\times\mathbf{b}}{\sin(\theta)}.$$

1.26 The Square Root of a Versor

Since $b\mathbf{v}b^{-1}$ rotates $\mathbf{v}$ by 2θ, if we want to rotate by just θ then we need to take the square root of b as

$$\begin{aligned} b^{\frac{1}{2}} &= e^{\frac{\theta}{2}\mathbf{n}} \\ b^{-\frac{1}{2}} &= e^{-\frac{\theta}{2}\mathbf{n}} \end{aligned}$$

resulting in the formula for rotation by θ in the plane normal to unit vector $\mathbf{n}$

$$\begin{aligned} \mathbf{v}_\theta &= e^{\frac{\theta}{2}\mathbf{n}}\mathbf{v}e^{-\frac{\theta}{2}\mathbf{n}} \\ &= \left[\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\mathbf{n}\right]\mathbf{v}\left[\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\mathbf{n}\right]. \end{aligned}$$

Any quaternion rotation can be expressed as n successive reflections in n vectors that reduces to a single conical rotation around a single resultant versor axis $\mathbf{u}$ by a certain angle θ, which may be expressed as the versor operator

$$R_{\mathbf{u}}^{\theta}\mathbf{v} = e^{\frac{\theta}{2}\mathbf{u}}(\mathbf{v})e^{-\frac{\theta}{2}\mathbf{u}}.$$

To apply a rotation by θ in n steps, use the $(2n)$th-root of b, and its conjugate, successively n times

$$\begin{aligned} b^{\frac{1}{2n}} &= e^{\frac{\theta}{2n}\mathbf{n}} \\ \prod^{n} e^{\frac{\theta}{2n}\mathbf{n}} &= e^{\frac{\theta}{2}\mathbf{n}}. \end{aligned}$$

If $|\mathbf{a}| = |\mathbf{b}| = r$, then the versor $(\mathbf{b}/\mathbf{a})^{\frac{1}{2}}$ can also be written as

$$\begin{aligned} \mathbf{c} &= r\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|} \\ \mathbf{c}^{-1} &= \frac{-\mathbf{c}}{r^2} \\ \mathbf{b}/\mathbf{c} &= \frac{-\mathbf{bc}}{r^2} = r\frac{-(\mathbf{ba}-r^2)}{r^2|\mathbf{a}+\mathbf{b}|} \\ \mathbf{c}/\mathbf{a} &= \frac{-\mathbf{ca}}{r^2} = r\frac{-(\mathbf{ba}-r^2)}{r^2|\mathbf{a}+\mathbf{b}|} \\ \mathbf{b}/\mathbf{c} &= \mathbf{c}/\mathbf{a} \\ \mathbf{b}/\mathbf{a} &= (\mathbf{b}/\mathbf{c})(\mathbf{c}/\mathbf{a}) = (\mathbf{b}/\mathbf{c})^2 = (\mathbf{c}/\mathbf{a})^2 \\ (\mathbf{b}/\mathbf{a})^{\frac{1}{2}} &= \mathbf{b}/\mathbf{c} = \mathbf{c}/\mathbf{a} \end{aligned}$$

and the rotation of vector $\mathbf{v}$ by θ in the $\mathbf{ab}$-plane, or round its normal $\mathbf{n}$, can be written as

$$\begin{aligned} (\mathbf{b}/\mathbf{a})^{\frac{1}{2}}\mathbf{v}(\mathbf{b}/\mathbf{a})^{-\frac{1}{2}} &= (\mathbf{b}/\mathbf{c})\mathbf{v}(\mathbf{c}/\mathbf{b}) = (\mathbf{c}/\mathbf{a})\mathbf{v}(\mathbf{a}/\mathbf{c}) \\ &= (\mathbf{b}/\mathbf{c})\mathbf{v}(\mathbf{a}/\mathbf{c}) = (\mathbf{c}/\mathbf{a})\mathbf{v}(\mathbf{c}/\mathbf{b}) \end{aligned}$$

so that the rotation is seen as various reflections. For example, if $\mathbf{v}=\mathbf{a}$, then

$$\begin{aligned}(\mathbf{c}/\mathbf{a})\mathbf{a}(\mathbf{a}/\mathbf{c}) &= \mathbf{c}\mathbf{a}\mathbf{c}^{-1}\\ &= (\mathbf{a}+\mathbf{b})\mathbf{a}(\mathbf{a}+\mathbf{b})^{-1}\\ &= \mathbf{a}^{||\mathbf{c}}-\mathbf{a}^{\perp\mathbf{c}}\\ &= (\mathbf{b}/\mathbf{c})\mathbf{a}(\mathbf{a}/\mathbf{c})=(-r^2)\mathbf{b}\mathbf{c}^{-2}=\mathbf{b}\end{aligned}$$

and this rotation is the reflection of $\mathbf{a}$ in $\mathbf{c}$.

If $|\mathbf{a}|=|\mathbf{b}|$, then the sum $\mathbf{a}+\mathbf{b}$ is the diagonal vector that bisects the rhombus parallelogram framed by $\mathbf{a}$ and $\mathbf{b}$ and is the *reflector* of $\mathbf{a}$ into $\mathbf{b}$ or $\mathbf{b}$ into $\mathbf{a}$.

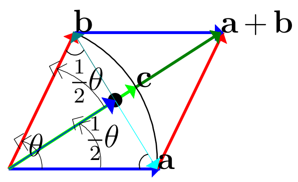


Figure 1.2. Rhombus bisector $\mathbf{a}+\mathbf{b}$

1.27 The Rodrigues Rotation Formula

Another formula for rotation can be derived without quaternions, using only vector analysis which includes vector addition and subtraction and the vector dot and cross products. The formula is known as the *Rodrigues Rotation Formula*

$$\mathbf{v}_\theta = \cos(\theta)\mathbf{v}+\sin(\theta)(\mathbf{n}\times\mathbf{v})+(1-\cos(\theta))(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$$

which rotates any vector $\mathbf{v}$ conically around the unit vector axis $\mathbf{n}$ by the angle θ radians counter-clockwise on right-handed axes or clockwise on left-handed axes according to the matching right-hand or left-hand rule.

This Rodrigues rotation formula is a known result that can be derived without too much difficulty by modeling the problem in vector analysis and finding the solution for $\mathbf{v}_\theta$. This formula can also be found from the quaternion rotation formula derived already in terms of biradials

$$\begin{aligned}\mathbf{v}_{2\theta} &= (\mathbf{b}/\mathbf{a})(\mathbf{v})(\mathbf{a}/\mathbf{b})=e^{\theta\mathbf{n}}(\mathbf{v}^{||\mathbf{n}}+\mathbf{v}^{\perp\mathbf{n}})e^{-\theta\mathbf{n}}\\ &= e^{\theta\mathbf{n}}\mathbf{v}^{||\mathbf{n}}e^{-\theta\mathbf{n}}+e^{\theta\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}e^{-\theta\mathbf{n}}\\ &= \mathbf{v}^{||\mathbf{n}}+e^{2\theta\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_\theta &= \mathbf{v}^{||\mathbf{n}}+e^{\theta\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}\\ &= \mathbf{v}^{||\mathbf{n}}+e^{\theta\mathbf{n}}(\mathbf{v}-\mathbf{v}^{||\mathbf{n}})=(1-e^{\theta\mathbf{n}})\mathbf{v}^{||\mathbf{n}}+e^{\theta\mathbf{n}}\mathbf{v}\\ &= (1-e^{\theta\mathbf{n}})(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+e^{\theta\mathbf{n}}\mathbf{v}\\ &= (1-c-s\mathbf{n})(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+(c+s\mathbf{n})\mathbf{v}\\ &= (1-c-s\mathbf{n})(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+c\mathbf{v}-s(\mathbf{n}\cdot\mathbf{v})+s(\mathbf{n}\times\mathbf{v})\\ &= c\mathbf{v}+s(\mathbf{n}\times\mathbf{v})+(1-c)(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\\ &= \cos(\theta)\mathbf{v}+\sin(\theta)(\mathbf{n}\times\mathbf{v})+(1-\cos(\theta))(\mathbf{v}\cdot\mathbf{n})\mathbf{n}.\end{aligned}$$

The identity of the quaternion rotation formula to the Rodrigues rotation formula is

$$\begin{aligned}\mathbf{v}_\theta &= R^\theta_\mathbf{n}\mathbf{v} = e^{\frac{1}{2}\theta\mathbf{n}}(\mathbf{v})e^{-\frac{1}{2}\theta\mathbf{n}} \\ &= \cos(\theta)\mathbf{v} + \sin(\theta)(\mathbf{n}\times\mathbf{v}) + (1-\cos(\theta))(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\end{aligned}$$

where again the unit vector $\mathbf{n}$ is the axis of conical rotation of vector $\mathbf{v}$ by θ according to the handedness described above.

1.28 Rotation Around Lines and Points

By proper choice of the rotation axis $\mathbf{n}$ and the angle θ, the conical rotation $R^\theta_\mathbf{n}\mathbf{v}$ can rotate $\mathbf{v}$ to any location on the *sphere* having center

$$\mathbf{o} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

and radius

$$r = \mathrm{T}\mathbf{v} = |\mathbf{v}|.$$

These conical rotations of $\mathbf{v}$ occur in *circles* located on the surface of the sphere, which we can call the *sphere of* $\mathbf{v}$.

There are two types of these circles. A *small circle* has a radius $r < |\mathbf{v}|$, and a *great circle* has radius $r = |\mathbf{v}|$. If $\mathbf{v}$ is perpendicular to $\mathbf{n}$, then the circle of rotation on the sphere is a great circle, and otherwise a small circle. The interior plane of a circle is the base of a rotational cone, with tip or apex at the origin $\mathbf{o}$. As $\mathbf{v}$ is rotated, the line of $\mathbf{v}$ moves on the sides of the cone, and $\mathbf{v}$ points to different points on the circle at the cone base.

The line $t\mathbf{n}$ of the rotation axis $\mathbf{n}$ intersects $\mathbf{o}$ and intersects the centers of a group of parallel circles in space that section the sphere of $\mathbf{v}$ by cuts perpendicular to $\mathbf{n}$. The line of $\mathbf{n}$ also intersects the sphere of $\mathbf{v}$ at the two points called the *poles* of the rotation $R^\theta_\mathbf{n}\mathbf{v}$ where the small circles vanish or degenerate into the poles. The great circle, on the sphere of $\mathbf{v}$ cut perpendicular to $\mathbf{n}$ through the center of the sphere of $\mathbf{v}$ (always the *origin* of a rotation $R^\theta_\mathbf{n}\mathbf{v}$), contains the points called the *polars* of the rotation $R^\theta_\mathbf{n}\mathbf{v}$.

When $\mathbf{v}$ is one of the two poles (a *pole vector*) of the rotation $R^\theta_\mathbf{n}\mathbf{v}$, then $\mathbf{v}$ is parallel to $\mathbf{n}$ and it rotates in the point (degenerated circle) of the pole and is unaffected by the rotation $R^\theta_\mathbf{n}\mathbf{v}$. When $\mathbf{v}$ is a polar (a *polar vector*) of the rotation $R^\theta_\mathbf{n}\mathbf{v}$, then $\mathbf{v}$ is perpendicular to $\mathbf{n}$ and is rotated in the great circle of the sphere of $\mathbf{v}$ perpendicular to $\mathbf{n}$, polar to pole $|\mathbf{v}|\mathbf{n}$. Otherwise, the rotation of $\mathbf{v}$ can be viewed in terms of its components parallel and perpendicular to the axis $\mathbf{n}$ as a *cylindrical rotation*

$$R^\theta_\mathbf{n}\mathbf{v} = R^\theta_\mathbf{n}\mathbf{v}^{||\mathbf{n}} + R^\theta_\mathbf{n}\mathbf{v}^{\perp\mathbf{n}}$$

where $\mathbf{v}^{\perp\mathbf{n}}$ is polar of the rotation $R^{\theta}_{\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}$ (planar rotation) and $\mathbf{v}^{||\mathbf{n}}$ is a pole of the rotation $R^{\theta}_{\mathbf{n}}\mathbf{v}^{||\mathbf{n}}$ (null rotation). The component $R^{\theta}_{\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}$ is rotated in the base of the cylinder in the plane perpendicular to $\mathbf{n}$ intersecting the origin of the rotation, and the component $R^{\theta}_{\mathbf{n}}\mathbf{v}^{||\mathbf{n}} = \mathbf{v}^{||\mathbf{n}}$ is the translation up the side of the cylinder to the small circle on the sphere of $\mathbf{v}$. The vector component $\mathbf{v}^{||\mathbf{n}}$ is also called an *eigenvector* ("eigen" is German for "own") of the rotation $R^{\theta}_{\mathbf{n}}\mathbf{v}^{||\mathbf{n}}$. The sum $R^{\theta}_{\mathbf{n}}\mathbf{v}^{||\mathbf{n}} + R^{\theta}_{\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}$ is still $\mathbf{v}$ rotated on a cone inside the cylinder and is a conical rotation of $\mathbf{v}$ round $\mathbf{n}$.

The quadrantal versor, unit vector $\mathbf{n}$, could also be called an *axial vector* or a pole vector of a rotation on the sphere of $\mathbf{n}$, the unit sphere. All vectors $\mathbf{x}^{\perp\mathbf{n}}$ perpendicular to the axial vector $\mathbf{n}$ could be called *planar-polar vectors* relative to all vectors $\mathbf{x}^{||\mathbf{n}}$ parallel to $\mathbf{n}$ which could be called *axial-pole vectors*, where $\mathbf{n}$ could also be called the *unit axial-pole vector*. Any circle around axis $\mathbf{n}$, in a plane perpendicular to $\mathbf{n}$ and centered on the line of $\mathbf{n}$, could be called a *polar circle* of $\mathbf{n}$. The rotation $R^{\theta}_{\mathbf{n}}\mathbf{x}^{\perp\mathbf{n}}$ of a planar-polar vector of the rotation is a planar rotation

$$e^{\frac{1}{2}\theta\mathbf{n}}\mathbf{x}^{\perp\mathbf{n}}e^{-\frac{1}{2}\theta\mathbf{n}} \;=\; e^{\theta\mathbf{n}}\mathbf{x}^{\perp\mathbf{n}}$$

and the rotation $R^{\theta}_{\mathbf{n}}\mathbf{x}^{||\mathbf{n}}$ of an axial-pole vector (eigenvector) of the rotation is a null rotation

$$e^{\frac{1}{2}\theta\mathbf{n}}\mathbf{x}^{||\mathbf{n}}e^{-\frac{1}{2}\theta\mathbf{n}} \;=\; e^{0\mathbf{n}}\mathbf{x}^{||\mathbf{n}}.$$

The pole round which counter-clockwise rotation is seen by right-hand rule on right-handed axes (holding the pole vector in right-hand) can be called the *north geometric pole* of the rotation, and the inverse pole called the *south geometric pole* of the rotation where left-hand rule on left-handed axes *would* see clockwise rotation (holding the inverse pole vector in left-hand as if it were the positive direction of the axis). A pole, as a *point* on the sphere, or as a *vector* from the sphere center to the pole, are synonymous for the purpose considered here, but they can be made as distinct mathematical values in conformal and quadric geometric algebra. With this terminology, the quadrantal versor $\mathbf{n}$ is a unit axial-pole vector of the unit sphere of $\mathbf{n}$ (the unit sphere) and is directed toward the point of the north geometric pole of the rotation on right-handed axes, or directed toward the south geometric pole on left-handed axes where positive rotations are then taken clockwise. This convention or orientation of north and south geometric poles matches the geometric poles and rotation of the Earth.

If the *sphere of* $\mathbf{v}$ *relative to* an origin $\mathbf{o}$ is considered the set

$$\mathbf{S}_{\mathbf{v}-\mathbf{o}} \;=\; \{\mathbf{x}\colon |\mathbf{x}-\mathbf{o}| = |\mathbf{v}-\mathbf{o}| = \mathrm{T}(\mathbf{x}-\mathbf{o}) = \mathrm{T}(\mathbf{v}-\mathbf{o})\}$$

of vectors $\mathbf{x}$, then $R^{\theta}_{\mathbf{n}}(\mathbf{S}_{\mathbf{v}-\mathbf{o}} - \mathbf{o}) + \mathbf{o}$ can be considered the rotation of the entire sphere. This notation is meant to convey that subtraction by $\mathbf{o}$, followed by application of rotation $R^{\theta}_{\mathbf{n}}$, followed by addition of $\mathbf{o}$ is to occur on each element $\mathbf{x}$ of the set $\mathbf{S}_{\mathbf{v}-\mathbf{o}}$ of vectors that are located on the surface of the sphere defined by $\mathbf{v}$ and $\mathbf{o}$ by the invariant radial distance $|\mathbf{v}-\mathbf{o}| = |\mathbf{x}-\mathbf{o}|$ that any other point $\mathbf{x}$ on the sphere surface must also have.

The rotations $R^{\theta}_{\mathbf{n}}\mathbf{S}_{\mathbf{v}}$ are limited to rotating spheres of radius $r = |\mathbf{v}|$, with surface points $\{\mathbf{x}\colon \mathrm{T}\mathbf{x} = \mathrm{T}\mathbf{v}\}$, and that are concentric spheres centered on the origin $\mathbf{o} = (0, 0, 0) = 0$ of space.

The rotation of spheres concentric on any arbitrary origin $\mathbf{o}' = o_x\mathbf{i} + o_y\mathbf{j} + o_z\mathbf{k}$ is the rotation $R^\theta_\mathbf{n}(\mathbf{S}_{\mathbf{v}-\mathbf{o}'} - \mathbf{o}') + \mathbf{o}'$. The spheres are centered at $\mathbf{o}'$, and $\mathbf{v}$ determines a particular radius $r = |\mathbf{v} - \mathbf{o}'|$.

The rotation of an entire sphere, as a set of vector elements $\mathbf{x}$, shows that translation (to and from) relative to the sphere origin is generally required for rotation of vectors round the surface of an arbitrary sphere located in space.

The usual concern is not rotating entire spheres, but is rotating a particular point $\mathbf{v}$ perpendicularly round an arbitrary line that does not necessarily intersect the origin $\mathbf{o} = 0$, or is to rotate $\mathbf{v}$ round an arbitrary point $\mathbf{c}$.

An arbitrary line can be represented by $\mathbf{a} = t\mathbf{n} + \mathbf{o}'$. We want to rotate $\mathbf{v}$ round this line as the rotational axis. The rotation will occur in a circle in space, and this circle is on the sphere centered at $\mathbf{o}'$ with radius $r = |\mathbf{v} - \mathbf{o}'|$, and the rotational axis is parallel to the unit vector $\mathbf{n}$. The center of the circle is at $\mathbf{c} = \mathcal{P}_\mathbf{n}(\mathbf{v} - \mathbf{o}') + \mathbf{o}' = [(\mathbf{v} - \mathbf{o}') \cdot \mathbf{n}]\mathbf{n} + \mathbf{o}'$. The location of $\mathbf{v}$ relative to the circle center is $\mathbf{v} - \mathbf{c}$.

The rotation of $\mathbf{v}$ round the line $\mathbf{a}$ is

$$\begin{aligned} \mathbf{v}' &= R^\theta_\mathbf{n}(\mathbf{v} - \mathbf{o}') + \mathbf{o}' \\ &= R^\theta_\mathbf{n}\mathbf{v} - R^\theta_\mathbf{n}\mathbf{o}' + \mathbf{o}'. \end{aligned}$$

If we viewed $\mathbf{c}$ as an arbitrary point that we wished to rotate round, then we still needed to specify a rotational axis $\mathbf{n}$ through $\mathbf{c}$, and the rotation is

$$\begin{aligned} \mathbf{v}' &= R^\theta_\mathbf{n}(\mathbf{v} - \mathbf{c}) + \mathbf{c} \\ &= R^\theta_\mathbf{n}[\mathbf{v} - \mathcal{P}_\mathbf{n}(\mathbf{v} - \mathbf{o}') - \mathbf{o}'] + \mathbf{c} \\ &= R^\theta_\mathbf{n}\mathbf{v} - R^\theta_\mathbf{n}\mathcal{P}_\mathbf{n}(\mathbf{v} - \mathbf{o}') - R^\theta_\mathbf{n}\mathbf{o}' + \mathbf{c} \\ &= R^\theta_\mathbf{n}\mathbf{v} - \mathcal{P}_\mathbf{n}(\mathbf{v} - \mathbf{o}') - R^\theta_\mathbf{n}\mathbf{o}' + \mathbf{c} \\ &= R^\theta_\mathbf{n}\mathbf{v} - (\mathbf{c} - \mathbf{o}') - R^\theta_\mathbf{n}\mathbf{o}' + \mathbf{c} \\ &= R^\theta_\mathbf{n}\mathbf{v} - R^\theta_\mathbf{n}\mathbf{o}' + \mathbf{o}'. \end{aligned}$$

1.29 Vector-arcs

A vector-arc $\mathbf{c}$ can be represented by the vector chord $\mathbf{c} = \mathbf{b} - \mathbf{a}$ in the circle where $\mathbf{a}$ is rotated to $\mathbf{b}$ by a quaternion biradial $\mathbf{b}/\mathbf{a}$ through an arc $r\theta$ of the circle that can be given by the *Law of Sines* as

$$\begin{aligned} \frac{|\mathbf{b} - \mathbf{a}|}{\sin(\theta)} &= \frac{r}{\sin\left(\frac{\pi - \theta}{2}\right)} \\ r\theta &= \frac{|\mathbf{c}|\sin\left(\frac{\pi - \theta}{2}\right)}{\sin(\theta)}\theta. \end{aligned}$$

The rotation can also be seen as the translation of $\mathbf{a}$ by vector-arc $\mathbf{c}$ across the chord $\mathbf{b} - \mathbf{a}$. The operation $\mathbf{a}$ to $\mathbf{b}$ can be written multiplicatively or additively

$$\begin{aligned} \mathbf{b} &= (\mathbf{b}/\mathbf{a})\mathbf{a} \\ \mathbf{b} &= (\mathbf{b} - \mathbf{a}) + \mathbf{a} \end{aligned}$$

as quaternion multiplication or vector-arc addition.

The vector-arc $\mathbf{c}=\mathbf{c}_0$ can be added to another different vector $\mathbf{v}=\mathbf{v}_0$ as $\mathbf{v}_1=\mathbf{c}_0+\mathbf{v}_0$, where $\mathbf{v}_1$ is viewed as being $\mathbf{v}_0$ rotated in the same relative direction $\mathbf{c}_0$ and by the same arc $r\theta$ on a sphere centered at a same relative position as to $\mathbf{a}$ and $\mathbf{b}$. The vector-arcs, or chords, of the rotations $\mathbf{v}_1-\mathbf{v}_0=\mathbf{b}-\mathbf{a}$ correspond. Some n successive rotations applied to $\mathbf{a}$ produces n successive vector-arcs $\mathbf{c}_k$, $k\in\{0,\ldots,n\}$. The vector-arcs $\mathbf{c}_k$ can be added successively to vector $\mathbf{v}_0$ as $\mathbf{v}_{k+1}=\mathbf{c}_k+\mathbf{v}_k$ and each one of these vectors remains on a sphere or path corresponding to the sphere or path of $\mathbf{a}$.

Since the order, or sequence, of adding vector-arcs onto $\mathbf{v}$ (adding successively) is important for replaying a sequence of vector-arc translations on a sphere or path, when a vector-arc $\mathbf{c}_k$ is added to $\mathbf{v}_k$, the next vector-arc $\mathbf{c}_{k+1}$ added to $\mathbf{v}_{k+1}$ is a specific vector-arc representing the next rotation and $\mathbf{c}_{k+1}$ is called the *provector-arc* of the prior *vector-arc* $\mathbf{c}_k$. The sum of a vector-arc $\mathbf{c}_k$ and its provector-arc $\mathbf{c}_{k+1}$ is their *transvector-arc* $\mathbf{t}_k=\mathbf{c}_{k+1}+\mathbf{c}_k$ representing an arcual sum.

The addition of vector-arcs equals successive rotations in the same sphere only if the vector-arcs are successive corresponding chords in the same sphere. The successive chords form a *gauche* or *skew polygon* inscribed in the sphere. To remain on the same sphere, the addition of the vector-arcs to $\mathbf{v}$ must be in rotations order, translating across a specific sequence of chords representing the rotation arcs and sides of a gauche polygon inscribed in the sphere. As chords are added to $\mathbf{v}$ in rotations sequence, the resulting positions of vertices of the polygon in the sphere are found. The effect is to replay a specific sequence of translations starting at $\mathbf{v}$, with vector-arcs added head to tail as the polygon sides or chords in the sphere. Direct rotations among the found points of the inscribed polygon can be computed as other transvector-arcs.

A sequence of vector-arcs can be pre-generated and applied in sequence forward (adding the provector-arc) and backward (subtracting the prior provector-arc). The sequences of vector-arcs could be applied to multiple points using additions or their pre-additions (transvector-arcs), which may be much faster than multiplying each point using a quaternion operator.

1.30 Mean Versor

The *mean versor* problem can be described as follows: If given n versors

$$\begin{aligned} R_i &= e^{\theta_i \mathbf{r}_i}=\cos(\theta_i)+\sin(\theta_i)\mathbf{r}_i \\ i &= 1,\ldots,n \end{aligned}$$

where each versor rotates by the angle θ_i in the plane perpendicular to the unit vector $\mathbf{r}_i$, then the vector $\mathbf{v}$ can be rotated into i different points using the half-angle (or full-angle) versors

$$\mathbf{v}_i = R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}.$$

For $i=3$, the three vectors $\mathbf{v}_i$ are the vertices of a spherical triangle T. The location of the spherical centroid $\bar{\mathbf{v}}$ of T represents the true mean rotation of $\mathbf{v}$ relative to the three vertices.

As a first approach to trying to find $\bar{\mathbf{v}}$ and the mean versor $\bar{\mathbf{v}}/\mathbf{v}$, we can consider the average of the vectors $\mathbf{v}_i$, scaled to touch the sphere, as approximately $\bar{\mathbf{v}}$. With this approach, the spherical centroid $\bar{\mathbf{v}}$ of T is approximately

$$\bar{\mathbf{v}} \approx |\mathbf{v}|\frac{\frac{1}{n}\sum_{i=1}^{n}\mathbf{v}_i}{|\frac{1}{n}\sum_{i=1}^{n}\mathbf{v}_i|} = |\mathbf{v}|\frac{\sum_{i=1}^{n}\mathbf{v}_i}{|\sum_{i=1}^{n}\mathbf{v}_i|}$$

on the condition that the vertices $\mathbf{v}_i$ are clustered closely together such that the angle between any two vertices does not exceed $\frac{\pi}{2}$ radians. For any $\mathbf{v}$, this condition is satisfied if

$$|\theta_i| \leq \frac{\pi}{4}$$

when using the half-angle versors, or if

$$|\theta_i| \leq \frac{\pi}{8}$$

when using the full-angle versors.

Under this condition, this formula for the spherical centroid of T has good accuracy. For angles exceeding these conditions, the error increases rapidly up to the error extremum or worse-possible error when two vertices reach π radians apart!

For $i>3$, the vector $\bar{\mathbf{v}}$ is generally *not* the centroid of a spherical polygon of the vertices $\mathbf{v}_i$, but $\bar{\mathbf{v}}$ does approximate the true mean rotation of $\mathbf{v}$.

The approximate mean versor is

$$\begin{aligned} R_m &= \bar{\mathbf{v}}/\mathbf{v} = \bar{\mathbf{v}}\mathbf{v}^{-1} \\ R_m^{-1} &= \mathbf{v}/\bar{\mathbf{v}} = \mathbf{v}\bar{\mathbf{v}}^{-1}. \end{aligned}$$

The approximate mean versor with half the angle is

$$\begin{aligned} R_m^{\frac{1}{2}} &= \left(|\mathbf{v}|\frac{\bar{\mathbf{v}}+\mathbf{v}}{|\bar{\mathbf{v}}+\mathbf{v}|}\right)/\mathbf{v} = \frac{-(\bar{\mathbf{v}}+\mathbf{v})\mathbf{v}}{|\mathbf{v}||\bar{\mathbf{v}}+\mathbf{v}|} \\ R_m^{-\frac{1}{2}} &= \mathbf{v}/\left(|\mathbf{v}|\frac{\bar{\mathbf{v}}+\mathbf{v}}{|\bar{\mathbf{v}}+\mathbf{v}|}\right) = \frac{-\mathbf{v}(\bar{\mathbf{v}}+\mathbf{v})}{|\mathbf{v}||\bar{\mathbf{v}}+\mathbf{v}|} \end{aligned}$$

The rotation of $\mathbf{v}$ by the approximate mean versor R_m is

$$\begin{aligned} \bar{\mathbf{v}} &= R_m^{\frac{1}{2}}\mathbf{v}R_m^{-\frac{1}{2}} = \frac{-(\bar{\mathbf{v}}+\mathbf{v})\mathbf{v}}{|\mathbf{v}||\bar{\mathbf{v}}+\mathbf{v}|}\mathbf{v}\frac{-\mathbf{v}(\bar{\mathbf{v}}+\mathbf{v})}{|\mathbf{v}||\bar{\mathbf{v}}+\mathbf{v}|} \\ &= \frac{(\bar{\mathbf{v}}+\mathbf{v})\mathbf{v}(\bar{\mathbf{v}}+\mathbf{v})}{(\bar{\mathbf{v}}+\mathbf{v})^2} = (\bar{\mathbf{v}}+\mathbf{v})\mathbf{v}(\bar{\mathbf{v}}+\mathbf{v})^{-1}. \end{aligned}$$

Using the identities

$$\begin{aligned} \mathbf{v}\mathbf{v} &= \bar{\mathbf{v}}\bar{\mathbf{v}} = -|\mathbf{v}|^2 \\ \bar{\mathbf{v}}(\bar{\mathbf{v}}+\mathbf{v})^2 &= \bar{\mathbf{v}}\bar{\mathbf{v}}\bar{\mathbf{v}} + \bar{\mathbf{v}}\bar{\mathbf{v}}\mathbf{v} + \bar{\mathbf{v}}\mathbf{v}\bar{\mathbf{v}} + \bar{\mathbf{v}}\mathbf{v}\mathbf{v} \\ \bar{\mathbf{v}}\mathbf{v}\bar{\mathbf{v}} &= \bar{\mathbf{v}}(\bar{\mathbf{v}}+\mathbf{v})^2 - \bar{\mathbf{v}}\bar{\mathbf{v}}\bar{\mathbf{v}} - \bar{\mathbf{v}}\bar{\mathbf{v}}\mathbf{v} - \bar{\mathbf{v}}\mathbf{v}\mathbf{v} \\ &= \bar{\mathbf{v}}(\bar{\mathbf{v}}+\mathbf{v})^2 - 2\mathbf{v}\mathbf{v}\bar{\mathbf{v}} - \mathbf{v}\mathbf{v}\mathbf{v} \end{aligned}$$

we can verify that

$$\begin{aligned}\bar{\mathbf{v}} &= \frac{\bar{\mathbf{v}}\mathbf{v}\bar{\mathbf{v}}+\mathbf{v}\mathbf{v}\bar{\mathbf{v}}+\bar{\mathbf{v}}\mathbf{v}\mathbf{v}+\mathbf{v}\mathbf{v}\mathbf{v}}{(\bar{\mathbf{v}}+\mathbf{v})^2}=\frac{\bar{\mathbf{v}}\mathbf{v}\bar{\mathbf{v}}+2\mathbf{v}\mathbf{v}\bar{\mathbf{v}}+\mathbf{v}\mathbf{v}\mathbf{v}}{(\bar{\mathbf{v}}+\mathbf{v})^2}\\ &= \frac{\bar{\mathbf{v}}(\bar{\mathbf{v}}+\mathbf{v})^2}{(\bar{\mathbf{v}}+\mathbf{v})^2}=\bar{\mathbf{v}}.\end{aligned}$$

By this first approach, this approximate mean versor R_m is dependent on a particular $\mathbf{v}$ and adapts well to a wide range versors R_i. It is computationally expensive since it requires computing all the rotations and averaging them. A potential problem of this approach is the possibly unacceptable error when two different $\mathbf{v}_i$ are very close to π radians apart, or are nearly inversions.

As a second approach to trying to find $\bar{\mathbf{v}}$ and the mean versor $\bar{\mathbf{v}}\,/\,\mathbf{v}$, we can consider using the average vector-arc of the vector-arcs $\theta_i\mathbf{r}_i$ represented by the versors R_i. By this approach, we try the geometric mean of the versors

$$\begin{aligned}\hat{R} &= \sqrt[n]{\prod_{i=1}^{n} R_i}\\ &= e^{(\theta_1\mathbf{r}_1+\theta_2\mathbf{r}_2+\cdots+\theta_n\mathbf{r}_n)/n}\end{aligned}$$

as approximating the true mean versor. Each $\theta_i\mathbf{r}_i$ can be interpretted as a vector-arc, and

$$(\theta_1\mathbf{r}_1+\theta_2\mathbf{r}_2+\cdots+\theta_n\mathbf{r}_n)\,/\,n$$

as the average vector-arc.

By this method, the approximate mean rotation of $\mathbf{v}$ is

$$\bar{\mathbf{v}} \approx \hat{R}^{\frac{1}{2}}\mathbf{v}\hat{R}^{-\frac{1}{2}}.$$

Good accuracy is achieved using the same condition $|\theta_i| \le \frac{\pi}{4}$ as in the first approach.

As a third approach to trying to find $\bar{\mathbf{v}}$ and the mean versor $\bar{\mathbf{v}}/\mathbf{v}$, the solution is to find the versor $\bar{R}$ that minimizes the distance d, where

$$\begin{aligned}d_i &= \left|R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}-\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}\right|\\ d &= \sum_{i=1}^{n} d_i.\end{aligned}$$

We can also choose to minimize the sum of squares

$$\begin{aligned}d_i^2 &= \left|R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}-\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}\right|^2\\ &= -\left(R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}-\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}\right)^2\\ \Delta &= \sum_{i=1}^{n} d_i^2.\end{aligned}$$

We can write d_i^2 as

$$\begin{aligned}
d_i^2 &= -\left(R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}} - \bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}\right)^2 \\
&= \left(\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}} - R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}\right)\left(R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}} - \bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}\right) \\
&= \bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}} - 2\mathbf{v}^2 + R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}.
\end{aligned}$$

Now we can write Δ as

$$\begin{aligned}
\Delta &= \\
&= \sum_{i=1}^{n}\left(\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}} - 2\mathbf{v}^2 + R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}\right) \\
&= \bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}\left(\sum_{i=1}^{n} R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}\right) - 2n\mathbf{v}^2 + \left(\sum_{i=1}^{n} R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}}\right)\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}.
\end{aligned}$$

If

$$\sum_{i=1}^{n} R_i^{\frac{1}{2}}\mathbf{v}R_i^{-\frac{1}{2}} = n\,\bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}$$

then $\Delta = 0$ and is minimized. This can be expanded as

$$\sum_{i=1}^{n} R_i\mathbf{v}^{\perp\mathbf{r}_i} + \sum_{i=1}^{n}\mathbf{v}^{||\mathbf{r}_i} = n(\bar{R}\mathbf{v}^{\perp\bar{R}} + \mathbf{v}^{||\bar{R}})$$

which shows more clearly that the closer $\bar{R}$ is to each R_i, then the closer Δ is to zero. This minimization is achieved by the arithmetic mean

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i.$$

This result does not depend on a particular $\mathbf{v}$. Technically, a versor should always have unit magnitude, but generally $|\bar{R}| \neq 1$. The quaternion $\bar{R}$ can be normalized to have unit magnitude if necessary, but when used in the following versor rotation operation, any magnitude of $\bar{R}$ cancels anyway.

By this approach, the approximate mean rotation of $\mathbf{v}$ is

$$\bar{\mathbf{v}} \approx \bar{R}^{\frac{1}{2}}\mathbf{v}\bar{R}^{-\frac{1}{2}}.$$

Once again, good accuracy is achieved using the same condition $|\theta_i| \leq \frac{\pi}{4}$ as in the first and second approaches.

Each approach has a different behavior as angles are increased. For $|\theta_i| \leq \frac{\pi}{4}$, they give nearly identical results. Each approach may be worth comparing to the others for a particular problem.

Testing the approaches: One way to experiment interactively with these formulas is to use the program GAViewer[1] and have it open and run the following file `mean-versor.g`:

```
/* mean-versor.g Mean Versor Demo */
// Use ctrl-rightmousebutton to drag vectors v r1 r2 r3
// around the sphere and see rotations change dynamically.
// Sliders for angles t1 t2 t3 also trigger dynamic changes.

// Lines ending with , ARE rendered
// Lines ending with ; ARE NOT rendered

// Reset the viewer.
reset();
set_window_title("Mean Versor Demo");

// Run a .geo format command.
// Certain things can only be done with geo commands.
// Set the text parsing/formatting mode for the text of labels.
cmd("tsmode equation");
// Equation mode formats text like Latex equation mode does.

// Draw basic x,y,z direction unit vectors.
x=e1,
y=e2,
z=e3,
label(x),
label(y),
label(z),

// Pseudoscalar I.
// Needed to compute duals; i.e,
// to convert vectors to pure quaternions in geometric algebra.
I = e1 e2 e3;
// Draw a unit ball or sphere.
B = 4/3 pi pow(0.80,2) e1 e2 e3;
B = color(B,1,1,1,0.25),

// Initial point v on unit radius sphere.
// This is the vector we are rotating.
v = (e1+e2+e3);
v = v / norm(v);
label(v),

// Make angles t1,t2,t3 in degrees here
// but converted to radians in formulas later.
ctrl_range(t1= 40,-180,180);
ctrl_range(t2=-25,-180,180);
ctrl_range(t3=-15,-180,180);
// Use half-angle versor rotation formulas below.
```

```
// Make versors R1 R2 R3 that rotate round axes r1 r2 r3
// that are initially close to e1 e2 e3 (but can be dragged).
r1 = (5 e1 +   e2 +   e3);
r2 = (  e1 + 5 e2 +   e3);
r3 = (  e1 +   e2 + 5 e3);
r1 = r1 / norm(r1),
r2 = r2 / norm(r2),
r3 = r3 / norm(r3),
label(r1),
label(r2),
label(r3),
R1 = exp( t1/2 pi/180 r1/I );
R2 = exp( t2/2 pi/180 r2/I );
R3 = exp( t3/2 pi/180 r3/I );

// Rotate v into v1 v2 v3 using R1 R2 R3.
v1 = R1 v / R1,
v2 = R2 v / R2,
v3 = R3 v / R3,
label(v1),
label(v2),
label(v3),

// Locate the mean rotation of v directly from the mean of v1 v2
v3.
// This is the accurate approximation of the centroid or mean
rotation of v.
// This is dependent on a particular v and on the versors R1 R2
R3.
m = (v1+v2+v3)/norm(v1+v2+v3),
label(m),

// Locate the mean rotation of v using the geometric mean of R1
R2 R3
// which is thought of as using the mean vector-arc.
G = exp( ((t1/2 pi/180 r1)+(t2/2 pi/180 r2)+(t3/2 pi/180 r3))/
(3I) );
g = G v / G,
label(g),

// Locate the mean rotation of v using the arithmetic mean of R1
R2 R3
// normalized to unit norm for a proper versor.
A = (R1+R2+R3)/norm(R1+R2+R3);
a = A v / A,
```

```
label(a),

// Begin dynamic element.
// Simply recalculate everything in here.
dynamic{d1:

// The basis should not be moved.
x=e1,
y=e2,
z=e3,
x= color(x,0,0,0,0.5),
y= color(y,0,0,0,0.5),
z= color(z,0,0,0,0.5),

// The sphere.
B = 4/3 pi pow(0.80,2) e1 e2 e3;
B = color(B,1,1,1,0.25),

// The vector we are rotating around a unit radius sphere.
v = v / norm(v),

// Make versors R1 R2 R3 that rotate round axes r1 r2 r3
// and that are initially set close to e1 e2 e3.
r1 = r1 / norm(r1),
r2 = r2 / norm(r2),
r3 = r3 / norm(r3),
r1 = color(r1,1,1,0,0.5),
r2 = color(r2,1,1,0,0.5),
r3 = color(r3,1,1,0,0.5),
R1 = exp( t1/2 pi/180 r1/I );
R2 = exp( t2/2 pi/180 r2/I );
R3 = exp( t3/2 pi/180 r3/I );

// Rotate v into v1 v2 v3 using R1 R2 R3.
v1 = R1 v / R1,
v2 = R2 v / R2,
v3 = R3 v / R3,
v1 = color(v1,0,0,1,0.5),
v2 = color(v2,0,0,1,0.5),
v3 = color(v3,0,0,1,0.5),

// Locate the mean rotation of v directly from the mean of v1 v2
v3.
m = norm(v) (v1+v2+v3)/norm(v1+v2+v3),
m = color(m,1,0,1,0.5),
```

```
// Locate the mean rotation of v using the geometric mean of R1
R2 R3.
G = exp( ((t1/2 pi/180 r1)+(t2/2 pi/180 r2)+(t3/2 pi/180 r3))/
(3I) );
g = G v / G,
g = color(g,1,0,1,0.5),

// Locate the mean rotation of v using the arithmetic mean of R1
R2 R3.
// The magnitude of A does not matter here.
A = (R1+R2+R3);
a = A v / A,
a = color(a,1,0,1,0.5),
}
```

1.31 Matrix Forms of Quaternions and Rotations

A quaternion can be written as a 4-tuple of scalar components or as a 4-element column vector, and it can also be written as a 4 × 4 matrix by expanding the quaternion product

$$
\begin{aligned}
p &= p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} = p_w + \mathbf{p} = (p_w, p_x, p_y, p_z) \\
q &= q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k} = q_w + \mathbf{q} = (q_w, q_x, q_y, q_z) \\
pq &= (p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k})(q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) \\
&= p_wq_w + p_wq_x\mathbf{i} + p_wq_y\mathbf{j} + p_wq_z\mathbf{k} + \\
&\quad p_xq_w\mathbf{i} - p_xq_x + p_xq_y\mathbf{k} - p_xq_z\mathbf{j} + \\
&\quad p_yq_w\mathbf{j} - p_yq_x\mathbf{k} - p_yq_y + p_yq_z\mathbf{i} + \\
&\quad p_zq_w\mathbf{k} + p_zq_x\mathbf{j} - p_zq_y\mathbf{i} - p_zq_z \\
&= p_wq_w - \mathbf{p}\cdot\mathbf{q} + p_w\mathbf{q} + q_w\mathbf{p} + \mathbf{p}\times\mathbf{q}
\end{aligned}
$$

$$
\begin{aligned}
[\, pq \,] &= \begin{pmatrix} p_wq_w - p_xq_x - p_yq_y - p_zq_z \\ p_xq_w + p_wq_x - p_zq_y + p_yq_z \\ p_yq_w + p_zq_x + p_wq_y - p_xq_z \\ p_zq_w - p_yq_x + p_xq_y + p_wq_z \end{pmatrix} \\
\{\, p \,\}[\, q \,] &= \begin{pmatrix} p_w & -p_x & -p_y & -p_z \\ p_x & p_w & -p_z & p_y \\ p_y & p_z & p_w & -p_x \\ p_z & -p_y & p_x & p_w \end{pmatrix} \begin{pmatrix} q_w \\ q_x \\ q_y \\ q_z \end{pmatrix}
\end{aligned}
$$

and factoring the resulting column vector form $[\, pq \,]$ into a matrix-vector product $\{\, p \,\}[\, q \,]$.

Quaternions in the matrix form $\{\ p\ \}$ can be added as $\{\ p\ \}+\{\ q\ \}=\{\ p+q\ \}$, and can be multiplied together using non-commutative matrix multiplication giving products of the form $\{\ p\ \}\{\ q\ \}=\{\ pq\ \}$. Quaternions in column vector form $[\ q\ ]$ can be added, and can be the multiplicand of a quaternion multiplier in matrix form $\{\ p\ \}$, giving products $\{\ p\ \}[\ q\ ]=[\ pq\ ]$ in column vector form.

For two vectors, their product is

$$\begin{aligned}\mathbf{pq} &= -\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\times\mathbf{q}\\ &= \frac{1}{2}(\mathbf{pq}+\mathbf{qp})+\frac{1}{2}(\mathbf{pq}-\mathbf{qp})\end{aligned}$$

and their matrix-vector product is

$$\begin{aligned}\{\ \mathbf{p}\ \}[\ \mathbf{q}\ ] &= \begin{pmatrix} 0 & -p_x & -p_y & -p_z \\ p_x & 0 & -p_z & p_y \\ p_y & p_z & 0 & -p_x \\ p_z & -p_y & p_x & 0 \end{pmatrix}\begin{pmatrix} 0 \\ q_x \\ q_y \\ q_z \end{pmatrix}\\ &= \begin{pmatrix} -p_xq_x-p_yq_y-p_zq_z \\ -p_zq_y+p_yq_z \\ p_zq_x-p_xq_z \\ -p_yq_x+p_xq_y \end{pmatrix} = \begin{pmatrix} -(\mathbf{p}\cdot\mathbf{q}) \\ [\ \mathbf{p}\times\mathbf{q}\ ]_x \\ [\ \mathbf{p}\times\mathbf{q}\ ]_y \\ [\ \mathbf{p}\times\mathbf{q}\ ]_z \end{pmatrix}\\ \{\ \mathbf{p}\ \}^{\bullet} &= \begin{pmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & 0 & 0 \\ -p_y & 0 & 0 & 0 \\ -p_z & 0 & 0 & 0 \end{pmatrix}\\ \{\ \mathbf{p}\ \}^{\times} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -p_z & p_y \\ 0 & p_z & 0 & -p_x \\ 0 & -p_y & p_x & 0 \end{pmatrix}\\ \{\ \mathbf{p}\ \} &= -\{\ \mathbf{p}\ \}^{\bullet}+\{\ \mathbf{p}\ \}^{\times}\\ [\ \mathbf{p}\cdot\mathbf{q}\ ] &= \{\ \mathbf{p}\ \}^{\bullet}[\ \mathbf{q}\ ]\\ [\ \mathbf{p}\times\mathbf{q}\ ] &= \{\ \mathbf{p}\ \}^{\times}[\ \mathbf{q}\ ]\\ [\ \mathbf{pq}\ ] &= (-\{\ \mathbf{p}\ \}^{\bullet}+\{\ \mathbf{p}\ \}^{\times})[\ \mathbf{q}\ ]=-[\ \mathbf{p}\cdot\mathbf{q}\ ]+[\ \mathbf{p}\times\mathbf{q}\ ]\end{aligned}$$

which gives matrices representing dot and cross product operators on column vectors.

The dot and cross products for two vectors in matrix forms are

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= -\frac{1}{2}(\mathbf{pq}+\mathbf{qp})\\ \{\ \mathbf{p}\ \}\cdot\{\ \mathbf{q}\ \} &= -\frac{1}{2}(\{\ \mathbf{p}\ \}\{\ \mathbf{q}\ \}+\{\ \mathbf{q}\ \}\{\ \mathbf{p}\ \})\\ \mathbf{p}\times\mathbf{q} &= \frac{1}{2}(\mathbf{pq}-\mathbf{qp})\\ \{\ \mathbf{p}\ \}\times\{\ \mathbf{q}\ \} &= \frac{1}{2}(\{\ \mathbf{p}\ \}\{\ \mathbf{q}\ \}-\{\ \mathbf{q}\ \}\{\ \mathbf{p}\ \})\end{aligned}$$

which take the symmetric and anti-symmetric parts of the vector product $\mathbf{pq}$.

The matrix form $\{\ p\ \}$ can be decomposed into a sum of matrices to identify the matrix representations of $\mathbf{i},\mathbf{j},\mathbf{k}$ as

$$\begin{aligned}
\{\ p\ \} &= p_w\{\ 1\ \}+p_x\{\ \mathbf{i}\ \}+p_y\{\ \mathbf{j}\ \}+p_z\{\ \mathbf{k}\ \}\\
\{\ 1\ \} &= \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}\\
\{\ \mathbf{i}\ \} &= \begin{pmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{pmatrix}\\
\{\ \mathbf{j}\ \} &= \begin{pmatrix} 0&0&-1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix}\\
\{\ \mathbf{k}\ \} &= \begin{pmatrix} 0&0&0&-1\\ 0&0&-1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}.
\end{aligned}$$

A quaternion rotation, by θ radians, of vector $\mathbf{v}$ round the unit vector axis $\mathbf{n}$ can be written as

$$\begin{aligned}
\mathbf{v}_\theta &= e^{\frac{1}{2}\theta\mathbf{n}}(\mathbf{v}^{||\mathbf{n}}+\mathbf{v}^{\perp\mathbf{n}})e^{-\frac{1}{2}\theta\mathbf{n}}\\
&= \mathbf{v}^{||\mathbf{n}}+e^{\theta\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}.
\end{aligned}$$

To write this rotation in matrix-vector multiplication form, we first need some more identities.

The vector component of $\mathbf{v}$ parallel to the unit vector $\mathbf{n}$ is

$$\begin{aligned}
\mathbf{v}^{||\mathbf{n}} &= \mathbf{n}(\mathbf{n}\cdot\mathbf{v})=\mathbf{n}(\mathbf{n}\times\mathbf{v}-\mathbf{n}\mathbf{v})\\
&= \mathbf{n}(\mathbf{n}\times\mathbf{v})+\mathbf{v}=e^{\frac{\pi}{2}\mathbf{n}}(\mathbf{n}\times\mathbf{v})+\mathbf{v}
\end{aligned}$$

$$\left[\ \mathbf{v}^{||\mathbf{n}}\ \right] = [\ \mathbf{n}\ ][\ \mathbf{n}\ ]^{\mathrm{T}}[\ \mathbf{v}\ ]$$

$$\begin{aligned}
&= \begin{pmatrix} 0\\ n_x\\ n_y\\ n_z \end{pmatrix}\begin{pmatrix} 0 & n_x & n_y & n_z \end{pmatrix}\begin{pmatrix} 0\\ v_x\\ v_y\\ v_z \end{pmatrix}\\
&= \{\ \mathbf{n}\ \}\{\ \mathbf{n}\ \}^{\times}[\ \mathbf{v}\ ]+[\ \mathbf{v}\ ]=(\{\ \mathbf{n}\ \}\{\ \mathbf{n}\ \}^{\times}+\{\ 1\ \})[\ \mathbf{v}\ ]\\
&= \{\ \mathbf{n}\ \}\{\ \mathbf{n}\ \}^{\bullet}[\ \mathbf{v}\ ]\\
&= \{\ \mathbf{n}\ \}^{||}[\ \mathbf{v}\ ]=\begin{pmatrix} 1&0&0&0\\ 0&n_xn_x&n_xn_y&n_xn_z\\ 0&n_yn_x&n_yn_y&n_yn_z\\ 0&n_zn_x&n_zn_y&n_zn_z \end{pmatrix}\begin{pmatrix} 0\\ v_x\\ v_y\\ v_z \end{pmatrix}
\end{aligned}$$

The vector component of $\mathbf{v}$ perpendicular to the unit vector $\mathbf{n}$ is

$$\begin{aligned}\mathbf{v}^{\perp\mathbf{n}} &= \mathbf{v}-\mathbf{v}^{||\mathbf{n}}\\ &= \mathbf{v}-(\mathbf{n}(\mathbf{n}\times\mathbf{v})+\mathbf{v})=-\mathbf{n}(\mathbf{n}\times\mathbf{v})=e^{-\frac{\pi}{2}\mathbf{n}}(\mathbf{n}\times\mathbf{v})\end{aligned}$$

$$\left[\ \mathbf{v}^{\perp\mathbf{n}}\ \right] = [\ \mathbf{v}\ ]-\left[\ \mathbf{v}^{||\mathbf{n}}\ \right]=[\ \mathbf{v}\ ]-\{\ \mathbf{n}\ \}^{||}[\ \mathbf{v}\ ]$$

$$\begin{aligned} &= (\{\ 1\ \}-\{\ \mathbf{n}\ \}^{||})[\ \mathbf{v}\ ]\\ &= \{\ \mathbf{n}\ \}^{\perp}[\ \mathbf{v}\ ]=\begin{pmatrix}0&0&0&0\\0&1-n_xn_x&-n_xn_y&-n_xn_z\\0&-n_yn_x&1-n_yn_y&-n_yn_z\\0&-n_zn_x&-n_zn_y&1-n_zn_z\end{pmatrix}\begin{pmatrix}0\\v_x\\v_y\\v_z\end{pmatrix}\\ &= -\{\ \mathbf{n}\ \}\{\ \mathbf{n}\ \}^{\times}[\ \mathbf{v}\ ].\end{aligned}$$

The rotor $e^{\theta\mathbf{n}}$ for the rotation round axis $\mathbf{n}$ by angle θ is

$$\begin{aligned} e^{\theta\mathbf{n}} &= \cos(\theta)+\sin(\theta)\mathbf{n}=c+s\mathbf{n}\\ \left\{\ e^{\theta\mathbf{n}}\ \right\} &= \{\ c\ \}+s\{\ \mathbf{n}\ \}=\begin{pmatrix}c&-sn_x&-sn_y&-sn_z\\sn_x&c&-sn_z&sn_y\\sn_y&sn_z&c&-sn_x\\sn_z&-sn_y&sn_x&c\end{pmatrix}.\end{aligned}$$

The rotation can also be written

$$\begin{aligned}\mathbf{v}_\theta &= \mathbf{v}^{||\mathbf{n}}+e^{\theta\mathbf{n}}\mathbf{v}^{\perp\mathbf{n}}=R^\theta_{\mathbf{n}}\mathbf{v}\\ &= \mathbf{n}(\mathbf{n}\times\mathbf{v})+\mathbf{v}-(c+s\mathbf{n})\mathbf{n}(\mathbf{n}\times\mathbf{v})\\ &= \mathbf{v}+(\mathbf{n}-c\mathbf{n}+s)(\mathbf{n}\times\mathbf{v})\\ &= \mathbf{v}+((1-c)\mathbf{n}+s)(\mathbf{n}\times\mathbf{v})\\ &= \mathbf{v}+((1-\cos\theta)\mathbf{n}+\sin\theta)(\mathbf{n}\times\mathbf{v})\end{aligned}$$

which appears to be a smaller form of the Rodrigues rotation formula. Starting here, we have enough identities to switch into matrices and column vectors as

$$\begin{aligned}[\ \mathbf{v}_\theta\ ] &= \{\ \mathbf{n}\ \}^{||}[\ \mathbf{v}\ ]+\left\{\ e^{\theta\mathbf{n}}\ \right\}\{\ \mathbf{n}\ \}^{\perp}[\ \mathbf{v}\ ]\\ &= \left(\{\ \mathbf{n}\ \}^{||}+\left\{\ e^{\theta\mathbf{n}}\ \right\}\{\ \mathbf{n}\ \}^{\perp}\right)[\ \mathbf{v}\ ]\\ &= (\{\ \mathbf{n}\ \}^{||}+(\{\ c\ \}+s\{\ \mathbf{n}\ \})(-\{\ \mathbf{n}\ \}\{\ \mathbf{n}\ \}^{\times}))[\ \mathbf{v}\ ]\\ &= (\{\ \mathbf{n}\ \}^{||}-c\{\ \mathbf{n}\ \}\{\ \mathbf{n}\ \}^{\times}+s\{\ \mathbf{n}\ \}^{\times})[\ \mathbf{v}\ ]\\ &= (\{\ \mathbf{n}\ \}^{||}+c(\{\ 1\ \}-\{\ \mathbf{n}\ \}^{||})+s\{\ \mathbf{n}\ \}^{\times})[\ \mathbf{v}\ ]\\ &= (\{\ \mathbf{n}\ \}^{||}+\{\ c\ \}-c\{\ \mathbf{n}\ \}^{||}+s\{\ \mathbf{n}\ \}^{\times})[\ \mathbf{v}\ ]\\ &= ((1-c)\{\ \mathbf{n}\ \}^{||}+\{\ c\ \}+s\{\ \mathbf{n}\ \}^{\times})[\ \mathbf{v}\ ]\\ &= \left\{\ R^\theta_{\mathbf{n}}\ \right\}[\ \mathbf{v}\ ]\end{aligned}$$

which can be written as the matrix-vector product

$$
\begin{aligned}
[\,\mathbf{v}_\theta\,] &= \{\, R^\theta_{\mathbf{n}} \,\}[\,\mathbf{v}\,] \\
&= \left(\begin{array}{c} (1-c)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_x n_x & n_x n_y & n_x n_z \\ 0 & n_y n_x & n_y n_y & n_y n_z \\ 0 & n_z n_x & n_z n_y & n_z n_z \end{pmatrix} + \\ \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} + s\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -n_z & n_y \\ 0 & n_z & 0 & -n_x \\ 0 & -n_y & n_x & 0 \end{pmatrix} \end{array} \right)[\,\mathbf{v}\,] \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-c)n_x n_x + c & (1-c)n_x n_y - s n_z & (1-c)n_x n_z + s n_y \\ 0 & (1-c)n_y n_x + s n_z & (1-c)n_y n_y + c & (1-c)n_y n_z - s n_x \\ 0 & (1-c)n_z n_x - s n_y & (1-c)n_z n_y + s n_x & (1-c)n_z n_z + c \end{pmatrix}[\,\mathbf{v}\,]
\end{aligned}
$$

where this matrix represents the rotation operator. The lower 3×3 part of the matrix is all that is needed to rotate vectors in 3D. The full 4×4 matrix can be used to rotate quaternions, where any scalar part is unaffected by the rotation and the vector part is rotated in the usual way.

Rotations round the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ axes are found by setting $\mathbf{n}$ as one of them, which gives

$$
\begin{aligned}
\{\, R^\theta_{\mathbf{i}} \,\} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \\
\{\, R^\theta_{\mathbf{j}} \,\} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \\
\{\, R^\theta_{\mathbf{k}} \,\} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

A rotation by a negative angle is the same as a change of basis onto axes rotated by the positive angle. *Cramer's Rule* provides a more generalized way to compute vectors relative to a new or changed basis.

1.32 Quaternions and Complex Numbers

Quaternions are the extension of complex numbers from 2D to 4D. However, this extension can be interpretted in different ways that suggest quaternions either are or are not actually an extension of complex numbers. In one interpretation, an extension of complex numbers is expected to also have commutative multiplication, which quaternions do not have. This can lead to a conclusion that quaternions are not really an extension of complex number, but are something else.

The relationship between complex numbers and quaternions, with respect to planar rotations by 2θ radians in the vector plane perpendicular to $\mathbf{u}$, *might* be written as

$$\begin{aligned} e^{2\theta\mathbf{u}} &= (e^{\pi\mathbf{u}})^{\frac{2\theta}{\pi}} = (-1)^{\frac{2\theta}{\pi}} = \sqrt{-1}^{\frac{4\theta}{\pi}} \\ &= i^{\frac{4\theta}{\pi}} = (i^4)^{\frac{\theta}{\pi}} = 1^{\frac{\theta}{\pi}} \\ 1 &= i^4 = e^{2\pi\mathbf{u}} = e^0. \end{aligned}$$

This *seems* to show that 1 must be identified with i^4 representing a planar rotation by 2π or a null rotation round $\mathbf{u}$, otherwise the results degenerate. The value i (not to be confounded as the quaternion unit $\mathbf{i}$) seems to represent rotation by $\pi/2$ around $\mathbf{u}$ since

$$e^{\frac{\pi}{2}\mathbf{u}} = (-1)^{\frac{1}{2}} = \sqrt{-1} = i.$$

In general, i^t seems to represent t quadrants of rotation around $\mathbf{u}$. However, this interpretation of i holds for any arbitrary axis $\mathbf{u}$, such that i can always be interpreted as a versor that is perpendicular to any and all *geometrical* (generally non-commutative multiplication) planes spanned by arbitrary biradials, or that i is parallel and equivalent to all axes of rotation on the unit-sphere.

On a *metrical* (generally commutative multiplication) complex number plane where *points* (a, b) are represented by complex numbers $c = a + bi = |c|\, e^{\theta i}$, the value i is a quadrantal rotation operator on the points

$$i^t(a + bi) = |c|\, e^{\frac{\pi}{2}ti} e^{\theta i} = |c|\, e^{(\frac{\pi}{2}t + \theta)i}.$$

If $\mathbf{u}$ is temporarily identified with i (as an isomorphism), then $\mathbf{u}^t = \mathbf{b}/\mathbf{a}$ can be interpreted simultaneously as both a *geometrical* rotation operator on *geometrical vectors* $\alpha\mathbf{a} + \beta\mathbf{b}$ in the $\mathbf{ab}$-plane and as a *metrical* rotation operator on *metrical points* (a, b) represented by complex numbers $a + bi \Leftrightarrow a + b\mathbf{u}$.

In standard quaternions, the scalars q_w, q_x, q_y, q_z are limited to real numbers so that complex numbers $a + bi$ are not elements of the set of standard quaternion numbers. This means that using i within standard or real quaternions is not permitted. Therefore, the identification of $\mathbf{u}$ with i, or trying $\mathbf{u} = i$ is not valid. In the above, the words *"might"* and *"seems"* are emphasized because $i^t = \sqrt{-1}^t$, or any power of -1, is not a valid value to construct within standard quaternions. Stated more clearly, the power

$$(-1)^t = (e^{\pi i})^t = e^{t\pi i} = \cos(t\pi) + \sin(t\pi)i$$

is generally a complex number, and therefore it is *not generally* permitted in real quaternions.

Whenever a value similar to $\sqrt{-1}^t$ is needed, a specific versor $\mathbf{u}^t$ having specific geometrical significance on the unit-sphere must be used as a metrical rotation operator on the ab-plane of pseudo-complex numbers $a + b\mathbf{u}$.

In practice, an exception can be made for $(-1)^t$ and allow this power if t is an integer, since the result remains a real number. Expressions of the form $(-1)^t$ are often used to give the sign or orientation of certain results. In general, this exception can cause problems because it is an operation outside of the proper algebra of quaternions.

Consider the general case of taking the power t of a quaternion q

$$\begin{aligned}
\mathbf{u} &= \mathrm{UVU}q = \mathrm{UV}\left(\frac{q}{\mathrm{T}q}\right) = \mathrm{U}\left(\frac{\mathbf{q}}{\mathrm{T}q}\right) = \frac{\mathrm{T}q}{\mathrm{T}\mathbf{q}}\mathbf{q} \\
0 \le \theta &= \cos^{-1}(\mathrm{SU}q) = \cos^{-1}\left(\frac{q_w}{\mathrm{T}q}\right) \le \pi \\
q^t &= (\mathrm{T}q\mathrm{U}q)^t = (\mathrm{T}q)^t e^{t\theta\mathbf{u}} = (\mathrm{T}q)^t[\cos(t\theta) + \sin(t\theta)\mathbf{u}]
\end{aligned}$$

and notice that tensors are positive and the power of a versor should never require taking the power $(-1)^t$ so long as the axis $\mathbf{u}$ is known.

If $\theta = 0$ or $\theta = \pi$ then $e^{\theta\mathbf{u}} = \pm 1$ and $\mathbf{u}$ can be lost, which is a problem where it can be said that a scalar $\pm\mathrm{T}q$, or any scalar by itself, is *not* a quaternion. It is important to not lose the axis $\mathbf{u}$ of the quaternion or else the quaternion geometric algebra is not closed, or it has exited into the metrical algebra of real or complex numbers if that is acceptable to your application. Whatever the angle of θ, the versor expression $e^{\theta\mathbf{u}}$ should be retained and not allowed to degenerate into a scalar if remaining in the quaternion algebra is important.

It would appear that quaternions do not really extend metrical real or complex numbers, but forms its very own special geometric algebra that is not closed if the axis (the geometric part) of a quaternion is lost. The paper *Planar Complex Numbers in Even n Dimensions* and the monograph *Complex Numbers in n Dimensions*, both by SILVIU OLARIU and published in 2000, present a possible extension of complex numbers that have commutative multiplication. Higher dimensional extensions of complex numbers are also known as hypercomplex numbers.

In geometric algebras, such as quaternions and Clifford geometric algebras, the value $\sqrt{-1}$ is usually avoided due to its ambiguous or indeterminate geometrical interpretation. It is avoided by limiting scalars to real numbers and using a sub-algebra isomorphic to complex numbers when needed. Again, it is also possible to allow the geometric algebra to exit into the metrical real or complex number algebra if that is acceptable. And again, an exception is integer t powers $(-1)^t$ that are commonly used in formulas to compute signs or orientation.

Hamilton's *biquaternions* include i as an element of its algebra, but it might be well to consider using the Clifford geometric algebra $\mathcal{G}_{4,1}$, wherein complex numbers, quaternions, biquaternions, and dual quaternions are all embedded isomorphically as subalgebras.

Chapter 2
Geometric Algebra

2.1 Quaternion Biradials in Geometric Algebra

Quaternions are almost entirely a subject about biradials and rotation operations. Studying the algebra of rotations also teaches a lot about the extension of quaternions, the multivectors of geometric algebra. In geometric algebra, the products and quotients of Euclidean vectors are also biradials or transition operators, and are very similar to quaternions. In fact, they are quaternions, but in a different form, with different terminology, and in a different, more generalized algebra. The quotient $\mathbf{b}/\mathbf{a}$ is still the transition operator that takes $\mathbf{a}$ to $\mathbf{b}$ as $\mathbf{b} = (\mathbf{b}/\mathbf{a})\mathbf{a}$. The transition of $\mathbf{a}$ to $\mathbf{b}$ is a composition of a rotation into the line of $\mathbf{b}$ followed by a scaling to the length of $\mathbf{b}$.

This chapter will revisit the quaternion rotation operation as it is represented in geometric algebra. This chapter also serves as a general introduction to geometric algebras. Many important general formulas and identities of geometric algebra are given in the sections of this chapter.

2.2 Imaginary Directed Quantities

A *real quaternion*

$$\begin{aligned} q &= q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k} \\ &= q_w + \mathbf{q} \end{aligned}$$

has a *pure quaternion* or *quaternion vector* part

$$\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$$

and a *scalar* part q_w, where the scalars q_w, q_x, q_y, q_z are limited to *real numbers*. The real quaternions are simply the *quaternions* commonly used.

The *quaternion vector units* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are used as a *basis* for representing points and directed quantities in a three-dimensional space. However, the square $\mathbf{q}^2$ of a vector quantity $\mathbf{q}$ in quaternions is $\mathbf{q}^2 = -|\mathbf{q}|^2$. In effect, the directed quantity $\mathbf{q}$ is treated as a pure imaginary quantity, even though its components q_x, q_y, q_z have been limited to real numbers and the quantity should be considered real and have a real positive squared quantity. This is one of the problems with quaternions and is part of the motivation for avoiding quaternion vector quotients and products in ordinary vector calculus or analysis, as introduced in the 1901 book *Vector Analysis* by E. B. Wilson that is founded upon the lectures of J. W. Gibbs.

2.3 Real Directed Quantities

A different orthonormal basis of base vector units $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for three-dimensional space can be used instead of the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ basis of quaternions. On this basis, the directed quantity $\mathbf{q}$ can be rewritten as

$$\mathbf{q} = q_x\mathbf{e}_1 + q_y\mathbf{e}_2 + q_z\mathbf{e}_3.$$

If we now require that $\mathbf{q}^2 = |\mathbf{q}|^2$, then we also require that the units have signature $+1$ as

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1.$$

The vector $\mathbf{q}$ with signature $+1$ is called a *Euclidean vector*, and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the Euclidean vector units of a three-dimensional Euclidean *space*, denoted $\mathcal{E}^3$ or $\mathcal{G}^3$. The scalars q_x, q_y, q_z are still limited to real numbers.

2.4 Clifford Geometric Algebras

The *algebra*, variously denoted $\mathcal{E}_3$ or $\mathcal{G}_3$, of the vector units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is called the *Clifford algebra* or *geometric algebra* of Euclidean 3-space, and can also be denoted $\mathcal{C}\ell(\mathcal{E}^3)$. Most of this chapter is concerned with this particular Clifford algebra of Euclidean 3D space.

This $\mathcal{G}_3$ geometric algebra of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ differs from the *quaternion geometric algebra* of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. However, the quaternion units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are isomorphic to the three *unit 2-blades*

$$\begin{aligned}
\mathbf{E_i} &= \mathbf{e}_3/\mathbf{e}_2 = \mathbf{e}_3 \wedge \mathbf{e}_2 \\
\mathbf{E_j} &= \mathbf{e}_1/\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_3 \\
\mathbf{E_k} &= \mathbf{e}_2/\mathbf{e}_1 = \mathbf{e}_2 \wedge \mathbf{e}_1
\end{aligned}$$

of this Clifford algebra, allowing quaternion concepts to be used within the *subalgebra* of these three unit 2-blades.

Clifford algebras also allow more vector units than three, and also to have some units that square positive and some that square negative. An example is the Clifford algebra having the vector units

$$\begin{aligned}
\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{e}_+^2 &= 1 \\
\mathbf{e}_-^2 &= -1
\end{aligned}$$

where the space is denoted $\mathcal{G}^{4,1}$ and its algebra is denoted $\mathcal{G}_{4,1}$. The algebra $\mathcal{G}_{4,1}$ is very useful and is called the *conformal geometric algebra of Euclidean physical space* (CGA). Its generalization to an n-dimensional basis is called the *homogeneous model of $\mathcal{E}^n$* or CnGA. Quaternion rotation concepts are central to using CGA, wherein operators of the form QAQ^{-1} generate all the ordinary transformations on A, including rotations, translations, and dilations.

Clifford geometric algebra is also commonly called just *geometric algebra*. Geometric algebra, in the form discussed in this book, was introduced in the 1984 book *Clifford Algebra to Geometric Calculus, A Unified Language for Mathematics and Physics*[7] by authors David Hestenes and Garret Sobczyk. Geometric algebra derives from earlier work on the abstract algebras of *Clifford numbers* that assign little or no geometrical meanings or interpretations. The emphasis of geometric algebra is on geometrical interpretations and applications. Clifford geometric algebra is named after William Kingdon Clifford. Clifford wrote papers, including *Applications of Grassmann's Extensive Algebra* (1878) and *On The Classification of Geometric Algebras* (1876), that introduced geometric algebra as a unification of the earlier geometric algebras of Jean-Robert Argand, August Ferdinand Möbius, Carl Friedrich Gauss, George Peacock, William Rowan Hamilton, Hermann Grassmann, Adhemar Jean Claude Barre de Saint-Venant, Thomas Kirkman, Augustin-Louis Cauchy, and Benjamin Peirce. Clifford died in 1879 before he could fully develop and publish his ideas on geometric algebra.

2.5 Rules for Euclidean Vector Units

While the quaternion units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ have the *rule* or formula

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} \;=\; -1,$$

the Clifford units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathcal{G}_3$ have the formula

$$\begin{aligned}
-\mathbf{e}_1^2 = -\mathbf{e}_2^2 = -\mathbf{e}_3^2 &= -\mathbf{e}_1 \cdot \mathbf{e}_1 = -\mathbf{e}_2 \cdot \mathbf{e}_2 = -\mathbf{e}_3 \cdot \mathbf{e}_3 \\
&= (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 = (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)^2 = \mathbf{I}^2 \\
&= (\mathbf{e}_3\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_3)(\mathbf{e}_2\mathbf{e}_1) \\
&= (\mathbf{e}_3/\mathbf{e}_2)(\mathbf{e}_1/\mathbf{e}_3)(\mathbf{e}_2/\mathbf{e}_1) \\
&= (\mathbf{e}_3 \wedge \mathbf{e}_2)(\mathbf{e}_1 \wedge \mathbf{e}_3)(\mathbf{e}_2 \wedge \mathbf{e}_1) \\
&= \mathbf{E_i}\mathbf{E_j}\mathbf{E_k} = \mathbf{E_i}^2 = \mathbf{E_j}^2 = \mathbf{E_k}^2 = -1
\end{aligned}$$

which is also showing some extra identities that map to quaternion quadrantal versors.

The quaternion units map into $\mathcal{G}_3$ as the Euclidean vector biradials, which are called unit *bivectors* or 2-*blades* in Clifford algebra

$$\begin{aligned}
\mathbf{i} = \mathbf{k}/\mathbf{j} &\;\Leftrightarrow\; \mathbf{e}_3/\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_3 \wedge \mathbf{e}_2 = \mathbf{E_i} \\
\mathbf{j} = \mathbf{i}/\mathbf{k} &\;\Leftrightarrow\; \mathbf{e}_1/\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_3 = \mathbf{E_j} \\
\mathbf{k} = \mathbf{j}/\mathbf{i} &\;\Leftrightarrow\; \mathbf{e}_2/\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_2 \wedge \mathbf{e}_1 = \mathbf{E_k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{E_i} &= \mathbf{E_k}/\mathbf{E_j} = -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_3\mathbf{e}_2 \\
\mathbf{E_j} &= \mathbf{E_i}/\mathbf{E_k} = -\mathbf{e}_3\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_3 \\
\mathbf{E_k} &= \mathbf{E_j}/\mathbf{E_i} = -\mathbf{e}_1\mathbf{e}_3\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_2\mathbf{e}_1.
\end{aligned}$$

The mapping is fairly simple when we view the quaternion units as quadrantal versors or rotation axes being held in right-hand on right-handed axes model, or held in left-hand on left-handed axes model while fingers curl around the rotational direction as thumb points in the positive direction of the axis.

The inverse of a quaternion unit is its negative or conjugate, while the inverse of a Euclidean unit is itself. The square of a unit bivector is -1, as for example

$$\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = -1.$$

For $i \neq j$, different units multiply anti-commutatively as

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$$

and are identical to their outer product

$$\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i$$

where the outer product symbol ($\wedge$) between different base vector units symbolizes the perpendicularity, or angle $\pi/2$ between the units. The inner product symbol ($\cdot$) between same base vector units symbolizes the parallelism, or zero angle between units.

An important definition is the formula for the geometric product of vector units

$$\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_i \wedge \mathbf{e}_j$$

which is a true formula since either $i = j$ and results in $\mathbf{e}_i \cdot \mathbf{e}_j + 0 = 1$, or $i \neq j$ and results in $0 + \mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$.

2.6 The Outer Product

The product ($\wedge$) is called the *outer product* and it constructs extensive quantities, geometric entities or objects, or *blades* from vectors. The outer product $\mathbf{a} \wedge \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ has the same magnitude $|\mathbf{a}||\mathbf{b}|\sin(\theta)$ as the cross product ($\times$) of the vectors, but the result of the outer product is *not* a mutually perpendicular vector $\mathbf{n}^{\perp\{\mathbf{a},\mathbf{b}\}}$ but is the value itself $\mathbf{a} \wedge \mathbf{b}$ and is called a 2-blade, 2-*vector*, or *bivector*. The vector cross product can be defined using the outer product.

The *cross product of vectors* $\mathbf{a}$ and $\mathbf{b}$ is given by the formula

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (\mathbf{a} \wedge \mathbf{b})/\mathbf{I} \\
&= |\mathbf{a}||\mathbf{b}|\sin(\theta)\mathbf{n}^{\perp\{\mathbf{a},\mathbf{b}\}} \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= (a_y b_z - a_z b_y)\mathbf{e}_1 + (a_z b_x - a_x b_z)\mathbf{e}_2 + (a_x b_y - a_y b_x)\mathbf{e}_3.
\end{aligned}$$

The outer product of parallel ($\theta = 0$) vectors is zero ($\sin\theta = 0$), and the outer product of perpendicular ($\theta = \pi/2$) vectors is an irreducible symbolic outer product expression of the perpendicular vector components.

The outer product of n vectors is a new extensive n-dimensional *geometrical entity* or object representing a *geometrical measure* of their perpendicularity or space.

The outer ($\wedge$) and inner ($\cdot$) product symbols between the different and same orthonormal base vector units in the rule formulas, and the relation to the value $\mathbf{I}$ and -1, can also serve to define these products in terms of simple algebraic multiplications of the units in ways consistent with the rule formula. The product of algebraic multiplication of two values in terms of only base vector units and scalars, obeying the basic rule formula, is known as the geometric product. The absence of any product symbol means algebraic multiplication consistent with rules, i.e. a geometric product.

An important result *in* the outer product of vectors is

$$\begin{aligned}
\mathbf{a}\wedge\mathbf{b} &= \mathbf{a}\wedge(\mathbf{b}^{||\mathbf{a}}+\mathbf{b}^{\perp\mathbf{a}}) = \mathbf{a}\wedge\mathbf{b}^{||\mathbf{a}}+\mathbf{a}\wedge\mathbf{b}^{\perp\mathbf{a}} = 0+\mathbf{a}\wedge\mathbf{b}^{\perp\mathbf{a}} = \mathbf{a}\mathbf{b}^{\perp\mathbf{a}}\\
&= (\mathbf{a}^{||\mathbf{b}}+\mathbf{a}^{\perp\mathbf{b}})\wedge\mathbf{b} = \mathbf{a}^{||\mathbf{b}}\wedge\mathbf{b}+\mathbf{a}^{\perp\mathbf{b}}\wedge\mathbf{b} = 0+\mathbf{a}^{\perp\mathbf{b}}\wedge\mathbf{b} = \mathbf{a}^{\perp\mathbf{b}}\mathbf{b}\\
&= \mathbf{a}\wedge(s\mathbf{a}+\mathbf{b}) = (t\mathbf{b}+\mathbf{a})\wedge\mathbf{b}\\
&= |\mathbf{a}|\,|\mathbf{b}|\sin(\theta)\frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|\sin(\theta)} = |\mathbf{a}|\,|\mathbf{b}|\sin(\theta)\mathbf{N}\\
&= -|\mathbf{a}|\,|\mathbf{b}|\sin(-\theta)\,\mathbf{N} = -\mathbf{b}\wedge\mathbf{a}
\end{aligned}$$

that *one* of the two parallel components is always *arbitrarily scaled* or *deleted*, such that $\mathbf{a}$ *or* $\mathbf{b}$ can be moved or translated parallel to $\mathbf{b}$ *or* $\mathbf{a}$ (perhaps until intersecting or becoming collinear with another object or line of interest) without affecting the result *of* their outer product. The value $\mathbf{N}$ is the irreducible unit bivector of the outer product $\mathbf{a}\wedge\mathbf{b}$. It will be shown that $\mathbf{N}$ represents the plane containing $\mathbf{a}$ and $\mathbf{b}$ as a unit of directed and oriented area and that it also acts as the quadrantal versor of the plane, similar to a unit vector in quaternions.

2.7 Extended Imaginaries

The outer product expression $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\wedge\mathbf{a}_n$ can generally be written

$$\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\wedge\mathbf{a}_n = \left|\,\mathbf{a}_1\ \ \mathbf{a}_2\ \ \cdots\ \ \mathbf{a}_n\,\right|\mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_n$$

where the determinant $|\,\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n\,|$ of the matrix of n vectors $\mathbf{a}_i$ in an n-dimensional space is the n-volume of a n-parallelotope formed by the n-frame of the n vectors $\mathbf{a}_i$. The outer product of the n vector-units $\mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_n$ can be regarded as a type of *imaginary value* or symbol, representing physical perpendicularity of the unit vectors in up to $n=3$ physical dimensions, and representing *imaginary perpendicularity* in higher dimensions $n>3$. Similar to the imaginary value symbol $\sqrt{-1}$, the *imaginary product* symbol $\mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_n$ is irreducible to a *real value* by itself. However, the square of an imaginary product, such as $(\mathbf{e}_1\wedge\mathbf{e}_2)^2=-1$, is a real value. Generally, it is shown that

$$\begin{aligned}
(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_n)^2 &= (\sqrt{-1})^{2[n(n-1)/2]} = (-1)^{n(n-1)/2} = \pm 1\\
(1,2,3,4,5,6,\ldots,n) &\rightarrow (+1,+1,-1,-1,+1,+1,\ldots,\pm 1)
\end{aligned}$$

holds good for n base unit vectors of an n-dimensional Euclidean space where $\mathbf{e}_i^2=1$. Taking square roots, we *might* (but should not) write $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n = (\sqrt{-1})^{n(n-1)/2}$ and find that the right side has *lost* all of the vectors or axes of the left side (a geometrical value), and that the right side is *generally* a kind of complex number (a metrical value). However, this direct identification of a geometrical value to a metrical value is *not valid*. Nevertheless, it can be said that the geometrical n-parallelotope unit $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$ may represent a *particular* extension or *instance* of the ordinary complex number symbol $\sqrt{-1}$ when $(-1)^{n(n-1)/2} = -1$, and of the split-complex or hyperbolic number symbol $\sqrt{+1}$ when $(-1)^{n(n-1)/2} = +1$. Multivectors of the form $s + \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n$, being a scalar s plus n-blade, behave either as ordinary complex values on a complex plane or as hyperbolic values on a hyperbolic plane. The unit n-parallelotopes $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$ can be interpretted as geometrically *specific* objects that represent the metrical values $\sqrt{-1}$ and $\sqrt{+1}$ which by themselves represent unknown, lost, or *arbitrary* geometrical objects. Similar to quaternions and losing an axis of rotation at angles $\theta=\pi$ and $\theta=0$, it could be a problem to lose a 2-blade or 2-vector used as a representation of a quaternion vector or axis of rotation.

2.8 Scalar Multiplication

The outer product with scalars is identical to ordinary scalar multiplication. Scalars are considered to be perpendicular to all values, including to another scalar so that $st=s\cdot t+s\wedge t=s\wedge t$ represents the n-volume of an unknown n-parallelotope. It is not certain that st is an area since an area is represented by $\mathbf{s}\wedge\mathbf{t}$, the outer product of two vectors. All we can say about st is that it is a scalar product, or pure inner product (this may seem to be a backwards statement), or pure measure of parallelism between geometrical objects in the same dimensions by non-zero amounts, or the result of a geometrical object *contracted* to zero dimensions by inner product with another geometrical object. *Contraction* is the generalization of the vector dot product to dot products of higher-dimensional (or grade) geometrical objects represented by outer product expressions. An outer product $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ of n scalars e_i can be reduced to $e_1 e_2 \cdots e_n (1 \wedge 1 \wedge \cdots \wedge 1)$, where $(1 \wedge 1 \wedge \cdots \wedge 1) = 1$ and it represents an identity matrix that has lost its vectors and has an indeterminate dimension, or has become dimensionless (non-vectorial) and allowed to be identical to the unit 0-vector 1.

As an additional comment, it should be noted that the formulas for the inner and outer products with scalars have not been fully agreed upon by all mathematicians in geometric algebra. New or alternative formulas are beyond the scope of this article. Most of the currently available software for geometric algebra continues to use the rules that the inner product with a scalar is zero, and the outer product with a scalar is identical to ordinary scalar multiplication, and these are the rules followed in this article.

A proposed change to the inner product says that the new inner product with a scalar is to be also identical to scalar multiplication, giving the formulas[8, pp. 279]

$$\begin{aligned} \alpha\mathbf{a} &= \mathbf{a}\cdot\alpha = \mathbf{a}\wedge\alpha \\ \mathbf{a}A &= \mathbf{a}\cdot A + \mathbf{a}\wedge A - \mathbf{a}\langle A\rangle \end{aligned}$$

where α is any scalar, $\mathbf{a}$ is any vector, and A is any multivector. If the multivector A contains a scalar component, represented by the scalar grade selector $\langle A\rangle$, then $-\mathbf{a}\langle A\rangle$ subtracts one of the two copies. If $A = \mathbf{A}$, where $\mathbf{A}$ is a blade (no scalar component), then the *"Chevalley formula"*

$$\mathbf{aA} = \mathbf{a}\cdot\mathbf{A} + \mathbf{a}\wedge\mathbf{A}$$

continues to be valid. However, in general the Chevalley formula with any multivector A is no longer valid under this change. This proposed change is essentially the same as the (fat) dot product, but it tries not to introduce a new product symbol.

2.9 Grades, Blades, and Vectors

The outer product expressions $t\wedge s$, $t\wedge\mathbf{a}$, $\mathbf{a}\wedge\mathbf{b}$, and $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ represent, respectively, the geometrical values or objects: a 0-*blade* or metrical *scalar* ts in a 1-dimensional scalar subspace, a 1-*vector* or geometrical *vector* directed-length $t\mathbf{a}$ in the $\mathbf{a}$-line, a 2-*vector* or geometrical *bivector* directed-area of parallelogram in the $\mathbf{ab}$-plane (or $\mathbf{N}$-plane), and a 3-*vector* or geometrical *trivector* directed-volume of parallelepiped in the $\mathbf{abc}$-space (or $\mathbf{I}$-space).

A sum of linearly independent k-*blades* is called a k-*vector*, where the number k is called the *grade*. The three vector units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the *unit* 1-*blades* and their linear combinations are grade-1 vectors in a 3-dimensional subspace of $\mathcal{G}_3$. The three *unit* 2-*blades* are $\mathbf{E_i} = \mathbf{e}_3\wedge\mathbf{e}_2$, $\mathbf{E_j} = \mathbf{e}_1\wedge\mathbf{e}_3$, and $\mathbf{E_k} = \mathbf{e}_2\wedge\mathbf{e}_1$, and a 2-*vector* has the general form $x\mathbf{E_i} + y\mathbf{E_j} + z\mathbf{E_k}$ representing a pure quaternion vector or grade-2 vector in a 3-dimensional subspace of $\mathcal{G}_3$. The single *unit* 3-*blade* $\mathbf{I} = \mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$, called the *unit pseudoscalar*, is a grade-3 vector in a 1-dimensional subspace of $\mathcal{G}_3$. There is a total of 8 k-blade dimensions in $\mathcal{G}_3$.

The outer product of a j-blade and k-blade is a $(k+j)$-blade, or 0 if any vector is common between the blades multiplied since they do not span any area or volume (no perpendicular component).

A *multivector* is a sum or linear combination of linearly independent k-vectors of various grades k.

There are no 4-*blades* or larger outer product expressions or objects than the 3-blade pseudoscalar in $\mathcal{G}_3$, since there are only three base vector units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Outer product expressions construct representations of *directed* geometric objects or values that are within a certain *attitude* (a space spanned by a set of unit k-vectors), have a positive scalar *magnitude*, and have a direction or *orientation* (relative to the attitude or space) that is represented by a positive or negative sign on the magnitude.

The value of an outer product expression can also be represented by a matrix containing the vectors as elements where the determinant gives the signed magnitude relative to the orientation of the basis.

2.10 The inner product

The product $(\cdot)$ called the *inner product* is the complementary product to the outer product and is similar to the dot product of vector analysis and linear algebra, *except* that the inner product of quaternion vectors, if represented by $\mathbf{i}, \mathbf{j}, \mathbf{k} \Leftrightarrow \mathbf{E_i}$, $\mathbf{E_j}$, $\mathbf{E_k}$ as *pseudovectors*, is the *negative* of the value given by the dot product in actual quaternions. The Clifford algebra extends the inner product to take inner products of multivectors. The inner product of parallel $(\theta = 0)$ components is a scalar $(\cos\theta = 1)$ similar to multiplying numbers on a real or complex number line or axis, and the inner product of perpendicular $(\theta = \pi/2)$ components is zero $(\cos\theta = 0)$. The inner product is a *geometrical measure* (a multivector) of parallelism. Between two k-vectors of the same grade, the inner product reduces to a pure *metrical measure* (a scalar) of their parallelism, and no contracted (grade reduced) geometrical outer product expressions, objects, or vector components will remain.

An important result *in* the inner product of vectors is

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot(\mathbf{b}^{||\mathbf{a}} + \mathbf{b}^{\perp\mathbf{a}}) = \mathbf{a}\cdot\mathbf{b}^{||\mathbf{a}} + \mathbf{a}\cdot\mathbf{b}^{\perp\mathbf{a}} = \mathbf{a}\cdot\mathbf{b}^{||\mathbf{a}} + 0 = \mathbf{a}\mathbf{b}^{||\mathbf{a}} \\
&= (\mathbf{a}^{||\mathbf{b}} + \mathbf{a}^{\perp\mathbf{b}})\cdot\mathbf{b} = \mathbf{a}^{||\mathbf{b}}\cdot\mathbf{b} + \mathbf{a}^{\perp\mathbf{b}}\cdot\mathbf{b} = \mathbf{a}^{||\mathbf{b}}\cdot\mathbf{b} + 0 = \mathbf{a}^{||\mathbf{b}}\mathbf{b} \\
&= \mathbf{a}\cdot(\mathbf{b}^{||\mathbf{a}} + t\mathbf{b}^{\perp\mathbf{a}}) = (\mathbf{a}^{||\mathbf{b}} + s\mathbf{a}^{\perp\mathbf{b}})\cdot\mathbf{b} \\
&= |\mathbf{a}|\,|\mathbf{b}|\cos(\theta) \\
&= |\mathbf{a}|\,|\mathbf{b}|\cos(-\theta) = \mathbf{b}\cdot\mathbf{a}
\end{aligned}$$

that *one* of the two perpendicular components is always *arbitrarily scaled* or *deleted*, such that **a** *or* **b** can be moved or translated perpendicular to **b** *or* **a** without affecting the result *of* their inner product.

The inner product of a j-blade and k-blade is a $|k - j|$-blade, which is often a 0-blade scalar. Scalars are considered to be perpendicular to all values, even to other scalars; therefore, the inner product involving a scalar factor is zero. Inner products that produce a scalar can be represented by the determinant of a matrix formed by the multiplication of two other matrices representing outer products of equal grade.

A note on terminology may be appropriate here. The "inner" and "outer" products discussed in this article are those of the Clifford algebra. In the terminology of matrix linear algebra, an "inner product", "dot product", or "scalar product" is an ordinary matrix multiplication of a row vector with a column vector that produces a single scalar element, and is the means by which the dot product of vectors is computed in vector calculus. Also in the terminology of matrix linear algebra, the "outer product" is the multiplication of a column vector with a row vector that produces a matrix. In the terminology of Hermann Grassmann's exterior algebra, there are "interior" and "exterior" products, which are very closely related to the Clifford geometric algebra's inner and outer products. Geometric algebra has many distinct products to represent scalar products, inner products, outer products, dot products, contraction products, and other specialized products that can relate to products or concepts used in other algebras. Although distinct, the various products in geometric algebra are all derived from, or are parts of, the *geometric product*.

The geometric product of two Euclidean vectors is the sum of their inner and outer products. This is similar to the quaternion product of two pure quaternions (quaternion vectors), which is their cross product minus their dot product. The quaternion product is the geometric product for quaternions. In Clifford geometric algebra, the geometric product generalizes the quaternion product to more than four dimensions.

2.11 The Sum of Inner and Outer Products

While the outer and inner products are of complementary magnitude according to the trigonometric functions, neither is a complete product of vectors since each deletes a parallel or perpendicular component in one of the factors. Outer and inner products are geometric measures of perpendicularity and parallelism, respectively. The complementary magnitudes suggest that they are the projected components of yet some other value that is vectorially equal to their sum

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos(\theta)+|\mathbf{a}|\,|\mathbf{b}|\sin(\theta)\,\mathbf{N} \\ &= |\mathbf{a}|\,|\mathbf{b}|(\cos(\theta)+\sin(\theta)\,\mathbf{N}) \end{aligned}$$

This sum appears very similar to the quaternion vector biradial $\mathbf{b}/\mathbf{a}=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\times\mathbf{b}$, and they can be shown to be related as *duals* by dividing by the unit pseudoscalar. Considering unit vectors, and quaternion vectors only on the left of $\Leftrightarrow$ and Euclidean vectors for the rest, we have

$$\begin{aligned} \mathbf{b}/\mathbf{a}=-\mathbf{b}\mathbf{a}=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\times\mathbf{b} \;\Leftrightarrow\; \mathbf{b}/\mathbf{a} &= \mathbf{b}\mathbf{a}=\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\wedge\mathbf{a} \\ \mathbf{a}\times\mathbf{b} &= (\mathbf{a}\wedge\mathbf{b})/\mathbf{I}=-(\mathbf{a}\wedge\mathbf{b})\mathbf{I}=(\mathbf{b}\wedge\mathbf{a})\mathbf{I} \\ (\mathbf{a}\times\mathbf{b})/\mathbf{I} &= \mathbf{b}\wedge\mathbf{a}=|\mathbf{a}|\,|\mathbf{b}|\sin(-\theta)\,\mathbf{N}. \end{aligned}$$

Since we started looking at the product $\mathbf{a}\wedge\mathbf{b}$ first, the angle θ is now seen to represent the angle measured positive from $\mathbf{b}$ to $\mathbf{a}$, and $-\theta$ goes the reverse way from $\mathbf{a}$ to $\mathbf{b}$. Unit bivector $\mathbf{N}$ represents a quadrantal versor in the direction from $\mathbf{b}$ to $\mathbf{a}$, or $\mathbf{a}=e^{\theta\mathbf{N}}\mathbf{b}$ and $\mathbf{b}=e^{-\theta\mathbf{N}}\mathbf{a}$. If we would agree to go back and reverse the sign on θ, we would have the usual rotational orientation used in quaternion rotation biradials. However, unit bivectors like $\mathbf{N}$ do in general represent the negative rotational direction compared to the similar looking cross product. For example, in quaternions $\mathbf{n}=\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is the quadrantal rotation versor or axis that rotates from $\mathbf{a}$ toward $\mathbf{b}$, while in Euclidean vectors the similar looking unit bivector $\mathbf{N}=\frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{a}\wedge\mathbf{b}|}$ is a quadrantal versor, or geometric multiplier that effects planar rotation by $\pi/2$, that rotates from $\mathbf{b}$ towards $\mathbf{a}$, according to

$$\begin{aligned} \frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{a}\wedge\mathbf{b}|} &= \frac{\mathbf{a}}{|\mathbf{a}|}\wedge\frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|}=\frac{\mathbf{a}}{|\mathbf{a}|}\frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|} \\ \frac{\mathbf{a}}{|\mathbf{a}|}\frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|}\frac{\mathbf{b}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|} &= \frac{\mathbf{a}}{|\mathbf{a}|}\frac{|\mathbf{b}^{\perp\mathbf{a}}|^2}{|\mathbf{b}^{\perp\mathbf{a}}|^2}=\frac{\mathbf{a}}{|\mathbf{a}|}. \end{aligned}$$

For unit vectors, the direct expression of this rotation is $(\mathbf{ab})\mathbf{b} = (\mathbf{a}/\mathbf{b})\mathbf{b} = \mathbf{a}$. We also could *again* agree to reverse angles and then make $\mathbf{N} = \frac{\mathbf{b}\wedge\mathbf{a}}{|\mathbf{b}\wedge\mathbf{a}|}$ as quadrantal versor from $\mathbf{a}$ towards $\mathbf{b}$ to match the Euclidean biradial $\mathbf{b}/\mathbf{a}$ where the angle θ would be positive from $\mathbf{a}$ to $\mathbf{b}$. The choice of *orientation*, to measure an angle positive or negative from $\mathbf{a}$ to $\mathbf{b}$ or $\mathbf{b}$ to $\mathbf{a}$, is the *sign* on the angle θ used in the sine function $\sin(\theta) = -\sin(-\theta)$ of the outer product and this is why $\mathbf{b}\wedge\mathbf{a} = -\mathbf{a}\wedge\mathbf{b}$ holds in general. The choice of orientation could also be viewed as a choice between axis or inverse axis, or between angle or reverse angle.

When the angle $\theta = 0$ between vectors $\mathbf{a}$ and $\mathbf{b}$, the product or sum $\mathbf{ab} = \mathbf{a}(t\mathbf{a}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$ reduces to just the *positive* scalar $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$ which is the expected product of parallel vectors; in quaternions, this product would be negative since quaternion vectors square negative and quaternions treat vectors as imaginary directed quantities in a space that is spanned by three imaginary units. Here, we have a real Euclidean space spanned by real units.

When $\theta = \pi/2$, the sum reduces to $|\mathbf{a}|\,|\mathbf{b}|\,\mathbf{N}$ which is a directed-area parallelogram of magnitude $|\mathbf{a}|\,|\mathbf{b}|$ in the plane of unit bivector $\mathbf{N}$, and so the product represents the expected geometrical object.

For other values of θ, the result is a multivector value where the scalar part represents the projection of the multivector onto a scalar axis, and the bivector part represents the projection of the multivector onto a bivector axis.

Since $\mathbf{N}$ is a unit bivector, it squares to -1 in the same way as a quaternion unit vector; and therefore, the sum of the inner and outer products of two Euclidean vectors has the form and characteristics of an ordinary complex number or a quaternion. The magnitude of this number is found by taking the square root of the product of conjugates, by which we can see that the magnitude really is $|\mathbf{a}|\,|\mathbf{b}|$. Also by similarity to complex numbers or quaternions, we know this must be the product $\mathbf{ab}$ from which the magnitudes and angle are taken. Yet, the Euclidean vectors $\mathbf{a}$ and $\mathbf{b}$ themselves do not have characteristics of ordinary complex numbers. Since $\mathbf{a}$ and $\mathbf{b}$ square positive, they do have the characteristics of split-complex numbers or hyperbolic numbers.

The multivector $\mathbf{ba}$ can be written as

$$\begin{aligned}
\mathbf{ba} &= \mathbf{b}(\mathbf{a}^{\|\mathbf{b}} + \mathbf{a}^{\perp\mathbf{b}}) = \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\wedge\mathbf{a} \\
&= \mathbf{b}(\mathbf{ab}/\mathbf{b}) \\
&= \mathbf{b}(\mathbf{a}\cdot\mathbf{b})\mathbf{b}^{-1} + \mathbf{b}(\mathbf{a}\wedge\mathbf{b})\mathbf{b}^{-1} \\
&= \mathbf{a}\cdot\mathbf{b} + (\mathbf{a}^{\|\mathbf{b}} - \mathbf{a}^{\perp\mathbf{b}})\wedge\mathbf{b} \\
&= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}^{\perp\mathbf{b}}\wedge\mathbf{b} \\
&= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\wedge\mathbf{b} \\
\mathbf{b}\cdot\mathbf{a} &= \mathbf{ba}^{\|\mathbf{b}} = \mathbf{a}\cdot\mathbf{b} \\
\mathbf{b}\wedge\mathbf{a} &= \mathbf{ba}^{\perp\mathbf{b}} = -\mathbf{a}\wedge\mathbf{b}
\end{aligned}$$

which shows more properties of the algebra, and is the reflection of the scalar and bivector components of the product $\mathbf{ab}$ in the vector $\mathbf{b}$. Each vector in the 2-blade $\mathbf{a}\wedge\mathbf{b}$ is rotated or reflected individually, and that is the general procedure to rotate blades.

2.12 The reverse of a blade

The outer product, that constructed the bivector $\mathbf{N}$, has the characteristic of constructing unit blades that square $+1$ or -1 in a certain pattern. In general, it can be shown by a sequence of anti-commutative identities, that any blade and its *reverse* are related by

$$\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n \;=\; (-1)^{n(n-1)/2} \mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1$$

where n is the grade of the blade. The square of any n-blade has the sign given by $(-1)^{n(n-1)/2}$. For $n = 0, 1, 2, 3, 4, 5\ldots$, the pattern looks like $+1, +1, -1, -1, +1, +1, \ldots$ so that at $n = 2$ for $\mathbf{N}$, then $\mathbf{N}^2 = -1$.

The value $n(n-1)/2$ can represent a triangular area that is equal to the total number of bivector reverses. For example, to reverse the $\mathbf{a}_k$ for $n = 6$, it takes 5 bivector reverses to move $\mathbf{a}_1$ to the right side

$$\begin{aligned}
\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \mathbf{a}_4 \wedge \mathbf{a}_5 \wedge \mathbf{a}_6 &= (-1)^1 \mathbf{a}_2 \wedge \mathbf{a}_1 \wedge \mathbf{a}_3 \wedge \mathbf{a}_4 \wedge \mathbf{a}_5 \wedge \mathbf{a}_6 \\
&= (-1)^2 \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \mathbf{a}_1 \wedge \mathbf{a}_4 \wedge \mathbf{a}_5 \wedge \mathbf{a}_6 \\
&= (-1)^3 \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \mathbf{a}_4 \wedge \mathbf{a}_1 \wedge \mathbf{a}_5 \wedge \mathbf{a}_6 \\
&= (-1)^4 \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \mathbf{a}_4 \wedge \mathbf{a}_5 \wedge \mathbf{a}_1 \wedge \mathbf{a}_6 \\
&= (-1)^5 \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \mathbf{a}_4 \wedge \mathbf{a}_5 \wedge \mathbf{a}_6 \wedge \mathbf{a}_1
\end{aligned}$$

and it takes $4, 3, 2, 1$ bivector reverses, respectively, to move $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ into their reversed spots. The last vector $\mathbf{a}_6$ takes zero reverses. A triangle of these bivector reverse counts can be constructed as the triangle of numbers

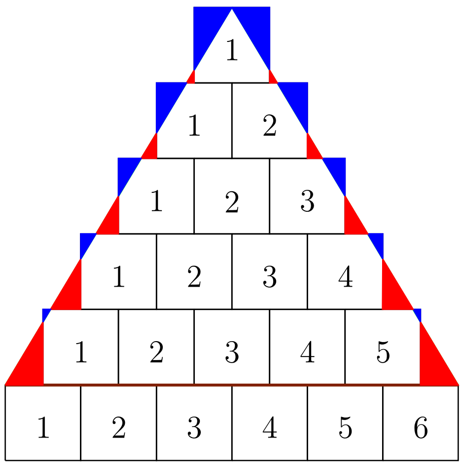


Figure 2.1. Triangular numbers.

$$\begin{array}{c|ccccccccccc}
5 & & & & & & \hat{1} & & & & & \\
4 & & & & & 1 & & 2 & & & & \\
3 & & & & 1 & & 2 & & 3 & & & \\
2 & & & 1 & & 2 & & 3 & & 4 & & \\
1 & & 1 & & 2 & & 3 & & 4 & & 5 & \\
0 & \bar{1} & & \bar{2} & & \bar{3} & & \bar{4} & & \bar{5} & & \bar{6}
\end{array}$$

that has a total areal count of 1/2 the base 6 times the height $(6-1)$

$$\begin{aligned} n(n-1)/2 &= 6(6-1)/2 \\ &= 15 \\ &= 5+4+3+2+1 \\ &= [n-1]([n-1]+1)/2 \end{aligned}$$

which is the triangular number or sum of integers formula for $[n-1]=5$.

The sum of integers is proved algebraically by adding the numbers counting up, plus adding the numbers counting down as follows.

$$\begin{aligned} & (1+2+3+\cdots+n)+ \\ & (n+(n-1)+(n-2)+\cdots+1) \\ = {} & 2\sum_{i=1}^{n} i = \sum^{n} (n+1) = n(n+1) \\ & \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \end{aligned}$$

2.13 Reversion

Reversion is an operation that reverses the order of products. Reversion is a primitive operation that has many uses in the construction of other more complicated operations.

Given an r-blade

$$\mathbf{A}_r = \bigwedge_{i=1}^{r} \mathbf{a}_i = \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_{r-1} \wedge \mathbf{a}_r$$

the reversion, or reverse, of $\mathbf{A}_r$ is

$$\begin{aligned} \tilde{\mathbf{A}}_r &= \bigwedge_{i=1}^{r} \mathbf{a}_{r-i+1} = \mathbf{a}_r \wedge \mathbf{a}_{r-1} \wedge \ldots \wedge \mathbf{a}_1 \\ &= (-1)^{r(r-1)/2} \mathbf{A}_r. \end{aligned}$$

The reverse of an r-vector $V_{\langle r\rangle}$, which is a sum of r-blades, is

$$\widetilde{V_{\langle r\rangle}} = (-1)^{r(r-1)/2} V_{\langle r\rangle},$$

the sum of the reverses of the r-blade terms.

More generally, given a multivector of grade r

$$A_r = \prod_{i=1}^{r} \mathbf{a}_i = \mathbf{a}_1 \ldots \mathbf{a}_{r-1} \mathbf{a}_r$$

the reverse of A_r is

$$\begin{aligned}\widetilde{A_r} &= \prod_{i=1}^{r} \mathbf{a}_{r-i+1} = \mathbf{a}_r \mathbf{a}_{r-1} \ldots \mathbf{a}_1 \\ &= \sum_{i=0}^{r} \widetilde{\langle A_r \rangle_i}\end{aligned}$$

which is the sum of the reverses of the i-vector terms.

Properties of the reverse, for any multivectors A and B:

$$\begin{aligned}(AB)^{\sim} &= \tilde{B}\tilde{A} \\ (\tilde{A})^{\sim} &= A.\end{aligned}$$

2.14 Conjugation of blades

The conjugation of blades $\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_{r-1} \wedge \mathbf{a}_r$ in a geometric algebra generalizes the conjugations of the imaginary unit i of complex numbers and the pure quaternion vectors $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of quaternions. Blades are a generalization of imaginaries to higher dimensions or grades.

In a $\mathcal{G}_{p,q}$ geometric algebra, there are $p+q=n$ orthogonal unit vectors where the first p unit vectors have positive signature and the last q unit vectors have negative signature

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} +1 & | \quad i=j; 1 \le i \le p \\ -1 & | \quad i=j; p+1 \le i \le n \\ 0 & | \quad i \ne j. \end{cases}$$

An r-blade $\mathbf{A}_r$ in this algebra is a scalar α multiple of the outer product of some combination of r elements taken from the set of n vector units $\mathbf{e}_i$

$$\mathbf{A}_r = \alpha \bigwedge_{i=1}^{r} \mathbf{e}_{\sigma(i)} = \alpha \mathbf{e}_{\sigma(1)} \wedge \ldots \wedge \mathbf{e}_{\sigma(r-1)} \wedge \mathbf{e}_{\sigma(r)}$$

where $\sigma(i) \in \sigma\{1, \ldots, p, p+1, \ldots, n\}$ is the ith element from some permutation of the ordered indices.

The number, or grade, of negative-signature vector units $\mathbf{e}_{\sigma(i)}$ in $\mathbf{A}_r$ is denoted $\mathrm{gr}_-(\mathbf{A}_r)$, and the number of positive-signature vector units $\mathbf{e}_{\sigma(i)}$ in $\mathbf{A}_r$ is denoted $\mathrm{gr}_+(\mathbf{A}_r)$.

An r-blade $\mathbf{A}_r$ can also be written as the outer product of r linearly independent vectors $\mathbf{a}_i$ as

$$\mathbf{A}_r = \bigwedge_{i=1}^{r} \mathbf{a}_i = \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_{r-1} \wedge \mathbf{a}_r.$$

An r-blade $\mathbf{A}_r$ is called a *null r-blade* $\mathbf{A}_r \in \mathcal{G}^{\circ r}_{p,q}$ if $\mathbf{A}_r\mathbf{A}_r = 0$. A null-blade *does not* have an inverse $\mathbf{A}_r^{-1}$ but *does have* a pseudoinverse $\mathbf{A}_r^{+}$ with respect to the inner product.

The *conjugate*[9] of any r-blade $\mathbf{A}_r$ is defined as

$$\begin{aligned}\mathbf{A}_r^{\dagger} &= (-1)^{\mathrm{gr}_-(\mathbf{A}_r)}\tilde{\mathbf{A}}_r \\ &= (-1)^{\mathrm{gr}_-(\mathbf{A}_r)+r(r-1)/2}\mathbf{A}_r.\end{aligned}$$

In any $\mathcal{G}_n$ geometric algebra of n positive-signature vector units,

$$\mathbf{A}_r^{\dagger} = \tilde{\mathbf{A}}_r.$$

Properties of the conjugate of a *non-null* r-blade $\mathbf{A}_r \in \mathcal{G}^{\varnothing r}_{p,q}$:

$$\begin{aligned}\mathbf{A}_r^{\dagger}\mathbf{A}_r &= \mathbf{A}_r\mathbf{A}_r^{\dagger} \\ &= |\mathbf{A}_r|^2 \\ &= \mathbf{A}_r^{\dagger}\cdot\mathbf{A}_r \\ \mathbf{A}_r^{-1} &= \frac{\mathbf{A}_r^{\dagger}}{\mathbf{A}_r^{\dagger}\mathbf{A}_r} = \frac{\tilde{\mathbf{A}}_r}{\tilde{\mathbf{A}}_r\mathbf{A}_r}.\end{aligned}$$

The inverse $\mathbf{A}_r^{-1}$ of a non-null r-blade $\mathbf{A}_r \in \mathcal{G}^{\varnothing r}_{p,q}$ is with respect to either the geometric product or inner product, as $\mathbf{A}_r^{-1}\mathbf{A} = \mathbf{A}_r^{-1}\cdot\mathbf{A}_r = +1$.

Properties of the conjugate of a *null* r-blade $\mathbf{A}_r \in \mathcal{G}^{\circ r}_{p,q}$:

$$\begin{aligned}\mathbf{A}_r^{\dagger}\cdot\mathbf{A}_r &= \mathbf{A}_r\cdot\mathbf{A}_r^{\dagger} \\ &= |\mathbf{A}_r|^2 \\ &\neq \mathbf{A}_r^{\dagger}\mathbf{A}_r \\ \mathbf{A}_r^{+} &= \frac{\mathbf{A}_r^{\dagger}}{\mathbf{A}_r^{\dagger}\cdot\mathbf{A}_r}\end{aligned}$$

A null r-blade $\mathbf{A}_r \in \mathcal{G}^{\circ r}_{p,q}$ has a *pseudoinverse* $\mathbf{A}_r^{+}$ with respect to the inner product $\mathbf{A}_r^{+}\cdot\mathbf{A}_r = +1$, but does not have an inverse $\mathbf{A}_r^{-1}$ with respect to the geometric product.

For a non-null blade, $\mathbf{A}_r^{+} = \mathbf{A}_r^{-1}$. For any r-blade, $\mathbf{A}_r^{+}\cdot\mathbf{A}_r = +1$ such that the pseudoinverse of an r-blade is always an inverse with respect to the inner product.

If $\mathbf{E}_i$ is the ith basis blade in a ordered set of canonical bases for any $\mathcal{G}_{p,q}$ geometric algebra, then $\mathbf{E}_i^{\dagger} = \mathbf{E}_i^{-1}$.

The conjugation of blades is also called *spatial reversion* in Space-Time Algebra, and can also be identified with *Hermitian conjugation* of complex matrices.

2.15 The square of a blade

In geometric algebra, squaring a blade is an important algebraic operation. It extends the concept of squaring a vector to more dimensions or to higher grades of vectors.

Consider vectors written in component form as

$$\mathbf{a}_k \;=\; a_{k1}\mathbf{e}_1 + a_{k2}\mathbf{e}_2 + \cdots + a_{kk}\mathbf{e}_k + \cdots + a_{kn}\mathbf{e}_n.$$

Using the formula for the reverse of a n-blade

$$\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n \;=\; (-1)^{n(n-1)/2}\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1$$

we can write the square of a n-blade as

$$(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)^2 \;=\;$$

$$(-1)^{n(n-1)/2}(\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n).$$

Using the recursive formulas for the inner products of blades, we can expand this inner product as

$$(-1)^{n(n-1)/2}(\mathbf{a}_n \cdot \ldots \cdot (\mathbf{a}_k \cdot \ldots \cdot (\mathbf{a}_1 \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n))\cdots)\cdots)$$

$$= \; (-1)^{n(n-1)/2}\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_1\cdot\mathbf{a}_k & \cdots & \mathbf{a}_1\cdot\mathbf{a}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_k\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_k\cdot\mathbf{a}_k & \cdots & \mathbf{a}_k\cdot\mathbf{a}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_n\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_n\cdot\mathbf{a}_k & \cdots & \mathbf{a}_n\cdot\mathbf{a}_n \end{vmatrix}$$

$$= \; (-1)^{n(n-1)/2}\begin{vmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_n \end{vmatrix} \begin{vmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_k & \ldots & \mathbf{a}_n \end{vmatrix}$$

$$= \; (-1)^{n(n-1)/2}\begin{vmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix}^2.$$

To demonstrate this result for $n = 3$, let's take

$$(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)^2 \;=\; (-1)^{3(3-1)/2}(\mathbf{a}_3 \wedge \mathbf{a}_2 \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)$$

$$= \; (-1)(\mathbf{a}_3 \wedge \mathbf{a}_2) \cdot ((\mathbf{a}_1\cdot\mathbf{a}_1)\mathbf{a}_2 \wedge \mathbf{a}_3 - (\mathbf{a}_1\cdot\mathbf{a}_2)\mathbf{a}_1 \wedge \mathbf{a}_3 + (\mathbf{a}_1\cdot\mathbf{a}_3)\mathbf{a}_1 \wedge \mathbf{a}_2)$$

$$= \; (-1)(\mathbf{a}_3) \cdot \begin{pmatrix} (\mathbf{a}_1\cdot\mathbf{a}_1)(\mathbf{a}_2\cdot\mathbf{a}_2)\mathbf{a}_3 - (\mathbf{a}_1\cdot\mathbf{a}_1)(\mathbf{a}_2\cdot\mathbf{a}_3)\mathbf{a}_2 & - \\ ((\mathbf{a}_1\cdot\mathbf{a}_2)(\mathbf{a}_2\cdot\mathbf{a}_1)\mathbf{a}_3 - (\mathbf{a}_1\cdot\mathbf{a}_2)(\mathbf{a}_2\cdot\mathbf{a}_3)\mathbf{a}_1) & + \\ (\mathbf{a}_1\cdot\mathbf{a}_3)(\mathbf{a}_2\cdot\mathbf{a}_1)\mathbf{a}_2 - (\mathbf{a}_1\cdot\mathbf{a}_3)(\mathbf{a}_2\cdot\mathbf{a}_2)\mathbf{a}_1 & \end{pmatrix}$$

$$= \; (-1)\begin{pmatrix} (\mathbf{a}_1\cdot\mathbf{a}_1)(\mathbf{a}_2\cdot\mathbf{a}_2)(\mathbf{a}_3\cdot\mathbf{a}_3) - (\mathbf{a}_1\cdot\mathbf{a}_1)(\mathbf{a}_2\cdot\mathbf{a}_3)(\mathbf{a}_3\cdot\mathbf{a}_2) & - \\ (\mathbf{a}_1\cdot\mathbf{a}_2)(\mathbf{a}_2\cdot\mathbf{a}_1)(\mathbf{a}_3\cdot\mathbf{a}_3) + (\mathbf{a}_1\cdot\mathbf{a}_2)(\mathbf{a}_2\cdot\mathbf{a}_3)(\mathbf{a}_3\cdot\mathbf{a}_1) & + \\ (\mathbf{a}_1\cdot\mathbf{a}_3)(\mathbf{a}_2\cdot\mathbf{a}_1)(\mathbf{a}_3\cdot\mathbf{a}_2) - (\mathbf{a}_1\cdot\mathbf{a}_3)(\mathbf{a}_2\cdot\mathbf{a}_2)(\mathbf{a}_3\cdot\mathbf{a}_1) & \end{pmatrix}$$

$$= \; (-1)\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{a}_1 & \mathbf{a}_1\cdot\mathbf{a}_2 & \mathbf{a}_1\cdot\mathbf{a}_3 \\ \mathbf{a}_2\cdot\mathbf{a}_1 & \mathbf{a}_2\cdot\mathbf{a}_2 & \mathbf{a}_2\cdot\mathbf{a}_3 \\ \mathbf{a}_3\cdot\mathbf{a}_1 & \mathbf{a}_3\cdot\mathbf{a}_2 & \mathbf{a}_3\cdot\mathbf{a}_3 \end{vmatrix}$$

and factoring this determinant gives

$$
\begin{aligned}
&= (-1)\begin{vmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{vmatrix} \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{vmatrix} \\
&= (-1)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \\
&= (-1)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^2 .
\end{aligned}
$$

The square of a n-blade has a sign given by $(-1)^{n(n-1)/2}$, and has a positive magnitude, or absolute value, given by the square of the determinant of the scalar coordinates.

The determinant of the scalar coordinates

$$
\begin{vmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix}
$$

is the signed, or oriented, n-volume of the directed n-parallelotope framed by the n vectors. Multiplying this determinant by $(1/n!)$ gives the signed n-volume of the directed n-simplex framed by the n vectors.

It can also be seen that the product of two parallel n-blades, on the same unit n-blade or in the same n-space, is the scalar product

$$
\begin{aligned}
&(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n) \cdot (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k \wedge \cdots \wedge \mathbf{b}_n) \\
&= (-1)^{n(n-1)/2}(\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k \wedge \cdots \wedge \mathbf{b}_n) \\
&= (-1)^{n(n-1)/2} \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_k & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_k \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_k \cdot \mathbf{b}_k & \cdots & \mathbf{a}_k \cdot \mathbf{b}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_n \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_n \cdot \mathbf{b}_k & \cdots & \mathbf{a}_n \cdot \mathbf{b}_n \end{vmatrix} \\
&= (-1)^{n(n-1)/2} \begin{vmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} & \cdots & b_{k1} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{1k} & \cdots & b_{kk} & \cdots & b_{nk} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{kn} & \cdots & b_{nn} \end{vmatrix} \\
&= \begin{vmatrix} a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \end{vmatrix} \begin{vmatrix} b_{11} & \cdots & b_{k1} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{1k} & \cdots & b_{kk} & \cdots & b_{nk} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{kn} & \cdots & b_{nn} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{vmatrix} a_{n1} & \cdots & a_{k1} & \cdots & a_{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{nk} & \cdots & a_{kk} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{nn} & \cdots & a_{kn} & \cdots & a_{1n} \end{vmatrix} \begin{vmatrix} b_{11} & \cdots & b_{k1} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{1k} & \cdots & b_{kk} & \cdots & b_{nk} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{kn} & \cdots & b_{nn} \end{vmatrix} \\
&= \left| \mathbf{a}_n \ \cdots \ \mathbf{a}_k \ \cdots \ \mathbf{a}_1 \right| \left| \mathbf{b}_1 \ \cdots \ \mathbf{b}_k \ \cdots \ \mathbf{b}_n \right|.
\end{aligned}
$$

The sign factor is eliminated by reversing rows in the first determinant.

More generally, the inner product of two l-blades $\mathbf{A}_l$ and $\mathbf{B}_l$ of vectors in a d-dimensional base space or d-space spanned by orthonormal units $\mathbf{e}_1$, ..., $\mathbf{e}_d$ is the determinant

$$
\begin{aligned}
\mathbf{A}_l &= \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_l \\
\mathbf{B}_l &= \mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k \wedge \cdots \wedge \mathbf{b}_l \\
\mathbf{a}_k &= a_{k1}\mathbf{e}_1 + \cdots + a_{kd}\mathbf{e}_d \\
\mathbf{A}_l \cdot \mathbf{B}_l &= (-1)^{l(l-1)/2} \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_k & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_l \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_k \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_k \cdot \mathbf{b}_k & \cdots & \mathbf{a}_k \cdot \mathbf{b}_l \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_l \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_l \cdot \mathbf{b}_k & \cdots & \mathbf{a}_l \cdot \mathbf{b}_l \end{vmatrix}.
\end{aligned}
$$

If $l = d$ then this determinant can be factored into the product of two determinants, but otherwise it cannot since the factored matrices would be $l \times d$ and $d \times l$ which are not square and have no determinants.

The free software SymPy, for symbolic mathematics using python, includes a Geometric Algebra Module and interactive calculator console `isympy`. The `isympy` console can be used demonstrate these results. A simple example of console interaction follows to compute the product of parallel 3-blades

$$
\begin{aligned}
\mathbf{a}_1 &= 3\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3 \\
\mathbf{a}_2 &= 2\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3 \\
\mathbf{a}_3 &= 9\mathbf{e}_1 + 6\mathbf{e}_2 + 9\mathbf{e}_3 \\
\mathbf{b}_1 &= 9\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \\
\mathbf{b}_2 &= 6\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_3 \\
\mathbf{b}_3 &= 2\mathbf{e}_1 + 4\mathbf{e}_2 + 7\mathbf{e}_3
\end{aligned}
$$

$$(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3).$$

```
$isympy
>>> from sympy.galgebra.ga import *
>>> (e1,e2,e3) = MV.setup('e*1|2|3',metric='[1,1,1]')
>>> (a1,a2,a3) = symbols('a1 a2 a3')
>>> (b1,b2,b3) = symbols('b1 b2 b3')
>>> a1 = 3*e1 + 4*e2 + 5*e3
>>> a2 = 2*e1 + 4*e2 + 5*e3
>>> a3 = 9*e1 + 6*e2 + 9*e3
```

```
>>> b1 = 9*e1 + 2*e2 + 3*e3
>>> b2 = 6*e1 + 5*e2 + 8*e3
>>> b3 = 2*e1 + 4*e2 + 7*e3
>>> (a1^a2^a3)|(b1^b2^b3)
-102
>>> (A,B) = symbols('A B')
>>> A = Matrix([[9,6,9],[2,4,5],[3,4,5]])
>>> B = Matrix([[9,2,3],[6,5,8],[2,4,7]])
>>> A.det() * B.det()
-102
```

2.16 The geometric product ab of Euclidean vectors

For Euclidean vectors $\mathbf{a}$ and $\mathbf{b}$, we have the products

$$\begin{aligned}
\mathbf{ab} &= (a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3)(b_x\mathbf{e}_1 + b_y\mathbf{e}_2 + b_z\mathbf{e}_3) \\
&= a_xb_x\mathbf{e}_1^2 + a_yb_y\mathbf{e}_2^2 + a_zb_z\mathbf{e}_3^2 + \\
&\quad a_xb_y\mathbf{e}_1\mathbf{e}_2 + a_xb_z\mathbf{e}_1\mathbf{e}_3 + \\
&\quad a_yb_x\mathbf{e}_2\mathbf{e}_1 + a_yb_z\mathbf{e}_2\mathbf{e}_3 + \\
&\quad a_zb_x\mathbf{e}_3\mathbf{e}_1 + a_zb_y\mathbf{e}_3\mathbf{e}_2 \\
&= a_xb_x + a_yb_y + a_zb_z + \\
&\quad (a_zb_y - a_yb_z)\mathbf{e}_3\mathbf{e}_2 + \\
&\quad (a_xb_z - a_zb_x)\mathbf{e}_1\mathbf{e}_3 + \\
&\quad (a_yb_x - a_xb_y)\mathbf{e}_2\mathbf{e}_1 + \\
&= \mathbf{a}\cdot\mathbf{b} + \begin{vmatrix} \mathbf{E_i} & \mathbf{E_j} & \mathbf{E_k} \\ b_x & b_y & b_z \\ a_x & a_y & a_z \end{vmatrix} \\
&= \mathbf{a}\cdot\mathbf{b} + (b_x\mathbf{E_i} + b_y\mathbf{E_j} + b_z\mathbf{E_k}) \times (a_x\mathbf{E_i} + a_y\mathbf{E_j} + a_z\mathbf{E_k}) \\
&= \mathbf{a}\cdot\mathbf{b} - (a_x\mathbf{E_i} + a_y\mathbf{E_j} + a_z\mathbf{E_k}) \times (b_x\mathbf{E_i} + b_y\mathbf{E_j} + b_z\mathbf{E_k}) \\
&= \mathbf{a}\cdot\mathbf{b} - (\mathbf{a}/\mathbf{I}) \times (\mathbf{b}/\mathbf{I}) = |\mathbf{a}|\,|\mathbf{b}|[\cos(\theta) - \sin(\theta)\,\mathbf{N}] \\
&= |\mathbf{a}|\,|\mathbf{b}|\,e^{-\theta\mathbf{N}} = \mathrm{K}(\mathbf{ba}) = \mathbf{a}\frac{\mathbf{b}^2}{\mathbf{b}} = |\mathbf{b}|^2\mathbf{a}/\mathbf{b} \\
\mathbf{ba} &= \mathbf{b}\frac{\mathbf{a}^2}{\mathbf{a}} = |\mathbf{a}|^2\mathbf{b}/\mathbf{a} = |\mathbf{a}|\,|\mathbf{b}|\,e^{\theta\mathbf{N}} = \mathrm{K}(\mathbf{ab})
\end{aligned}$$

which are called the *geometric product* of vectors. The notation and concept of the quaternion conjugate K operation can be usefully adapted to the Euclidean biradials, such that where there was a pure quaternion unit vector part $\mathbf{n}$ to take negative (as axis or angle) there is now a similar unit bivector part $\mathbf{N}$. The unit bivector $\mathbf{N}$ is now being taken as quadrantal versor from $\mathbf{a}$ toward $\mathbf{b}$, and angle θ is positive from $\mathbf{a}$ toward $\mathbf{b}$.

The geometric product can also be shown to hold for any product of a vector and multivector, such as $\mathbf{a}B = \mathbf{a}\cdot B + \mathbf{a}\wedge B$ and $A\mathbf{b} = A\cdot\mathbf{b} + A\wedge\mathbf{b}$, although the evaluation of these inner and outer products of a vector and multivector require usage of the formulas for expansion of inner and outer products. The geometric product usually has no special symbol, and the product is invoked, as shown, by spaces and juxtaposition of the values to be multiplied.

2.17 The quaternion product PQ of even-grade multivectors

The geometric product is found to be the same as the quaternion product when multiplying values that isomorphically represent quaternions in $\mathcal{G}_3$. The quaternion representation and multiplications are

$$
\begin{aligned}
\mathbf{q} &= q_x\mathbf{e}_1 + q_y\mathbf{e}_2 + q_z\mathbf{e}_3\\
\mathbf{p} &= p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3\\
\mathbf{Q} &= \mathbf{q}/\mathbf{I} = (q_x\mathbf{E}_\mathbf{i} + q_y\mathbf{E}_\mathbf{j} + q_z\mathbf{E}_\mathbf{k})\\
\mathbf{P} &= \mathbf{p}/\mathbf{I} = (p_x\mathbf{E}_\mathbf{i} + p_y\mathbf{E}_\mathbf{j} + p_z\mathbf{E}_\mathbf{k})\\
Q &= q_w + \mathbf{Q}\\
P &= p_w + \mathbf{P}\\
\mathbf{p}\cdot\mathbf{q} &= (\mathbf{PI})\cdot(\mathbf{QI})\\
&= \frac{1}{2}(\mathbf{PIQI} + \mathbf{QIPI}) = \frac{1}{2}(\mathbf{I}^2\mathbf{PQ} + \mathbf{I}^2\mathbf{QP})\\
&= -\frac{1}{2}(\mathbf{PQ} + \mathbf{QP}) = -\mathbf{P}\cdot\mathbf{Q}\\
\mathbf{p}\times\mathbf{q} &= (\mathbf{p}\wedge\mathbf{q})/\mathbf{I} = -(\mathbf{p}\wedge\mathbf{q})\mathbf{I}\\
&= -\frac{1}{2}(\mathbf{pq} - \mathbf{qp})\mathbf{I} = -\frac{1}{2}(\mathbf{PIQI} - \mathbf{QIPI})\mathbf{I}\\
&= \frac{1}{2}(\mathbf{PQ} - \mathbf{QP})\mathbf{I} = (\mathbf{P}\times\mathbf{Q})\mathbf{I}\\
\mathbf{P}\cdot\mathbf{Q} &= \frac{1}{2}(\mathbf{PQ} + \mathbf{QP})\\
\mathbf{P}\times\mathbf{Q} &= \frac{1}{2}(\mathbf{PQ} - \mathbf{QP})\\
\mathbf{P}\wedge\mathbf{Q} &= 0\\
\mathbf{PQ} &= \frac{1}{2}(\mathbf{PQ} + \mathbf{QP}) + \frac{1}{2}(\mathbf{PQ} - \mathbf{QP}) = \mathbf{P}\cdot\mathbf{Q} + \mathbf{P}\times\mathbf{Q}\\
PQ &= p_wq_w + p_w\mathbf{Q} + q_w\mathbf{P} + \mathbf{PQ}\\
&= p_wq_w + p_w\mathbf{Q} + q_w\mathbf{P} + \mathbf{P}\cdot\mathbf{Q} + \mathbf{P}\times\mathbf{Q}\\
&= (p_wq_w + \mathbf{P}\cdot\mathbf{Q}) + (p_w\mathbf{Q} + q_w\mathbf{P} + \mathbf{P}\times\mathbf{Q})\\
&= \langle PQ\rangle_0 + \langle PQ\rangle_2
\end{aligned}
$$

where the pseudoscalar $\mathbf{I}$ multiplies commutatively (like a scalar) with all other values, and $\mathbf{I}^2 = -1$, causing some changes in signs.

2.18 Dual and undual to convert to and from quaternions

Taking the *dual* (dividing by $\mathbf{I}$ on the right) of the Euclidean vector $\mathbf{p}$ as $\mathbf{p}/\mathbf{I}=\mathbf{P}$ produces its pure quaternion vector representation $\mathbf{P}$. Taking the *undual* (multiplying by $\mathbf{I}$ on the right) of the bivector $\mathbf{P}$ as $\mathbf{PI}=\mathbf{p}$ reproduces the Euclidean vector represented by the pure quaternion vector $\mathbf{P}$.

2.19 The inner product of quaternionic bivectors has negative sign

In the equation that represents quaternion multiplication PQ, the sign on $\mathbf{P}\cdot\mathbf{Q}$ is positive, making the equation appear different here than in the actual quaternion formula for quaternion multiplication pq that PQ represents. Therefore, $\mathbf{P}\cdot\mathbf{Q}=-\mathbf{p}\cdot\mathbf{q}$, and $-\mathbf{P}\cdot\mathbf{Q}$ is the correct dot product of the undual vectors represented. This difference in sign is an issue to be aware of, but it is technically not a flaw or problem. The other values, and the values taken by grades $\langle PQ\rangle_0$ and $\langle PQ\rangle_2$, are completely consistent with quaternion multiplication.

2.20 The commutator product

The product symbol ($\times$) is being used in two different contexts or meanings. In $\mathbf{p}\times\mathbf{q}$ it is the Euclidean *vector cross product* and is the dual of the vector outer product. In $\mathbf{P}\times\mathbf{Q}$ it is called the *commutator product*, and is representing a quaternion vector cross product which is then taken undual to be the Euclidean vector cross product result. The commutator product is defined generally, in terms of any multivectors, by the identity

$$A\times B \;=\; \frac{1}{2}(AB-BA)=-B\times A$$

where we see this form for $\mathbf{P}\times\mathbf{Q}$. For vectors, we happen to have $\mathbf{a}\wedge\mathbf{b}=\frac{1}{2}(\mathbf{ab}-\mathbf{ba})\neq\mathbf{a}\times\mathbf{b}$. Writing $\mathbf{a}\times\mathbf{b}=(\mathbf{a}\wedge\mathbf{b})/\mathbf{I}$ is not the commutator product; it is the vector cross product or dual of the vector outer product. The meaning of the symbol ($\times$) is overloaded in the context of vectors. While outer products are associative $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}=\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$, commutator products are non-associative and obey the *Jacobi identity*

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) \;=\; 0.$$

In general, the commutator product has the *derivative property*

$$\begin{aligned}
A\times(BC) &= (A\times B)C+B(A\times C)\\
A\times(B\times C) &= (A\times B)\times C+B\times(A\times C)\\
A\times(B\wedge C) &= (A\times B)\wedge C+B\wedge(A\times C)\\
A\times(B\cdot C) &= (A\times B)\cdot C+B\cdot(A\times C).
\end{aligned}$$

This non-associative algebra of commutator products is mostly avoided, but there is a formula

$$\begin{aligned} BA &= B\cdot A + B\times A + B\wedge A \\ B\cdot A + B\wedge A &= \frac{1}{2}(BA + AB) \end{aligned}$$

with the conditions $B = \langle B\rangle_2$ and $\langle A\rangle_1 = 0$, that relates multiplying bivectors B and multivectors that contain no 1-vector components A. However, this conditional formula is of little interest for our needs.

We are only concerned with the representation of quaternions as bivectors, and the formulas of interest are the ones representing quotients and products of quaternion unit vectors, namely the biradials or versors for rotation from vector $\mathbf{a}$ to $\mathbf{b}$

$$\begin{aligned} |\mathbf{a}| = |\mathbf{b}| &= \mathbf{a}^2 = \mathbf{b}^2 = 1 \\ \mathbf{B} &= \mathbf{b}/\mathbf{I} = -\mathbf{b}\mathbf{I} \\ \mathbf{A} &= \mathbf{a}/\mathbf{I} = -\mathbf{a}\mathbf{I} \\ \mathbf{A}\times\mathbf{B} &= \frac{1}{2}(\mathbf{AB} - \mathbf{BA}) = \sin(\theta)\mathbf{N} \\ \theta &= \cos^{-1}(\langle\mathbf{b}/\mathbf{a}\rangle_0) = \cos^{-1}(\langle\mathbf{ba}\rangle_0) \\ \mathbf{B}/\mathbf{A} = -\mathbf{BA} = \mathbf{ba} &= -\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B} \\ &= \cos(\theta) + \sin(\theta)\mathbf{N} = e^{\theta\mathbf{N}} \\ \mathbf{A}/\mathbf{B} = -\mathbf{AB} = \mathbf{ab} &= -\mathbf{A}\cdot\mathbf{B} - \mathbf{A}\times\mathbf{B} \\ &= \cos(-\theta) + \sin(-\theta)\mathbf{N} = e^{-\theta\mathbf{N}} \\ \mathbf{AB} = -\mathbf{A}/\mathbf{B} = -\mathbf{ab} &= \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B} \\ &= \cos(\pi-\theta) + \sin(\pi-\theta)\mathbf{N} = e^{(\pi-\theta)\mathbf{N}} \\ \mathbf{BA} = -\mathbf{B}/\mathbf{A} = -\mathbf{ba} &= \mathbf{A}\cdot\mathbf{B} - \mathbf{A}\times\mathbf{B} \\ &= \cos(\theta-\pi) + \sin(\theta-\pi)\mathbf{N} = e^{(\theta-\pi)\mathbf{N}} \end{aligned}$$

where duals $\mathbf{a}/\mathbf{I}$ and $\mathbf{b}/\mathbf{I}$ take the Euclidean vectors into quaternion representations $\mathbf{A}$ and $\mathbf{B}$. An undual $\mathbf{XI} = \mathbf{x}$ takes a quaternion vector $\mathbf{X}$ representation back to a Euclidean vector $\mathbf{x}$.

Note well, that the inner product of two quaternion vectors represented as bivectors

$$\begin{aligned} (\mathbf{AI})\cdot(\mathbf{BI}) &= \mathbf{a}\cdot\mathbf{b} \\ &= (\mathbf{AI})\cdot(\mathbf{IB}) \\ &= -(\mathbf{A}\cdot\mathbf{B}) \\ \mathbf{A}\cdot\mathbf{B} &= -(\mathbf{a}\cdot\mathbf{b}) \end{aligned}$$

is the *negative* of a dot product, or inner product of two Euclidean vectors. Two *quaternion vectors* multiply as

$$\mathbf{ab} = -\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b}$$

and two *quaternion vectors represented as bivectors* multiply as

$$\mathbf{AB} = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B}.$$

This difference in sign, $(+)$ on the inner product and $(-)$ on dot product, must be paid careful attention.

2.21 Evaluating expressions in Clifford geometric algebra

The products of Clifford geometric algebra are somewhat complicated, and only careful usage of formulas and identities will give a correct evaluation. The geometric product is often written as the sum of the inner and outer products, but that formula only applies generally when at least one of the values being multiplied is a vector. Identities are used to change a product that doesn't involve a vector into one that does, and these identities are used recursively until the product is expanded enough to complete the product evaluation using other identities for products of vectors and multivectors.

For example, the inner product $\mathbf{e}_1\cdot(\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3) = \mathbf{e}_1\cdot(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)$ evaluates *almost* as if $\mathbf{e}_2\mathbf{e}_3$ are merely scalars once they are arranged on the right side by using the anti-commutative multiplication identity of perpendicular vectors; the result is $-\mathbf{e}_2\mathbf{e}_3$. The evaluation of the geometric product $(\mathbf{e}_3\wedge\mathbf{e}_2)(\mathbf{e}_1\wedge\mathbf{e}_3)$ might at first be thought to be $(\mathbf{e}_3\wedge\mathbf{e}_2)\cdot(\mathbf{e}_1\wedge\mathbf{e}_3)$ or $(\mathbf{e}_3\wedge\mathbf{e}_2)\wedge(\mathbf{e}_1\wedge\mathbf{e}_3)$, or even $(\mathbf{e}_3\wedge\mathbf{e}_2)\cdot(\mathbf{e}_1\wedge\mathbf{e}_3)+(\mathbf{e}_3\wedge\mathbf{e}_2)\wedge(\mathbf{e}_1\wedge\mathbf{e}_3)$, but it is none of them since neither operand of this geometric product is a vector. The correct evaluation using a recursive expansion identity (see identities below) is

$$\begin{aligned}
(\mathbf{e}_3\wedge\mathbf{e}_2)(\mathbf{e}_1\wedge\mathbf{e}_3) &= (\mathbf{e}_3)(\mathbf{e}_2(\mathbf{e}_1\wedge\mathbf{e}_3))\\
&= (\mathbf{e}_3)(\mathbf{e}_2\cdot(\mathbf{e}_1\wedge\mathbf{e}_3)+\mathbf{e}_2\wedge(\mathbf{e}_1\wedge\mathbf{e}_3))\\
&= (\mathbf{e}_3)(0+\mathbf{e}_2\wedge\mathbf{e}_1\wedge\mathbf{e}_3)\\
&= \mathbf{e}_3\cdot(\mathbf{e}_2\wedge\mathbf{e}_1\wedge\mathbf{e}_3)+\mathbf{e}_3\wedge(\mathbf{e}_2\wedge\mathbf{e}_1\wedge\mathbf{e}_3)\\
&= \mathbf{e}_2\wedge\mathbf{e}_1+0=\mathbf{e}_2\wedge\mathbf{e}_1.
\end{aligned}$$

This geometric product evaluation was made more complicated by it not being a simpler product of a vector with a multivector. The evaluation of $(\mathbf{e}_2\wedge\mathbf{e}_1)(\mathbf{e}_2\wedge\mathbf{e}_1)$ is

$$\begin{aligned}
(\mathbf{e}_2\wedge\mathbf{e}_1)(\mathbf{e}_2\wedge\mathbf{e}_1) &= (\mathbf{e}_2)(\mathbf{e}_1(\mathbf{e}_2\wedge\mathbf{e}_1))\\
&= (\mathbf{e}_2)(\mathbf{e}_1\cdot(\mathbf{e}_2\wedge\mathbf{e}_1)+\mathbf{e}_1\wedge(\mathbf{e}_2\wedge\mathbf{e}_1))\\
&= (\mathbf{e}_2)(-\mathbf{e}_2+0)\\
&= (\mathbf{e}_2)\cdot(-\mathbf{e}_2)+(\mathbf{e}_2)\wedge(-\mathbf{e}_2)\\
&= -1+0
\end{aligned}$$

which is the correct result that matches the rule formula above. The rules or formulas applied here are listed in the identities below. Of course, these products can also be evaluated more simply by using the identities already above and applying the rule for *perpendicular* vectors $\mathbf{ab} = -\mathbf{ba}$ as needed to move values to the left or right in order to square vectors into scalars. The evaluation then looks like this

$$\begin{aligned}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 &= \mathbf{I}^2 = (\mathbf{e}_3\wedge\mathbf{e}_2)(\mathbf{e}_1\wedge\mathbf{e}_3)(\mathbf{e}_2\wedge\mathbf{e}_1) = -1\\ (\mathbf{e}_3\wedge\mathbf{e}_2)(\mathbf{e}_1\wedge\mathbf{e}_3) &= \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_1\mathbf{e}_3\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_1\\ (\mathbf{e}_2\wedge\mathbf{e}_1)(\mathbf{e}_2\wedge\mathbf{e}_1) &= \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = -1.\end{aligned}$$

2.22 The general form of a multivector A

While looking below at some of the fundamental identities of this Clifford algebra $\mathcal{G}_3$, it may be helpful to follow along with the following additional explanations or interpretations of the values in the identities. Additional reading in the literature on Clifford geometric algebra is required to more fully understand all the concepts.

The value

$$A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \langle A\rangle_3$$

is the general form of a value, called a *multivector*, in $\mathcal{G}_3$. The multivector A has parts $\langle A\rangle_k$ called the *k-vector part* of A, similar to taking the scalar Sq and vector Vq parts of a quaternion except that there are *four* different linearly independent *parts*. The number k itself is called the *grade*, and $\langle A\rangle_k$ is the grade k part operator or selector on A. The values in a part $\langle A\rangle_k$ are also said to be *of grade k* or *have grade k*.

The part

$$\langle A\rangle_0 = \langle A\rangle = c$$

is a *real number* scalar, sometimes called a *0-vector* or *0-blade*, and is often associated with the cosine of an angle as in quaternions.

The part

$$\langle A\rangle_1 = \mathbf{a} = a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3$$

with real scalars a_x, a_y, a_z, is a Euclidean vector, called a *1-vector* or just a *vector*, and its three linearly independent orthogonal components $a_x\mathbf{e}_1, a_y\mathbf{e}_2, a_z\mathbf{e}_3$ are called *1-blades*.

The part

$$\begin{aligned}\langle A\rangle_2 &= \mathbf{A} = \alpha_x\mathbf{E_i} + \alpha_y\mathbf{E_j} + \alpha_z\mathbf{E_k}\\ &= \alpha_x(\mathbf{e}_3\wedge\mathbf{e}_2) + \alpha_y(\mathbf{e}_1\wedge\mathbf{e}_3) + \alpha_z(\mathbf{e}_2\wedge\mathbf{e}_1)\end{aligned}$$

with real scalars $\alpha_x, \alpha_y, \alpha_z$, is called a *bivector* (not the bivector in biquaternions) or *2-vector*, and its three linearly independent orthogonal components $\alpha_x\mathbf{E_i}$, $\alpha_y\mathbf{E_j}$, $\alpha_z\mathbf{E_k}$ are called *2-blades*, and this part also represents a pure quaternion vector.

The part

$$\langle A\rangle_3 = \gamma\mathbf{I} = \gamma(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3)$$

with real scalar γ, is called a *3-vector* or *3-blade*, or a *pseudoscalar*.

There are a total of $2^3 = 8$ linearly independent *components* or blades in all the 4 parts of A.

In a Clifford geometric algebra $\mathcal{G}(\mathcal{V}^n)$ with n orthonormal base vector units, there are 2^n unique blades, and therefore the algebra has values in a 2^n-dimensional real number space. One way to see how there are 2^n unique blades is by considering the expression

$$\mathbf{e}_1^{p_1} \cdots \mathbf{e}_k^{p_k} \cdots \mathbf{e}_n^{p_n}$$

where the $p_k \in \{0,\ 1\}$ act as presence bits. The corresponding binary number $p_1 \cdots p_k \cdots p_n$ has 2^n possible values. When all presence bits are zero, the resulting number 1 is the grade-0 base unit. The grade of a blade is the sum of the presence bits. If the ordering of the units is changed in this expression, then by anti-commutativity there would only be differences in signs but no additional unique blades. The space of the 2^n blades is called the *Grassmann space* $\Lambda(\mathcal{V}^n)$ generated by the vector space $\mathcal{V}^n$.

2.23 The algebra of the even-grade parts is quaternions

The algebra of multivectors having only the even grade parts $A_+ = \langle A \rangle_0 + \langle A \rangle_2$ is a closed algebra of values and isomorphic to quaternions in general, so it is a subalgebra of $\mathcal{G}_3$. The algebra of multivectors having only the odd grade parts $A_- = \langle A \rangle_1 + \langle A \rangle_3$ is not closed since their products or quotients are in the even grade parts, which can be thought of again as quaternions as just mentioned. The product of an even part and odd part is an odd part. This relationship between odd and even parts can be notated as $A_+ A_+ = A_+$, and $A_- A_- = A_+$, and $A_+ A_- = A_- A_+ = A_-$, and $A_+ A_- A_+ = A_-$. If the odd parts $A_- = \langle A \rangle_1 + \langle A \rangle_3$ are taken to represent only *pure quaternions* (vectors), letting $\langle A \rangle_3 = 0$, and the even parts $A_+ = \langle A \rangle_0 + \langle A \rangle_2$ are taken to represent only *impure quaternions* (quaternions with non-zero scalar part), then we will find that $A_-' = A_+ A_- A_+^{-1}$ can (abstractly) represent the quaternion versor operator $R_{\mathbf{n}}^{\theta} \mathbf{v} = e^{\frac{1}{2}\theta \mathbf{n}} \mathbf{v} e^{-\frac{1}{2}\theta \mathbf{n}}$, which will take the Clifford algebra form $R_{\mathbf{N}}^{\theta} \mathbf{v} = e^{\frac{1}{2}\theta \mathbf{N}} \mathbf{v} e^{-\frac{1}{2}\theta \mathbf{N}}$.

In this form, $\mathbf{N}$ is a *unit bivector* that directly represents a *plane* of rotation, or represents a quaternion unit vector *axis* of rotation $\mathbf{N} = \mathbf{n} / \mathbf{I}$, and it is the *quadrantal versor* of the plane represented by $\mathbf{N}$. The value of $\mathbf{N}$ is still, in terms of quaternion biradials, the product or quotient of two *perpendicular* unit vectors. However, $\mathbf{N}$ is now given as the product or quotient of two *perpendicular* Euclidean unit 1-vectors $\mathbf{N} = \mathbf{b} / \mathbf{a}$, *or* as the product or quotient of two perpendicular quaternion unit vectors represented by unit 2-vectors

$$\begin{aligned} \mathbf{N} &= (\mathbf{b}/\mathbf{I})/(\mathbf{a}/\mathbf{I}) = \mathbf{B}/\mathbf{A} \\ &= \mathbf{b}(-\mathbf{I})(\mathbf{a}\mathbf{I}^{-1})^{-1} = \mathbf{b}(-\mathbf{I})\mathbf{I}\mathbf{a}^{-1} \\ &= \mathbf{b}/\mathbf{a} = \mathbf{b}\mathbf{a} \end{aligned}$$

or as the dual $\mathbf{N}=\mathbf{n}/\mathbf{I}$ of a unit vector rotational axis $\mathbf{n}$. More generally, by allowing $\mathbf{b}$ and $\mathbf{a}$ to again have any angle θ separating them, and where we had the biradial b in quaternions, we now have the *unit multivector of even grade*

$$\begin{aligned} B &= \mathbf{b}/\mathbf{a}=\mathbf{ba} \\ &= \mathbf{b}\cdot\mathbf{a}+\mathbf{b}\wedge\mathbf{a} \\ &= \cos(\theta)+\sin(\theta)\frac{\mathbf{b}\wedge\mathbf{a}}{|\mathbf{b}\wedge\mathbf{a}|} \\ &= \cos(\theta)+\sin(\theta)\mathbf{N} \\ &= \langle B\rangle_0+\langle B\rangle_2 \\ &= e^{\theta\mathbf{N}} \end{aligned}$$

representing the same versor, and

$$\mathbf{N} = \mathbf{b}\wedge\frac{\mathbf{a}^{\perp\mathbf{b}}}{|\mathbf{a}^{\perp\mathbf{b}}|}=\frac{\mathbf{ba}^{\perp\mathbf{b}}}{|\mathbf{a}^{\perp\mathbf{b}}|}$$

is the *unit bivector* of the plane of the Euclidean unit vectors $\mathbf{a}$ to $\mathbf{b}$. $\mathbf{N}=e^{\frac{\pi}{2}\mathbf{N}}$ operates as quadrantal versor directly on Euclidean vectors in the plane of $\mathbf{N}$, such that $\mathbf{N}(\alpha\mathbf{a}+\beta\mathbf{b})$ is rotated by $\pi/2$ in the plane and in the direction $\mathbf{a}$ toward $\mathbf{b}$.

The *reversor*,

$$e^{-\theta\mathbf{N}} = \mathbf{a}/\mathbf{b}=\mathbf{ab}=\mathrm{K}(\mathbf{ba})$$

is the *reverse* of B denoted $B^{\dagger}=\mathbf{ab}$.

The quadrantal reversor is

$$-\mathbf{N} = \frac{\mathbf{ab}^{\perp\mathbf{a}}}{|\mathbf{b}^{\perp\mathbf{a}}|}.$$

2.24 Representing a quaternion versor using an odd-grade part

It is also possible for the odd part $A_{-}=\langle A\rangle_1+\langle A\rangle_3$ to represent a quaternion if it takes the form

$$\begin{aligned} Q\mathbf{I} &= |Q|[\cos(\theta/2)\mathbf{I}+\sin(\theta/2)\mathbf{n}] \\ Q &= |Q|[\cos(\theta/2)+\sin(\theta/2)(\mathbf{n}/\mathbf{I})] \end{aligned}$$

where the Euclidean unit vector $\mathbf{n}$ is the rotational axis, and θ is the rotational angle according to a right-hand or left-hand rule on right-handed or left-handed axes. A Euclidean vector $\mathbf{a}$ can then be rotated by θ round $\mathbf{n}$ as

$$(Q\mathbf{I})\mathbf{a}(Q\mathbf{I})^{-1} = Q\mathbf{IaI}^{-1}Q^{-1}=Q\mathbf{a}Q^{-1}.$$

2.25 The rotation (b/a)v(a/b) in Euclidean vectors

In general, the operation $\mathbf{n}/\mathbf{I}$, called taking the *dual* of $\mathbf{n}$, transforms a Euclidean *planar-polar* vector $\mathbf{n}$ into an orthogonal 2-vector in the *dual space* that represents the same vector $\mathbf{n}$ but as a quaternion *axial-pole* vector (also called a *pseudovector*) that can operate as a quadrantal versor if it has unit scale. The operation $(\mathbf{n}/\mathbf{I})\mathbf{I}$ can be called taking the *undual* of $\mathbf{n}/\mathbf{I}$. By taking the dual of Euclidean vectors to transform them into quaternions, ordinary quaternion rotation, in terms of biradials, can be performed and then the undual can be used to transform the rotated result back to a Euclidean vector. Assuming unit vectors $\mathbf{b}, \mathbf{a}$, then rotation of $\mathbf{v}$ by twice the angle from $\mathbf{a}$ to $\mathbf{b}$ is

$$\begin{aligned} Q(\mathbf{v}/\mathbf{I})Q^{-1}\mathbf{I} &= ([\mathbf{b}/\mathbf{I}]/[\mathbf{a}/\mathbf{I}])(\mathbf{v}/\mathbf{I})([\mathbf{b}/\mathbf{I}]/[\mathbf{a}/\mathbf{I}])^{-1}\mathbf{I} \\ &= (\mathbf{b}[-\mathbf{I}]\mathbf{a}\mathbf{I})(\mathbf{v}[-\mathbf{I}])(\mathbf{b}[-\mathbf{I}]\mathbf{a}\mathbf{I})^{-1}\mathbf{I} \\ &= (\mathbf{b}\mathbf{a})(\mathbf{v}[-\mathbf{I}])(\mathbf{b}\mathbf{a})^{-1}\mathbf{I} \\ &= (\mathbf{b}/\mathbf{a})\mathbf{v}(\mathbf{a}/\mathbf{b}) \end{aligned}$$

which shows that the products or quotients of Euclidean unit vectors represent quaternion versors that can directly operate on a Euclidean vector to perform a rotation in the Euclidean vector space. Therefore, it is not required to take duals to transform to quaternion representations of rotations, nor then undual any rotated results.

The following are some of the most important identities for the Clifford geometric algebra of Euclidean 3-space $\mathcal{G}_3$.

2.26 Identities: Euclidean base unit vectors

Euclidean base vector units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in $\mathcal{G}_3$:

$$\begin{aligned} n &= 3 \\ \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 &= \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1 \\ \mathbf{e}_i^{-1} &= \frac{1}{\mathbf{e}_i} = \mathbf{e}_i \\ \mathbf{e}_i/\mathbf{e}_j &= \mathbf{e}_i\mathbf{e}_j^{-1} = \mathbf{e}_i\mathbf{e}_j \\ \mathbf{e}_i \cdot \mathbf{e}_j &= \begin{cases} \mathbf{e}_i\mathbf{e}_j = 1 = \delta_{ij} & \mid \ i = j \\ 0 = \delta_{ij} & \mid \ i \neq j \end{cases} \\ \mathbf{e}_i \wedge \mathbf{e}_j &= \begin{cases} 0 & \mid \ i = j \\ \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i & \mid \ i \neq j \end{cases} \\ \mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_i\mathbf{e}_j &= \begin{cases} \mathbf{e}_i \cdot \mathbf{e}_j = 1 & \mid \ i = j \\ \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i & \mid \ i \neq j \end{cases} \\ (\mathbf{e}_i\mathbf{e}_j)^{-1} &= \mathbf{e}_j\mathbf{e}_i \\ (\mathbf{e}_i\mathbf{e}_j)^2 &= \begin{cases} (\mathbf{e}_i \cdot \mathbf{e}_j)^2 = 1^2 = 1 & \mid \ i = j \\ (\mathbf{e}_i \wedge \mathbf{e}_j)^2 = -\mathbf{e}_j\mathbf{e}_i\mathbf{e}_i\mathbf{e}_j = -1 & \mid \ i \neq j \end{cases} \\ \mathbf{I} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 &= (\mathbf{e}_1 \wedge \mathbf{e}_2)\mathbf{e}_3 = \mathbf{e}_1(\mathbf{e}_2 \wedge \mathbf{e}_3) \end{aligned}$$

$$
\begin{aligned}
&= (\mathbf{e}_1 \wedge \mathbf{e}_2) \wedge \mathbf{e}_3 = \mathbf{e}_1 \wedge (\mathbf{e}_2 \wedge \mathbf{e}_3) \\
&= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\
\mathbf{I}^{-1} &= \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 \\
\mathbf{I}^2 &= (-1)^{n(n-1)/2}\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&= (-1)^{n(n-1)/2} = (-1)^{3(3-1)/2} = -1
\end{aligned}
$$

2.27 Identities: Euclidean 1-vectors

Euclidean 1-blades $a_i\mathbf{e}_i$ and 1-vector (vector) $\mathbf{a}$ in $\mathcal{G}_3$:

$$
\begin{aligned}
\mathbf{a}_1 &= a_1\mathbf{e}_1 = a_x\mathbf{e}_1 = \mathbf{a}_x \\
\mathbf{a}_2 &= a_2\mathbf{e}_2 = a_y\mathbf{e}_2 = \mathbf{a}_y \\
\mathbf{a}_3 &= a_3\mathbf{e}_3 = a_z\mathbf{e}_3 = \mathbf{a}_z \\
\mathbf{a} &= a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3 \\
\mathbf{a}^2 &= a_x^2 + a_y^2 + a_z^2 \\
&= \sum_{i,j=1}^{n} a_i a_j \delta_{ij} = \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2
\end{aligned}
$$

2.28 Identities: The geometric product ba of 1-vectors

The products of vectors in terms of the units. The geometric, inner, and outer products are evaluated directly in terms of the base units, and verifies that the geometric product of vectors is the sum of the inner and outer products of vectors. The outer product is found to be equal to the negative of the commutator product of their duals, which represents the negative cross product of their quaternion representations. The product **ba** is shown as a scaled versor that rotates θ from $\mathbf{a}$ to $\mathbf{b}$:

$$
\begin{aligned}
\mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \\
\mathbf{ba} &= b_1a_1 + b_1a_2\mathbf{e}_1\mathbf{e}_2 + b_1a_3\mathbf{e}_1\mathbf{e}_3 + \\
&\quad b_2a_1\mathbf{e}_2\mathbf{e}_1 + b_2a_2 + b_2a_3\mathbf{e}_2\mathbf{e}_3 + \\
&\quad b_3a_1\mathbf{e}_3\mathbf{e}_1 + b_3a_2\mathbf{e}_3\mathbf{e}_2 + b_3a_3 \\
&= (b_1a_1 + b_2a_2 + b_3a_3) + \\
&\quad (b_3a_2 - b_2a_3)\mathbf{e}_3\mathbf{e}_2 - (b_3a_1 - b_1a_3)\mathbf{e}_1\mathbf{e}_3 + (b_2a_1 - b_1a_2)\mathbf{e}_2\mathbf{e}_1 \\
\mathbf{b}\cdot\mathbf{a} &= (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)\cdot(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \\
&= b_1a_1\mathbf{e}_1\cdot\mathbf{e}_1 + b_2a_2\mathbf{e}_2\cdot\mathbf{e}_2 + b_3a_3\mathbf{e}_3\cdot\mathbf{e}_3 \\
&= b_1a_1 + b_2a_2 + b_3a_3 \\
&= \sum_{i,j=1}^{3} b_i a_j \delta_{ij} \\
\mathbf{b}\wedge\mathbf{a} &= (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)\wedge(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \\
&= b_1a_2\mathbf{e}_1\wedge\mathbf{e}_2 + b_1a_3\mathbf{e}_1\wedge\mathbf{e}_3 +
\end{aligned}
$$

$$
\begin{aligned}
& b_2a_1\mathbf{e}_2\wedge\mathbf{e}_1 + b_2a_3\mathbf{e}_2\wedge\mathbf{e}_3 + \\
& b_3a_1\mathbf{e}_3\wedge\mathbf{e}_1 + b_3a_2\mathbf{e}_3\wedge\mathbf{e}_2 \\
= \; & (b_3a_2 - b_2a_3)\mathbf{e}_3\mathbf{e}_2 - (b_3a_1 - b_1a_3)\mathbf{e}_1\mathbf{e}_3 + (b_2a_1 - b_1a_2)\mathbf{e}_2\mathbf{e}_1 \\
= \; & \begin{vmatrix} \mathbf{e}_3/\mathbf{e}_2 & \mathbf{e}_1/\mathbf{e}_3 & \mathbf{e}_2/\mathbf{e}_1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (\mathbf{a}/\mathbf{I}) \times (\mathbf{b}/\mathbf{I}) = \mathbf{A}\times\mathbf{B}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{ba} &= \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\wedge\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \mathbf{B}\times\mathbf{A} \\
&= |\mathbf{B}|\,|\mathbf{A}|\cos(\theta) + |\mathbf{B}|\,|\mathbf{A}|\sin(\theta)\frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{B}|\,|\mathbf{A}|\sin(\theta)} \\
&= |\mathbf{B}|\,|\mathbf{A}|[\cos(\theta) + \sin(\theta)\mathbf{N}] = |\mathbf{B}|\,|\mathbf{A}|\, e^{\theta\mathbf{N}} \\
&= |\mathbf{b}|\,|\mathbf{a}|\left[\cos(\theta) + \sin(\theta)\frac{\mathbf{b}\wedge\mathbf{a}}{|\mathbf{b}|\,|\mathbf{a}|\sin(\theta)}\right] \\
&= |\mathbf{b}|\,|\mathbf{a}|[\cos(\theta) + \sin(\theta)\mathbf{N}] \\
&= |\mathbf{b}|\,|\mathbf{a}|\, e^{\theta\mathbf{N}}
\end{aligned}
$$

2.29 Identities: The quaternion unit 2-vectors

Isomorphic (1 to 1) map between unit 2-blades and quaternion units, where the dual $\mathbf{b}/\mathbf{I}$ of a planar-polar Euclidean vector $\mathbf{b}$ still represents $\mathbf{b}$ but as an axial-pole quaternion vector $\mathbf{B}$, which is a quadrantal versor $e^{\frac{\pi}{2}\mathbf{B}}$ of the plane perpendicular to $\mathbf{b}$ if $|\mathbf{b}| = |\mathbf{B}| = 1$:

$$
\begin{aligned}
\mathbf{i} = \mathbf{k}/\mathbf{j} \;&\Leftrightarrow\; \mathbf{E_i} = \mathbf{e}_3/\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_3\wedge\mathbf{e}_2 = \mathbf{e}_1/\mathbf{I} \\
\mathbf{j} = \mathbf{i}/\mathbf{k} \;&\Leftrightarrow\; \mathbf{E_j} = \mathbf{e}_1/\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_1\wedge\mathbf{e}_3 = \mathbf{e}_2/\mathbf{I} \\
\mathbf{k} = \mathbf{j}/\mathbf{i} \;&\Leftrightarrow\; \mathbf{E_k} = \mathbf{e}_2/\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_2\wedge\mathbf{e}_1 = \mathbf{e}_3/\mathbf{I} \\
\mathbf{e}_1/\mathbf{I} \;&=\; \mathbf{e}_1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_1(-\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2) = \mathbf{e}_1(\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2) = \mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_3/\mathbf{e}_2 \\
\mathbf{e}_2/\mathbf{I} \;&=\; \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_2(-\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1) = -\mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_1/\mathbf{e}_3 \\
\mathbf{e}_3/\mathbf{I} \;&=\; \mathbf{e}_3\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_2/\mathbf{e}_1 \\
\mathbf{b}/\mathbf{I} \;&=\; (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)/\mathbf{I} = b_1\mathbf{E_i} + b_2\mathbf{E_j} + b_3\mathbf{E_k} = \mathbf{B} \Leftrightarrow \mathbf{b}.
\end{aligned}
$$

It is also important to show that the inner products of the quaternion mapping give the correct results

$$
\begin{aligned}
\mathbf{E_i}\cdot\mathbf{E_k} &= (\mathbf{e}_3\wedge\mathbf{e}_2)\cdot(\mathbf{e}_2\wedge\mathbf{e}_1) \\
&= \mathbf{e}_3\cdot([\mathbf{e}_2\cdot\mathbf{e}_2]\mathbf{e}_1 - \mathbf{e}_2[\mathbf{e}_2\cdot\mathbf{e}_1]) = \mathbf{e}_3\cdot\mathbf{e}_1 = 0 \\
&= \mathbf{E_i}\cdot\mathbf{E_j} = \mathbf{E_j}\cdot\mathbf{E_k} \\
\mathbf{E_i}\cdot\mathbf{E_i} &= (\mathbf{e}_3\wedge\mathbf{e}_2)\cdot(\mathbf{e}_3\cdot\mathbf{e}_2) \\
&= \mathbf{e}_3\cdot([\mathbf{e}_2\cdot\mathbf{e}_3]\mathbf{e}_2 - \mathbf{e}_3[\mathbf{e}_2\cdot\mathbf{e}_2]) = -\mathbf{e}_3\cdot\mathbf{e}_3 = -1 \\
&= \mathbf{E_i}\mathbf{E_i} = \mathbf{E_j}\mathbf{E_j} = \mathbf{E_k}\mathbf{E_k} \\
\mathbf{a}\cdot\mathbf{b} &= (\mathbf{AI})\cdot(\mathbf{BI}) = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \\
&= \frac{1}{2}(\mathbf{AIBI} + \mathbf{BIAI}) = -\frac{1}{2}(\mathbf{AB} + \mathbf{BA}) \\
&= -\mathbf{A}\cdot\mathbf{B} = -\mathbf{A}\,\bar{\times}\,\mathbf{B}
\end{aligned}
$$

except for a sign change, which is expected since quaternion vector units square negative or have negative signature. The vector analysis vector dot product and the geometric algebra inner product are not computed in the same way. The vector dot product is the product of a row and a column of scalar components, as in linear algebra matrix products, and ignores the vector units and their signatures. The geometric algebra inner product of blades is in the form of a Laplace expansion, as in computing determinants, and also is in terms of vector-valued factors where the signatures or squares of the vector units contribute to the result.

Cross products are

$$\begin{aligned}
\mathbf{A}\times\mathbf{B} &= \frac{1}{2}(\mathbf{AB}-\mathbf{BA})=\frac{1}{2}(\mathbf{ba}-\mathbf{ab})\\
&= \begin{vmatrix} \mathbf{e}_3\mathbf{e}_2 & \mathbf{e}_1\mathbf{e}_3 & \mathbf{e}_2\mathbf{e}_1\\ a_x & a_y & a_z\\ b_x & b_y & b_z \end{vmatrix} = \mathbf{b}\wedge\mathbf{a}\\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_x & a_y & a_z\\ b_x & b_y & b_z \end{vmatrix} /\mathbf{I} = (\mathbf{a}\times\mathbf{b})/\mathbf{I}\\
\mathbf{a}\times\mathbf{b} &= (\mathbf{a}\wedge\mathbf{b})/\mathbf{I}.
\end{aligned}$$

2.30 Quaternions mapped into a two dimensional space

Quaternions can be mapped into a two dimensional space with negative signatures as

$$\begin{aligned}
\mathbf{e}_1^2=\mathbf{e}_2^2 &= -1\\
\mathbf{i} &\Leftrightarrow \mathbf{e}_1\\
\mathbf{j} &\Leftrightarrow \mathbf{e}_2\\
\mathbf{k} &\Leftrightarrow \mathbf{e}_1\mathbf{e}_2
\end{aligned}$$

and then have the products

$$\begin{aligned}
\mathbf{ij} &\Leftrightarrow \mathbf{e}_1\mathbf{e}_2\Leftrightarrow\mathbf{k}\\
\mathbf{ik} &\Leftrightarrow \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2=-\mathbf{e}_2\Leftrightarrow-\mathbf{j}\\
\mathbf{jk} &\Leftrightarrow \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2=\mathbf{e}_1\Leftrightarrow\mathbf{i}
\end{aligned}$$

which appear to work as expected. However, there is a possible problem since, while the map gives three linearly independent units, it does not give three mutually orthogonal units according to the inner products

$$\begin{aligned}
\mathbf{i}\cdot\mathbf{j} &\Leftrightarrow \mathbf{e}_1\cdot\mathbf{e}_2=0\\
\mathbf{i}\cdot\mathbf{k} &\Leftrightarrow \mathbf{e}_1\cdot(\mathbf{e}_1\wedge\mathbf{e}_2)=-\mathbf{e}_2\Leftrightarrow-\mathbf{j}\neq 0\\
\mathbf{k}\cdot\mathbf{i} &\Leftrightarrow (\mathbf{e}_1\wedge\mathbf{e}_2)\cdot\mathbf{e}_1=\mathbf{e}_2\Leftrightarrow\mathbf{j}\neq 0\\
\mathbf{j}\cdot\mathbf{k} &\Leftrightarrow \mathbf{e}_2\cdot(\mathbf{e}_1\wedge\mathbf{e}_2)=\mathbf{e}_1\Leftrightarrow\mathbf{i}\neq 0\\
\mathbf{k}\cdot\mathbf{j} &\Leftrightarrow (\mathbf{e}_1\wedge\mathbf{e}_2)\cdot\mathbf{e}_2=-\mathbf{e}_1\Leftrightarrow-\mathbf{i}\neq 0.
\end{aligned}$$

For this mapping, the inner product fails to give the desired scalar results. The vector dot product, as a matrix product of a row and a column of scalar components, can be used. The outer products are also not useful in this mapping.

The mapping of vector dot products to the *negative* of the anti-commutator (symmetric) products

$$\begin{aligned}
\mathbf{i}\bullet\mathbf{j} &\Leftrightarrow -\mathbf{e}_1 \bar{\times} \mathbf{e}_2 = -\frac{1}{2}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) = 0 \\
\mathbf{i}\bullet\mathbf{k} &\Leftrightarrow -\mathbf{e}_1 \bar{\times} (\mathbf{e}_1\mathbf{e}_2) = -\frac{1}{2}(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1) = 0 \\
\mathbf{j}\bullet\mathbf{k} &\Leftrightarrow -\mathbf{e}_2 \bar{\times} (\mathbf{e}_1\mathbf{e}_2) = -\frac{1}{2}(\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2) = 0 \\
\mathbf{j}\bullet\mathbf{j} = \mathbf{k}\bullet\mathbf{k} = \mathbf{i}\bullet\mathbf{i} = -\mathbf{i}\mathbf{i} &\Leftrightarrow -\mathbf{e}_1 \bar{\times} \mathbf{e}_1 = -\frac{1}{2}(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_1) = 1
\end{aligned}$$

gives good results.

The mapping of vector cross products to commutator (anti-symmetric) products

$$\begin{aligned}
\mathbf{ij} = \mathbf{i}\times\mathbf{j} &\Leftrightarrow \mathbf{e}_1 \times \mathbf{e}_2 = \frac{1}{2}(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) = \mathbf{e}_1\mathbf{e}_2 \Leftrightarrow \mathbf{k} \\
\mathbf{ik} = \mathbf{i}\times\mathbf{k} &\Leftrightarrow \mathbf{e}_1 \times (\mathbf{e}_1\mathbf{e}_2) = \frac{1}{2}(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1) = -\mathbf{e}_2 \Leftrightarrow -\mathbf{j} \\
\mathbf{jk} = \mathbf{j}\times\mathbf{k} &\Leftrightarrow \mathbf{e}_2 \times (\mathbf{e}_1\mathbf{e}_2) = \frac{1}{2}(\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2) = \mathbf{e}_1 \Leftrightarrow \mathbf{i}
\end{aligned}$$

gives good results.

By combining these two good results, we can write the formula for quaternion products on this mapping as

$$\begin{aligned}
p &= p_w + \mathbf{p} = p_w + p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_1\mathbf{e}_2 \\
q &= q_w + \mathbf{q} = q_w + q_x\mathbf{e}_1 + q_y\mathbf{e}_2 + q_z\mathbf{e}_1\mathbf{e}_2 \\
\mathbf{pq} &= \mathbf{p} \bar{\times} \mathbf{q} + \mathbf{p} \times \mathbf{q} \\
pq &= (p_w + \mathbf{p})(q_w + \mathbf{q}) = p \bar{\times} q + p \times q \\
&= p_w q_w + p_w\mathbf{q} + q_w\mathbf{p} + \mathbf{pq} \\
&= (p_w q_w + \mathbf{p} \bar{\times} \mathbf{q}) + (p_w\mathbf{q} + q_w\mathbf{p} + \mathbf{p} \times \mathbf{q}) \\
&= \langle pq\rangle_0 + (\langle pq\rangle_1 + \langle pq\rangle_2)
\end{aligned}$$

which shows that the geometric product gives all the expected quaternion results, even though the inner and outer products were not useful. For this mapping, the useful products are the geometric product and its decomposition into the anti-commutator and commutator products.

2.31 Identities: General form of multivector A by grade-k parts

General form of a multivector A and its graded k-vector grade-k parts $\langle A\rangle_k$ in $\mathcal{G}_3$:

$$\begin{aligned}
A &= \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \langle A\rangle_3 \\
c &= \langle A\rangle_0 = \langle A\rangle \\
\mathbf{a} &= \langle A\rangle_1 = a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3 \\
\mathbf{A} &= \langle A\rangle_2 = \alpha_x\mathbf{E_i} + \alpha_y\mathbf{E_j} + \alpha_z\mathbf{E_k} \\
\gamma\mathbf{I} &= \langle A\rangle_3 = \gamma\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \gamma(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)
\end{aligned}$$

2.32 Identities: Squares, inverses, and commutative property of I

Squares, inverses, and commutative property of unit 3-blade pseudoscalar $\mathbf{I}$:

$$\begin{aligned}
\mathbf{e}_i^2 &= 1\\
\mathbf{a}^2 &= |\mathbf{a}|^2\\
\mathbf{A}^2 &= -|\mathbf{A}|^2\\
\mathbf{I}^2 &= -1\\
\mathbf{e}_i^{-1} &= \mathbf{e}_i\\
\mathbf{a}^{-1} &= \frac{\mathbf{a}}{\mathbf{a}^2} = \frac{\mathbf{a}}{|\mathbf{a}|^2}\\
\mathbf{A}^{-1} &= \frac{\mathbf{A}}{\mathbf{A}^2} = \frac{-\mathbf{A}}{|\mathbf{A}|^2}\\
\mathbf{I}^{-1} &= \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3 = -\mathbf{I}\\
\mathbf{I}\mathbf{e}_i &= \mathbf{e}_i\mathbf{I}\\
\mathbf{I}\mathbf{a} &= \mathbf{a}\mathbf{I}\\
\mathbf{I}\mathbf{A} &= \mathbf{A}\mathbf{I}\\
\mathbf{I}A &= A\mathbf{I}
\end{aligned}$$

2.33 Identities: Product ba and rotation operator R

Geometric, inner, outer, and cross products of vectors $\mathbf{a}$ and $\mathbf{b}$ with positive angle θ measured from $\mathbf{a}$ toward $\mathbf{b}$ in the $\mathbf{ab}$-plane (angle θ causes preference to give the products in order $\mathbf{ba}$):

Recap: The dual (dividing by the pseudoscalar $\mathbf{I}$ or multiplying $\mathbf{I}^{-1}$ or $-\mathbf{I}$ on right side) of a Euclidean planar-polar 1-vector represents it as a Quaternion axial-pole 2-vector often useful, at the same time and with the same orientation, for rotations of both Euclidean 1-vectors and other vectors represented in Quaternion 2-vector form. The undual (multiplying on right side by $\mathbf{I}$) of a 2-vector (representing a Quaternion vector) is a 1-vector that represents it as a Euclidean planar-polar vector having a positive square. The two representations of a vector by undual or dual, i.e. Euclidean 1-vector or Quaternion 2-vector, are equivalent vectors with respect to the Euclidean or Quaternion vector operations within their grades or complementary orthogonal dual spaces. The dual of a 1-vector and undual of a 2-vector can be done any time it is convenient to use the vector or bivector as Quaternion or Euclidean vector so long as appropriate care is taken not to mix incompatible formulas for one vector type with formulas for the other vector type, such as squaring since 1-vectors square positive and 2-vectors square negative. The rotation operator $R_{\mathbf{N}}^{\theta}$, in terms of a unit 2-vector $\mathbf{N}$, is compatible between the two such that the same operator rotates both 1-vectors and 2-vectors in the expected way within their grades. In the formulas or identities below, the dual and undual are used in the way just described, and this may allow easier understanding of how the formulas are derived and work. For example, the planar-polar Eucliden 1-vector $\mathbf{a}\times\mathbf{b}$ can partially (four possible angles θ, $-\theta$, $\pi-\theta$, $\theta-\pi$) represent rotation $\mathbf{b}=R^{\theta}_{\mathbf{N}}\mathbf{a}$ from $\mathbf{a}$ toward $\mathbf{b}$, and its dual

$$(\mathbf{a}\times\mathbf{b})/\mathbf{I} \;=\; \mathbf{b}\wedge\mathbf{a}=\mathbf{b}^{\perp\mathbf{a}}\mathbf{a}=|\mathbf{b}^{\perp\mathbf{a}}\mathbf{a}|\,\mathbf{N}$$

also partially represents that rotation as a Quaternion axial-pole 2-vector. The bivector $\mathbf{N}$ can always be created as $\mathbf{n}/\mathbf{I}$ where $\mathbf{n}$ is the Euclidean unit vector axis of rotation. Also notable is that the vector cross product $\mathbf{a}\times\mathbf{b}$ is computed by a determinant of exactly the same form as in quaternions or vector analysis by using the direct mapping (not duals) $\mathbf{i}\Leftrightarrow\mathbf{e}_1$, $\mathbf{j}\Leftrightarrow\mathbf{e}_2$, $\mathbf{k}\Leftrightarrow\mathbf{e}_3$ in the determinant.

$$\begin{aligned}
\mathbf{ba} &= \frac{1}{2}(\mathbf{ba}+\mathbf{ab})+\frac{1}{2}(\mathbf{ba}-\mathbf{ab})\\
&= \mathbf{b}\cdot\mathbf{a}+\mathbf{b}\wedge\mathbf{a}\\
&= |\mathbf{b}|\,|\mathbf{a}|\cos(\theta)+|\mathbf{b}|\,|\mathbf{a}|\sin(\theta)\mathbf{N}\\
&= |\mathbf{b}|\,|\mathbf{a}|[\cos(\theta)+\sin(\theta)\mathbf{N}]=|\mathbf{b}|\,|\mathbf{a}|\,e^{\theta\mathbf{N}}\\
&= \mathrm{K}(\mathbf{ab})\\
\mathbf{b}\cdot\mathbf{a} &= \frac{1}{2}(\mathbf{ba}+\mathbf{ab})=|\mathbf{b}|\,|\mathbf{a}|\cos(\theta)\\
&= |\mathbf{b}|\,|\mathbf{a}|\cos(-\theta)=\mathbf{a}\cdot\mathbf{b}\\
&= \sum_{i,j=1}^{3} b_i a_j\delta_{ij}=b_1a_1+b_2a_2+b_3a_3\\
\mathbf{b}\wedge\mathbf{a} &= \frac{1}{2}(\mathbf{ba}-\mathbf{ab})=|\mathbf{b}|\,|\mathbf{a}|\sin(\theta)\frac{\mathbf{b}\wedge\mathbf{a}}{|\mathbf{b}|\,|\mathbf{a}|\sin(\theta)}=|\mathbf{b}|\,|\mathbf{a}|\sin(\theta)\mathbf{N}\\
&= -|\mathbf{b}|\,|\mathbf{a}|\sin(-\theta)\mathbf{N}=-\mathbf{a}\wedge\mathbf{b}\\
&= \mathbf{b}\wedge(\mathbf{a}^{||\mathbf{b}}+\mathbf{a}^{\perp\mathbf{b}})=\mathbf{b}\wedge\mathbf{a}^{\perp\mathbf{b}}=\mathbf{ba}^{\perp\mathbf{b}}\\
&= (\mathbf{b}^{||\mathbf{a}}+\mathbf{b}^{\perp\mathbf{a}})\wedge\mathbf{a}=\mathbf{b}^{\perp\mathbf{a}}\wedge\mathbf{a}=\mathbf{b}^{\perp\mathbf{a}}\mathbf{a}\\
&= (\mathbf{a}\times\mathbf{b})/\mathbf{I}=(\mathbf{a}/\mathbf{I})\times(\mathbf{b}/\mathbf{I})\\
&= \begin{vmatrix}\mathbf{e}_1/\mathbf{I} & \mathbf{e}_2/\mathbf{I} & \mathbf{e}_3/\mathbf{I}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}\mathbf{e}_3\mathbf{e}_2 & \mathbf{e}_1\mathbf{e}_3 & \mathbf{e}_2\mathbf{e}_1\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\
\mathbf{a}\times\mathbf{b} &= (\mathbf{b}\wedge\mathbf{a})\mathbf{I}=(\mathbf{a}\wedge\mathbf{b})/\mathbf{I}\\
&= [(\mathbf{a}/\mathbf{I})\times(\mathbf{b}/\mathbf{I})]\mathbf{I}=\frac{1}{2}(\mathbf{ba}-\mathbf{ab})\mathbf{I}=\frac{1}{2}(\mathbf{ab}-\mathbf{ba})/\mathbf{I}\\
&= \begin{vmatrix}\mathbf{e}_1/\mathbf{I} & \mathbf{e}_2/\mathbf{I} & \mathbf{e}_3/\mathbf{I}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\mathbf{I}=\begin{vmatrix}\mathbf{e}_3/\mathbf{e}_2 & \mathbf{e}_1/\mathbf{e}_3 & \mathbf{e}_2/\mathbf{e}_1\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\mathbf{I}\\
&= \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\
R^{\theta}_{\mathbf{N}}\mathbf{v} &= (\mathbf{ba})^{\frac{1}{2}}(\mathbf{v})(\mathbf{ba})^{-\frac{1}{2}}=(\mathbf{b}/\mathbf{a})^{\frac{1}{2}}(\mathbf{v})(\mathbf{a}/\mathbf{b})^{\frac{1}{2}}=e^{\frac{1}{2}\theta\mathbf{N}}(\mathbf{v})e^{-\frac{1}{2}\theta\mathbf{N}}
\end{aligned}$$

2.34 Identities: Vector multivector products aB and Ab

Inner product with a scalar is zero, and outer product with scalar is the same as scalar multiplication.

$$\begin{aligned} c\cdot\mathbf{a}=\mathbf{a}\cdot c &= 0 \\ c\wedge\mathbf{a}=\mathbf{a}\wedge c &= c\mathbf{a} \end{aligned}$$

In the following, r_- denotes that r is an odd integer, and r_+ denotes that r is an even integer.

For vector $\mathbf{a}$ and s-blade $\mathbf{B}_s$ their products are

$$\begin{aligned} \mathbf{a}\mathbf{B}_s &= \langle\mathbf{a}\mathbf{B}_s\rangle_{|s-1|}+\langle\mathbf{a}\mathbf{B}_s\rangle_{s+1}=\mathbf{a}\cdot\mathbf{B}_s+\mathbf{a}\wedge\mathbf{B}_s \\ &= \mathbf{a}\,\bar{\times}\,\mathbf{B}_s+\mathbf{a}\times\mathbf{B}_s \\ &= \frac{1}{2}(\mathbf{a}\mathbf{B}_s+\mathbf{B}_s\mathbf{a})+\frac{1}{2}(\mathbf{a}\mathbf{B}_s-\mathbf{B}_s\mathbf{a}) \\ \mathbf{a}\wedge\mathbf{B}_s &= \frac{1}{2}(\mathbf{a}\mathbf{B}_s+(-1)^s\mathbf{B}_s\mathbf{a})=\begin{cases}\mathbf{a}\,\bar{\times}\,\mathbf{B}_s & | \; s_+ \\ \mathbf{a}\times\mathbf{B}_s & | \; s_-\end{cases} \\ \mathbf{a}\cdot\mathbf{B}_s &= \frac{1}{2}(\mathbf{a}\mathbf{B}_s-(-1)^s\mathbf{B}_s\mathbf{a})=\begin{cases}\mathbf{a}\,\bar{\times}\,\mathbf{B}_s & | \; s_- \\ \mathbf{a}\times\mathbf{B}_s & | \; s_+\end{cases} \\ (-1)^{s-1}\mathbf{a}\cdot\mathbf{B}_s &= \frac{1}{2}(\mathbf{B}_s\mathbf{a}-(-1)^s\mathbf{a}\mathbf{B}_s)=\mathbf{B}_s\cdot\mathbf{a}. \end{aligned}$$

For r-blade $\mathbf{A}_r$ and vector $\mathbf{b}$ their products are

$$\begin{aligned} \mathbf{A}_r\mathbf{b} &= \langle\mathbf{A}_r\mathbf{b}\rangle_{|r-1|}+\langle\mathbf{A}_r\mathbf{b}\rangle_{r+1}=\mathbf{A}_r\cdot\mathbf{b}+\mathbf{A}_r\wedge\mathbf{b} \\ &= \mathbf{A}_r\,\bar{\times}\,\mathbf{b}+\mathbf{A}_r\times\mathbf{b} \\ &= \frac{1}{2}(\mathbf{A}_r\mathbf{b}+\mathbf{b}\mathbf{A}_r)+\frac{1}{2}(\mathbf{A}_r\mathbf{b}-\mathbf{b}\mathbf{A}_r) \\ \mathbf{A}_r\wedge\mathbf{b} &= \frac{1}{2}(\mathbf{A}_r\mathbf{b}+(-1)^r\mathbf{b}\mathbf{A}_r)=\begin{cases}\mathbf{A}_r\,\bar{\times}\,\mathbf{b} & | \; r_+ \\ \mathbf{A}_r\times\mathbf{b} & | \; r_-\end{cases} \\ \mathbf{A}_r\cdot\mathbf{b} &= \frac{1}{2}(\mathbf{A}_r\mathbf{b}-(-1)^r\mathbf{b}\mathbf{A}_r)=\begin{cases}\mathbf{A}_r\,\bar{\times}\,\mathbf{b} & | \; r_- \\ \mathbf{A}_r\times\mathbf{b} & | \; r_+\end{cases} \\ (-1)^{r-1}\mathbf{A}_r\cdot\mathbf{b} &= \frac{1}{2}(\mathbf{b}\mathbf{A}_r-(-1)^r\mathbf{A}_r\mathbf{b})=\mathbf{b}\cdot\mathbf{A}_r. \end{aligned}$$

For the geometric product $\mathbf{a}B$ of vector $\mathbf{a}$ and multivector B, the vector $\mathbf{a}$ distributes into B and multiplies with each blade within B.

$$\begin{aligned} B_s &= \mathbf{b}_1\cdots\mathbf{b}_s=\sum_{i=0}^{s}\langle B\rangle_i \\ &= \begin{cases}\langle B\rangle_1+\langle B\rangle_3+\cdots+\langle B\rangle_{s-2}+\langle B\rangle_s & | \; s_- \\ \langle B\rangle+\langle B\rangle_2+\cdots+\langle B\rangle_{s-2}+\langle B\rangle_s & | \; s_+\end{cases} \\ \mathbf{a}B_s &= \mathbf{a}\cdot B_s+\mathbf{a}\wedge B_s \\ A_r\mathbf{b} &= A_r\cdot\mathbf{b}+A_r\wedge\mathbf{b}. \end{aligned}$$

More generally, a multivector is not always a product of vectors and may contain terms or parts of any grade. See also, the recursive formula for the inner product of blades.

2.35 Operator precedence convention for ambiguous products

The otherwise ambiguous products

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c} &= \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) \\ \mathbf{a}\wedge\mathbf{b}\cdot\mathbf{c} &= (\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} \end{aligned}$$

or take these meanings of products (or operator precedence) since vectors and outer products are the geometrical objects most often being operated on, measured, or compared by inner products.

The products

$$\begin{aligned} \mathbf{ab}\wedge\mathbf{c} &= \mathbf{a}(\mathbf{b}\wedge\mathbf{c}) \\ \mathbf{a}\wedge\mathbf{bc} &= (\mathbf{a}\wedge\mathbf{b})\mathbf{c} \end{aligned}$$

take these meanings for a similar reason.

Another operator precedence is

$$\begin{aligned} \mathbf{a}\cdot\mathbf{bc} &= (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\ \mathbf{ab}\cdot\mathbf{c} &= \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \end{aligned}$$

and since the inner product is a scalar factor that is seen frequently.

2.36 Identities: Projection and rejection operators for vectors

Projection, rejection, and orthogonalization operations:

$$\begin{aligned} \mathcal{P}_{\mathbf{b}}(\mathbf{a}) = \mathbf{a}^{||\mathbf{b}} = \mathbf{a} - \mathbf{a}^{\perp\mathbf{b}} &= (\mathbf{a}\cdot\mathbf{b})/\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} \\ \mathcal{P}_{\mathbf{b}}^{\perp}(\mathbf{a}) = \mathbf{a}^{\perp\mathbf{b}} = \mathbf{a} - \mathbf{a}^{||\mathbf{b}} &= (\mathbf{a}\wedge\mathbf{b})/\mathbf{b} = \frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{\mathbf{a}\wedge\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} \\ \mathbf{a} = (\mathbf{ab})/\mathbf{b} = (\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b})/\mathbf{b} &= \mathcal{P}_{\mathbf{b}}(\mathbf{a}) + \mathcal{P}_{\mathbf{b}}^{\perp}(\mathbf{a}) = \mathbf{a}^{||\mathbf{b}} + \mathbf{a}^{\perp\mathbf{b}} \end{aligned}$$

2.37 Associative, distributive algebraic properties of products

The *associative properties*:

$$\begin{aligned} (\cdots AB)C &= A(BC\cdots) = \cdots ABC\cdots \\ (\cdots\wedge A\wedge B)\wedge C &= A\wedge(B\wedge C\wedge\cdots) = \cdots\wedge A\wedge B\wedge C\wedge\cdots \end{aligned}$$

Note that the inner product is not associative.

The *distributive properties*:

$$\begin{aligned}
A(B+C+\cdots) &= AB+AC+\cdots \\
(\cdots+B+C)A &= \cdots+BA+CA \\
A\wedge(B+C+\cdots) &= A\wedge B+A\wedge C+\cdots \\
(A+B+\cdots)\wedge C &= A\wedge C+B\wedge C+\cdots \\
A\cdot(B+C+\cdots) &= A\cdot B+A\cdot C+\cdots \\
(A+B+\cdots)\cdot C &= A\cdot C+B\cdot C+\cdots \\
A\times(B+C+\cdots) &= A\times B+A\times C+\cdots
\end{aligned}$$

The commutator product has the *derivative properties*:

$$\begin{aligned}
A\times(B\times C) &= (A\times B)\times C+B\times(A\times C) \\
A\times(\mathrm{BC}) &= (A\times B)C+B(A\times C).
\end{aligned}$$

If A is a bivector, then these additional identities hold for the commutator:

$$\begin{aligned}
A\times(B\cdot C) &= (A\times B)\cdot C+B\cdot(A\times C) \\
A\times(B\wedge C) &= (A\times B)\wedge C+B\wedge(A\times C).
\end{aligned}$$

2.38 The inner product of blades

Consider an r-blade $\mathbf{A}_r$ and an s-blade $\mathbf{B}_s$, where $r\le s$,

$$\begin{aligned}
\mathbf{A}_r = \bigwedge_{i=1}^{r}\mathbf{a}_i &= \mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_r \\
\mathbf{B}_s = \bigwedge_{i=1}^{s}\mathbf{b}_i &= \mathbf{b}_1\wedge\cdots\wedge\mathbf{b}_s.
\end{aligned}$$

Their inner product $\mathbf{A}_r\cdot\mathbf{B}_s$ is defined as the grade $|s-r|$ part of their geometric product,

$$\mathbf{A}_r\cdot\mathbf{B}_s = \langle\mathbf{A}_r\mathbf{B}_s\rangle_{|s-r|}.$$

If $r\ge s$, then we can write this as $(-1)^{s(r-1)}\mathbf{B}_s\cdot\mathbf{A}_r$, and the situation is similar. The inner product is always the *grade-reducing* product of grade $|s-r|$. Although this definition of the inner product, as the grade $|s-r|$ projection of the geometric product, is correct and sometimes useful, it does not provide an explicit formula for computing $\mathbf{A}_r\cdot\mathbf{B}_s$.

An explicit formula, given in references [7] and [5], is the *reduction formula* for expanding the inner product of blades,

$$\begin{aligned}
\mathbf{A}_r\cdot\mathbf{B}_s &= (\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r)\cdot(\mathbf{b}_1\wedge\ldots\wedge\mathbf{b}_s) \\
&= (\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_{r-1})\cdot(\mathbf{a}_r\cdot(\mathbf{b}_1\wedge\ldots\wedge\mathbf{b}_s)).
\end{aligned}$$

Here, we can make use of another well-known form of the reduction formula

$$\mathbf{a}\cdot\mathbf{B}_s = \sum_{i=1}^{s}(-1)^{i+1}(\mathbf{a}\cdot\mathbf{b}_i)(\mathbf{B}_s\backslash\mathbf{b}_i),$$

where

$$\mathbf{B}_s \backslash \mathbf{b}_i = \bigwedge_{j \neq i} \mathbf{b}_j = \mathbf{b}_1 \wedge \ldots \wedge \check{\mathbf{b}}_i \wedge \ldots \wedge \mathbf{b}_s,$$

with $\check{\mathbf{b}}_i$ indicating that $\mathbf{b}_i$ is removed from the blade. Using the reduction formula repeatedly leads to

$$\mathbf{A}_r \cdot \mathbf{B}_s = \mathbf{a}_1 \cdot (\ldots(\mathbf{a}_{r-2} \cdot (\mathbf{a}_{r-1} \cdot (\mathbf{a}_r \cdot \mathbf{B}_s)))\ldots).$$

This also leads to the following formula, for the inner product of two l-blades $\mathbf{A}_l \cdot \mathbf{B}_l$,

$$\mathbf{A}_l \cdot \mathbf{B}_l = (-1)^{l(l-1)/2} \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_k & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_l \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_k \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_k \cdot \mathbf{b}_k & \cdots & \mathbf{a}_k \cdot \mathbf{b}_l \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_l \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_l \cdot \mathbf{b}_k & \cdots & \mathbf{a}_l \cdot \mathbf{b}_l \end{vmatrix}, \tag{2.1}$$

which can be verified by working the algebra. The reduction formula is also discussed in reference [3].

Another explicit formula for the computation of the inner product $\mathbf{A}_r \cdot \mathbf{B}_s$ is given in reference [7] as

$$\mathbf{A}_r \cdot \mathbf{B}_s = \sum_{k_1 < \ldots < k_r} \epsilon_{k_1 \ldots k_s} (\mathbf{A}_r \cdot (\mathbf{b}_{k_1} \wedge \ldots \wedge \mathbf{b}_{k_r})) \mathbf{b}_{k_{r+1}} \wedge \ldots \wedge \mathbf{b}_{k_s}.$$

Using Sweedler's notation (see reference [8]), this formula can be written as

$$\mathbf{A}_r \cdot \mathbf{B}_s = \sum_{(r, s-r) \vdash \mathbf{B}_s} (\mathbf{A}_r \cdot \mathbf{B}_{s(1)}) \mathbf{B}_{s(2)}.$$

These formulas are not quite as simple as the reduction formula, but they lead to a more general expansion formula called the expansion of the geometric product of blades (EGPB), which is discussed later in the next section. The Levi-Civita symbol $\epsilon_{k_1 \ldots k_s}$, where $\epsilon_{k_1 \ldots k_s} = 1$ $(=-1)$ for even (odd) permutation of the indices $k_1 \ldots k_s$ in $\mathbf{B}_s = \epsilon_{k_1 \ldots k_s} \mathbf{B}_{s(1)_k} \wedge \mathbf{B}_{s(2)_k}$, gives the sign of the combinatorial permutation of $\mathbf{B}_s$ into the two partitions $\mathbf{B}_{s(1)_k}$ and $\mathbf{B}_{s(2)_k}$. By Sweedler's notation $(r, s-r) \vdash \mathbf{B}_s$, the blade $\mathbf{B}_s$ is combinatorially permutated for every combination having the first r indices in ascending order $k_1 < \ldots < k_r$, and partitioned into two sub-blades $\mathbf{B}_{s(1)_k} = \epsilon_{k_1 \ldots k_s} \mathbf{b}_{k_1} \wedge \ldots \wedge \mathbf{b}_{k_r}$ and $\mathbf{B}_{s(2)_k} = \mathbf{b}_{k_{r+1}} \wedge \ldots \wedge \mathbf{b}_{k_s}$, where the sign of permutation $\epsilon_{k_1 \ldots k_s}$ is multiplied into $\mathbf{B}_{s(1)_k}$ (or into $\mathbf{B}_{s(2)_k}$). The required ascending order $k_1 < \ldots < k_r$ of the first r indices implies taking all combinations of r-blades from $\mathbf{B}_s$, but not all permutations. The order of the other $s-r$ indices $k_{r+1} \ldots k_s$ can be arbitrary, but must match the order in $\epsilon_{k_1 \ldots k_s}$ (the ascending order $k_{r+1} < \ldots < k_s$ is recommended). The summation range $k_1 < \ldots < k_r$, which implies taking combinations, then also implies a summation of k terms, $k = 1 \ldots C(s, r)$, where $C(s, r)$ is the number of combinations of r from s (still assuming $r \leq s$). By Sweedler's notation, the summation range $(r, s-r) \vdash \mathbf{B}_s$ implies the same summation of k terms as the summation range $k_1 < \ldots < k_r$. For the notation $(r, s-r) \vdash \mathbf{B}_s$, the r-blade $\mathbf{B}_{s(1)_k}$ runs over all k combinations. Note that, if we had instead used the notation $(s-r, r) \vdash \mathbf{B}_s$, then the r-blade $\mathbf{B}_{s(2)_k}$ would run over all k combinations; we will use this other notation in the next section on the expansion of the geometric product of blades (EGPB).

Example. Expand $\mathbf{A}_r \cdot \mathbf{B}_s$, given that $r = 2$, $s = 4$, $\mathbf{A}_r = \mathbf{A}_2 = \mathbf{a}_1 \wedge \mathbf{a}_2$, and $\mathbf{B}_s = \mathbf{B}_4 = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3 \wedge \mathbf{b}_4$. The number k of summation terms is $k = 1 \ldots C(s, r) = 1 \ldots C(4, 2) = 1 \ldots 6$. We denote the k partitionings of $\mathbf{B}_s$ by $\mathbf{B}_{s(1)_k}$ and $\mathbf{B}_{s(2)_k}$, where $\mathbf{B}_s = \epsilon_{k_1 \ldots k_s} \mathbf{B}_{s(1)_k} \wedge \mathbf{B}_{s(2)_k}$ for each k.

The $k = 1 \ldots 6$ partitions of $\mathbf{B}_4$, and their signs of permutation, are

$$
\begin{array}{lll}
\mathbf{B}_{4(1)_1} = \mathbf{b}_1 \wedge \mathbf{b}_2 & , \quad \mathbf{B}_{4(2)_1} = \mathbf{b}_3 \wedge \mathbf{b}_4 & , \quad \epsilon_{1234} = 1 \\
\mathbf{B}_{4(1)_2} = \mathbf{b}_1 \wedge \mathbf{b}_3 & , \quad \mathbf{B}_{4(2)_2} = \mathbf{b}_2 \wedge \mathbf{b}_4 & , \quad \epsilon_{1324} = -1 \\
\mathbf{B}_{4(1)_3} = \mathbf{b}_1 \wedge \mathbf{b}_4 & , \quad \mathbf{B}_{4(2)_3} = \mathbf{b}_2 \wedge \mathbf{b}_3 & , \quad \epsilon_{1423} = 1 \\
\mathbf{B}_{4(1)_4} = \mathbf{b}_2 \wedge \mathbf{b}_3 & , \quad \mathbf{B}_{4(2)_4} = \mathbf{b}_1 \wedge \mathbf{b}_4 & , \quad \epsilon_{2314} = 1 \\
\mathbf{B}_{4(1)_5} = \mathbf{b}_2 \wedge \mathbf{b}_4 & , \quad \mathbf{B}_{4(2)_5} = \mathbf{b}_1 \wedge \mathbf{b}_3 & , \quad \epsilon_{2413} = -1 \\
\mathbf{B}_{4(1)_6} = \mathbf{b}_3 \wedge \mathbf{b}_4 & , \quad \mathbf{B}_{4(2)_6} = \mathbf{b}_1 \wedge \mathbf{b}_2 & , \quad \epsilon_{3412} = 1.
\end{array}
$$

Notice, the ascending order of the indices $k_1 < \ldots < k_r$ in $\mathbf{B}_{s(1)_k}$. We also choose $k_{r+1} < \ldots < k_r$ in $\mathbf{B}_{s(2)_k}$. Now, we can write the summation

$$
\begin{array}{rl}
\displaystyle\sum_{(r,s-r)\vdash \mathbf{B}_s} (\mathbf{A}_r \cdot \mathbf{B}_{s(1)}) \mathbf{B}_{s(2)} & = \\
\displaystyle\sum_{k=1}^{6} (\mathbf{A}_2 \cdot \mathbf{B}_{4(1)_k}) \mathbf{B}_{4(2)_k} & = \\
\epsilon_{1234}(\mathbf{A}_2 \cdot \mathbf{B}_{4(1)_1}) \mathbf{B}_{4(2)_1} + \epsilon_{1324}(\mathbf{A}_2 \cdot \mathbf{B}_{4(1)_2}) \mathbf{B}_{4(2)_2} & + \\
\epsilon_{1423}(\mathbf{A}_2 \cdot \mathbf{B}_{4(1)_3}) \mathbf{B}_{4(2)_3} + \epsilon_{2314}(\mathbf{A}_2 \cdot \mathbf{B}_{4(1)_4}) \mathbf{B}_{4(2)_4} & + \\
\epsilon_{2413}(\mathbf{A}_2 \cdot \mathbf{B}_{4(1)_5}) \mathbf{B}_{4(2)_5} + \epsilon_{3412}(\mathbf{A}_2 \cdot \mathbf{B}_{4(1)_6}) \mathbf{B}_{4(2)_6} & = \\
((\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2)) \mathbf{b}_3 \wedge \mathbf{b}_4 - ((\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_3)) \mathbf{b}_2 \wedge \mathbf{b}_4 & + \\
((\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_4)) \mathbf{b}_2 \wedge \mathbf{b}_3 + ((\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_2 \wedge \mathbf{b}_3)) \mathbf{b}_1 \wedge \mathbf{b}_4 & - \\
((\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_2 \wedge \mathbf{b}_4)) \mathbf{b}_1 \wedge \mathbf{b}_3 + ((\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_3 \wedge \mathbf{b}_4)) \mathbf{b}_1 \wedge \mathbf{b}_2 & .
\end{array}
$$

Each of the inner products could be further expanded using (2.1), which should then compare equally to the same result obtained using the simpler reduction formula.

2.39 Expansion of the geometric product of blades

The following formula (2.2), found in reference [8], is called the *expansion of the geometric product of blades* (EGPB).

$$
\langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s-2l} = \sum_{\substack{(r-l,l)\vdash \mathbf{A}_r, \\ (l,s-l)\vdash \mathbf{B}_s}} \langle \mathbf{A}_{r(2)} \mathbf{B}_{s(1)} \rangle \, \mathbf{A}_{r(1)} \wedge \mathbf{B}_{s(2)}. \tag{2.2}
$$

The EGPB formula algebraically expands the *geometric product* $\mathbf{A}_r \mathbf{B}_s$ of an r-blade

$$
\mathbf{A}_r = \bigwedge_{i=1}^{r} \mathbf{a}_i = \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r
$$

and s-blade

$$
\mathbf{B}_s = \bigwedge_{i=1}^{s} \mathbf{b}_i = \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_s
$$

for only the terms of grade $r+s-2l$ in $\mathbf{A}_r\mathbf{B}_s$, as indicated by the *grade projection*

$$\langle \mathbf{A}_r\mathbf{B}_s\rangle_{r+s-2l}.$$

The notation used in the double summation (double sum, or series) is called *Sweedler's notation* (see [8]). The EGPB formula can be written more explicitly as

$$\langle \mathbf{A}_r\mathbf{B}_s\rangle_{r+s-2l} = \sum_{\substack{j_{r-l+1}<\ldots<j_r,\\ k_1<\ldots<k_l}} \mathbf{A}_{r(1)_j}\wedge(\mathbf{A}_{r(2)_j}\cdot\mathbf{B}_{s(1)_k})\wedge\mathbf{B}_{s(2)_k},$$

Partition (1), $(r-l)$-blade	: $\mathbf{A}_{r(1)_j}$	$=$ $\epsilon_{j_1\ldots j_r}(\mathbf{a}_{j_1}\wedge\ldots\wedge\mathbf{a}_{j_{r-l}})$	: $j_1<\ldots<j_{r-l}$
Partition (2), l-blade	: $\mathbf{A}_{r(2)_j}$	$=$ $(\mathbf{a}_{j_{r-l+1}}\wedge\ldots\wedge\mathbf{a}_{j_r})$	: $j_{r-l+1}<\ldots<j_r$
Partition (1), l-blade	: $\mathbf{B}_{s(1)_k}$	$=$ $(\mathbf{b}_{k_1}\wedge\ldots\wedge\mathbf{b}_{k_l})$	: $k_1<\ldots<k_l$
Partition (2), $(s-l)$-blade	: $\mathbf{B}_{s(2)_k}$	$=$ $\epsilon_{k_1\ldots k_s}(\mathbf{b}_{k_{l+1}}\wedge\ldots\wedge\mathbf{b}_{k_s})$	: $k_{l+1}<\ldots<k_s$
Combinatorially permutated	: $\mathbf{A}_r$	$=$ $\mathbf{A}_{r(1)_j}\wedge\mathbf{A}_{r(2)_j}$	
Combinatorially permutated	: $\mathbf{B}_s$	$=$ $\mathbf{B}_{s(1)_k}\wedge\mathbf{B}_{s(2)_k}$.	

The ascending orderings of the indices, $j_{r-l+1}<\ldots<j_r$ and $k_1<\ldots<k_l$, on the l-blade partitions $\mathbf{A}_{r(2)_j}$ and $\mathbf{B}_{s(1)_k}$, respectively, imply summing over all possible *combinations* of the l-blades $\mathbf{A}_{r(2)_j}$ and $\mathbf{B}_{s(1)_k}$ taken (partitioned) from the original r-blade $\mathbf{A}_r$ and s-blade $\mathbf{B}_s$, respectively. For each summation term, $\mathbf{A}_r=\mathbf{A}_{r(1)_j}\wedge\mathbf{A}_{r(2)_j}$ and $\mathbf{B}_s=\mathbf{B}_{s(1)_k}\wedge\mathbf{B}_{s(2)_k}$ are certain *combinatorial permutations* of $\mathbf{A}_r$ and $\mathbf{B}_s$, respectively, with permutation signs $\epsilon_{j_1\ldots j_r}$ and $\epsilon_{k_1\ldots k_s}$ (Levi-Civita symbols), respectively. The Levi-Civita symbol $\epsilon_{j_1\ldots j_r}$, also called the permutation symbol or antisymmetric symbol, is defined as

$$\epsilon_{j_1\ldots j_r}=\begin{cases} 1 & : \text{ if } j_1\ldots j_r \text{ is an even permutation of } 1\ldots r\\ -1 & : \text{ if } j_1\ldots j_r \text{ is an odd permutation of } 1\ldots r\\ 0 & : \text{ otherwise, or if any } j_i \text{ repeats in } j_1\ldots j_r.\end{cases}$$

The case of any repeating index j_i is not relevant here. It is recommended to also take ascending indices $j_1<\ldots<j_{r-l}$ and $k_{l+1}<\ldots<k_s$ for the outer partitions $\mathbf{A}_{r(1)_j}$ and $\mathbf{B}_{s(2)_k}$ that hold the remaining vectors that are wedged outside of the inner product of the two l-blades. We will use the notation $C(n,k)$ as the number of combinations of k elements taken from n elements. The total number of double summation terms is the product $C(r,\,l)C(s,\,l)$, which is implied by the double summation ranges $j_{r-l+1}<\ldots<j_r$, $k_1<\ldots<k_l$, or by the equivalent double summation ranges $(r-l,l)\vdash\mathbf{A}_r$, $(l,s-l)\vdash\mathbf{B}_s$ in Sweedler's notation.

The grade, $r+s-2l$, of the expansion can be chosen in steps of 2, by $2l$, since the products of canonical basis orthonormal vectors $\{\mathbf{e}_1,\ldots,\mathbf{e}_{n=p+q}\}$ in a geometric algebra $\mathcal{G}_{p,q}$ either square to a scalar (0-blade) when equal (e.g., $\mathbf{e}_i^2=\pm1$) or form a bivector (2-blade) when different (e.g., $\mathbf{e}_i\mathbf{e}_j=\mathbf{e}_i\wedge\mathbf{e}_j$, $i\neq j$). Any one of the possible grades $r+s-2l$ in the geometric product of two blades can be selected by choosing a suitable value for l. For an r-blade and s-blade, the lowest possible grade in their geometric product is $|r-s|$, the highest possible grade is $(r+s)$, and other possible grades are between in steps of 2 grades. Setting $l=0$ makes an outer product. Setting $l=(r+s-|r-s|)/2$ makes an inner product. Therefore, the EGPB formula provides an alternative to using the reduction formula for the inner product of blades, as discussed in the prior section.

In a geometric algebra $\mathcal{G}_{p,q}$ with $n = p + q$ canonical basis orthonormal vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, the unit pseudoscalar is $\mathbf{I}_n = \mathbf{e}_1 ... \mathbf{e}_n$. If $r = n$ and/or $s = n$, then $\mathbf{A}_n = \alpha \mathbf{I}_n$ and/or $\mathbf{B}_n = \beta \mathbf{I}_n$ are pseudoscalars in $\mathcal{G}_{p,q}$. The geometric product of any blade with any pseudoscalar of the algebra will produce a blade in the dual space having grade $|r - s|$. Therefore, in this case, the geometric product $\langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s-2l}$ is always the inner product $\langle \mathbf{A}_r \mathbf{B}_s \rangle_{|r-s|} = \mathbf{A}_r \cdot \mathbf{B}_s$, and all the other grades that could be chosen by setting l will result in zero. In summary, $\mathbf{A}_r \beta \mathbf{I}_n = \beta \mathbf{A}_r \cdot \mathbf{I}_n$, $\alpha \mathbf{I}_n \mathbf{B}_s = \alpha \mathbf{I}_n \cdot \mathbf{B}_s$, and $\mathbf{A}_r \cdot \mathbf{I}_n = (-1)^{r(n-1)} \mathbf{I}_n \cdot \mathbf{A}_r$, where for n odd the pseudoscalars commute without any sign changes. Note that, what has been said so far with respect to an algebra $\mathcal{G}_{p,q}$ can, most likely, also hold in an algebra $\mathcal{G}_{p,q,r} = \mathcal{G}(\mathbb{R}^{p,q,r}) = \mathcal{G}(\mathbb{R}^{p,q} \oplus \mathbb{N}^r) = \mathcal{G}_{p,q} \otimes \mathbb{N}_r$ that includes a null or degenerate vector space $\mathbb{N}^r$ without a metric (or a zero metric), where the algebra $\mathbb{N}_r = \Lambda(\mathbb{R}^r)$ is the Grassmann (Exterior) algebra of $\mathbb{R}^r$ with only the outer (exterior) product (all inner products are zero in $\mathbb{N}_r$), but this is beyond the scope of this book.

The Sweedler's notation can be explained further as follows. Blade $\mathbf{A}_r$ is partitioned ($\vdash$) into a *2-partition* of *shape* $(r - l, l)$, and blade $\mathbf{B}_s$ is partitioned into a 2-partition of shape $(l, s - l)$. The first number in a partition shape is the grade of a sub-blade that is referenced as a partition number (1), and the second number in a shape is the grade of a sub-blade that is referenced as a partition number (2). The partition (1) of a partitioning of $\mathbf{A}_r$ is denoted $\mathbf{A}_{r(1)}$, and the partition (2) is denoted $\mathbf{A}_{r(2)}$. In the double summation, the grade l partitions, $\mathbf{A}_{r(2)}$ and $\mathbf{B}_{s(1)}$ range over l-blade combinations taken from $\mathbf{A}_r$ and $\mathbf{B}_s$, respectively. By requiring that indices are in ascending order, it is assured (and implied) that only combinations are taken in the partitioned sub-blades, since the EGPB formula does not require all permutations. However, it is possible to define the EGPB formula where it sums over all permutations and then averages the combinatorially repeating terms, but that form of its definition is computationally inefficient and unnecessary. The sign of permutation $\epsilon_{j_1 \dots j_r} \epsilon_{k_1 \dots k_s}$ on a particular summation term depends on the particular order of the indices $j_1 \dots j_r$ and $k_1 \dots k_s$ in the combinatorial permutations of $\mathbf{A}_r$ and $\mathbf{B}_s$, respectively, for the term.

We can also write the EGPB formula a little more explicitly as

$$\langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s-2l} = \sum_{j=1}^{C(r,l)} \sum_{k=1}^{C(s,l)} \mathbf{A}_{r(1)_j} \wedge (\mathbf{A}_{r(2)_j} \cdot \mathbf{B}_{s(1)_k}) \wedge \mathbf{B}_{s(2)_k} =$$

$$\sum_{j=1}^{C(r,l)} \sum_{k=1}^{C(s,l)} ($$

$$\epsilon_{j_1 \dots j_r} \epsilon_{k_1 \dots k_s} (\mathbf{a}_{j_1} \wedge \dots \wedge \mathbf{a}_{j_{r-l}}) \wedge [(\mathbf{a}_{j_{r-l+1}} \wedge \dots \wedge \mathbf{a}_{j_r}) \cdot (\mathbf{b}_{k_1} \wedge \dots \wedge \mathbf{b}_{k_l})] \wedge (\mathbf{b}_{k_{l+1}} \wedge \dots \wedge \mathbf{b}_{k_s})$$

$$),$$

where we still require ascending (combinatorial) orderings of the indices in each partition sub-blade: $j_1 < \dots < j_{r-l}$ for each $(r - l)$-blade $\mathbf{A}_{r(1)_j}$, $j_{r-l+1} < \dots < j_r$ for each l-blade $\mathbf{A}_{r(2)_j}$, $k_1 < \dots < k_l$ for each l-blade $\mathbf{B}_{s(1)_k}$, and $k_{l+1} < \dots < k_s$ for each $(s - l)$-blade $\mathbf{B}_{s(2)_k}$.

Note that, in these more explicit formulas, we are using the fact that the outer product $a \wedge A$ of a scalar a and any multivector A is just scalar multiplication, $a \wedge A = aA$, and this arrangement makes it a little easier to see the overall ordering of the indices $j_1 \ldots j_r$ and $k_1 \ldots k_s$ in the combinatorially permutated blades, $\mathbf{A}_r = \mathbf{A}_{r(1)_j} \wedge \mathbf{A}_{r(2)_j}$ and $\mathbf{B}_s = \mathbf{B}_{s(1)_k} \wedge \mathbf{B}_{s(2)_k}$, and how these combinatorially permutated index orderings should exactly match the index orderings in the sign factors $\epsilon_{j_1 \ldots j_r} \epsilon_{k_1 \ldots k_s}$, which is the product of two Levi-Civita symbols, where $\epsilon_{j_1 \ldots j_r}$ is the sign for the combinatorial permutation of $\mathbf{A}_r$, and $\epsilon_{k_1 \ldots k_s}$ is the sign for the combinatorial permutation of $\mathbf{B}_s$, per term of the double summation ranging over j and then k. There is a total of $C(r,l)C(s,l)$ summation terms, ranging over index $j = 1 \ldots C(r,l)$ and then over index $k = 1 \ldots C(s,l)$ for each j. The double summation is over all combinations of the l-blades $\mathbf{A}_{r(2)_j}$ and $\mathbf{B}_{s(1)_k}$ within the inner product, which means summing over all ascending orderings $j_{r-l+1} < \ldots < j_r$ and $k_1 < \ldots < k_l$ of the l-blade indices, and conventionally choosing to also use ascending orderings $j_1 < \ldots < j_{r-l}$ and $k_{l+1} < \ldots < k_s$ for the indices on the remaining two outer blades $\mathbf{A}_{r(1)_j}$ and $\mathbf{B}_{s(2)_k}$ that sandwich the middle inner product, which is a scalar-valued inner product of two l-blades $\mathbf{A}_{r(2)_j} \cdot \mathbf{B}_{s(1)_k}$. For each value of summation range index $j = 1 \ldots C(r,l)$, there are combinatorially permutated ascending indices $j_{r-l+1} < \ldots < j_r$ over the l-blade $\mathbf{A}_{r(2)_j}$, and also ascending indices (by convention or choice) $j_1 < \ldots < j_{r-l}$ over the leading $(r-l)$-blade $\mathbf{A}_{r(1)_j}$ (e.g., for $j = 1$, the multi-indices $j_1 \ldots j_r$ are denoted $1_1 \ldots 1_r$). For each value of summation range index $k = 1 \ldots C(s,l)$, there are combinatorially permutated ascending indices $k_1 < \ldots < k_l$ over the l-blade $\mathbf{B}_{s(1)_k}$, and also ascending indices (by convention or choice) $k_{l+1} < \ldots < k_s$ over the trailing $(s-l)$-blade $\mathbf{B}_{s(2)_k}$ (e.g., for $k = 1$, the multi-indices $k_1 \ldots k_s$ are denoted $1_1 \ldots 1_s$). The following example may help to demonstrate the usage of the EGPB formula.

Example. Expand $\langle \mathbf{A}_3 \mathbf{B}_4 \rangle_3 = \langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s-2l}$, where $r = 3$, $s = 4$, and $l = 2$. For the partitioning $(r-l, l) \vdash \mathbf{A}_r = (1,2) \vdash \mathbf{A}_3$, there are $j = 1 \ldots C(r,l) = 1 \ldots C(3,2) = 1 \ldots 3$ combinatorial permutations of the l-blade $\mathbf{A}_{r(2)_j}$ taken from the original r-blade $\mathbf{A}_r$, and each has a certain sign of permutation $\epsilon_{j_1 \ldots j_r}$ relative to the original r-blade $\mathbf{A}_r$. Similarly, for the partitioning $(l, s-l) \vdash \mathbf{B}_s = (2, 2) \vdash \mathbf{B}_4$, there are $k = 1 \ldots C(s,l) = 1 \ldots C(4,2) = 1 \ldots 6$ combinatorial permutations of the l-blade $\mathbf{B}_{s(1)_k}$ taken from the original s-blade $\mathbf{B}_s$, and each has a certain sign of permutation $\epsilon_{k_1 \ldots k_s}$ relative to the original s-blade $\mathbf{B}_s$. Over the summation ranges of j and k, we can write out these partitions and their signs as follows.

First, we take combinations of indices $j_{r-l+1} < \ldots < j_r$ for the inner l-blades $\mathbf{A}_{3(2)_j}$:

$$
\begin{aligned}
\mathbf{A}_{3(2)_1} &= \mathbf{a}_{1_2} \wedge \mathbf{a}_{1_3} = \mathbf{a}_1 \wedge \mathbf{a}_2 \\
\mathbf{A}_{3(2)_2} &= \mathbf{a}_{2_2} \wedge \mathbf{a}_{2_3} = \mathbf{a}_1 \wedge \mathbf{a}_3 \\
\mathbf{A}_{3(2)_3} &= \mathbf{a}_{3_2} \wedge \mathbf{a}_{3_3} = \mathbf{a}_2 \wedge \mathbf{a}_3.
\end{aligned}
$$

Then, we take the remaining combinations for the outer leading $(r-l)$-blades $\mathbf{A}_{3(1)_j}$:

$$
\begin{aligned}
\mathbf{A}_{3(1)_1} &= \epsilon_{1_1 1_2 1_3} \mathbf{a}_{1_1} = \epsilon_{312} \mathbf{a}_3 = \mathbf{a}_3 \\
\mathbf{A}_{3(1)_2} &= \epsilon_{2_1 2_2 2_3} \mathbf{a}_{2_1} = \epsilon_{213} \mathbf{a}_2 = -\mathbf{a}_2 \\
\mathbf{A}_{3(1)_3} &= \epsilon_{3_1 3_2 3_3} \mathbf{a}_{3_1} = \epsilon_{123} \mathbf{a}_1 = \mathbf{a}_1.
\end{aligned}
$$

Again, we first take combinations of indices $k_1 < \ldots < k_l$ for the inner l-blades $\mathbf{B}_{4(1)_k}$:

$$\begin{aligned}
\mathbf{B}_{4(1)_1} &= \mathbf{b}_{1_1} \wedge \mathbf{b}_{1_2} = \mathbf{b}_1 \wedge \mathbf{b}_2\\
\mathbf{B}_{4(1)_2} &= \mathbf{b}_{2_1} \wedge \mathbf{b}_{2_2} = \mathbf{b}_1 \wedge \mathbf{b}_3\\
\mathbf{B}_{4(1)_3} &= \mathbf{b}_{3_1} \wedge \mathbf{b}_{3_2} = \mathbf{b}_1 \wedge \mathbf{b}_4\\
\mathbf{B}_{4(1)_4} &= \mathbf{b}_{4_1} \wedge \mathbf{b}_{4_2} = \mathbf{b}_2 \wedge \mathbf{b}_3\\
\mathbf{B}_{4(1)_5} &= \mathbf{b}_{5_1} \wedge \mathbf{b}_{5_2} = \mathbf{b}_2 \wedge \mathbf{b}_4\\
\mathbf{B}_{4(1)_6} &= \mathbf{b}_{6_1} \wedge \mathbf{b}_{6_2} = \mathbf{b}_3 \wedge \mathbf{b}_4.
\end{aligned}$$

Then, we take the remaining combinations for the outer trailing $(s-l)$-blades $\mathbf{B}_{4(2)_k}$, where we choose (by our convention) to use ascending indices $k_{l+1} < \ldots < k_s$ for these as well:

$$\begin{aligned}
\mathbf{B}_{4(2)_1} &= \epsilon_{1_11_21_31_4}\mathbf{b}_{1_3} \wedge \mathbf{b}_{1_4} = \epsilon_{1234}\mathbf{b}_3 \wedge \mathbf{b}_4 = \mathbf{b}_3 \wedge \mathbf{b}_4\\
\mathbf{B}_{4(2)_2} &= \epsilon_{2_12_22_32_4}\mathbf{b}_{2_3} \wedge \mathbf{b}_{2_4} = \epsilon_{1324}\mathbf{b}_2 \wedge \mathbf{b}_4 = -\mathbf{b}_2 \wedge \mathbf{b}_4\\
\mathbf{B}_{4(2)_3} &= \epsilon_{3_13_23_33_4}\mathbf{b}_{3_3} \wedge \mathbf{b}_{3_4} = \epsilon_{1423}\mathbf{b}_2 \wedge \mathbf{b}_3 = \mathbf{b}_2 \wedge \mathbf{b}_3\\
\mathbf{B}_{4(2)_4} &= \epsilon_{4_14_24_34_4}\mathbf{b}_{4_3} \wedge \mathbf{b}_{4_4} = \epsilon_{2314}\mathbf{b}_1 \wedge \mathbf{b}_4 = \mathbf{b}_1 \wedge \mathbf{b}_4\\
\mathbf{B}_{4(2)_5} &= \epsilon_{5_15_25_35_4}\mathbf{b}_{5_3} \wedge \mathbf{b}_{5_4} = \epsilon_{2413}\mathbf{b}_1 \wedge \mathbf{b}_3 = -\mathbf{b}_1 \wedge \mathbf{b}_3\\
\mathbf{B}_{4(2)_6} &= \epsilon_{6_16_26_36_4}\mathbf{b}_{6_3} \wedge \mathbf{b}_{6_4} = \epsilon_{3412}\mathbf{b}_1 \wedge \mathbf{b}_2 = \mathbf{b}_1 \wedge \mathbf{b}_2.
\end{aligned}$$

Using these partitioned sub-blades $\mathbf{A}_{3(1)_j}$, $\mathbf{A}_{3(2)_j}$, $\mathbf{B}_{4(1)_k}$, and $\mathbf{B}_{4(2)_k}$ that include the signs of permutation $\epsilon_{j_1j_2j_3}$ and $\epsilon_{k_1k_2k_3k_4}$, it is possible to write the sum of all $C(r, l)C(s-l) = C(3, 2)C(4, 2) = 3*6 = 18$ of the 3-blade terms of the double summation, but this is left as an exercise for the reader.□

Beyond a simple example, such as the example just given, it seems evident that using the EGPB formula by manual calculation (by hand, or with pencil and paper) is very tedious and error-prone. The EGPB formula is better suited to an implementation as a computer algorithm. The EGPB formula, implemented as a computer algorithm, could constitute part of the core logic of an efficient computer algebra software (CAS) that implements Geometric Algebra. In general, the geometric product of blades $\mathbf{A}_r\mathbf{B}_s$ followed by a grade projection $\langle\mathbf{A}_r\mathbf{B}_s\rangle_{r+s-2l}$, which is just a selection or filtering for terms of the given grade, is an inefficient "brute-force" method of computing the part of grade $r+s-2l$ in a geometric product. However, the EGPB formula reduces the amount of computation and yields a more efficient computational method. The inner product of the two l-blades in the EGPB formula can be computed by very efficient matrix methods. The method of computation by the EGPB formula is likely to be optimal (most efficient) since it directly generates only the terms of interest, of grade $r+s-2l$.

Other than reference [8], and the foregoing discussion, it seems that (at this time) there is very little other literature that dicusses the EGPB formula.

2.40 Definitions of products by graded parts of geometric product

The following are definitions of multivector products by associative, distributive, non-commutative multiplication of each component blade in A with each blade in B, followed by a grade selection (except for the geometric product). All products are based on selecting a certain grade of blades from the geometric product. The inner product has a special rule (to treat scalars as perpendicular to all values) that first loses or deletes any scalar part from the factors (to make sure they can't be selected), and it produces results only of absolute grade difference $|s-r|$; a scalar results from inner product only when $r=s$, and 0 if the factors are perpendicular. The scalar product simply selects the scalar part of any geometric product. The outer product also produces a scalar in the special case of the outer product of two scalars or 0-blades, where the grades add as $r+s=0+0=0$. The outer product is zero if $r+s>n$ where n is the number of vector units of the Clifford geometric algebra (in the case considered here, $n=3$ with vector units $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$). The outer product is zero when the two blades multiplied have a common vector, since parallel vectors span zero area or volume.

Geometric product (evaluated using only rules of units; $\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_j\mathbf{e}_i$ and $\mathbf{e}_i\mathbf{e}_i=1$):

$$AB = \sum_{r,s} \langle A\rangle_r \langle B\rangle_s = A \mathbin{\bar{\times}} B + A \times B$$

Scalar product:

$$A * B = \sum_{r,s} \langle \langle A\rangle_r \langle B\rangle_s\rangle_0$$

Inner product (standard inner product, where $\alpha \cdot B = 0$ for scalar α):

$$A \cdot B = \sum_{r\neq 0, s\neq 0} \langle \langle A\rangle_r \langle B\rangle_s\rangle_{|s-r|}$$

Outer product ($\alpha \wedge B = \alpha B$ for scalar α; $\langle X\rangle_{r+s}=0$ if $r+s>n$ of $\mathcal{G}_n$):

$$A \wedge B = \sum_{r,s} \langle \langle A\rangle_r \langle B\rangle_s\rangle_{r+s}$$

Commutator or anti-symmetric product:

$$A \times B = \frac{1}{2}(AB - BA) = -B \times A$$

Symmetric or anti-commutator product:

$$A \mathbin{\bar{\times}} B = \frac{1}{2}(AB + BA) = B \mathbin{\bar{\times}} A$$

The following three products work together as a modification of the inner product.

Dot product (modified non-standard inner product, where $\alpha \bullet B = \alpha B$):

$$A \bullet B = \sum_{r,s} \langle \langle A\rangle_r \langle B\rangle_s\rangle_{|s-r|}$$

Left contraction (A contraction onto B; when $r>s$ the result is 0):

$$A \rfloor B = \sum_{r,s} \langle \langle A\rangle_r \langle B\rangle_s\rangle_{s-r}$$

Right contraction (A contraction by B; when $s > r$ the result is 0):

$$A \lfloor B = \sum_{r,s} \langle \langle A \rangle_r \langle B \rangle_s \rangle_{r-s}$$

There are many more identities elsewhere in the literature on Clifford geometric algebras. An important concept these formulas show is that the inner product lowers grade, and outer product increases grade in definite amounts. Recognizing when a complicated expression will reduce to a scalar can sometimes save significant work, as shown some in the next subsection on the triple vector cross product.

2.41 Identities: Inner products

First, we make use of the following in the identities of this subsection:

Let $\mathbf{A}_r^{\sim}$ denote the *reversion* or reverse of an r-blade $\mathbf{A}_r$, where

$$\begin{aligned}
\mathbf{A}_r &= \bigwedge_{i=1}^{r} \mathbf{a}_i = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{r-1} \wedge \mathbf{a}_r \\
\mathbf{A}_r^{\sim} &= \bigwedge_{i=1}^{r} \mathbf{a}_{r-i+1} = \mathbf{a}_r \wedge \mathbf{a}_{r-1} \wedge \cdots \wedge \mathbf{a}_1 \\
\mathbf{A}_r &= (-1)^{r(r-1)/2} \mathbf{A}_r^{\sim}.
\end{aligned}$$

Properties of integer powers of -1:

$$\begin{aligned}
(-1)^a &= (-1)^{-a} \\
(-1)^a(-1)^b &= (-1)^{a+b} = (-1)^{-a-b} \\
&= (-1)^{a-b} = (-1)^{b-a}. \\
(-1)^{ab} &= ((-1)^a)^b = ((-1)^b)^a
\end{aligned}$$

Identity 1. *Commutation of the inner product of blades*. Given r-blade $\mathbf{A}_r$, s-blade $\mathbf{B}_s$, and the condition $r \leq s$, then the following identity holds good:

$$\begin{aligned}
\mathbf{A}_r \cdot \mathbf{B}_s &= (-1)^{r(s-1)} \mathbf{B}_s \cdot \mathbf{A}_r \\
&= \langle \mathbf{A}_r \mathbf{B}_s \rangle_{s-r}.
\end{aligned}$$

The inner product of two l-blades is generally commutative: $\mathbf{A}_l \cdot \mathbf{B}_l = \mathbf{B}_l \cdot \mathbf{A}_l$. Either l is even or $(l-1)$ is even; therefore, the sign $(-1)^{l(l-1)}$ is always positive.

Proof 1. Using reversion gives

$$\begin{aligned}
\mathbf{A}_r \cdot \mathbf{B}_s &= (-1)^{(s-r)(s-r-1)/2 - r(r-1)/2 - s(s-1)/2} (\mathbf{A}_r^{\sim} \cdot \mathbf{B}_s^{\sim})^{\sim} \\
&= (-1)^{(s^2 - 2rs - s + r^2 + r - r^2 + r - s^2 + s)/2} (\mathbf{A}_r^{\sim} \cdot \mathbf{B}_s^{\sim})^{\sim} \\
&= (-1)^{r - rs} (\mathbf{A}_r^{\sim} \cdot \mathbf{B}_s^{\sim})^{\sim} = (-1)^{rs - r} (\mathbf{A}_r^{\sim} \cdot \mathbf{B}_s^{\sim})^{\sim} \\
&= (-1)^{r(s-1)} (\mathbf{A}_r^{\sim} \cdot \mathbf{B}_s^{\sim})^{\sim} = (-1)^{r(s+1)} (\mathbf{A}_r^{\sim} \cdot \mathbf{B}_s^{\sim})^{\sim}.
\end{aligned}$$

The last step, is to notice another useful identity (with no condition on it):

$$(\mathbf{A}_r^{\sim} \cdot \mathbf{B}_s^{\sim})^{\sim} = \mathbf{B}_s \cdot \mathbf{A}_r.$$

In this identity, the LHS and RHS inner products have the same pairing of vector inner products when recursively reduced, but they are reverses that may differ in sign. The LHS is reversed to match the sign of the RHS. This identity is also a known property of the reversion operation, usually stated generally for the entire geometric product (not just the inner product part) of any two multivectors as

$$\begin{aligned} A &= \mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_r \\ B &= \mathbf{b}_1\mathbf{b}_2\ldots\mathbf{b}_s \\ (AB)^{\sim} &= B^{\sim}A^{\sim} \\ (A^{\sim})^{\sim} &= A \end{aligned}$$

therefore $(A^{\sim}B^{\sim})^{\sim} = BA$, with the inner product part included. □

Identity 2. *Conditional associativity of the inner product of blades*. Given r-blade $\mathbf{A}_r$, s-blade $\mathbf{B}_s$, t-blade $\mathbf{C}_t$, and the condition $r + s \leq t$, then the following identity holds good:

$$\begin{aligned} (\mathbf{A}_r \cdot \mathbf{C}_r) \cdot \mathbf{B}_s &= \mathbf{A}_r \cdot (\mathbf{C}_t \cdot \mathbf{B}_s) \\ &= \langle \mathbf{A}_r\mathbf{C}_t\mathbf{B}_s\rangle_{t-r-s}. \end{aligned}$$

Intuitively, it is equivalent to first contract $\mathbf{A}_r$ then $\mathbf{B}_s$ or first $\mathbf{B}_s$ then $\mathbf{A}_r$ on $\mathbf{C}_t$, provided that no factors commute in the change.

The product is zero if any vector in $\mathbf{A}_r$ or in $\mathbf{B}_s$ is orthogonal to all vectors in $\mathbf{C}_t$. If $\mathbf{A}_r$ and $\mathbf{B}_s$ have any common vector then the product is again zero.

It may be tempting to drop the parentheses and just write $\mathbf{A}_r \cdot \mathbf{C}_t \cdot \mathbf{B}_s$ when the condition $r + s \leq t$ is satisfied. However, if Identity 1 is applied as follows, then the conditional associativity must be explicity written again such as

$$\mathbf{A}_r \cdot \mathbf{C}_t \cdot \mathbf{B}_s = (-1)^{s(t-1)}\mathbf{A}_r \cdot (\mathbf{B}_s \cdot \mathbf{C}_t).$$

Therefore, writing $\mathbf{A}_r \cdot \mathbf{C}_t \cdot \mathbf{B}_s$ is conditionally correct but *generally* incorrect and a choice of the association should be made with parentheses to avoid mistakes.

Proof 2. Using inner product commutation (Identity 1) gives

$$(\mathbf{A}_r \cdot \mathbf{C}_t) \cdot \mathbf{B}_s$$

$$= ((-1)^{r(t-1)}\mathbf{C}_t \cdot \mathbf{A}_r) \cdot \mathbf{B}_s$$

then using inner product reduction (Identity 3) in reverse gives

$$= (-1)^{r(t-1)}\mathbf{C}_t \cdot (\mathbf{A}_r \wedge \mathbf{B}_s)$$

then using inner product commutation again gives

$$\begin{aligned} &= (-1)^{(r+s)(t-1)}(-1)^{r(t-1)}(\mathbf{A}_r \wedge \mathbf{B}_s)\cdot \mathbf{C}_t \\ &= (-1)^{s(t-1)}(\mathbf{A}_r \wedge \mathbf{B}_s)\cdot \mathbf{C}_t \end{aligned}$$

then using inner product reduction gives

$$= (-1)^{s(t-1)}\mathbf{A}_r \cdot (\mathbf{B}_s \cdot \mathbf{C}_t)$$

and finally, using inner product commutation again gives

$$\begin{aligned} &= (-1)^{s(t-1)}(-1)^{s(t-1)}\mathbf{A}_r \cdot (\mathbf{C}_t \cdot \mathbf{B}_s) \\ &= \mathbf{A}_r \cdot (\mathbf{C}_t \cdot \mathbf{B}_s) \end{aligned}$$

which was to be shown (Q.E.D.). □

Identity 3. *Reduction of the inner product of blades*. Given r-blade $\mathbf{A}_r$, s-blade $\mathbf{B}_s$, t-blade $\mathbf{C}_t$, and condition $r+s \le t$, then the following identity holds good:

$$(\mathbf{A}_r \wedge \mathbf{B}_s)\cdot \mathbf{C}_t = \mathbf{A}_r \cdot (\mathbf{B}_s \cdot \mathbf{C}_t).$$

A proof of this identity is given by Perwass [9, pp. 75] using grade projections.

For $r \le s$, this identity, applied recursively, produces the nested inner products identity

$$\begin{aligned} \mathbf{A}_r \cdot \mathbf{B}_s &= (\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_{r-1} \wedge \mathbf{a}_r)\cdot \mathbf{B}_s \\ &= \mathbf{a}_1 \cdot (\ldots \cdot (\mathbf{a}_{r-1} \cdot (\mathbf{a}_r \cdot \mathbf{B}_s))\ldots). \end{aligned}$$

2.42 Duality of inner and outer products

For any multivectors A and B, and commutative pseudoscalar $\mathbf{I}^2 = -1$ of odd grade (e.g., in $\mathcal{G}_3$),

$$\begin{aligned} (\mathbf{I}A)\wedge B &= -\mathbf{I}\mathbf{I}((\mathbf{I}A)\wedge B) \\ &= -\mathbf{I}((\mathbf{I}\cdot(\mathbf{I}A))\cdot B) \\ &= -\mathbf{I}((\mathbf{I}\mathbf{I}A)\cdot B) \\ &= \mathbf{I}(A\cdot B). \end{aligned}$$

Similarly,

$$\begin{aligned} A\wedge(\mathbf{I}B) &= -(A\wedge(\mathbf{I}B))\mathbf{I}\mathbf{I} \\ &= -(A\cdot((\mathbf{I}B)\cdot \mathbf{I}))\mathbf{I} \\ &= -(A\cdot(\mathbf{I}B\mathbf{I}))\mathbf{I} \\ &= (A\cdot B)\mathbf{I}. \end{aligned}$$

Then also,

$$(\mathbf{I}A)\wedge B \;=\; A\wedge(\mathbf{I}B).$$

2.43 Triple vector cross product

The triple vector cross product

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\wedge[(\mathbf{b}\wedge\mathbf{c})/\mathbf{I}])/\mathbf{I}\\
&= ([(\mathbf{c}\wedge\mathbf{b})\mathbf{I}]\wedge\mathbf{a})\cdot\mathbf{I}\\
&= [(\mathbf{c}\wedge\mathbf{b})\mathbf{I}]\cdot(\mathbf{a}\cdot\mathbf{I})\\
&= [(\mathbf{c}\wedge\mathbf{b})\mathbf{I}]\cdot(\mathbf{I}\mathbf{a})\\
&= \langle(\mathbf{c}\wedge\mathbf{b})\mathbf{I}\mathbf{I}\mathbf{a}\rangle_{|1-2|}\\
&= \langle(\mathbf{b}\wedge\mathbf{c})\mathbf{a}\rangle_{|1-2|}\\
&= (\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a}\\
&= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}
\end{aligned}$$

is one of the more confusing products to evaluate in vector calculus. Here, it is converted into geometric algebra, where the inner product of blades is used to derive the expanded identity. It may be hard to see this, but the pseudoscalar $\mathbf{I}$ commutes in the geometric product with any other value, and it also commutes across scalar, inner, and outer product symbols if it is next to the symbol. Perhaps this can be seen easier when the inner product is viewed as the geometric product (wherein $\mathbf{I}$ clearly commutes) followed by the grade selection.

The triple cross product can also be seen as two rejections and two scaled rotations in the quaternion dual mapping where the final result is taken undual back to a Euclidean vector as

$$\begin{aligned}
\mathbf{b}\times\mathbf{c} &= \Big(|\mathbf{b}|e^{\frac{\pi}{2}\frac{\mathbf{b}/\mathbf{I}}{|\mathbf{b}|}}\mathbf{c}^{\perp\mathbf{b}}/\mathbf{I}\Big)\mathbf{I} = (\mathbf{b}/\mathbf{I})[(\mathbf{c}\wedge\mathbf{b})\mathbf{b}^{-1}/\mathbf{I}]\mathbf{I}\\
&= \mathbf{b}(\mathbf{b}\wedge\mathbf{c})\mathbf{b}^{-1}\mathbf{I}\\
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \mathbf{a}(\mathbf{a}\wedge[\mathbf{b}(\mathbf{b}\wedge\mathbf{c})\mathbf{b}^{-1}\mathbf{I}])\mathbf{a}^{-1}\mathbf{I}\\
&= \mathbf{a}(\mathbf{a}\wedge[((\mathbf{b}\cdot\mathbf{b})\mathbf{c}-\mathbf{b}(\mathbf{b}\cdot\mathbf{c}))\mathbf{b}^{-1}\mathbf{I}])\mathbf{a}^{-1}\mathbf{I}\\
&= \mathbf{a}(\mathbf{a}\wedge[(\mathbf{c}\mathbf{b}-\mathbf{b}\cdot\mathbf{c})\mathbf{I}])\mathbf{a}^{-1}\mathbf{I}\\
&= \mathbf{a}(\mathbf{a}\wedge[(\mathbf{c}\wedge\mathbf{b})\cdot\mathbf{I}])\mathbf{a}^{-1}\mathbf{I}\\
&= \mathbf{a}(\mathbf{a}\wedge[\mathbf{c}\cdot(\mathbf{b}\cdot\mathbf{I})])\mathbf{a}^{-1}\mathbf{I}\\
&= \mathbf{a}(\mathbf{a}\wedge[\mathbf{c}\cdot(\mathbf{b}\mathbf{I})])\mathbf{a}^{-1}\mathbf{I}\\
&= ((\mathbf{a}\cdot\mathbf{a})[\mathbf{c}\cdot(\mathbf{b}\mathbf{I})]-\mathbf{a}(\mathbf{a}\cdot[\mathbf{c}\cdot(\mathbf{b}\mathbf{I})]))\mathbf{a}^{-1}\mathbf{I}\\
&= ([\mathbf{c}\cdot(\mathbf{b}\mathbf{I})]\cdot\mathbf{a}+[\mathbf{c}\cdot(\mathbf{b}\mathbf{I})]\wedge\mathbf{a}-\mathbf{a}\cdot[\mathbf{c}\cdot(\mathbf{b}\mathbf{I})])\mathbf{I}\\
&= ([\mathbf{c}\cdot(\mathbf{b}\mathbf{I})]\wedge\mathbf{a})\cdot\mathbf{I}\\
&= ((\mathbf{c}\wedge\mathbf{b})\cdot\mathbf{I})\cdot(\mathbf{a}\cdot\mathbf{I})\\
&= ((\mathbf{c}\wedge\mathbf{b})\mathbf{I})\cdot(\mathbf{I}\mathbf{a})\\
&= (\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a}\\
&= \mathbf{b}(\mathbf{c}\cdot\mathbf{a})-(\mathbf{b}\cdot\mathbf{a})\mathbf{c}.
\end{aligned}$$

Perhaps the above are overly-complicated ways to evaluate the triple cross product, but demonstrate different ways to view the expression. The simple way to evaluate the triple cross product is to use the recursive formula for the inner product of blades and the identity that the geometric product with a pseudoscalar is always just an inner product. Let's also continue to assume a 3D space.

$$\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

$$\begin{aligned}
&\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \\
&= (\mathbf{a} \wedge ((\mathbf{b} \wedge \mathbf{c})\mathbf{I}^{-1}))\mathbf{I}^{-1} \\
&= (\mathbf{a} \wedge ((\mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{I}^{-1})) \cdot \mathbf{I}^{-1} \\
&= \mathbf{a} \cdot (((\mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{I}^{-1}) \cdot \mathbf{I}^{-1}) \\
&= \mathbf{a} \cdot ((\mathbf{b} \wedge \mathbf{c})\mathbf{I}^{-1}\mathbf{I}^{-1}) \\
&= \mathbf{a} \cdot (\mathbf{c} \wedge \mathbf{b}) \\
&= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
\end{aligned}$$

2.44 Rotation of blades

Rotating a blade is the same as rotating each vector in the blade as

$$\begin{aligned}
R_{\mathbf{N}}^{\theta}\mathbf{A}_{\langle k \rangle} &= e^{\frac{1}{2}\theta\mathbf{N}}(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_i \wedge \cdots \wedge \mathbf{a}_k)e^{-\frac{1}{2}\theta\mathbf{N}} \\
&= R_{\mathbf{N}}^{\theta}\mathbf{a}_1 \wedge \cdots \wedge R_{\mathbf{N}}^{\theta}\mathbf{a}_i \wedge \cdots \wedge R_{\mathbf{N}}^{\theta}\mathbf{a}_k
\end{aligned}$$

where unit bivector $\mathbf{N} = \mathbf{n}/\mathbf{I}$ is the dual of a Euclidean unit vector rotation axis $\mathbf{n}$, or is some other unit bivector $\mathbf{N} = \frac{\mathbf{b} \wedge \mathbf{a}}{|\mathbf{b} \wedge \mathbf{a}|}$, representing a quaternion rotation axis or plane of rotation.

This result, a kind of outermorphism called *versor outermorphism*[9, pp. 92], is true since rotations preserve lengths and angles in rotated objects. Blade objects are rotated as rigid bodies. As an object, a blade represents a k-parallelotope, or a k-simplex if scaled by $(1/k!)$.

In the previous section, this formula was not assumed and we had

$$\mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{b} \wedge \mathbf{c})\mathbf{b}^{-1}\mathbf{I}$$

as a rotation of a 2-blade. This can be evaluated and simplified as

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \mathbf{b}(\mathbf{b} \wedge \mathbf{c})\mathbf{b}^{-1}\mathbf{I} \\
&= (\mathbf{b}\mathbf{b}\mathbf{b}^{-1}) \wedge (\mathbf{b}\mathbf{c}\mathbf{b}^{-1})\mathbf{I} \\
&= (\mathbf{b} \wedge [\mathbf{b}(\mathbf{c}^{||\mathbf{b}} + \mathbf{c}^{\perp\mathbf{b}})\mathbf{b}^{-1}])\mathbf{I} \\
&= (\mathbf{b} \wedge [\mathbf{c}^{||\mathbf{b}} - \mathbf{c}^{\perp\mathbf{b}}])\mathbf{I} \\
&= (-\mathbf{b} \wedge \mathbf{c}^{\perp\mathbf{b}})\mathbf{I} \\
&= (-\mathbf{b} \wedge \mathbf{c})\mathbf{I} \\
&= (\mathbf{b} \wedge \mathbf{c})/\mathbf{I}
\end{aligned}$$

giving the known formula for the vector cross product as the dual of the outer product.

2.45 Cramer's Rule and Reciprocal Bases

2.45.1 Systems of scalar equations

Consider the system of three scalar equations in three unknown scalars x_1, x_2, x_3

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3 \end{aligned}$$

and assign an orthonormal vector basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for $\mathcal{G}_3$ as

$$\begin{aligned} a_{11}\mathbf{e}_1x_1 + a_{12}\mathbf{e}_1x_2 + a_{13}\mathbf{e}_1x_3 &= c_1\mathbf{e}_1 \\ a_{21}\mathbf{e}_2x_1 + a_{22}\mathbf{e}_2x_2 + a_{23}\mathbf{e}_2x_3 &= c_2\mathbf{e}_2 \\ a_{31}\mathbf{e}_3x_1 + a_{32}\mathbf{e}_3x_2 + a_{33}\mathbf{e}_3x_3 &= c_3\mathbf{e}_3. \end{aligned}$$

Let the vectors

$$\begin{aligned} \mathbf{a}_1 &= a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 \\ \mathbf{a}_2 &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 \\ \mathbf{a}_3 &= a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3. \end{aligned}$$

Adding the system of equations, it is seen that

$$\begin{aligned} \mathbf{c} &= c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3. \end{aligned}$$

Using the exterior product, each unknown scalar x_k can be solved as

$$\begin{aligned} \mathbf{c}\wedge\mathbf{a}_2\wedge\mathbf{a}_3 &= x_1\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3 \\ \mathbf{c}\wedge\mathbf{a}_1\wedge\mathbf{a}_3 &= x_2\mathbf{a}_2\wedge\mathbf{a}_1\wedge\mathbf{a}_3 \\ \mathbf{c}\wedge\mathbf{a}_1\wedge\mathbf{a}_2 &= x_3\mathbf{a}_3\wedge\mathbf{a}_1\wedge\mathbf{a}_2 \\ x_1 &= \frac{\mathbf{c}\wedge\mathbf{a}_2\wedge\mathbf{a}_3}{\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3} \\ x_2 &= \frac{\mathbf{c}\wedge\mathbf{a}_1\wedge\mathbf{a}_3}{\mathbf{a}_2\wedge\mathbf{a}_1\wedge\mathbf{a}_3} = \frac{\mathbf{a}_1\wedge\mathbf{c}\wedge\mathbf{a}_3}{\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3} \\ x_3 &= \frac{\mathbf{c}\wedge\mathbf{a}_1\wedge\mathbf{a}_2}{\mathbf{a}_3\wedge\mathbf{a}_1\wedge\mathbf{a}_2} = \frac{\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{c}}{\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3}. \end{aligned}$$

For n equations in n unknowns, the solution for the kth unknown x_k generalizes to

$$
\begin{aligned}
x_k &= \frac{\mathbf{a}_1 \wedge \cdots \wedge (\mathbf{c})_k \wedge \cdots \wedge \mathbf{a}_n}{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n} \\
&= (\mathbf{a}_1 \wedge \cdots \wedge (\mathbf{c})_k \wedge \cdots \wedge \mathbf{a}_n)(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)^{-1} \\
&= \frac{(\mathbf{a}_1 \wedge \cdots \wedge (\mathbf{c})_k \wedge \cdots \wedge \mathbf{a}_n)(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)}{(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)} \\
&= \frac{(\mathbf{a}_1 \wedge \cdots \wedge (\mathbf{c})_k \wedge \cdots \wedge \mathbf{a}_n) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)}{(-1)^{n(n-1)/2}(\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)} \\
&= \frac{(\mathbf{a}_n \wedge \cdots \wedge (\mathbf{c})_k \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)}{(\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)}.
\end{aligned}
$$

If the $\mathbf{a}_k$ are linearly independent, then the x_k can be expressed in determinant form identical to Cramer's Rule as

$$
\begin{aligned}
x_k &= \frac{(\mathbf{a}_n \wedge \cdots \wedge (\mathbf{c})_k \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)}{(\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_n)} \\
&= \frac{\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_1\cdot(\mathbf{c})_k & \cdots & \mathbf{a}_1\cdot\mathbf{a}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_k\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_k\cdot(\mathbf{c})_k & \cdots & \mathbf{a}_k\cdot\mathbf{a}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_n\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_n\cdot(\mathbf{c})_k & \cdots & \mathbf{a}_n\cdot\mathbf{a}_n \end{vmatrix}}{\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_1\cdot\mathbf{a}_k & \cdots & \mathbf{a}_1\cdot\mathbf{a}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_k\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_k\cdot\mathbf{a}_k & \cdots & \mathbf{a}_k\cdot\mathbf{a}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_n\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_n\cdot\mathbf{a}_k & \cdots & \mathbf{a}_n\cdot\mathbf{a}_n \end{vmatrix}} = \frac{\begin{vmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_n \end{vmatrix} \begin{vmatrix} \mathbf{a}_1 & \cdots & (\mathbf{c})_k & \cdots & \mathbf{a}_n \end{vmatrix}}{\begin{vmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_n \end{vmatrix} \begin{vmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_k & \cdots & \mathbf{a}_n \end{vmatrix}} \\
&= \frac{\begin{vmatrix} \mathbf{a}_1 & \cdots & (\mathbf{c})_k & \cdots & \mathbf{a}_n \end{vmatrix}}{\begin{vmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_k & \cdots & \mathbf{a}_n \end{vmatrix}} = \frac{\begin{vmatrix} a_{11} & \dots & c_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & c_k & \cdots & a_{kn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & c_n & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \dots & a_{1k} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix}}
\end{aligned}
$$

where $(\mathbf{c})_k$ denotes the substitution of vector $\mathbf{a}_k$ with vector $\mathbf{c}$ in the kth numerator position.

2.45.2 Systems of vector equations

Consider the system of n vector equations in n unknown vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k, \ldots,$ $\mathbf{x}_n$

$$
\begin{aligned}
a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + a_{13}\mathbf{x}_3 + \cdots + a_{1k}\mathbf{x}_k + \cdots + a_{1n}\mathbf{x}_n &= \mathbf{c}_1 \\
a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + a_{23}\mathbf{x}_3 + \cdots + a_{2k}\mathbf{x}_k + \cdots + a_{2n}\mathbf{x}_n &= \mathbf{c}_2 \\
a_{31}\mathbf{x}_1 + a_{32}\mathbf{x}_2 + a_{33}\mathbf{x}_3 + \cdots + a_{3k}\mathbf{x}_k + \cdots + a_{3n}\mathbf{x}_n &= \mathbf{c}_3 \\
&\vdots \\
a_{k1}\mathbf{x}_1 + a_{k2}\mathbf{x}_2 + a_{k3}\mathbf{x}_3 + \cdots + a_{kk}\mathbf{x}_k + \cdots + a_{kn}\mathbf{x}_n &= \mathbf{c}_k \\
&\vdots \\
a_{n1}\mathbf{x}_1 + a_{n2}\mathbf{x}_2 + a_{n3}\mathbf{x}_3 + \cdots + a_{nk}\mathbf{x}_k + \cdots + a_{nn}\mathbf{x}_n &= \mathbf{c}_n
\end{aligned}
$$

where we want to solve for each unknown vector $\mathbf{x}_k$ in terms of the given scalar constants a_{rc} and vector constants $\mathbf{c}_k$.

2.45.3 Solving for unknown vectors.

Using the Clifford algebra (or geometric algebra) of Euclidean vectors, the vectors $\mathbf{x}_k$ and $\mathbf{c}_k$ are in a vector space having d dimensions spanned by a basis of d orthonormal base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, ..., $\mathbf{e}_d$. This d-dimensional space can be extended to be a subspace of a larger $(d+n)$-dimensional space $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_d, \ldots, \mathbf{e}_{d+k}, \ldots, \mathbf{e}_{d+n}$.

Multiply the kth equation by the $(d+k)$th orthonormal base unit $\mathbf{e}_{d+k}$, using the exterior product on the right, as

$$\begin{aligned}
(a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + a_{13}\mathbf{x}_3 + \cdots + a_{1k}\mathbf{x}_k + \cdots + a_{1n}\mathbf{x}_n) \wedge \mathbf{e}_{d+1} &= \mathbf{c}_1 \wedge \mathbf{e}_{d+1} \\
(a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + a_{23}\mathbf{x}_3 + \cdots + a_{2k}\mathbf{x}_k + \cdots + a_{2n}\mathbf{x}_n) \wedge \mathbf{e}_{d+2} &= \mathbf{c}_2 \wedge \mathbf{e}_{d+2} \\
(a_{31}\mathbf{x}_1 + a_{32}\mathbf{x}_2 + a_{33}\mathbf{x}_3 + \cdots + a_{3k}\mathbf{x}_k + \cdots + a_{3n}\mathbf{x}_n) \wedge \mathbf{e}_{d+3} &= \mathbf{c}_3 \wedge \mathbf{e}_{d+3} \\
&\vdots \\
(a_{k1}\mathbf{x}_1 + a_{k2}\mathbf{x}_2 + a_{k3}\mathbf{x}_3 + \cdots + a_{kk}\mathbf{x}_k + \cdots + a_{kn}\mathbf{x}_n) \wedge \mathbf{e}_{d+k} &= \mathbf{c}_k \wedge \mathbf{e}_{d+k} \\
&\vdots \\
(a_{n1}\mathbf{x}_1 + a_{n2}\mathbf{x}_2 + a_{n3}\mathbf{x}_3 + \cdots + a_{nk}\mathbf{x}_k + \cdots + a_{nn}\mathbf{x}_n) \wedge \mathbf{e}_{d+n} &= \mathbf{c}_n \wedge \mathbf{e}_{d+n}.
\end{aligned}$$

The original system of equations in grade-1 vectors is now transformed into a system of equations in grade-2 vectors, and no parallel components have been deleted by the exterior products since they multiply on perpendicular extended base units.

Let the vectors

$$\begin{aligned}
\mathbf{a}_1 &= a_{11}\mathbf{e}_{d+1} + a_{21}\mathbf{e}_{d+2} + a_{31}\mathbf{e}_{d+3} + \cdots + a_{k1}\mathbf{e}_{d+k} + \cdots + a_{n1}\mathbf{e}_{d+n} \\
\mathbf{a}_2 &= a_{12}\mathbf{e}_{d+1} + a_{22}\mathbf{e}_{d+2} + a_{32}\mathbf{e}_{d+3} + \cdots + a_{k2}\mathbf{e}_{d+k} + \cdots + a_{n2}\mathbf{e}_{d+n} \\
\mathbf{a}_3 &= a_{13}\mathbf{e}_{d+1} + a_{23}\mathbf{e}_{d+2} + a_{33}\mathbf{e}_{d+3} + \cdots + a_{k3}\mathbf{e}_{d+k} + \cdots + a_{n3}\mathbf{e}_{d+n} \\
&\vdots \\
\mathbf{a}_k &= a_{1k}\mathbf{e}_{d+1} + a_{2k}\mathbf{e}_{d+2} + a_{3k}\mathbf{e}_{d+3} + \cdots + a_{kk}\mathbf{e}_{d+k} + \cdots + a_{nk}\mathbf{e}_{d+n} \\
&\vdots \\
\mathbf{a}_n &= a_{1n}\mathbf{e}_{d+1} + a_{2n}\mathbf{e}_{d+2} + a_{3n}\mathbf{e}_{d+3} + \cdots + a_{kn}\mathbf{e}_{d+k} + \cdots + a_{nn}\mathbf{e}_{d+n}.
\end{aligned}$$

Adding the transformed system of equations gives

$$\begin{aligned}
\mathbf{C} &= \mathbf{c}_1 \wedge \mathbf{e}_{d+1} + \mathbf{c}_2 \wedge \mathbf{e}_{d+2} + \mathbf{c}_3 \wedge \mathbf{e}_{d+3} + \cdots + \mathbf{c}_k \wedge \mathbf{e}_{d+k} + \cdots + \mathbf{c}_n \wedge \mathbf{e}_{d+n} \\
&= \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 + \cdots + \mathbf{C}_k + \cdots + \mathbf{C}_n \\
&= \mathbf{x}_1 \wedge \mathbf{a}_1 + \mathbf{x}_2 \wedge \mathbf{a}_2 + \mathbf{x}_3 \wedge \mathbf{a}_3 + \cdots + \mathbf{x}_k \wedge \mathbf{a}_k + \cdots + \mathbf{x}_n \wedge \mathbf{a}_n
\end{aligned}$$

which is a 2-vector equation. These exterior (wedge) products are equal to Clifford products since the factors are perpendicular.

For $n=3$, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are solved by multiplying $\mathbf{a}_2\wedge\mathbf{a}_3$, $\mathbf{a}_1\wedge\mathbf{a}_3$, and $\mathbf{a}_1\wedge\mathbf{a}_2$, respectively, on the right with exterior products

$$
\begin{aligned}
\mathbf{C}\wedge\mathbf{a}_2\wedge\mathbf{a}_3 &= \mathbf{x}_1\wedge\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3=\mathbf{x}_1(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)\\
\mathbf{C}\wedge\mathbf{a}_1\wedge\mathbf{a}_3 &= \mathbf{x}_2\wedge\mathbf{a}_2\wedge\mathbf{a}_1\wedge\mathbf{a}_3=\mathbf{x}_2(\mathbf{a}_2\wedge\mathbf{a}_1\wedge\mathbf{a}_3)\\
\mathbf{C}\wedge\mathbf{a}_1\wedge\mathbf{a}_2 &= \mathbf{x}_3\wedge\mathbf{a}_3\wedge\mathbf{a}_1\wedge\mathbf{a}_2=\mathbf{x}_3(\mathbf{a}_3\wedge\mathbf{a}_1\wedge\mathbf{a}_2)\\
\mathbf{x}_1 &= (\mathbf{C}\wedge\mathbf{a}_2\wedge\mathbf{a}_3)(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)^{-1}\\
&= \frac{(\mathbf{C}\wedge\mathbf{a}_2\wedge\mathbf{a}_3)\cdot((-1)^{1-1}\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)}{(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)^2}\\
\mathbf{x}_2 &= (\mathbf{C}\wedge\mathbf{a}_1\wedge\mathbf{a}_3)(\mathbf{a}_2\wedge\mathbf{a}_1\wedge\mathbf{a}_3)^{-1}\\
&= \frac{(\mathbf{a}_1\wedge\mathbf{C}\wedge\mathbf{a}_3)\cdot((-1)^{2-1}\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)}{(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)^2}\\
\mathbf{x}_3 &= (\mathbf{C}\wedge\mathbf{a}_1\wedge\mathbf{a}_2)(\mathbf{a}_3\wedge\mathbf{a}_1\wedge\mathbf{a}_2)^{-1}\\
&= \frac{(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{C})\cdot((-1)^{3-1}\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)}{(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)^2}.
\end{aligned}
$$

In the solution of $\mathbf{x}_1$, and similarly for $\mathbf{x}_2$ and $\mathbf{x}_3$, $\mathbf{C}\wedge\mathbf{a}_2\wedge\mathbf{a}_3$ is a 4-blade having 3 of its 4 dimensions in the extended dimensions $\mathbf{e}_{d+k}$, and the remaining one dimension is in the solution space of the vectors $\mathbf{x}_k$ and $\mathbf{c}_k$. The 3-blade $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3$ is in the problem space, or the extended dimensions. The inner product $(\mathbf{C}\wedge\mathbf{a}_2\wedge\mathbf{a}_3)\cdot(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)$ reduces, or contracts, to a 1-vector in the d-dimensional solution space. The divisor $(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3)^2$, the square of a blade, is a scalar product that can be computed by a determinant. Since $\mathbf{C}$ is a 2-vector, it commutes $\mathbf{C}\wedge\mathbf{a}_k=\mathbf{a}_k\wedge\mathbf{C}$ with the vectors $\mathbf{a}_k$ without sign change and is conveniently shifted into the vacant kth spot. A sign change $(-1)^{k-1}$ occurs in every even(+) kth solution $\mathbf{x}_+$, such as $\mathbf{x}_2$, due to commuting or shifting $\mathbf{a}_k$ right an odd number of times, in the dividend blade $\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_n$, into its kth spot.

In general, $\mathbf{x}_k$ is solved as

$$
\begin{aligned}
\mathbf{x}_k &= (\mathbf{a}_1\wedge\cdots\wedge(\mathbf{C})_k\wedge\cdots\wedge\mathbf{a}_n)\cdot((-1)^{k-1}\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_n)^{-1}\\
&= \frac{(\mathbf{a}_1\wedge\cdots\wedge(\mathbf{C})_k\wedge\cdots\wedge\mathbf{a}_n)\cdot((-1)^{k-1}\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_n)}{(\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_n)^2}\\
&= \frac{(-1)^{k-1}(\mathbf{a}_1\wedge\cdots\wedge(\mathbf{C})_k\wedge\cdots\wedge\mathbf{a}_n)\cdot(\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_n)}{(-1)^{n(n-1)/2}(\mathbf{a}_n\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_1)\cdot(\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_n)}\\
&= \frac{(-1)^{k-1}(\mathbf{a}_1\wedge\cdots\wedge(\mathbf{C})_k\wedge\cdots\wedge\mathbf{a}_n)\cdot(\mathbf{a}_n\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_1)}{(\mathbf{a}_n\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_1)\cdot(\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_n)}\\
&= \frac{(\mathbf{a}_1\wedge\cdots\wedge(\mathbf{C})_k\wedge\cdots\wedge\mathbf{a}_n)\cdot(\mathbf{a}_n\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_1)}{(-1)^{k-1}\begin{vmatrix}\mathbf{a}_1\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_1\cdot\mathbf{a}_k & \cdots & \mathbf{a}_1\cdot\mathbf{a}_n\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \mathbf{a}_k\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_k\cdot\mathbf{a}_k & \cdots & \mathbf{a}_k\cdot\mathbf{a}_n\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \mathbf{a}_n\cdot\mathbf{a}_1 & \cdots & \mathbf{a}_n\cdot\mathbf{a}_k & \cdots & \mathbf{a}_n\cdot\mathbf{a}_n\end{vmatrix}}\\
&= \frac{(\mathbf{a}_1\wedge\cdots\wedge(\mathbf{C})_k\wedge\cdots\wedge\mathbf{a}_n)\cdot(\mathbf{a}_n\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_1)}{(-1)^{k-1}\left|\,\mathbf{a}_1\ \cdots\ \mathbf{a}_k\ \cdots\ \mathbf{a}_n\,\right|^2}
\end{aligned}
$$

where $(\mathbf{C})_k$ denotes replacing the kth element $\mathbf{a}_k$ with $\mathbf{C}$. The factor $(-1)^{k-1}$ accounts for shifting the kth vector $\mathbf{a}_k$ by $k-1$ places. The $(n+1)$-blade $\mathbf{a}_1\wedge\cdots\wedge(\mathbf{C})_k\wedge\cdots\wedge\mathbf{a}_n$ is multiplied by inner product with the reversed n-blade $\mathbf{a}_n\wedge\cdots\wedge\mathbf{a}_k\wedge\cdots\wedge\mathbf{a}_1$, producing a 1-vector in the d-dimensional solution space.

Using this formula, for solving a system of n vector equations having n unknown vectors $\mathbf{x}_1$, ..., $\mathbf{x}_k$, ..., $\mathbf{x}_n$ in a d-dimensional space, requires extending the space to $(d + n)$ dimensions. The extended n dimensions are essentially used to hold the system of n equations represented by the scalar constants 1-vectors $\mathbf{a}_k$ and the vector constants 1-vectors $\mathbf{c}_k$. The n vector constants $\mathbf{c}_k$ are grade-increased to 2-vectors or grade-2 vectors $\mathbf{c}_k \wedge \mathbf{e}_{d+k} = \mathbf{C}_k$ that are partly in the extended space. Notice the similarity of form to Cramer's Rule for systems of scalar equations; a basis is added in both cases. The advantage of this formula is that it avoids scalar coordinates and the results are directly in terms of vectors.

The system of vector equations can also be solved in terms of coordinates, without using the geometric algebra formula above, by the usual process of expanding all the vectors in the system into their coordinate vector components. In each expanded equation, the parallel (like) components are summed into d groups that form d independent systems of n unknown coordinates in n equations. Each system solves for one dimension of coordinates. After solving the d systems, the solved vectors can be reassembled from the solved coordinates. It seems that few books explicitly discuss this process for systems of vector equations. This process is the application of the abstract concept of linear independence as it applies to linearly independent dimensions of vector components or unit vectors. The linear independence concept extends to multivectors in geometric algebra, where each unique unit blade is linearly independent of the others for the purpose of solving equations or systems of equations. An equation containing a sum of d linearly independent terms can be rewritten as d separate independent equations, each in the terms of one dimension.

2.45.4 Solving for unknown scalars

It is also noticed that, instead of solving for unknown vectors $\mathbf{x}_k$, the $\mathbf{x}_k$ *may* be known vectors and the vectors $\mathbf{a}_k$ *may* be unknown. The vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ *could* be solved as

$$\begin{aligned}
-\mathbf{C} \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 &= \mathbf{a}_1 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \mathbf{a}_1(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3) \\
-\mathbf{C} \wedge \mathbf{x}_1 \wedge \mathbf{x}_3 &= \mathbf{a}_2 \wedge \mathbf{x}_2 \wedge \mathbf{x}_1 \wedge \mathbf{x}_3 = \mathbf{a}_2(\mathbf{x}_2 \wedge \mathbf{x}_1 \wedge \mathbf{x}_3) \\
-\mathbf{C} \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 &= \mathbf{a}_3 \wedge \mathbf{x}_3 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 = \mathbf{a}_3(\mathbf{x}_3 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2) \\
\mathbf{a}_1 &= (-\mathbf{C} \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)^{-1} \\
&= \frac{(-\mathbf{C} \wedge \mathbf{x}_2 \wedge \mathbf{x}_3) \cdot ((-1)^{1-1}\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)}{(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)^2} \\
\mathbf{a}_2 &= (-\mathbf{C} \wedge \mathbf{x}_1 \wedge \mathbf{x}_3)(\mathbf{x}_2 \wedge \mathbf{x}_1 \wedge \mathbf{x}_3)^{-1} \\
&= \frac{(-\mathbf{x}_1 \wedge \mathbf{C} \wedge \mathbf{x}_3) \cdot ((-1)^{2-1}\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)}{(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)^2} \\
\mathbf{a}_3 &= (-\mathbf{C} \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)(\mathbf{x}_3 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)^{-1} \\
&= \frac{(-\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{C}) \cdot ((-1)^{3-1}\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)}{(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3)^2}.
\end{aligned}$$

In general, vector $\mathbf{a}_k$ *may* be solved as

$$\begin{aligned}
\mathbf{a}_k &= (-\mathbf{x}_1 \wedge \cdots \wedge (\mathbf{C})_k \wedge \cdots \wedge \mathbf{x}_n) \cdot ((-1)^{k-1}\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n)^{-1} \\
&= \frac{(-\mathbf{x}_1 \wedge \cdots \wedge (\mathbf{C})_k \wedge \cdots \wedge \mathbf{x}_n) \cdot ((-1)^{k-1}\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n)}{(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n)^2} \\
&= \frac{(-1)^k(\mathbf{x}_1 \wedge \cdots \wedge (\mathbf{C})_k \wedge \cdots \wedge \mathbf{x}_n) \cdot (\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n)}{(-1)^{n(n-1)/2}(\mathbf{x}_n \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_1) \cdot (\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n)} \\
&= \frac{(-1)^k(\mathbf{x}_1 \wedge \cdots \wedge (\mathbf{C})_k \wedge \cdots \wedge \mathbf{x}_n) \cdot (\mathbf{x}_n \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_1)}{(\mathbf{x}_n \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_1) \cdot (\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n)} \\
&= \frac{(\mathbf{x}_1 \wedge \cdots \wedge (\mathbf{C})_k \wedge \cdots \wedge \mathbf{x}_n) \cdot (\mathbf{x}_n \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_1)}{(-1)^k \begin{vmatrix} \mathbf{x}_1\cdot\mathbf{x}_1 & \cdots & \mathbf{x}_1\cdot\mathbf{x}_k & \cdots & \mathbf{x}_1\cdot\mathbf{x}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{x}_k\cdot\mathbf{x}_1 & \cdots & \mathbf{x}_k\cdot\mathbf{x}_k & \cdots & \mathbf{x}_k\cdot\mathbf{x}_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{x}_n\cdot\mathbf{x}_1 & \cdots & \mathbf{x}_n\cdot\mathbf{x}_k & \cdots & \mathbf{x}_n\cdot\mathbf{x}_n \end{vmatrix}} \\
&= \frac{(\mathbf{x}_1 \wedge \cdots \wedge (\mathbf{C})_k \wedge \cdots \wedge \mathbf{x}_n) \cdot (\mathbf{x}_n \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_1)}{(-1)^k \left| \mathbf{x}_1 \ \cdots \ \mathbf{x}_k \ \cdots \ \mathbf{x}_n \right|^2}
\end{aligned}$$

and represents transforming or projecting the system, or each vector $\mathbf{c}_k$, onto the basis of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n$ which need not be orthonormal. *However*, solving for the vectors $\mathbf{a}_k$ by this formula is *unnecessary*, and unnecessarily requires n vectors $\mathbf{c}_1, \ldots, \mathbf{c}_k, \ldots, \mathbf{c}_n$ at a time. Solving each equation is independent in this case. This has been shown to clarify the usage, as far as what not to do, unless one has an unusual need to solve a particular vector $\mathbf{a}_k$. Instead, the following can be done in the case of projecting vectors $\mathbf{c}_k$ onto a new arbitrary basis $\mathbf{x}_k$.

2.45.5 Projecting a vector onto an arbitrary basis

Projecting any vector $\mathbf{c}$ onto a new arbitrary basis $\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n$ as

$$\begin{aligned}
\mathbf{c} &= c_1\mathbf{e}_1 + \cdots + c_k\mathbf{e}_k + \cdots + c_n\mathbf{e}_n \\
&= a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + \cdots + a_n\mathbf{x}_n
\end{aligned}$$

where each $\mathbf{x}_k$ is written in the form

$$\mathbf{x}_k = x_{k1}\mathbf{e}_1 + x_{k2}\mathbf{e}_2 + \cdots + x_{kk}\mathbf{e}_k + \cdots + x_{kn}\mathbf{e}_n$$

is a system of n scalar equations in n unknown coordinates a_k

$$\begin{aligned}
a_1x_{11} + \cdots + a_kx_{k1} + \cdots + a_nx_{n1} &= c_1 \\
&\vdots \\
a_1x_{1k} + \cdots + a_kx_{kk} + \cdots + a_nx_{nk} &= c_k \\
&\vdots \\
a_1x_{1n} + \cdots + a_kx_{kn} + \cdots + a_nx_{nn} &= c_n
\end{aligned}$$

and can be solved using the ordinary Cramer's rule for systems of scalar equations, where the step of adding a basis can be considered as already done. For $n=3$, the solutions for the scalars a_k are

$$\begin{aligned}
\mathbf{c}\wedge\mathbf{x}_2\wedge\mathbf{x}_3 &= a_1\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3\\
\mathbf{c}\wedge\mathbf{x}_1\wedge\mathbf{x}_3 &= a_2\mathbf{x}_2\wedge\mathbf{x}_1\wedge\mathbf{x}_3\\
\mathbf{c}\wedge\mathbf{x}_1\wedge\mathbf{x}_2 &= a_3\mathbf{x}_3\wedge\mathbf{x}_1\wedge\mathbf{x}_2\\
a_1 &= \frac{\mathbf{c}\wedge\mathbf{x}_2\wedge\mathbf{x}_3}{\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3}\\
a_2 &= \frac{\mathbf{c}\wedge\mathbf{x}_1\wedge\mathbf{x}_3}{\mathbf{x}_2\wedge\mathbf{x}_1\wedge\mathbf{x}_3}=\frac{\mathbf{x}_1\wedge\mathbf{c}\wedge\mathbf{x}_3}{\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3}\\
a_3 &= \frac{\mathbf{c}\wedge\mathbf{x}_1\wedge\mathbf{x}_2}{\mathbf{x}_3\wedge\mathbf{x}_1\wedge\mathbf{x}_2}=\frac{\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{c}}{\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3}.
\end{aligned}$$

For n basis vectors (n equations in n unknowns), the solution for the kth unknown scalar coordinate a_k generalizes to

$$\begin{aligned}
a_k &= \frac{\mathbf{x}_1\wedge\cdots\wedge(\mathbf{c})_k\wedge\cdots\wedge\mathbf{x}_n}{\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n}\\
&= \frac{\begin{vmatrix} x_{11} & \dots & c_1 & \cdots & x_{n1}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_{1k} & \cdots & c_k & \cdots & x_{nk}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_{1n} & \cdots & c_n & \cdots & x_{nn}\end{vmatrix}}{\begin{vmatrix} x_{11} & \dots & x_{k1} & \cdots & x_{n1}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_{1k} & \cdots & x_{kk} & \cdots & x_{nk}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_{1n} & \cdots & x_{kn} & \cdots & x_{nn}\end{vmatrix}}
\end{aligned}$$

the formula for Cramer's rule.

The remainder of this subsection outlines some additional concepts or applications that may be important to consider when using arbitrary bases, but otherwise you may skip ahead to the next subsection.

2.45.6 Reciprocal Bases

The *reciprocal basis* $\mathbf{x}'_1, \dots, \mathbf{x}'_k, \dots, \mathbf{x}'_n$ of the arbitrary basis $\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_n$ is such that $\mathbf{c}\cdot\mathbf{x}'_k = a_k$, while in general $\mathbf{c}\cdot\mathbf{x}_k \neq a_k$. The kth reciprocal base $\mathbf{x}'_k$ is

$$\begin{aligned}
\mathbf{c}\cdot\mathbf{x}'_k = a_k &= (-1)^{k-1}(\mathbf{c}\wedge\mathbf{x}_1\wedge\cdots\wedge()_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1}\\
&= (-1)^{k-1}\mathbf{c}\cdot((\mathbf{x}_1\wedge\cdots\wedge()_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1})\\
\mathbf{x}'_k &= (-1)^{k-1}(\mathbf{x}_1\wedge\cdots\wedge()_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1}
\end{aligned}$$

where $()_k$ denotes that the kth vector $\mathbf{x}_k$ is removed from the blade. In mathematics literature, the reciprocal basis $\mathbf{x}'_1, \dots, \mathbf{x}'_k, \dots, \mathbf{x}'_n$ is usually written using superscript indices as $\mathbf{x}^1, \dots, \mathbf{x}^k, \dots, \mathbf{x}^n$ which should not be confused as exponents or powers of the vectors. The reciprocal bases can be computed once and saved, and then any vector $\mathbf{c}$ can be projected onto the arbitrary basis as $\mathbf{c} = (\mathbf{c}\cdot\mathbf{x}^k)\mathbf{x}_k$ with Einstein notation, or implied summation, over the range of $k \in \{1, \cdots, n\}$.

Note that

$$\begin{aligned}
\mathbf{x}_k\cdot\mathbf{x}^k &= (-1)^{k-1}\mathbf{x}_k\cdot((\mathbf{x}_1\wedge\cdots\wedge()_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1})\\
&= (-1)^{k-1}(\mathbf{x}_k\wedge\mathbf{x}_1\wedge\cdots\wedge()_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1}\\
&= (\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1}=1=\mathbf{x}^k\cdot\mathbf{x}_k\\
\mathbf{x}^k\cdot\mathbf{x}_k &= (-1)^{k-1}\mathbf{x}^k\cdot((\mathbf{x}^1\wedge\cdots\wedge()^k\wedge\cdots\wedge\mathbf{x}^n)\cdot(\mathbf{x}^1\wedge\cdots\wedge\mathbf{x}^k\wedge\cdots\wedge\mathbf{x}^n)^{-1})
\end{aligned}$$

and that for $j\neq k$

$$\begin{aligned}
\mathbf{x}_j\cdot\mathbf{x}^k &= (-1)^{k-1}\mathbf{x}_j\cdot((\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_j\wedge\cdots\wedge()_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1})\\
&= (-1)^{k-1}(\mathbf{x}_j\wedge\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_j\wedge\cdots\wedge()_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1}\\
&= (\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_j\wedge\cdots\wedge(\mathbf{x}_j)_k\wedge\cdots\wedge\mathbf{x}_n)\cdot(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k\wedge\cdots\wedge\mathbf{x}_n)^{-1}=0=\mathbf{x}^k\cdot\mathbf{x}_j\\
\mathbf{x}^k\cdot\mathbf{x}_j &= (-1)^{j-1}\mathbf{x}^k\cdot((\mathbf{x}^1\wedge\cdots\wedge\mathbf{x}^k\wedge\cdots\wedge()^j\wedge\cdots\wedge\mathbf{x}^n)\cdot(\mathbf{x}^1\wedge\cdots\wedge\mathbf{x}^k\wedge\cdots\wedge\mathbf{x}^n)^{-1})
\end{aligned}$$

therefore if the $\mathbf{x}^k$ are the new arbitrary bases, then the $\mathbf{x}_k$ are the reciprocal bases and we also have

$$\mathbf{c} = (\mathbf{c}\cdot\mathbf{x}_k)\mathbf{x}^k$$

with the summation convention over k.

If we abandon the old basis $\mathbf{e}_k$ and old coordinates c_k and a_k of $\mathbf{c}$ and refer $\mathbf{c}$ only to the new basis $\mathbf{x}_k$ and its reciprocal $\mathbf{x}^k$, then we can newly *rename coordinates* for $\mathbf{c}$ on the new bases as

$$\begin{aligned}
\mathbf{c} &= (\mathbf{c}\cdot\mathbf{x}^k)\mathbf{x}_k=c^k\mathbf{x}_k\\
\mathbf{c} &= (\mathbf{c}\cdot\mathbf{x}_k)\mathbf{x}^k=c_k\mathbf{x}^k.
\end{aligned}$$

This is a coordinates naming convention that is often used implicitly such that $c^k=\mathbf{c}\cdot\mathbf{x}^k$ and $c_k=\mathbf{c}\cdot\mathbf{x}_k$ are understood as identities. Using this coordinates naming convention we can derive the expression

$$\mathbf{c}\cdot\mathbf{c} = c^k\mathbf{x}_k\cdot c_j\mathbf{x}^j=c^kc_j\mathbf{x}_k\cdot\mathbf{x}^j.$$

Since $\mathbf{x}_k\cdot\mathbf{x}^j=1$ for $j=k$ and $\mathbf{x}_k\cdot\mathbf{x}^j=0$ for $j\neq k$ (or $\mathbf{x}_k\cdot\mathbf{x}^j=\delta_k^j$ using Kronecker delta), this expression reduces to the identity

$$\mathbf{c}\cdot\mathbf{c} = c^kc_k=(\mathbf{c}\cdot\mathbf{x}^k)(\mathbf{c}\cdot\mathbf{x}_k).$$

Since $\mathbf{c}$ is an arbitrary vector, we can choose any two vectors $\mathbf{u}$ and $\mathbf{v}$ and find the identities

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= u^kv_k=u_kv^k\\
&= (\mathbf{u}\cdot\mathbf{x}^k)(\mathbf{v}\cdot\mathbf{x}_k)=(\mathbf{u}\cdot\mathbf{x}_k)(\mathbf{v}\cdot\mathbf{x}^k).
\end{aligned}$$

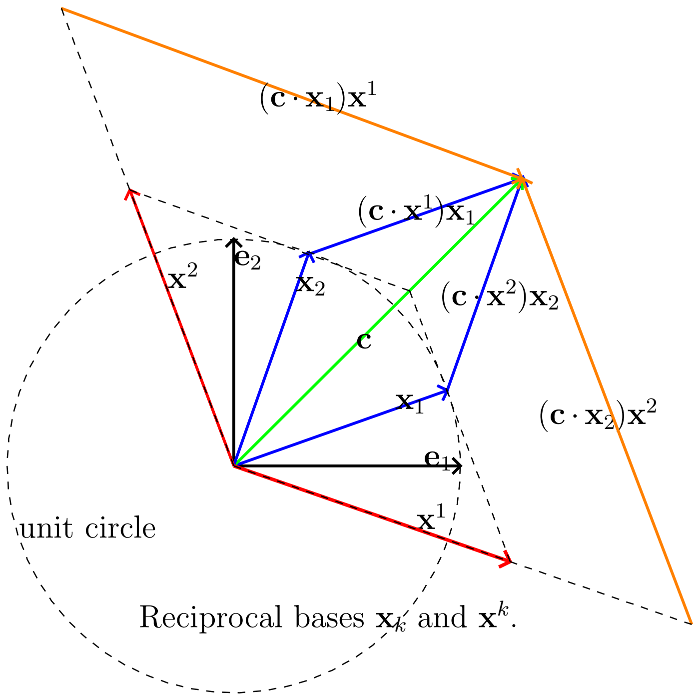


Figure 2.2. Reciprocal bases

2.45.7 Tensors

In terms of a basis $\mathbf{x}_k$ and its reciprocal basis $\mathbf{x}^k$, the inner or dot product $\mathbf{u}\cdot\mathbf{v}$ of two vectors can be written four ways

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= [(\mathbf{u}\cdot\mathbf{x}^j)\mathbf{x}_j]\cdot[(\mathbf{v}\cdot\mathbf{x}^k)\mathbf{x}_k] = u^j v^k \mathbf{x}_j\cdot\mathbf{x}_k = u^j v^k m_{jk}\\
&= [(\mathbf{u}\cdot\mathbf{x}_j)\mathbf{x}^j]\cdot[(\mathbf{v}\cdot\mathbf{x}_k)\mathbf{x}^k] = u_j v_k \mathbf{x}^j\cdot\mathbf{x}^k = u_j v_k m^{jk}\\
&= [(\mathbf{u}\cdot\mathbf{x}^j)\mathbf{x}_j]\cdot[(\mathbf{v}\cdot\mathbf{x}_k)\mathbf{x}^k] = u^j v_k m_j^k = u^j v_k \delta_j^k = u^k v_k\\
&= [(\mathbf{u}\cdot\mathbf{x}_j)\mathbf{x}^j]\cdot[(\mathbf{v}\cdot\mathbf{x}^k)\mathbf{x}_k] = u_j v^k m_k^j = u_j v^k \delta_k^j = u_k v^k.
\end{aligned}$$

In the language of tensors, m is called the metric tensor of the basis, δ is the Kronecker delta, an upper-indexed (superscripted) element is called contravariant, and a lower-indexed (subscripted) element is called covariant. Equating right-hand sides, we obtain the tensor contractions that are equivalent to the dot product

$$\begin{aligned}
u^j v^k m_{jk} &= u_k v^k = u^j v_j = \mathbf{u}\cdot\mathbf{v}\\
u_j v_k m^{jk} &= u_j v^j = u^k v_k = \mathbf{u}\cdot\mathbf{v}
\end{aligned}$$

where in the first equation either $u^j m_{jk} = u_k$ or $v^k m_{jk} = v_j$ (index-lowering contractions), and in the second equation either $u_j m^{jk} = u^k$ or $v_k m^{jk} = v^j$ (index-raising contractions). The contraction that lowers the index on u^j into u_k expands to the sum

$$\begin{aligned}
u^j m_{jk} &= u^1\mathbf{x}_1\cdot\mathbf{x}_k + u^2\mathbf{x}_2\cdot\mathbf{x}_k + \cdots + u^n\mathbf{x}_n\cdot\mathbf{x}_k\\
&= (u^1\mathbf{x}_1 + u^2\mathbf{x}_2 + \cdots + u^n\mathbf{x}_n)\cdot\mathbf{x}_k\\
&= (u^j\mathbf{x}_j)\cdot\mathbf{x}_k = \mathbf{u}\cdot\mathbf{x}_k = u_k.
\end{aligned}$$

Contractions are a form of inner product. Contractions such as these

$$\begin{aligned} u_k &= \mathbf{u}\cdot\mathbf{x}_k = u_j\mathbf{x}^j\cdot\mathbf{x}_k = u_j m^j_k = u_j\delta^j_k \\ u^k &= \mathbf{u}\cdot\mathbf{x}^k = u^j\mathbf{x}_j\cdot\mathbf{x}^k = u^j m^k_j = u^j\delta^k_j \end{aligned}$$

are called index renaming. Contractions involving m and δ have many relations such as

$$\begin{aligned} m_{1k}m^{1k} &= (\mathbf{x}_1\cdot\mathbf{x}_k)(\mathbf{x}^1\cdot\mathbf{x}^k) = (x_1)_k(x^1)^k = \mathbf{x}_1\cdot\mathbf{x}^1 = 1 \\ m_{jk}m^{jk} &= n = m^j_j = m^k_k = \delta^j_j = \delta^k_k \\ m^i_j m_{ik} &= (\mathbf{x}^i\cdot\mathbf{x}_j)(\mathbf{x}_i\cdot\mathbf{x}_k) = (x_j)^i(x_k)_i = \mathbf{x}_j\cdot\mathbf{x}_k = m_{jk} \\ m^j_i m^{ik} &= (\mathbf{x}^j\cdot\mathbf{x}_i)(\mathbf{x}^i\cdot\mathbf{x}^k) = (x^j)_i(x^k)^i = \mathbf{x}^j\cdot\mathbf{x}^k = m^{jk}. \end{aligned}$$

When viewed as $n\times n$ matrices, m_{jk} and m^{jk} are inverse matrices. The matrices m are symmetric, so the indices can be reversed. The contraction that computes the matrix product is

$$\begin{aligned} m^{ji}m_{ik} &= (\mathbf{x}^j\cdot\mathbf{x}^i)(\mathbf{x}_i\cdot\mathbf{x}_k) = (x^j)^i(x_k)_i = \mathbf{x}^j\cdot\mathbf{x}_k = m^j_k = \delta^j_k \\ [m^{jk}] &= [m_{jk}]^{-1}. \end{aligned}$$

The Kronecker delta δ^j_k, viewed as a matrix, is the identity matrix. From this matrix product identity, the reciprocal bases $\mathbf{x}^j$ can be computed as

$$\begin{aligned} m^{ji}\mathbf{x}_i\cdot\mathbf{x}_k &= \mathbf{x}^j\cdot\mathbf{x}_k \\ m^{ji}\mathbf{x}_i &= \mathbf{x}^j = (\mathbf{x}^j\cdot\mathbf{x}^i)\mathbf{x}_i = (x^j)^i\mathbf{x}_i. \end{aligned}$$

The formula $\mathbf{u}\cdot\mathbf{v} = u_iv^i = u^iv_i$ for the inner or dot product of vectors requires the terms to be products of covariant and contravariant component pairs. One of the vectors has to be expressed in terms of the reciprocal basis relative to the basis of the other vector. This requirement is satisfied when expressing vectors on an orthonormal basis that is self-reciprocal, but must be paid proper attention otherwise. The formula is often written $\mathbf{u}\cdot\mathbf{v} = \Sigma u_iv_i$, but this is valid only if the vectors are both expressed on the same orthonormal basis $\mathbf{e}^k = \mathbf{e}_k$ with $\mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk}$.

2.45.8 Vector Derivative Operators

The derivative operator ∇ called del is often written as

$$\nabla = \sum_{i=1}^{n} \mathbf{e}_i\frac{\partial}{\partial x_i} = \mathbf{e}_i\frac{\partial}{\partial x_i}$$

where the $\mathbf{e}_i$ are an orthonormal standard basis with vectors written in the Cartesian form $\mathbf{x} = x_j\mathbf{e}_j$. Del ∇ can be treated as a vector in computations. It can also be written as

$$\nabla = \mathbf{x}^i\frac{\partial}{\partial r^i} = \mathbf{x}_i\frac{\partial}{\partial r_i}$$

for a basis $\mathbf{x}_i$ and reciprocal basis $\mathbf{x}^i$, and position vector $\mathbf{r} = r^j \mathbf{x}_j = r_j \mathbf{x}^j$ written in the tensor forms. For example, the divergence of $\mathbf{r}$ can be computed several ways as

$$\begin{aligned}
\nabla \cdot \mathbf{r} &= \frac{\partial(\mathbf{x}^i \cdot \mathbf{r})}{\partial r^i} = \frac{\partial r^i}{\partial r^i} = \delta_i^i = n \\
\nabla \cdot \mathbf{r} &= \frac{\partial(\mathbf{x}_i \cdot \mathbf{r})}{\partial r_i} = \frac{\partial r_i}{\partial r_i} = \delta_i^i = n \\
\nabla \cdot \mathbf{r} &= \frac{\partial}{\partial r_i} \mathbf{x}_i \cdot (r^j \mathbf{x}_j) = \frac{\partial}{\partial r_i} r^j m_{ij} = \frac{\partial r_i}{\partial r_i} = \delta_i^i = n \\
\nabla \cdot \mathbf{r} &= \frac{\partial}{\partial r_i} \mathbf{x}_i \cdot (r_j \mathbf{x}^j) = \frac{\partial}{\partial r_i} r_j m_i^j = \frac{\partial r_i}{\partial r_i} = \delta_i^i = n \\
\nabla \cdot \mathbf{r} &= \frac{\partial}{\partial r^i} \mathbf{x}^i \cdot (r^j \mathbf{x}_j) = \frac{\partial}{\partial r^i} r^j m_j^i = \frac{\partial r^i}{\partial r^i} = \delta_i^i = n \\
\nabla \cdot \mathbf{r} &= \frac{\partial}{\partial r^i} \mathbf{x}^i \cdot (r_j \mathbf{x}^j) = \frac{\partial}{\partial r^i} r_j m^{ij} = \frac{\partial r^i}{\partial r^i} = \delta_i^i = n.
\end{aligned}$$

The derivative operator ∇ can be applied further in this way as a vector, where

$$\nabla \mathbf{r} = \nabla \cdot \mathbf{r} + \nabla \wedge \mathbf{r}$$

in geometric calculus for vectors in any number of dimensions n, and

$$\nabla \mathbf{r} = -\nabla \cdot \mathbf{r} + \nabla \times \mathbf{r}$$

in quaternions or vector analysis in three dimensions spanned by the orthonormal quaternion vector units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

In $n = 3$ dimensions, the product $\nabla \cdot \mathbf{r}$ is known as divergence, and the product $(\nabla \wedge \mathbf{r}) / \mathbf{I}_3 = \nabla \times \mathbf{r}$ is known as curl. The value $\mathbf{I}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ is the pseudoscalar of the Clifford algebra. Dividing the bivector $\nabla \wedge \mathbf{r}$ by the pseudoscalar $\mathbf{I}_3$ produces its spatial *dual* in the orthogonal vector space with the same magnitude, and oriented with sign in the expected direction for the curl vector. For a scalar field f, the product ∇f is known as the gradient vector, which generalizes the scalar-valued derivative of a single-variable function to a vector-valued derivative of a multi-variable function f.

2.45.9 Curvilinear Coordinate Systems

In the rectilinear coordinates system (or affine or oblique coordinate system) that has been considered so far, the metric tensor m has been a constant matrix containing constant ratios that relate to the amount of shearing that occurs in transforming from one rectilinear system to another. In a curvilinear coordinates system, the metric tensor m may be variable and varies with the position vector $\mathbf{r}$. The local frame or basis $\mathbf{x}_i$ at $\mathbf{r}$ can be defined as

$$\mathbf{x}_i = \frac{\partial \mathbf{r}}{\partial s^i} = \frac{\partial r^k}{\partial s^i} \mathbf{e}_k$$

where the position vector $\mathbf{r} = r^k \mathbf{e}_k$. It can be assumed that $\mathbf{e}_k$ is a standard basis. Each r^k is a function of the variables s^i, and each s^i is at least an implicit function of the variables r^k such that the transformation is invertible. The basis $\mathbf{x}_i$ is a frame local to each position of $\mathbf{r}$ in space, and may vary with position. The covariant metric tensor is

$$m_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j = \frac{\partial r^k \mathbf{e}_k}{\partial s^i} \cdot \frac{\partial r^l \mathbf{e}_l}{\partial s^j} = \frac{\partial r^k}{\partial s^i} \frac{\partial r^l}{\partial s^j} \delta_{kl} = \sum_k \frac{\partial r^k}{\partial s^i} \frac{\partial r^k}{\partial s^j}$$

and in terms of the Jacobian matrix $\mathbf{J}$, m_{ij} is expressed as the matrix

$$\begin{aligned} J_{ki} &= \frac{\partial r^k}{\partial s^i} \\ \mathbf{J} &= [J_{ki}] \\ [m_{ij}] &= \left[\sum_k J_{ki} J_{kj} \right] = \mathbf{J}^{\mathrm{T}} \mathbf{J}. \end{aligned}$$

The contravariant metric tensor m^{ij} is again the matrix inverse of the covariant metric tensor

$$[m^{ij}] = [m_{ij}]^{-1}$$

and the contravariant or reciprocal basis is

$$\mathbf{x}^i = m^{ij} \mathbf{x}_j.$$

In a cylindrical coordinate system or spherical coordinate system, m_{ij} is a diagonal matrix and m^{ij} is easily found as the matrix with each element inverted.

2.45.10 Projecting a vector onto an orthogonal basis.

Projections onto arbitrary bases $\mathbf{x}_k$, as solved using Cramer's rule as above, treats projections onto orthogonal bases as only a special case. Projections onto mutually *orthogonal* bases can be achieved using the ordinary projection operation

$$a_k = \mathbf{c} \cdot \frac{\mathbf{x}_k}{|\mathbf{x}_k|^2} = \frac{\mathbf{c} \cdot \mathbf{x}_k}{\mathbf{x}_k \cdot \mathbf{x}_k} = \frac{\mathbf{c}^{||\mathbf{x}_k} \cdot \mathbf{x}_k}{\mathbf{x}_k \cdot \mathbf{x}_k}$$

which is correct only if the $\mathbf{x}_k$ are mutually orthogonal. *If* the bases $\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n$ are constrained to be mutually perpendicular (orthogonal), then the formula for Cramer's rule becomes

$$\begin{aligned} a_k &= \frac{\mathbf{x}_1 \wedge \cdots \wedge (\mathbf{c}^{||\mathbf{x}_k} + \mathbf{c}^{\perp \mathbf{x}_k})_k \wedge \cdots \wedge \mathbf{x}_n}{\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n} \\ &= \frac{\mathbf{x}_1 \cdots (\mathbf{c}^{||\mathbf{x}_k})_k \cdots \mathbf{x}_n}{\mathbf{x}_1 \cdots \mathbf{x}_k \cdots \mathbf{x}_n} = \frac{\mathbf{c}^{||\mathbf{x}_k} \mathbf{x}_1 \cdots ()_k \cdots \mathbf{x}_n}{\mathbf{x}_k \mathbf{x}_1 \cdots ()_k \cdots \mathbf{x}_n} \\ &= \frac{\mathbf{c}^{||\mathbf{x}_k}}{\mathbf{x}_k} = \frac{\mathbf{c}^{||\mathbf{x}_k} \mathbf{x}_k}{\mathbf{x}_k \mathbf{x}_k} = \frac{\mathbf{c} \cdot \mathbf{x}_k}{\mathbf{x}_k \cdot \mathbf{x}_k} \end{aligned}$$

where $\mathbf{c}$ has been written as a sum of vector components parallel and perpendicular to $\mathbf{x}_k$. For any two perpendicular vectors $\mathbf{x}_j$,$\mathbf{x}_k$, their exterior product $\mathbf{x}_j \wedge \mathbf{x}_k = \mathbf{x}_j \mathbf{x}_k$ equals their Clifford product. The vector component $\mathbf{c}^{\perp \mathbf{x}_k}$ must be parallel to the other $\mathbf{x}_{j \neq k}$, therefore its outermorphism is zero. The result is Cramer's rule reduced to orthogonal projection of vector $\mathbf{c}$ onto base $\mathbf{x}_k$ such that $\mathbf{c}^{||\mathbf{x}_k} = a_k \mathbf{x}_k$.

In general, the bases $\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n$ are not necessarily mutually orthogonal and the projection to use is Cramer's rule, generalized projection, not the dot product specific to orthogonal projection.

An orthonormal basis is identical to its reciprocal basis since

$$\begin{aligned}
\mathbf{x}^k &= (-1)^{k-1}(\mathbf{x}_1 \wedge \cdots \wedge ()_k \wedge \cdots \wedge \mathbf{x}_n) \cdot (\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \wedge \cdots \wedge \mathbf{x}_n)^{-1} \\
&= (-1)^{k-1}(\mathbf{x}_1 \cdots ()_k \cdots \mathbf{x}_n)(\mathbf{x}_n \cdots ()_k \cdots \mathbf{x}_1)\mathbf{x}_k(-1)^{k-1} \\
&= \mathbf{x}_k
\end{aligned}$$

and $\mathbf{c} = (\mathbf{c} \cdot \mathbf{x}^k)\mathbf{x}_k$ with implied summation over the range of $k \in \{1, \cdots, n\}$. For an orthogonal basis, each reciprocal base is already shown to be

$$\mathbf{x}^k = \frac{\mathbf{x}_k}{|\mathbf{x}_k|^2} = \frac{\mathbf{x}_k}{\mathbf{x}_k \cdot \mathbf{x}_k} = \frac{\mathbf{x}_k}{\mathbf{x}_k \mathbf{x}_k} = \frac{1}{\mathbf{x}_k} = \mathbf{x}_k^{-1}$$

which suggests the name reciprocal basis.

2.45.11 Solving a system of vector equations using SymPy.

The free software SymPy, for symbolic mathematics using python, includes a Geometric Algebra Module and interactive calculator console `isympy`. The `isympy` console can be used to solve systems of vector equations using the formulas of this article. A simple example of console interaction follows to solve the system

$$\begin{aligned}
3\mathbf{v}_1 + 4\mathbf{v}_2 + 5\mathbf{v}_3 &= \mathbf{c}_1 = 9\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \\
2\mathbf{v}_1 + 3\mathbf{v}_2 + 7\mathbf{v}_3 &= \mathbf{c}_2 = 6\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_3 \\
9\mathbf{v}_1 + 6\mathbf{v}_2 + 9\mathbf{v}_3 &= \mathbf{c}_3 = 2\mathbf{e}_1 + 4\mathbf{e}_2 + 7\mathbf{e}_3.
\end{aligned}$$

```
$isympy
>>> from sympy.galgebra.ga import *
>>> (e1,e2,e3,e4,e5,e6) = MV.setup('e*1|2|3|4|5|6',metric='[1,1,
1,1,1,1]')
>>> (v1,v2,v3) = symbols('v1 v2 v3')
>>> (c1,c2,c3,C) = symbols('c1 c2 c3 C')
>>> (a1,a2,a3) = symbols('a1 a2 a3')
>>> a1 = 3*e4 + 2*e5 + 9*e6
>>> a2 = 4*e4 + 3*e5 + 6*e6
>>> a3 = 5*e4 + 7*e5 + 9*e6
>>> c1 = 9*e1 + 2*e2 + 3*e3
>>> c2 = 6*e1 + 5*e2 + 8*e3
>>> c3 = 2*e1 + 4*e2 + 7*e3
>>> C = (c1^e4) + (c2^e5) + (c3^e6)
>>> v1 = (C^a2^a3)|((-1)**(1-1)*MV.inv(a1^a2^a3))
>>> v2 = (a1^C^a3)|((-1)**(2-1)*MV.inv(a1^a2^a3))
>>> v3 = (a1^a2^C)|((-1)**(3-1)*MV.inv(a1^a2^a3))
>>> 3*v1 + 4*v2 + 5*v3
9*e_1 + 2*e_2 + 3*e_3
>>> 2*v1 + 3*v2 + 7*v3
```

```
6*e_1 + 5*e_2 + 8*e_3
>>> 9*v1 + 6*v2 + 9*v3
2*e_1 + 4*e_2 + 7*e_3
```

Chapter 3

Quadric Geometric Algebra

3.1 Introduction

Quadric Geometric Algebra (QGA)[10] is a geometrical application of the $\mathcal{G}_{6,3}$ Geometric Algebra. This algebra is also known as the $\mathcal{C}\ell_{6,3}$ Clifford Algebra. QGA is a super-algebra over $\mathcal{G}_{4,1}$ Conformal Geometric Algebra (CGA) and $\mathcal{G}_{1,3}$ Spacetime Algebra (STA), which can each be defined within sub-algebras of QGA.

CGA provides representations of spherical entities (points, spheres, planes, and lines) and a complete set of operations (translation, rotation, dilation, and intersection) that apply to them. QGA extends CGA to also include representations of some non-spherical entities: principal axes-aligned quadric surfaces and many of their degenerate forms such as planes, lines, and points.

General quadric surfaces are characterized by the homogeneous polynomial equation of degree 2

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J \;=\; 0$$

which can characterize quadric surfaces located at any center point and aligned along arbitrary axes. However, QGA includes vector entities that can represent only the principal axes-aligned quadric surfaces characterized by

$$Ax^2 + By^2 + Cz^2 + Gx + Hy + Iz + J \;=\; 0.$$

This is still a very significant advancement over CGA.

A possible performance issue with using QGA is the increased computation required to use a 9D vector space, as compared to the smaller 5D vector space of CGA. A 5D CGA subspace can be used when only CGA entities are involved in computations.

The operations that work correctly with the QGA axes-aligned quadric entities include translation, transposition, dilation, and intersection.

In general, the operation of rotation *does not work* correctly on non-spherical QGA quadric surface entities. Rotation also does not work correctly on the QGA point entities. Attempting to rotate a QGA quadric surface may result in a different type of quadric surface, or a quadric surface that is rotated and distorted in an unexpected way. Attempting to rotate a QGA point may produce a value that projects as the expected rotated vector, but the produced value is generally not a correct embedding of the rotated vector. The failure of QGA points to rotate correctly also leads to the inability to use outermorphisms to rotate dual Geometric Outer Product Null Space (GOPNS) entities. To rotate a QGA point, it must be projected to a vector or converted to a CGA point for rotation operations, then the rotated result can be re-embedded or converted back into a QGA point. A quadric surface rotated by an arbitrary angle cannot be represented by any known QGA entity. Representation of general quadric surfaces with useful operations will require an algebra (that appears to be unknown at this time) that extends QGA.

Although rotation is generally unavailable in QGA, the transposition operation is a special-case modification of rotation by $\pi/2$ that works correctly on all QGA GIPNS entities. Transpositions allow QGA GIPNS entities to be reflected in the six diagonal planes $y=\pm x$, $z=\pm x$, and $z=\pm y$.

Entities for all principal axes-aligned quadric surfaces can be defined in QGA. These include ellipsoids, cylinders, cones, paraboloids, and hyperboloids in all of their various forms.

A powerful feature of QGA is the ability to compute the intersections of axes-aligned quadric surfaces. With few exceptions, the outer product of QGA GIPNS surface entities represents their surfaces intersection(s). This method of computing intersections works the same as it does in CGA, where only spherical entities are available.

3.2 Points and GIPNS Surfaces

3.2.1 QGA Point

The algebra $\mathcal{G}_{6,3}$ has six unit-vector elements with positive signature, and three with negative signature:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & | \; i=j, 1 \le i \le 6 \\ -1 & | \; i=j, 7 \le i \le 9 \\ 0 & | \; i \ne j. \end{cases}$$

A QGA point $\mathbf{P}$ is the embedding of a 3D vector $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ into this 9D vector space. The embedding is similar to that used in CGA, but each of the three dimensions in $\mathbf{p}$ is embedded independently into a different subspace. The embeddings of the three axes are three independent embeddings of 1D spaces. Since each axis is embedded independently, most of the operations on QGA points and quadric surfaces operate on each axis embedding separately from the other two. The independent embeddings of the axes enables the representation of non-spherical quadric surface entities.

Each dimension of vector $\mathbf{p}$ is stereographically embedded into a 1-sphere or unit circle on a plane spanned with one of the additional positive-signature elements $\mathbf{e}_4$, $\mathbf{e}_5$ or $\mathbf{e}_6$. Then, the unit circle is homogenized or raised by adding one of the three negative-signature elements $\mathbf{e}_7$, $\mathbf{e}_8$ or $\mathbf{e}_9$. The plane spanned by a positive-signature vector and negative-signature vector is known as a Minkowski plane. By using negative-signature elements for the homogeneous dimensions, the points are defined as null vectors which square to zero. Same as for all CGA points, all QGA points are null vectors. The embedded points on the raised or homogenized circle can be scaled by any non-zero scalar, and the space spanned by scaled points through the raised circle has the shape of a cone called the null cone. All x, y or z embeded coordinates have a value somewhere on the x, y or z null cone, respectively. Only embedded points that are normalized back into the raised unit circles in the null cones can be projected back down to the original points they represent in Euclidean space by reverse homogenization and reverse stereographic embedding operations.

For the stereographic embedding of the $x\mathbf{e}_1$ component of $\mathbf{p}$, the $\mathbf{e}_4$ element serves as the stereographic pole of a 1-sphere or unit circle on the plane it spans with $\mathbf{e}_1$. The stereographic embedding of $x\mathbf{e}_1$ is the intersection $\alpha\mathbf{e}_1 + \beta\mathbf{e}_4$ of the unit circle on the $\mathbf{e}_1\mathbf{e}_4$-plane with the line through $\mathbf{e}_4$ and $x\mathbf{e}_1$. As $x \to \pm\infty$, the intersection approaches $\mathbf{e}_4$, which can be thought of as the north pole of the 1-sphere. For $x = 0$, the intersection is $-\mathbf{e}_4$. These intersections allow for special representations for a point at infinity and for a point at the origin. For the homogenization or raising, the element $\mathbf{e}_7$ is added so that the complete point embedding of $x\mathbf{e}_1$ is $\alpha\mathbf{e}_1 + \beta\mathbf{e}_4 + \mathbf{e}_7$. The values of α and β on the unit circle are solved as follows.

The initial relations are the unit circle $\alpha^2 + \beta^2 = 1$ and, by similar triangles, the line $\frac{1-\beta}{\alpha} = \frac{1}{x}$.

$$\begin{aligned} \alpha^2 &= 1-\beta^2 = (1+\beta)(1-\beta) = ((1-\beta)x)^2 \\ (1+\beta) &= (1-\beta)x^2 \\ \beta x^2 + \beta &= x^2 - 1 \\ \beta &= \frac{x^2-1}{x^2+1} \end{aligned}$$

$$\begin{aligned} \alpha &= (1-\beta)x \\ &= \left(1 - \frac{x^2-1}{x^2+1}\right)x \\ &= \left(\frac{x^2+1}{x^2+1} - \frac{x^2-1}{x^2+1}\right)x \\ &= \frac{2x}{x^2+1} \end{aligned}$$

The stereographic embedding of $x\mathbf{e}_1$, denoted $\mathcal{S}(x\mathbf{e}_1)$, can now be written as

$$\begin{aligned} \mathcal{S}(x\mathbf{e}_1) &= \alpha\mathbf{e}_1 + \beta\mathbf{e}_4 \\ &= \left(\frac{2x}{x^2+1}\right)\mathbf{e}_1 + \left(\frac{x^2-1}{x^2+1}\right)\mathbf{e}_4. \end{aligned}$$

The homogenization of $\mathcal{S}(x\mathbf{e}_1)$, denoted $\mathcal{H}_M(\mathcal{S}(x\mathbf{e}_1))$, can be written as

$$\mathbf{P}_x = \mathcal{H}_M(\mathcal{S}(x\mathbf{e}_1)) = \left(\frac{2x}{x^2+1}\right)\mathbf{e}_1 + \left(\frac{x^2-1}{x^2+1}\right)\mathbf{e}_4 + \mathbf{e}_7.$$

Since this point entity is homogeneous, and x^2+1 is never zero, it is permissible to scale it by $\frac{x^2+1}{2}$ and write

$$\begin{aligned}\mathbf{P}_x = \mathcal{H}_M(\mathcal{S}(x\mathbf{e}_1)) &= x\mathbf{e}_1 + \frac{x^2-1}{2}\mathbf{e}_4 + \frac{x^2+1}{2}\mathbf{e}_7 \\ &= x\mathbf{e}_1 + \frac{x^2}{2}(\mathbf{e}_4+\mathbf{e}_7) + \frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_7).\end{aligned}$$

When $x=0$,

$$\mathbf{P}_{x=0} = \frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_7) = \mathbf{e}_{ox}$$

representing the point at the origin on the x-axis. In the limit as $x \to \pm\infty$, we find that

$$\mathbf{P}_{x\to\infty} = \mathbf{e}_4 + \mathbf{e}_7 = \mathbf{e}_{\infty x}$$

represents the point at infinity on the x-axis. By taking inner products, it can be shown that these points are all null vectors on a null cone, and the inner product $\mathbf{e}_{ox}\cdot\mathbf{e}_{\infty x} = -1$. The embedding of the x-axis $\mathbf{P}_x$ can now be written as

$$\mathbf{P}_x = x\mathbf{e}_1 + \frac{1}{2}x^2\mathbf{e}_{\infty x} + \mathbf{e}_{ox}.$$

Similarly, it can be shown that the axes or components $y\mathbf{e}_2$ and $z\mathbf{e}_3$ can be embedded as the points

$$\begin{aligned}\mathbf{P}_y &= y\mathbf{e}_2 + \frac{1}{2}y^2\mathbf{e}_{\infty y} + \mathbf{e}_{oy} \\ \mathbf{P}_z &= z\mathbf{e}_3 + \frac{1}{2}z^2\mathbf{e}_{\infty z} + \mathbf{e}_{oz}\end{aligned}$$

where

$$\begin{aligned}\mathbf{e}_{\infty x} &= \mathbf{e}_4 + \mathbf{e}_7 \\ \mathbf{e}_{\infty y} &= \mathbf{e}_5 + \mathbf{e}_8 \\ \mathbf{e}_{\infty z} &= \mathbf{e}_6 + \mathbf{e}_9 \\ \mathbf{e}_{ox} &= \frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_7) \\ \mathbf{e}_{oy} &= \frac{1}{2}(-\mathbf{e}_5+\mathbf{e}_8) \\ \mathbf{e}_{oz} &= \frac{1}{2}(-\mathbf{e}_6+\mathbf{e}_9).\end{aligned}$$

Note that, in CGA a single stereographic embedding dimension $\mathbf{e}_4$ of positive signature, and a single homogeneous dimension $\mathbf{e}_5$ of negative signature is used to embed all three components of vector $\mathbf{p}$ into a 5D vector space such that $\mathbf{p}$ is stereographically embedded into a single 3-sphere with axis $\mathbf{e}_4$, and homogeneously raised by $\mathbf{e}_5$. In QGA, the stereographic embeddings are per axis or component into three independent 1-spheres that are raised into three independent homogeneous dimensions.

Each homogeneously normalized point component embedding $\mathbf{P}_x$, $\mathbf{P}_y$, and $\mathbf{P}_z$ is a parabola in the $\mathbf{e}_{ox}$, $\mathbf{e}_{oy}$, or $\mathbf{e}_{oz}$-plane, respectively. These are 1D parabolic sections of the null cones. Therefore, the embedding of a 1D Euclidean space (line or axis) is into a null parabola on a null cone.

The QGA embedding $\mathcal{Q}(\mathbf{p})$ of a 3D point $\mathbf{p}$ is the sum of the three independent 1D point embeddings,

$$\begin{aligned}
\mathbf{P} = \mathbf{P}_{\mathcal{Q}} = \mathcal{Q}(\mathbf{p}) &= \mathbf{P}_x + \mathbf{P}_y + \mathbf{P}_z \\
&= \mathbf{p} + \frac{1}{2}x^2\,\mathbf{e}_{\infty x} + \frac{1}{2}y^2\,\mathbf{e}_{\infty y} + \frac{1}{2}z^2\,\mathbf{e}_{\infty z} + \mathbf{e}_{ox} + \mathbf{e}_{oy} + \mathbf{e}_{oz} \\
&= \mathbf{p} + \frac{1}{2}x^2\,\mathbf{e}_{\infty x} + \frac{1}{2}y^2\,\mathbf{e}_{\infty y} + \frac{1}{2}z^2\,\mathbf{e}_{\infty z} + \mathbf{e}_o
\end{aligned}$$

where

$$\begin{aligned}
\lim_{x\to\infty} \mathbf{P} &= \mathbf{e}_{\infty x} \\
\lim_{y\to\infty} \mathbf{P} &= \mathbf{e}_{\infty y} \\
\lim_{z\to\infty} \mathbf{P} &= \mathbf{e}_{\infty z} \\
\mathbf{P}_{x,y,z=0} &= \mathbf{e}_o = \mathbf{e}_{ox} + \mathbf{e}_{oy} + \mathbf{e}_{oz}.
\end{aligned}$$

Note that, in CGA this simple addition of component embeddings is incorrect. In CGA, the embedding must be viewed as into a single 3-sphere with a single $\mathbf{e}_o$ and $\mathbf{e}_\infty$ shared in common among the three component dimensions in $\mathbf{p}$. In QGA, each component or axis has distinct origin and infinity point entities.

By definition,

$$\mathbf{e}_\infty = \frac{1}{3}(\mathbf{e}_{\infty x} + \mathbf{e}_{\infty y} + \mathbf{e}_{\infty z})$$

such that

$$\mathbf{e}_o \cdot \mathbf{e}_\infty = -1.$$

The component $\mathbf{e}_o$ serves as the homogeneous component of a point $\mathbf{P}$. The projection of a point $\mathbf{P}$ back to a 3D vector $\mathbf{p}$ is

$$\mathbf{p} = \frac{\mathbf{P}\cdot\mathbf{e}_1}{-\mathbf{P}\cdot\mathbf{e}_\infty}\mathbf{e}_1 + \frac{\mathbf{P}\cdot\mathbf{e}_2}{-\mathbf{P}\cdot\mathbf{e}_\infty}\mathbf{e}_2 + \frac{\mathbf{P}\cdot\mathbf{e}_3}{-\mathbf{P}\cdot\mathbf{e}_\infty}\mathbf{e}_3.$$

The projection normalizes the homogeneous component and then extracts the 3D vector part.

When computing inner products, it is helpful to remember the following inner product multiplication table for the base vectors and special points.

$\bullet$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{ox}$	$\mathbf{e}_{oy}$	$\mathbf{e}_{oz}$	$\mathbf{e}_{\infty x}$	$\mathbf{e}_{\infty y}$	$\mathbf{e}_{\infty z}$	$\mathbf{e}_o$	$\mathbf{e}_\infty$
$\mathbf{e}_1$	1	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_2$	0	1	0	0	0	0	0	0	0	0	0
$\mathbf{e}_3$	0	0	1	0	0	0	0	0	0	0	0
$\mathbf{e}_{ox}$	0	0	0	0	0	0	-1	0	0	0	$-\frac{1}{3}$
$\mathbf{e}_{oy}$	0	0	0	0	0	0	0	-1	0	0	$-\frac{1}{3}$
$\mathbf{e}_{oz}$	0	0	0	0	0	0	0	0	-1	0	$-\frac{1}{3}$
$\mathbf{e}_{\infty x}$	0	0	0	-1	0	0	0	0	0	-1	0
$\mathbf{e}_{\infty y}$	0	0	0	0	-1	0	0	0	0	-1	0
$\mathbf{e}_{\infty z}$	0	0	0	0	0	-1	0	0	0	-1	0
$\mathbf{e}_o$	0	0	0	0	0	0	-1	-1	-1	0	-1
$\mathbf{e}_\infty$	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	-1	0

Same as in CGA, it is also found that the inner product of two QGA points $\mathbf{P}_1$ and $\mathbf{P}_2$ is

$$\begin{aligned}
\mathbf{P}_1\cdot\mathbf{P}_2 &= \mathbf{p}_1\cdot\mathbf{p}_2-\frac{1}{2}(x_1^2+y_1^2+z_1^2)-\frac{1}{2}(x_2^2+y_2^2+z_2^2)\\
&= |\mathbf{p}_1||\mathbf{p}_2|\cos(\theta)-\frac{1}{2}|\mathbf{p}_1|^2-\frac{1}{2}|\mathbf{p}_2|^2\\
-2(\mathbf{P}_1\cdot\mathbf{P}_2) &= |\mathbf{p}_1|^2+|\mathbf{p}_2|^2-2|\mathbf{p}_1||\mathbf{p}_2|\cos(\theta)\\
&= (\mathbf{p}_2-\mathbf{p}_1)^2=d^2\\
&= \mathbf{p}_1^2+\mathbf{p}_2^2-(\mathbf{p}_2\mathbf{p}_1+\mathbf{p}_1\mathbf{p}_2)\\
&= |\mathbf{p}_1|^2+|\mathbf{p}_2|^2-2(\mathbf{p}_2\cdot\mathbf{p}_1)
\end{aligned}$$

such that the squared distance d^2 between $\mathbf{P}_1$ and $\mathbf{P}_2$ is $d^2=-2(\mathbf{P}_1\cdot\mathbf{P}_2)$.

3.2.2 Inner Product Null Space

The Inner Product Null Space (IPNS) of k-vector $\mathbf{n}$, denoted $\mathbb{NI}(\mathbf{n}\in\mathcal{G}^k)$, is defined as the set of vectors

$$\mathbb{NI}(\mathbf{n}\in\mathcal{G}^k) \;=\; \left\{\; \mathbf{t}\in\mathcal{G}^1 \;:\; \mathbf{t}\cdot\mathbf{n}=0 \;\right\}.$$

For example: in 3D Euclidean space, if vector $\mathbf{n}$ is a unit normal vector representing a plane through the origin, then its IPNS is the set of all vectors in the plane. In 3D Euclidean space, or in nD projective geometry, only certain types of geometric objects can be represented using IPNS.

The Geometric Inner Product Null Space (GIPNS) of k-vector $\mathbf{S}$, denoted $\mathbb{NI}_G(\mathbf{S}\in\mathcal{G}^k)$, is defined as the set of vectors

$$\mathbb{NI}_G(\mathbf{S}\in\mathcal{G}^k) \;=\; \left\{\; \mathbf{t}\in\mathcal{G}_3^1 \;:\; \mathcal{H}_M(\mathcal{S}(\mathbf{t}))\cdot\mathbf{S}=0 \;\right\}.$$

The k-vector $\mathbf{S}$ is an GIPNS entity that represents a geometric surface. The vectors $\mathbf{t}$ are restricted to 3D Euclidean vectors since the GIPNS is defined to represent a geometric surface in 3D. The embedding of $\mathbf{t}$ implies that $\mathbf{S}$ is an embedded representation of a geometric surface. The set $\mathbb{NI}_G(\mathbf{S}\in\mathcal{G}^k)$ is all 3D Euclidean vectors on the geometric surface represented by $\mathbf{S}$.

The following subsections show how various geometric surfaces in 3D Euclidean space are embedded as GIPNS entities in $\mathcal{G}_{6,3}$ QGA.

3.2.3 QGA GIPNS Ellipsoid

The homogeneous quadratic equation that characterizes a principal axes-aligned ellipsoid is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}+\frac{(z-p_x)^2}{r_z^2}-1 \;=\; 0$$

where $\mathbf{p}=p_x\,\mathbf{e}_1+p_y\,\mathbf{e}_2+p_z\,\mathbf{e}_3$ is the position (or shifted origin, or center) of the ellipsoid, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c). Expanding the squares, the equation can be written as

$$\frac{-2p_x x}{r_x^2}+\frac{-2p_y y}{r_y^2}+\frac{-2p_z z}{r_z^2}+\left(\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2}\right)+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right) \;=\; 0.$$

The embedding of a test point $\mathbf{t}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$ as a QGA point $\mathbf{T}$ is

$$\mathbf{T} \;=\; \mathbf{t}+\frac{1}{2}x^2\,\mathbf{e}_{\infty x}+\frac{1}{2}y^2\,\mathbf{e}_{\infty y}+\frac{1}{2}z^2\,\mathbf{e}_{\infty z}+\mathbf{e}_o.$$

If the ellipsoid surface entity is the vector $\mathbf{S}$, then any test point $\mathbf{T}$ that is on the ellipsoid surface must satisfy the Geometric Inner Product Null Space (GIPNS) condition $\mathbf{S}\cdot\mathbf{T}=0$. Under this condition, $\mathbf{S}$ is called the GIPNS representation or entity of an ellipsoid.

The construction of vector $\mathbf{S}$ follows from the GIPNS condition

$$\begin{aligned}\mathbf{S}\cdot\mathbf{T} \;&=\; 0\\ &=\; \frac{-2p_x x}{r_x^2}+\frac{-2p_y y}{r_y^2}+\frac{-2p_z z}{r_z^2}+\left(\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2}\right)+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right).\end{aligned}$$

Computing the inner product $\mathbf{S}\cdot\mathbf{T}$ with

$$\mathbf{S} \;=\; -\frac{2p_x}{r_x^2}\mathbf{e}_1-\frac{2p_y}{r_y^2}\mathbf{e}_2-\frac{2p_z}{r_z^2}\mathbf{e}_3-\frac{2}{r_x^2}\mathbf{e}_{ox}-\frac{2}{r_y^2}\mathbf{e}_{oy}-\frac{2}{r_z^2}\mathbf{e}_{oz}-\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right)\mathbf{e}_\infty$$

gives the correct homogeneous expression of an ellipsoid, matching the GIPNS condition. The construction of $\mathbf{S}$ is simply by inspection that the sought inner product is what is produced. The definition of a QGA point makes it possible to extract x, y, z and their squares and to form homogeneous expressions of axes-aligned quadric surfaces. With points and surface entities defined by these kinds of GIPNS conditions, the next challenge is to find useful operations that work on the entities within the algebra. Fortunately, many useful operations have been found within this particular algebra.

Since $\mathbf{T}$ is homogeneous, $\mathbf{S}$ is also homogeneous, where multiplication by any non-zero scalar does not affect the objects represented by the entities. For $\mathbf{S}$, the $\mathbf{e}_\infty$ component serves as the homogeneous component.

For $\mathbf{S}$ and $\mathbf{T}$ both homogeneously normalized, then

$$\frac{\mathbf{S}\cdot\mathbf{T}}{(\mathbf{S}\cdot\mathbf{e}_\infty)(\mathbf{T}\cdot\mathbf{e}_\infty)}\begin{cases}<0 & : \ \mathbf{t}\text{ inside ellipsoid,}\\ =0 & : \ \mathbf{t}\text{ on ellipsoid,}\\ >0 & : \ \mathbf{t}\text{ outside ellipsoid.}\end{cases}$$

3.2.4 QGA and CGA GIPNS Sphere

A sphere is a special case of an ellipsoid with $r = r_x = r_y = r_z$, that reduces to

$$\begin{aligned}\mathbf{S} &= p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \mathbf{e}_{ox} + \mathbf{e}_{oy} + \mathbf{e}_{oz} + \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 - r^2)\,\mathbf{e}_\infty \\ &= \mathbf{p} + \frac{1}{2}(\mathbf{p}^2 - r^2)\mathbf{e}_\infty + \mathbf{e}_o\end{aligned}$$

after multiplying through by $-\frac{1}{2}r^2$, while *assuming* $r \neq 0$. This appears to be the same form as the CGA sphere entity. If $r = 0$, then this CGA form of the sphere entity is *invalid* in QGA since it is not permissible to scale the homogeneous QGA sphere entity by zero.

Nevertheless, in the limit as $r \to 0$ it is seen that the sphere $\mathbf{S}$ approaches the form of a CGA point that represents the position $\mathbf{p}$. This CGA point is not a QGA point, but it is also a 6,3D vector, and it also projects back to a 3D vector $\mathbf{p}$ by the same method as a QGA point.

For $\mathbf{S}$ and $\mathbf{T}$ both homogeneously normalized, then

$$\frac{\mathbf{S}\cdot\mathbf{T}}{(\mathbf{S}\cdot\mathbf{e}_\infty)(\mathbf{T}\cdot\mathbf{e}_\infty)}\begin{cases} >0 & : \ \mathbf{t} \text{ inside sphere,} \\ =0 & : \ \mathbf{t} \text{ on sphere,} \\ <0 & : \ \mathbf{t} \text{ outside sphere.}\end{cases}$$

These sphere test inequalities are the reverses of the ellipsoid test inequalities since the sphere is derived from the ellipsoid after multiplying through by the negative value $-\frac{1}{2}r^2$.

3.2.5 CGA 6,3D Point

In $\mathcal{G}_{4,1}$ Conformal Geometric Algebra (CGA), the embedding of a 3D point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ starts with a stereographic embedding of $\mathbf{p}$ onto a hypersphere or 3-sphere using $\mathbf{e}_4$ as the stereographic 3-sphere pole. This requires finding the intersection of the line through $\mathbf{e}_4$ and $\mathbf{p}$ with the 3-sphere. The vectors $\mathbf{e}_4$ and $\mathbf{p}$ are perpendicular, and we can treat the embedding of $\mathbf{p}$ similarly to a 1D axis embedding into a stereographic 1-sphere or circle.

The identities

$$\begin{aligned}|\mathbf{p}| &= \sqrt{x^2 + y^2 + z^2} \\ \hat{\mathbf{p}} &= \frac{\mathbf{p}}{|\mathbf{p}|} \\ \mathbf{p} &= |\mathbf{p}|\hat{\mathbf{p}} \\ \mathbf{p}^2 &= |\mathbf{p}|^2 = x^2 + y^2 + z^2\end{aligned}$$

are used in the following.

The stereographic embedding of $|\mathbf{p}|\hat{\mathbf{p}}$ is the intersection $\alpha\hat{\mathbf{p}} + \beta\mathbf{e}_4$ of the unit 3-circle on the $\hat{\mathbf{p}}\,\mathbf{e}_4$-hyperplane with the line through $\mathbf{e}_4$ and $|\mathbf{p}|\hat{\mathbf{p}}$. The Minkowski homogenization is $\alpha\hat{\mathbf{p}} + \beta\mathbf{e}_4 + \mathbf{e}_5$. The point at the origin embeds to $\mathbf{e}_o = -\mathbf{e}_4 + \mathbf{e}_5$ and the point at infinity embeds to $\mathbf{e}_\infty = \mathbf{e}_4 + \mathbf{e}_5$. It is convenient to scale $\mathbf{e}_o$ as $\mathbf{e}_o = \frac{1}{2}(-\mathbf{e}_4 + \mathbf{e}_5)$ such that $\mathbf{e}_o \cdot \mathbf{e}_\infty = -1$. The values for α and β are solved as follows.

The initial relations are the unit circle $\alpha^2+\beta^2=1$ and, by similar triangles, the line $\frac{1-\beta}{\alpha}=\frac{1}{|\mathbf{p}|}$.

$$\begin{aligned}
\alpha^2 &= 1-\beta^2=(1+\beta)(1-\beta)=((1-\beta)|\mathbf{p}|)^2 \\
(1+\beta) &= (1-\beta)|\mathbf{p}|^2 \\
\beta\,|\mathbf{p}|^2+\beta &= |\mathbf{p}|^2-1 \\
\beta &= \frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}
\end{aligned}$$

$$\begin{aligned}
\alpha &= (1-\beta)|\mathbf{p}| \\
&= \left(1-\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)|\mathbf{p}| \\
&= \left(\frac{|\mathbf{p}|^2+1}{|\mathbf{p}|^2+1}-\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)|\mathbf{p}| \\
&= \frac{2\,|\mathbf{p}|}{|\mathbf{p}|^2+1}
\end{aligned}$$

The stereographic embedding of $|\mathbf{p}|\hat{\mathbf{p}}$, denoted $\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}})$, can now be written as

$$\begin{aligned}
\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}}) &= \alpha\hat{\mathbf{p}}+\beta\mathbf{e}_4 \\
&= \left(\frac{2\,|\mathbf{p}|}{|\mathbf{p}|^2+1}\right)\hat{\mathbf{p}}+\left(\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)\mathbf{e}_4.
\end{aligned}$$

The homogenization of $\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}})$, denoted $\mathcal{H}_M(\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}}))$, can be written as

$$\mathbf{P}=\mathcal{H}_M(\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}})) = \left(\frac{2|\mathbf{p}|}{|\mathbf{p}|^2+1}\right)\hat{\mathbf{p}}+\left(\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)\mathbf{e}_4+\mathbf{e}_5.$$

Since this point entity is homogeneous, and $|\mathbf{p}|^2+1$ is never zero, it is permissible to scale it by $\frac{|\mathbf{p}|^2+1}{2}$ and write

$$\begin{aligned}
\mathbf{P} &= \mathcal{H}_M(\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}}))=\mathcal{C}_{4,1}(\mathbf{p}) \\
&= |\mathbf{p}|\hat{\mathbf{p}}+\frac{|\mathbf{p}|^2-1}{2}\mathbf{e}_4+\frac{|\mathbf{p}|^2+1}{2}\mathbf{e}_5 \\
&= |\mathbf{p}|\hat{\mathbf{p}}+\frac{|\mathbf{p}|^2}{2}(\mathbf{e}_4+\mathbf{e}_5)+\frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_5) \\
&= \mathbf{p}+\frac{1}{2}\mathbf{p}^2(\mathbf{e}_4+\mathbf{e}_5)+\frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_5)
\end{aligned}$$

When $|\mathbf{p}|=0$,

$$\mathbf{P}_{|\mathbf{p}|=0} = \frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_5)=\mathbf{e}_o$$

representing the point at the origin. In the limit as $|\mathbf{p}|\to\pm\infty$, we find that

$$\mathbf{P}_{|\mathbf{p}|\to\infty} = \mathbf{e}_4+\mathbf{e}_5=\mathbf{e}_\infty$$

represents the point at infinity. By taking inner products, it can be shown that these points are all null vectors on a null 4-cone, and the inner product $\mathbf{e}_o\cdot\mathbf{e}_\infty=-1$. The CGA embedding of $\mathbf{p}$ as $\mathbf{P}$ can now be written as

$$\begin{aligned}
\mathbf{P} &= \mathcal{C}_{4,1}(\mathbf{p}) \\
&= \mathbf{p}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty+\mathbf{e}_o.
\end{aligned}$$

A form of CGA point, being called here a CGA 6,3D point, can be defined in the $\mathcal{G}_{6,3}$ geometric algebra by the conformal embedding of a 3D vector $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ as

$$\mathbf{P}_{\mathcal{C}} = \mathcal{C}_{6,3}(\mathbf{p}) \;=\; \mathbf{p} + \frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty + \mathbf{e}_o.$$

This matches the form used in 4,1D $\mathcal{G}_{4,1}$ CGA, except that $\mathbf{e}_\infty$ and $\mathbf{e}_o$ are

$$\begin{aligned}
\mathbf{e}_\infty &= \frac{1}{3}(\mathbf{e}_{\infty x} + \mathbf{e}_{\infty y} + \mathbf{e}_{\infty z}) \\
&= \frac{1}{3}(\mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8 + \mathbf{e}_9) \\
\mathbf{e}_o &= \mathbf{e}_{ox} + \mathbf{e}_{oy} + \mathbf{e}_{oz} \\
&= \frac{1}{2}(-\mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8 + \mathbf{e}_9)
\end{aligned}$$

as previously defined for QGA points. All CGA entities and operations are available based on this form of CGA point.

The projection of a CGA 6,3D point $\mathbf{P}_{\mathcal{C}}$ back to a 3D vector $\mathbf{p}$ is

$$\mathbf{p} \;=\; \frac{\mathbf{P}_{\mathcal{C}} \cdot \mathbf{e}_1}{-\mathbf{P}_{\mathcal{C}} \cdot \mathbf{e}_\infty}\mathbf{e}_1 + \frac{\mathbf{P}_{\mathcal{C}} \cdot \mathbf{e}_2}{-\mathbf{P}_{\mathcal{C}} \cdot \mathbf{e}_\infty}\mathbf{e}_2 + \frac{\mathbf{P}_{\mathcal{C}} \cdot \mathbf{e}_3}{-\mathbf{P}_{\mathcal{C}} \cdot \mathbf{e}_\infty}\mathbf{e}_3.$$

This is the same method by which a QGA point is projected, and the same as done in CGA.

The QGA or CGA 6,3D GIPNS sphere entity $\mathbf{S}$ can be written in terms of a conformal center point $\mathbf{P}_{\mathcal{C}}$ and radius r as

$$\begin{aligned}
\mathbf{S} &= \mathbf{p} + \frac{1}{2}(\mathbf{p} - r^2)\mathbf{e}_\infty + \mathbf{e}_o \\
&= \mathbf{P}_{\mathcal{C}} - \frac{1}{2}r^2\mathbf{e}_\infty.
\end{aligned}$$

QGA and CGA 6,3D points are compatible and interchangeable in all operations that include only spherical GIPNS entities of the kinds available in CGA. This is true since the spherical GIPNS entities are given by the same formulas in both QGA and CGA.

QGA spherical GOPNS entities are constructed by the outer products of QGA points according to QGA formulas. CGA 6,3D spherical GOPNS entities are constructed by the outer products of CGA 6,3D points according to CGA formulas. These two formulations of spherical GOPNS entities are not identical and give different entities that may represent the same geometrical objects.

QGA spherical GOPNS entities appear to be compatible in operations with CGA 6,3D points substituting for QGA points. However, CGA 6,3D spherical GOPNS entities do not appear to be compatible in operations with QGA points attempting to substitute for CGA 6,3D points.

QGA spherical GOPNS entities are compatible in operations with the super-set of both CGA 6,3D and QGA points. CGA 6,3D spherical GOPNS entities are only compatible with the CGA 6,3D sub-set of points.

Any operation with a non-spherical QGA entity is compatible with only QGA points.

Computations with CGA 6,3D spherical entities and points may be faster than the equivalent computations using QGA spherical entities and points. For the fastest CGA computations, a 4,1D CGA sub-algebra could be defined, but that algebra would be completely incompatible with the algebra of CGA 6,3D and QGA.

3.2.6 QGA and CGA GIPNS Line

The homogeneous equation for a line through two points $\mathbf{p}_1$ and $\mathbf{p}_2$ can be written

$$(\mathbf{t}-\mathbf{p}_1)\cdot((\mathbf{p}_2-\mathbf{p}_1)/\mathbf{I}_3) \;=\; 0$$

with

$$\begin{aligned}
\mathbf{p}_1 &= x_1\mathbf{e}_1+y_1\mathbf{e}_2+z_1\mathbf{e}_3\\
\mathbf{p}_2 &= x_2\mathbf{e}_1+y_2\mathbf{e}_2+z_2\mathbf{e}_3\\
\mathbf{t} &= x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3
\end{aligned}$$

where $\mathbf{t}$ is a test point. If $\mathbf{t}$ is on the line, then the equation holds good. The direction of the line can be defined by the unit vector $\mathbf{d}$

$$\mathbf{d} \;=\; \frac{\mathbf{p}_2-\mathbf{p}_1}{|\mathbf{p}_2-\mathbf{p}_1|}.$$

The unit bivector $\mathbf{D}$ that represents a plane perpendicular to the line is the dual of $\mathbf{d}$

$$\mathbf{D} \;=\; \mathbf{d}/\mathbf{I}_3=-\mathbf{d}\cdot\mathbf{I}_3.$$

We can choose a single point on the line, and call it $\mathbf{p}$. The equation for the line can now be expressed as

$$\begin{aligned}
(\mathbf{t}-\mathbf{p})\cdot\mathbf{D} &= 0\\
\mathbf{t}\cdot\mathbf{D}-\mathbf{p}\cdot\mathbf{D} &= 0.
\end{aligned}$$

If $\mathbf{t}$ is on the line, then $\mathbf{t}-\mathbf{p}$ has zero projection onto the perpendicular plane bivector $\mathbf{D}$.

The QGA embedding of the test point $\mathbf{t}$ is

$$\mathbf{T} \;=\; \mathbf{t}+\frac{1}{2}x^2\mathbf{e}_{\infty x}+\frac{1}{2}y^2\mathbf{e}_{\infty y}+\frac{1}{2}z^2\mathbf{e}_{\infty z}+\mathbf{e}_o.$$

If the line entity is $\mathbf{L}$, then any test point $\mathbf{T}$ on the line must satisfy the GIPNS condition

$$\mathbf{T}\cdot\mathbf{L} \;=\; 0=\mathbf{t}\cdot\mathbf{D}-\mathbf{p}\cdot\mathbf{D}.$$

The GIPNS line entity $\mathbf{L}$ can be constructed as

$$\mathbf{L} \;=\; \mathbf{D}-(\mathbf{p}\cdot\mathbf{D})\,\mathbf{e}_\infty.$$

This *bivector* form of GIPNS line entity $\mathbf{L}$ is the same in CGA and QGA.

The line entity is a bivector since it is actually the intersection of two plane entities. It will be shown that a GIPNS plane entity is represented by a vector, and the outer product of two plane entities is the bivector line entity. It will also be shown that a line is a circle with infinite radius and center at infinity, and that a circle is the intersection of two spheres that are each represented by vectors; thus, lines and circles are represented by bivectors.

The intersection of two line entities cannot be computed by their outer product. Any two lines that intersect are coplanar and their outer product is zero.

An embedded test point $\mathbf{T}$ is tested for intersection with GIPNS line entity $\mathbf{L}$ by evaluating their inner product as

$$\begin{aligned}\mathbf{T}\cdot\mathbf{L} &= \mathbf{t}\cdot\mathbf{D}-\mathbf{t}\cdot((\mathbf{p}\cdot\mathbf{D})\wedge\mathbf{e}_\infty)-\mathbf{e}_o\cdot((\mathbf{p}\cdot\mathbf{D})\wedge\mathbf{e}_\infty)\\ &= \mathbf{t}\cdot\mathbf{D}-(\mathbf{t}\cdot(\mathbf{p}\cdot\mathbf{D}))\wedge\mathbf{e}_\infty-\mathbf{p}\cdot\mathbf{D}\\ &= (\mathbf{t}-\mathbf{p})\cdot\mathbf{D}-((\mathbf{t}\wedge\mathbf{p})\cdot\mathbf{D})\mathbf{e}_\infty.\end{aligned}$$

The GIPNS condition $\mathbf{T}\cdot\mathbf{L}=0$ is satisfied if $(\mathbf{t}-\mathbf{p})\cdot\mathbf{D}=0$. Under this condition, it follows that $\mathbf{t}\wedge\mathbf{p}$ and $\mathbf{D}$ are orthogonal bivectors where $(\mathbf{t}\wedge\mathbf{p})\cdot\mathbf{D}=0$, and the unduals $(\mathbf{t}\wedge\mathbf{p})\mathbf{I}_3$ and $\mathbf{D}\mathbf{I}_3=\mathbf{d}$ and are orthogonal vectors.

The magnitude

$$\begin{aligned}|\mathbf{T}\cdot\mathbf{L}| &= \sqrt{((\mathbf{t}-\mathbf{p})\cdot\mathbf{D})^2+(((\mathbf{t}\wedge\mathbf{p})\cdot\mathbf{D})\mathbf{e}_\infty)^2}\\ &= \sqrt{((\mathbf{t}-\mathbf{p})\cdot\mathbf{D})^2}\\ &= \sqrt{(\mathcal{P}_{\mathbf{D}}(\mathbf{t}-\mathbf{p}))^2}\end{aligned}$$

is the distance of $\mathbf{T}$ from line $\mathbf{L}$.

3.2.7 QGA and CGA GIPNS Plane

The homogeneous equation of a plane through three Euclidean 3D points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ can be written as

$$(\mathbf{t}-\mathbf{p}_1)\wedge(\mathbf{p}_3-\mathbf{p}_1)\wedge(\mathbf{p}_2-\mathbf{p}_1) = 0$$

or as

$$(\mathbf{t}-\mathbf{p}_1)\cdot(((\mathbf{p}_3-\mathbf{p}_1)\wedge(\mathbf{p}_2-\mathbf{p}_1))\cdot\mathbf{I}_3) = 0$$

where $\mathbf{t}$ is a test point. If point $\mathbf{t}$ is on the plane, then the equation holds good. The unit bivector $\mathbf{N}$ of the plane can be defined by

$$\mathbf{N} = \frac{(\mathbf{p}_3-\mathbf{p}_1)\wedge(\mathbf{p}_2-\mathbf{p}_1)}{|(\mathbf{p}_3-\mathbf{p}_1)\wedge(\mathbf{p}_2-\mathbf{p}_1)|}.$$

The plane normal unit vector $\mathbf{n}$ can be defined by the undual

$$\mathbf{n} = \mathbf{N}\cdot\mathbf{I}_3.$$

The points $\mathbf{p}_1$ to $\mathbf{p}_2$ to $\mathbf{p}_3$ can be assumed as arranged counter-clockwise, right-handed, when viewed from *above* their plane, and $\mathbf{n}$ points into the *above* half-space of the plane. The bivector $\mathbf{N}$ represents the quaternion quadrantal versor, or rotor of the plane that rotates by half-angle $\pi/2$ or full-angle π in rotor operations.

We can choose any point in the plane, and call it $\mathbf{p}$. The inner product $\mathbf{p}\cdot\mathbf{n}=d$ is the distance of the plane from the origin and is a constant for all points $\mathbf{p}$ in the plane. The plane equation can now be written as

$$\begin{aligned}\mathbf{t}\cdot\mathbf{n}-\mathbf{p}\cdot\mathbf{n} &= (\mathbf{t}-\mathbf{p})\cdot\mathbf{n}=0\\ \mathbf{t}\cdot\mathbf{n}-d &= 0.\end{aligned}$$

The QGA embedding of test point $\mathbf{t}$ is

$$\mathbf{T} = \mathbf{t}+\frac{1}{2}x^2\mathbf{e}_{\infty x}+\frac{1}{2}y^2\mathbf{e}_{\infty y}+\frac{1}{2}z^2\mathbf{e}_{\infty z}+\mathbf{e}_o.$$

Let $\mathbf{\Pi}$ be the QGA GIPNS plane entity. Point $\mathbf{T}$ is on plane $\mathbf{\Pi}$ if the GIPNS condition $\mathbf{T}\cdot\mathbf{\Pi}=0$ is satisfied. The vector $\mathbf{\Pi}$ can be constructed as

$$\mathbf{\Pi} = \mathbf{n}+d\mathbf{e}_\infty$$

such that

$$\begin{aligned}\mathbf{T}\cdot\mathbf{\Pi} &= \mathbf{t}\cdot\mathbf{n}+\mathbf{e}_o\cdot(d\mathbf{e}_\infty)\\ &= \mathbf{t}\cdot\mathbf{n}-d.\end{aligned}$$

This form of the GIPNS plane entity $\mathbf{\Pi}$ is the same in both QGA and CGA. The magnitude $|\mathbf{T}\cdot\mathbf{\Pi}|$ is the distance of point $\mathbf{T}$ from plane $\mathbf{\Pi}$.

Given two planes $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$, their outer product represents their intersection line $\mathbf{L}$ as

$$\begin{aligned}\mathbf{L} &= \mathbf{\Pi}_1\wedge\mathbf{\Pi}_2\\ &= (\mathbf{n}_1+d_1\mathbf{e}_\infty)\wedge(\mathbf{n}_2+d_2\mathbf{e}_\infty)\\ &= \mathbf{n}_1\wedge\mathbf{n}_2+d_2\mathbf{n}_1\mathbf{e}_\infty+d_1\mathbf{e}_\infty\mathbf{n}_2\\ &= \mathbf{n}_1\wedge\mathbf{n}_2+(d_2\mathbf{n}_1-d_1\mathbf{n}_2)\mathbf{e}_\infty\\ &= \mathbf{D}-(\mathbf{p}\cdot\mathbf{D})\mathbf{e}_\infty\end{aligned}$$

where

$$\begin{aligned}\mathbf{p}\cdot\mathbf{D} &= (\mathbf{p}\cdot\mathbf{n}_1)\mathbf{n}_2-(\mathbf{p}\cdot\mathbf{n}_2)\mathbf{n}_1\\ &= d_1\mathbf{n}_2-d_2\mathbf{n}_1\end{aligned}$$

such that $\mathbf{p}$ is any point on both planes (the line), and $\mathbf{D}$ is the bivector perpendicular to the line. A test point $\mathbf{T}$ is on the line (both planes) only if

$$\mathbf{T}\cdot\mathbf{L} = \mathbf{T}\cdot(\mathbf{\Pi}_1\wedge\mathbf{\Pi}_2)=0.$$

A GIPNS line $\mathbf{L}$ is the GIPNS intersection entity $\mathbf{\Pi}_1\wedge\mathbf{\Pi}_2$ of two GIPNS planes. This serves as a specific example of the general formula or operation by which GIPNS intersection entities are constructed. The outer product of GIPNS entities forms a new GIPNS intersection entity that represents their geometrical intersection. The intersection operation does not produce a correct or useful intersection entity for all combinations of entities, but it is known to produce useful intersections for most combinations. As an example mentioned previously, the outer product of two lines cannot produce an intersection entity. If two lines intersect

$$\mathbf{L}_1\wedge\mathbf{L}_2 = (\mathbf{\Pi}_1\wedge\mathbf{\Pi}_2)\wedge(\mathbf{\Pi}_3\wedge\mathbf{\Pi}_4)$$

then the lines are coplanar, having at least one plane in common, and therefore have a zero or null outer product.

Sphere $\mathbf{S}$, of radius r and center position $\mathbf{p}=(d+r)\mathbf{n}$, touches the plane $\mathbf{\Pi}$ tangentially and is written as

$$\begin{aligned}\mathbf{S} &= \mathbf{p}+\frac{1}{2}(\mathbf{p}^2-r^2)\mathbf{e}_\infty+\mathbf{e}_o\\ &= (d+r)\mathbf{n}+\frac{1}{2}((d+r)^2-r^2)\mathbf{e}_\infty+\mathbf{e}_o\\ &= (d+r)\mathbf{n}+\frac{1}{2}(d^2+2dr)\mathbf{e}_\infty+\mathbf{e}_o.\end{aligned}$$

Homogeneously dividing $\mathbf{S}$ by r, and taking the limit as $r \to \infty$,

$$\begin{aligned}
\lim_{r\to\infty}\frac{\mathbf{S}}{r} &= \lim_{r\to\infty}\frac{(d+r)\mathbf{n}+\frac{1}{2}(d^2+2dr)\mathbf{e}_\infty+\mathbf{e}_o}{r}\\
&= \mathbf{n}+d\mathbf{e}_\infty\\
&= \mathbf{\Pi}
\end{aligned}$$

this sphere $\mathbf{S}$ becomes equal to the plane $\mathbf{\Pi}$.

3.2.8 QGA and CGA GIPNS Circle

A circle can be defined by a center position $\mathbf{p}$, radius r, and unit vector $\mathbf{n}$ that is normal to the plane of the circle. A test point $\mathbf{t}$ is on the circle if it satisfies the two equations

$$\begin{aligned}
(\mathbf{t}-\mathbf{p})^2 &= r^2\\
&= \mathbf{t}^2+\mathbf{p}^2-\mathbf{t}\mathbf{p}-\mathbf{p}\mathbf{t}\\
&= -2\mathbf{t}\cdot\mathbf{p}+\mathbf{t}^2+\mathbf{p}^2\\
&= -2(\mathbf{T}_\mathcal{C}\cdot\mathbf{P}_\mathcal{C})\\
(\mathbf{t}-\mathbf{p})\cdot\mathbf{n} &= 0\\
&= \mathbf{t}\cdot\mathbf{n}-\mathbf{p}\cdot\mathbf{n}\\
&= \mathbf{t}\cdot\mathbf{n}-d\\
&= \mathbf{T}_\mathcal{C}\cdot\mathbf{\Pi}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_\mathcal{C} &= \mathbf{t}+\frac{1}{2}\mathbf{t}^2\mathbf{e}_\infty+\mathbf{e}_o\\
\mathbf{P}_\mathcal{C} &= \mathbf{p}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty+\mathbf{e}_o\\
\mathbf{\Pi} &= \mathbf{n}+d\mathbf{e}_\infty=\mathbf{n}+(\mathbf{p}\cdot\mathbf{n})\mathbf{e}_\infty.
\end{aligned}$$

The two equations are not linear in $(\mathbf{t}-\mathbf{p})$, and cannot be added. The first equation can be rewritten as the homogeneous equation

$$\begin{aligned}
-2(\mathbf{T}_\mathcal{C}\cdot\mathbf{P}_\mathcal{C})-r^2 &= -2\mathbf{T}_\mathcal{C}\cdot\left(\mathbf{p}+\frac{1}{2}(\mathbf{p}^2-r^2)\,\mathbf{e}_\infty+\mathbf{e}_o\right)\\
&= -2\mathbf{T}_\mathcal{C}\cdot\mathbf{S}=0\\
\mathbf{T}_\mathcal{C}\cdot\mathbf{S} &= 0.
\end{aligned}$$

The two homogeneous equations now say that $\mathbf{T}_\mathcal{C}$ must be on the intersection of the sphere $\mathbf{S}$ and the plane $\mathbf{\Pi}$, which can be expressed as

$$\mathbf{T}_\mathcal{C}\cdot(\mathbf{S}\wedge\mathbf{\Pi}) = 0.$$

Therefore, the GIPNS circle entity $\mathbf{C}$ is

$$\mathbf{C} = \mathbf{S}\wedge\mathbf{\Pi}.$$

The plane $\mathbf{\Pi}$ is the limit of a sphere with infinite radius, and circle $\mathbf{C}$ can also be expressed as the intersection of two spheres

$$\mathbf{C} = \mathbf{S}_1 \wedge \mathbf{S}_2.$$

Given the plane $\mathbf{\Pi}_1$ through $\mathbf{a}$ and normal to unit vector $\mathbf{n}_1$,

$$\mathbf{\Pi}_1 = \mathbf{n}_1 + (\mathbf{a} \cdot \mathbf{n}_1)\mathbf{e}_\infty$$

and the sphere $\mathbf{S}$ also through $\mathbf{a}$ with radius r and center position $\mathbf{p} = \mathbf{a} + r\mathbf{n}_2$,

$$\begin{aligned}
\mathbf{S} &= (\mathbf{a} + r\mathbf{n}_2) + \frac{1}{2}((\mathbf{a} + r\mathbf{n}_2)^2 - r^2)\mathbf{e}_\infty + \mathbf{e}_o \\
&= (\mathbf{a} + r\mathbf{n}_2) + \frac{1}{2}(\mathbf{a}^2 + 2r\mathbf{a} \cdot \mathbf{n}_2)\mathbf{e}_\infty + \mathbf{e}_o
\end{aligned}$$

then the circle $\mathbf{C}$ through $\mathbf{a}$ is given by the intersection

$$\mathbf{C} = \mathbf{S} \wedge \mathbf{\Pi}_1.$$

In the limit as $r \to \infty$, the sphere $\mathbf{S}$ becomes the plane $\mathbf{\Pi}_2$

$$\begin{aligned}
\lim_{r\to\infty} \frac{\mathbf{S}}{r} &= \lim_{r\to\infty} \frac{(\mathbf{a} + r\mathbf{n}_2) + \frac{1}{2}(\mathbf{a}^2 + 2r\mathbf{a} \cdot \mathbf{n}_2)\mathbf{e}_\infty + \mathbf{e}_o}{r} \\
&= \mathbf{n}_2 + (\mathbf{a} \cdot \mathbf{n}_2)\mathbf{e}_\infty \\
&= \mathbf{\Pi}_2
\end{aligned}$$

and the circle $\mathbf{C}$ becomes the line $\mathbf{L}$ through $\mathbf{a}$

$$\begin{aligned}
\lim_{r\to\infty} \mathbf{C} &= \mathbf{L} = \mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2 \\
&= (\mathbf{n}_1 + (\mathbf{a} \cdot \mathbf{n}_1)\mathbf{e}_\infty) \wedge (\mathbf{n}_2 + (\mathbf{a} \cdot \mathbf{n}_2)\mathbf{e}_\infty) \\
&= \mathbf{n}_1 \wedge \mathbf{n}_2 + (\mathbf{a} \cdot \mathbf{n}_2)\mathbf{n}_1 \wedge \mathbf{e}_\infty + (\mathbf{a} \cdot \mathbf{n}_1)\mathbf{e}_\infty \wedge \mathbf{n}_2 \\
&= \mathbf{n}_1 \wedge \mathbf{n}_2 + ((\mathbf{a} \cdot \mathbf{n}_2)\mathbf{n}_1 - (\mathbf{a} \cdot \mathbf{n}_1)\mathbf{n}_2) \wedge \mathbf{e}_\infty \\
&= \mathbf{D} - (\mathbf{a} \cdot \mathbf{D})\mathbf{e}_\infty.
\end{aligned}$$

With the assumption that no radius r is zero, then QGA points can substitute for the CGA 6,3D points that have been used in this section, such that these results also hold good in QGA.

3.2.9 QGA GIPNS Cylinder

An axes-aligned elliptic cylinder is the limit of an ellipsoid as one of the semi-diameters approaches ∞. This limit eliminates the terms of the cylinder axis from the homogeneous ellipsoid equation.

The x-axis aligned cylinder takes $r_x \to \infty$, reducing the ellipsoid equation to

$$\frac{(y-p_y)^2}{r_y^2} + \frac{(z-p_z)^2}{r_z^2} - 1 = 0.$$

Similarly, the y-axis and z-axis aligned cylinders are

$$\begin{aligned}
\frac{(x-p_x)^2}{r_x^2} + \frac{(z-p_z)^2}{r_z^2} - 1 &= 0 \\
\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - 1 &= 0
\end{aligned}$$

where $\mathbf{p} = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3$ is the position (or shifted origin, or center) of the ellipsoid, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c).

Similarly to the GIPNS ellipsoid entity, the GIPNS cylinder entities are given by eliminating the cylinder axis components from the ellipsoid entity.

The GIPNS x-axis aligned cylinder entity is

$$\mathbf{H}^{||x} = -\frac{2p_y}{r_y^2}\mathbf{e}_2 - \frac{2p_z}{r_z^2}\mathbf{e}_3 - \frac{2}{r_y^2}\mathbf{e}_{oy} - \frac{2}{r_z^2}\mathbf{e}_{oz} - \left(\frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - 1\right)\mathbf{e}_\infty.$$

The GIPNS y-axis aligned cylinder entity is

$$\mathbf{H}^{||y} = -\frac{2p_x}{r_x^2}\mathbf{e}_1 - \frac{2p_z}{r_z^2}\mathbf{e}_3 - \frac{2}{r_x^2}\mathbf{e}_{ox} - \frac{2}{r_z^2}\mathbf{e}_{oz} - \left(\frac{p_x^2}{r_x^2} + \frac{p_z^2}{r_z^2} - 1\right)\mathbf{e}_\infty.$$

The GIPNS z-axis aligned cylinder entity is

$$\mathbf{H}^{||z} = -\frac{2p_x}{r_x^2}\mathbf{e}_1 - \frac{2p_y}{r_y^2}\mathbf{e}_2 - \frac{2}{r_x^2}\mathbf{e}_{ox} - \frac{2}{r_y^2}\mathbf{e}_{oy} - \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - 1\right)\mathbf{e}_\infty.$$

3.2.10 QGA GIPNS Cone

An axis-aligned elliptic cone is an axis-aligned cylinder that is linearly scaled along the axis.

The homogeneous equation for an x-axis aligned cone is

$$\frac{(y-p_y)^2}{r_y^2} + \frac{(z-p_z)^2}{r_z^2} - \frac{(x-p_x)^2}{r_x^2} = 0.$$

where $\mathbf{p} = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3$ is the position (or shifted origin, or center) of the cone apex, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c) of the ellipsoid upon which the cone is based. When

$$\frac{(x-p_x)^2}{r_x^2} = 1$$

the cross section of the cone is the size of the similar cylinder. When $x = p_x$ the cross section of the cone is degenerated into the cone apex point.

Similarly, the homogeneous equations for y-axis and z-axis aligned cones are

$$\begin{aligned}\frac{(x-p_x)^2}{r_x^2} + \frac{(z-p_z)^2}{r_z^2} - \frac{(y-p_y)^2}{r_y^2} &= 0\\ \frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)^2}{r_z^2} &= 0.\end{aligned}$$

The GIPNS cone entities are constructed similarly to the ellipsoid and cylinder entities.

The GIPNS x-axis aligned cone entity is

$$\mathbf{V}^{||x} = \frac{2p_x}{r_x^2}\mathbf{e}_1 - \frac{2p_y}{r_y^2}\mathbf{e}_2 - \frac{2p_z}{r_z^2}\mathbf{e}_3 + \frac{2}{r_x^2}\mathbf{e}_{ox} - \frac{2}{r_y^2}\mathbf{e}_{oy} - \frac{2}{r_z^2}\mathbf{e}_{oz} - \left(\frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - \frac{p_x^2}{r_x^2}\right)\mathbf{e}_\infty.$$

The GIPNS y-axis aligned cone entity is

$$\mathbf{V}^{\|y} = \frac{2p_y}{r_y^2}\mathbf{e}_2 - \frac{2p_x}{r_x^2}\mathbf{e}_1 - \frac{2p_z}{r_z^2}\mathbf{e}_3 + \frac{2}{r_y^2}\mathbf{e}_{oy} - \frac{2}{r_x^2}\mathbf{e}_{ox} - \frac{2}{r_z^2}\mathbf{e}_{oz} - \left(\frac{p_x^2}{r_x^2} + \frac{p_z^2}{r_z^2} - \frac{p_y^2}{r_y^2}\right)\mathbf{e}_\infty.$$

The GIPNS z-axis aligned cone entity is

$$\mathbf{V}^{\|z} = \frac{2p_z}{r_z^2}\mathbf{e}_3 - \frac{2p_x}{r_x^2}\mathbf{e}_1 - \frac{2p_y}{r_y^2}\mathbf{e}_2 + \frac{2}{r_z^2}\mathbf{e}_{oz} - \frac{2}{r_x^2}\mathbf{e}_{ox} - \frac{2}{r_y^2}\mathbf{e}_{oy} - \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2}\right)\mathbf{e}_\infty.$$

3.2.11 QGA GIPNS Elliptic Paraboloid

The elliptic paraboloid has a cone-like shape that opens up or down. The other paraboloid that would open the other way is imaginary with no real solution points.

The homogeneous equation of a z-axis aligned elliptic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)}{r_z} = 0.$$

The surface opens up the z-axis for $r_z > 0$, and opens down the z-axis for $r_z < 0$. Similar equations for x-axis and y-axis aligned elliptic paraboloids are

$$\begin{aligned}
\frac{(z-p_z)^2}{r_z^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(x-p_x)}{r_x} &= 0\\
\frac{(x-p_x)^2}{r_x^2} + \frac{(z-p_z)^2}{r_z^2} - \frac{(y-p_y)}{r_y} &= 0.
\end{aligned}$$

Expanding the squares, the z-axis aligned equation is

$$\frac{-2p_x x}{r_x^2} + \frac{-2p_y y}{r_y^2} + \frac{-z}{r_z} + \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) = 0$$

and the x-axis and y-axis aligned equations are

$$\begin{aligned}
\frac{-2p_z z}{r_z^2} + \frac{-2p_y y}{r_y^2} + \frac{-x}{r_x} + \frac{z^2}{r_z^2} + \frac{y^2}{r_y^2} + \left(\frac{p_z^2}{r_z^2} + \frac{p_y^2}{r_y^2} + \frac{p_x}{r_x}\right) &= 0\\
\frac{-2p_x x}{r_x^2} + \frac{-2p_z z}{r_z^2} + \frac{-y}{r_y} + \frac{x^2}{r_x^2} + \frac{z^2}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_z^2}{r_z^2} + \frac{p_y}{r_y}\right) &= 0.
\end{aligned}$$

The QGA embedding of the test point $\mathbf{t}$ is

$$\mathbf{T} = \mathbf{t} + \frac{1}{2}x^2\mathbf{e}_{\infty x} + \frac{1}{2}y^2\mathbf{e}_{\infty y} + \frac{1}{2}z^2\mathbf{e}_{\infty z} + \mathbf{e}_o.$$

The GIPNS x-axis aligned elliptic paraboloid entity is constructed as

$$\mathbf{K}^{\|x} = \frac{-2p_z}{r_z^2}\mathbf{e}_3 + \frac{-2p_y}{r_y^2}\mathbf{e}_2 + \frac{-1}{r_x}\mathbf{e}_1 + \frac{-2}{r_z^2}\mathbf{e}_{oz} + \frac{-2}{r_y^2}\mathbf{e}_{oy} - \left(\frac{p_z^2}{r_z^2} + \frac{p_y^2}{r_y^2} + \frac{p_x}{r_x}\right)\mathbf{e}_\infty.$$

The GIPNS y-axis aligned elliptic paraboloid entity is constructed as

$$\mathbf{K}^{\|y} = \frac{-2p_z}{r_z^2}\mathbf{e}_3 + \frac{-2p_x}{r_x^2}\mathbf{e}_1 + \frac{-1}{r_y}\mathbf{e}_2 + \frac{-2}{r_z^2}\mathbf{e}_{oz} + \frac{-2}{r_x^2}\mathbf{e}_{ox} - \left(\frac{p_z^2}{r_z^2} + \frac{p_x^2}{r_x^2} + \frac{p_y}{r_y}\right)\mathbf{e}_\infty.$$

The GIPNS z-axis aligned elliptic paraboloid entity is constructed as

$$\mathbf{K}^{\|z} = \frac{-2p_x}{r_x^2}\mathbf{e}_1 + \frac{-2p_y}{r_y^2}\mathbf{e}_2 + \frac{-1}{r_z}\mathbf{e}_3 + \frac{-2}{r_x^2}\mathbf{e}_{ox} + \frac{-2}{r_y^2}\mathbf{e}_{oy} - \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right)\mathbf{e}_\infty.$$

3.2.12 QGA GIPNS Hyperbolic Paraboloid

The hyperbolic paraboloid has a saddle shape. The saddle can be mounted or aligned on a *saddle* axis with another axis chosen as the *up* axis. The third axis may be called the *straddle* axis.

The homogeneous equation of a hyperbolic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)}{r_z} = 0.$$

This particular form of the equation has saddle x-axis, straddle y-axis, and up z-axis for $r_z > 0$ or up negative z-axis for $r_z < 0$. By its similarity to the z-axis aligned elliptic paraboloid with the elliptic y-axis inverted, this particular form can be seen as z-axis aligned. Other forms can be made by transposing axes using the transposition operations.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} + \frac{-z}{r_z} + \frac{x^2}{r_x^2} + \frac{-y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) = 0.$$

The QGA embedding of the test point $\mathbf{t}$ is

$$\mathbf{T} = \mathbf{t} + \frac{1}{2}x^2\mathbf{e}_{\infty x} + \frac{1}{2}y^2\mathbf{e}_{\infty y} + \frac{1}{2}z^2\mathbf{e}_{\infty z} + \mathbf{e}_o.$$

The QGA GIPNS hyperbolic paraboloid entity is constructed as

$$\mathbf{Q} = \frac{-2p_x}{r_x^2}\mathbf{e}_1 + \frac{2p_y}{r_y^2}\mathbf{e}_2 + \frac{-1}{r_z}\mathbf{e}_3 + \frac{-2}{r_x^2}\mathbf{e}_{ox} + \frac{2}{r_y^2}\mathbf{e}_{oy} - \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right)\mathbf{e}_\infty.$$

3.2.13 QGA GIPNS Hyperboloid of One Sheet

The hyperboloid of one sheet has a shape that is similar to an hourglass which continues to open both upward and downward. The homogeneous equation is

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)^2}{r_z^2} - 1 = 0.$$

This particular form opens up and down the z-axis. Planes parallel to the z-axis cut hyperbola sections. Planes perpendicular to the z-axis cut ellipse sections. At $z = p_z$, the ellipse section has a minimum size of the similar cylinder. Other forms can be made by transposing axes using the transposition operations.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2} + \frac{-2p_y y}{r_y^2} + \frac{2p_z z}{r_z^2} + \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{-z^2}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right) = 0.$$

The QGA embedding of the test point $\mathbf{t}$ is

$$\mathbf{T} = \mathbf{t} + \frac{1}{2}x^2\mathbf{e}_{\infty x} + \frac{1}{2}y^2\mathbf{e}_{\infty y} + \frac{1}{2}z^2\mathbf{e}_{\infty z} + \mathbf{e}_o.$$

The QGA GIPNS hyperboloid of one sheet entity is constructed as

$$\mathbf{Q} = 2\left(-\frac{p_x\mathbf{e}_1}{r_x^2} - \frac{p_y\mathbf{e}_2}{r_y^2} + \frac{p_z\mathbf{e}_3}{r_z^2} - \frac{\mathbf{e}_{ox}}{r_x^2} - \frac{\mathbf{e}_{oy}}{r_y^2} + \frac{\mathbf{e}_{oz}}{r_z^2}\right) - \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right)\mathbf{e}_\infty.$$

3.2.14 QGA GIPNS Hyperboloid of Two Sheets

The hyperboloid of two sheets has the shapes of two separate hyperbolic dishes; one opens upward, and the other one opens downward. The shape is like an hourglass that is pinched closed and the two halves are also separated by some distance. The homogeneous equation is

$$-\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)^2}{r_y^2} + \frac{(z-p_z)^2}{r_z^2} - 1 = 0.$$

This particular form has the two dishes opening up and down the z-axis. The dishes are separated by distance $2r_z$ centered at p_z. At $|z - p_z| = \sqrt{2}r_z$, the sections perpendicular to the z-axis are the size of the similar cylinder.

Expanding the squares, the equation is

$$\frac{2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} - \frac{2p_z z}{r_z^2} - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} + \frac{z^2}{r_z^2} + \left(\frac{-p_x^2}{r_x^2} + \frac{-p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - 1\right) = 0.$$

The QGA embedding of the test point $\mathbf{t}$ is

$$\mathbf{T} = \mathbf{t} + \frac{1}{2}x^2\mathbf{e}_{\infty x} + \frac{1}{2}y^2\mathbf{e}_{\infty y} + \frac{1}{2}z^2\mathbf{e}_{\infty z} + \mathbf{e}_o.$$

The QGA GIPNS hyperboloid of two sheets entity is constructed as

$$\mathbf{Q} = 2\left(\frac{p_x\mathbf{e}_1}{r_x^2} + \frac{p_y\mathbf{e}_2}{r_y^2} - \frac{p_z\mathbf{e}_3}{r_z^2} + \frac{\mathbf{e}_{ox}}{r_x^2} + \frac{\mathbf{e}_{oy}}{r_y^2} - \frac{\mathbf{e}_{oz}}{r_z^2}\right) - \left(\frac{-p_x^2}{r_x^2} + \frac{-p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - 1\right)\mathbf{e}_\infty.$$

3.2.15 QGA GIPNS Parabolic Cylinder

The homogeneous equation for the z-axis aligned parabolic cylinder is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)}{r_y} = 0.$$

The z coordinate is free, which creates a type of 2-surface or cylinder with parabolic sections that open up the y-axis for $r_y > 0$, and open down the y-axis for $r_y < 0$. The similar equations for x-axis and y-axis aligned parabolic cylinders are

$$\frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)}{r_z} = 0$$
$$\frac{(x-p_x)^2}{r_x^2} - \frac{(z-p_z)}{r_z} = 0$$

with parabolas that open up or down the z-axis. Other forms can be made by transpositions using the transposition operations.

Expanding the squares, the equations are

$$\begin{aligned}
\frac{-2p_x x}{r_x^2} - \frac{y}{r_y} + \frac{x^2}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y}{r_y}\right) &= 0 \\
\frac{-2p_y y}{r_y^2} - \frac{z}{r_z} + \frac{y^2}{r_y^2} + \left(\frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) &= 0 \\
\frac{-2p_x x}{r_x^2} - \frac{z}{r_z} + \frac{x^2}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_z}{r_z}\right) &= 0.
\end{aligned}$$

The QGA embedding of the test point $\mathbf{t}$ is

$$\mathbf{T} = \mathbf{t} + \frac{1}{2}x^2\mathbf{e}_{\infty x} + \frac{1}{2}y^2\mathbf{e}_{\infty y} + \frac{1}{2}z^2\mathbf{e}_{\infty z} + \mathbf{e}_o.$$

The GIPNS z-axis aligned parabolic cylinder opening up the y-axis is constructed as

$$\mathbf{B}^{\|z} = \frac{-2p_x}{r_x^2}\mathbf{e}_1 - \frac{\mathbf{e}_2}{r_y} + \frac{-2}{r_x^2}\mathbf{e}_{ox} - \left(\frac{p_x^2}{r_x^2} + \frac{p_y}{r_y}\right)\mathbf{e}_\infty.$$

The GIPNS x-axis aligned parabolic cylinder opening up the z-axis is constructed as

$$\mathbf{B}^{\|x} = \frac{-2p_y}{r_y^2}\mathbf{e}_2 - \frac{\mathbf{e}_3}{r_z} + \frac{-2}{r_y^2}\mathbf{e}_{oy} - \left(\frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right)\mathbf{e}_\infty.$$

The GIPNS y-axis aligned parabolic cylinder opening up the z-axis is constructed as

$$\mathbf{B}^{\|y} = \frac{-2p_x}{r_x^2}\mathbf{e}_1 - \frac{\mathbf{e}_3}{r_z} + \frac{-2}{r_x^2}\mathbf{e}_{ox} - \left(\frac{p_x^2}{r_x^2} + \frac{p_z}{r_z}\right)\mathbf{e}_\infty.$$

The intersection of a parabolic cylinder and plane $\mathbf{B}\wedge\mathbf{\Pi}$ is a GIPNS parabolic string or 1-surface.

3.2.16 QGA GIPNS Parallel Planes Pair

Parallel pairs of axes-aligned planes are represented by the simple quadratic equations in one variable

$$\begin{aligned}
(x - p_{x1})(x - p_{x2}) &= 0 \\
(y - p_{y1})(y - p_{y2}) &= 0 \\
(z - p_{z1})(z - p_{z2}) &= 0.
\end{aligned}$$

Each solution is a plane. Expanding the equations gives

$$\begin{aligned}
x^2 - (p_{x1} + p_{x2})x + p_{x1}p_{x2} &= 0 \\
y^2 - (p_{y1} + p_{y2})y + p_{y1}p_{y2} &= 0 \\
z^2 - (p_{z1} + p_{z2})z + p_{z1}p_{z2} &= 0.
\end{aligned}$$

The GIPNS parallel planes pair entity perpendicular to the x-axis is

$$\mathbf{\Pi}^{\perp x} = -(p_{x1} + p_{x2})\,\mathbf{e}_1 - 2\mathbf{e}_{ox} - (p_{x1}p_{x2})\,\mathbf{e}_\infty.$$

The GIPNS parallel planes pair entity perpendicular to the x-axis is

$$\mathbf{\Pi}^{\perp y} = -(p_{y1}+p_{y2})\,\mathbf{e}_2 - 2\mathbf{e}_{oy} - (p_{y1}p_{y2})\,\mathbf{e}_\infty.$$

The GIPNS parallel planes pair entity perpendicular to the x-axis is

$$\mathbf{\Pi}^{\perp z} = -(p_{z1}+p_{z2})\,\mathbf{e}_3 - 2\mathbf{e}_{oz} - (p_{z1}p_{z2})\,\mathbf{e}_\infty.$$

These surfaces can also be described as being types of cylinders with cross sections being two parallel lines.

3.2.17 QGA GIPNS Non-parallel Planes Pair

The homogeneous equation for a pair of intersecting, non-parallel planes that are parallel to the z-axis is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)^2}{r_y^2} = 0.$$

This equation can be written as

$$(y-p_y) = \pm\frac{r_y}{r_x}(x-p_x)$$

with the z coordinate free to range. This surface can also be described as a kind of cylinder with a cross section in plane z that is two lines with slopes $\pm\frac{r_y}{r_x}$ intersecting at $\mathbf{p} = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + z\,\mathbf{e}_3$.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} + \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2}\right) = 0.$$

The GIPNS *non-parallel planes pair* entity parallel to the z-axis is

$$\mathbf{X}^{\|z} = \frac{-2p_x}{r_x^2}\,\mathbf{e}_1 + \frac{2p_y}{r_y^2}\,\mathbf{e}_2 - \frac{2}{r_x^2}\,\mathbf{e}_{ox} + \frac{2}{r_y^2}\,\mathbf{e}_{oy} - \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2}\right)\mathbf{e}_\infty.$$

The entities parallel to the x-axis and y-axis are constructed similarly, or can be created by transpositions.

3.2.18 QGA GIPNS Ellipse

The GIPNS ellipse entity is similar to the GIPNS circle entity. An ellipse can be constructed by the intersection of a plane with an ellipsoidal entity.

The GIPNS ellipse $\mathbf{C}$, constructed by intersection of GIPNS ellipsoid $\mathbf{S}$ and GIPNS plane $\mathbf{\Pi}$, is

$$\mathbf{C} = \mathbf{S} \wedge \mathbf{\Pi}.$$

The GIPNS ellipse $\mathbf{C}$, constructed by intersection of GIPNS elliptic cone $\mathbf{V}$ and GIPNS plane $\mathbf{\Pi}$, is

$$\mathbf{C} = \mathbf{V} \wedge \mathbf{\Pi}.$$

The GIPNS ellipse $\mathbf{C}$, constructed by intersection of GIPNS elliptic cylinder $\mathbf{H}$ and GIPNS plane $\mathbf{\Pi}$, is

$$\mathbf{C} = \mathbf{H} \wedge \mathbf{\Pi}.$$

The GIPNS ellipse $\mathbf{C}$, constructed by intersection of GIPNS elliptic paraboloid $\mathbf{K}$ and GIPNS plane $\mathbf{\Pi}$, is

$$\mathbf{C} = \mathbf{K} \wedge \mathbf{\Pi}.$$

When neither of the intersecting surfaces is a plane, the intersections of the quadric surfaces are quadric strings that may or may not be planar. There are some special cases in which two quadric surfaces intersect in circles or ellipses.

QGA GIPNS entities for other conic sections, such as parabola and hyperbola, are the GIPNS intersections of various GIPNS 2D surfaces.

3.2.19 QGA GIPNS Parallel Circles Pair

In general, the intersection of two ellipsoids is one or two quadric strings. However, there is a special case where two ellipsoids of revolution intersect in one circle or in a pair of parallel circles that may have different radii.

Given two ellipsoids of revolution $\mathbf{S}_1$ and $\mathbf{S}_2$ that are circular in the same plane and situated at intersecting positions along the same axial line, their intersection $\mathbf{C}$ is

$$\mathbf{C} = \mathbf{S}_1 \wedge \mathbf{S}_2$$

where $\mathbf{C}$ represents one circle or a pair of parallel circles that may have differing radii. For the case where the intersection is in a parallel circles pair, then $\mathbf{C}$ can be called a GIPNS parallel circles pair entity.

Using a symbolic test point $\mathbf{T}$, a symbolic computation $\mathbf{T} \cdot \mathbf{C}$ shows that $\mathbf{C}$ also represents the intersections of various *other* surfaces that are related to surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ by having the same circular intersections.

3.3 GOPNS Surfaces

3.3.1 Outer Product Null Space

The Outer Product Null Space (OPNS) of k-vector $\mathbf{N}$, denoted $\mathbb{NO}(\mathbf{N} \in \mathcal{G}^k)$, is defined as

$$\begin{aligned} \mathbb{NO}(\mathbf{N} \in \mathcal{G}^k) &= \left\{ \mathbf{t} \in \mathcal{G}^1 \ : \ \mathbf{t} \wedge \mathbf{N} = 0 \right\} \\ &= \mathrm{span}(\mathbf{N}). \end{aligned}$$

For example, if $\mathbf{N}$ is a bivector in 3D Euclidean space, then its OPNS is the set of all vectors in the plane spanned by $\mathbf{N}$.

The duality between OPNS and IPNS is

$$\mathbb{NI}(\mathbf{n} \in \mathcal{G}^k_{p,q}) = \mathbb{NO}((\mathbf{n}^* = \mathbf{N}) \in \mathcal{G}^{n-k}_{p,q})$$

where the $(n-k)$-vector $\mathbf{n}^* = \mathbf{N}$ is the dual of k-vector $\mathbf{n}$, and $n = p + q$ is the dimension of the vector space that contains all k-vectors $\mathbf{n}$.

The Geometric Outer Product Null Space (GOPNS) of k-vector $\mathbf{S}^*$, denoted $\mathbb{NO}_G(\mathbf{S}^* \in \mathcal{G}^k)$, is defined as

$$\begin{aligned}\mathbb{NO}_G(\mathbf{S}^* \in \mathcal{G}^k) &= \left\{ \mathbf{t} \in \mathcal{G}_3^1 \;:\; \mathcal{H}_M(\mathcal{S}(\mathbf{t})) \wedge \mathbf{S}^* = 0 \right\}\\ &= \operatorname{span}_G(\mathbf{S}^*).\end{aligned}$$

The GOPNS entity $\mathbf{S}^*$ is the dual of the GIPNS entity $\mathbf{S}$ that represents the same geometric surface. The set is restricted to 3D Euclidean vectors since the GOPNS represents a geometric surface in 3D. Although $\mathbf{S}^*$ is generally a k-vector, it is constructed as a k-blade wedge of k points. The GOPNS is also the geometric span over the surface represented by $\mathbf{S}^*$.

3.3.2 QGA and CGA GOPNS Points

The QGA GOPNS point entity is simply an QGA embedded point $\mathbf{P}$, which can also be named $\mathbf{P}_{\mathcal{Q}}$ to more clearly indicate a QGA point. The CGA 6,3D GOPNS point entity is also simply an CGA 6,3D embedded point $\mathbf{P}_{\mathcal{C}}$. Points are null vectors

$$\begin{aligned}\mathbf{PP} &= \mathbf{P} \cdot \mathbf{P} + \mathbf{P} \wedge \mathbf{P}\\ &= 0\end{aligned}$$

and are both a GIPNS and GOPNS representation of a point. The dual of a point gives another form of the point, but it is just another GIPNS and GOPNS representation of the same point. Points could also be called Geometric Product Null Space (GPNS) entities.

3.3.3 CGA GOPNS Point Pair

The CGA 6,3D GOPNS *point pair* 2-vector entity $\mathbf{2}_{\mathcal{C}}$ is the wedge of two CGA 6,3D points $\mathbf{P}_{\mathcal{C}_i}$

$$\begin{aligned}\mathbf{2}_{\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2}\\ &= (\mathbf{S}_1 \wedge \mathbf{S}_2 \wedge \mathbf{S}_3)/\mathbf{I}_{\mathcal{C}}\end{aligned}$$

and is the conformal dual of the GIPNS intersection of three GIPNS sphere 1-vector entities $\mathbf{S}_i$.

3.3.4 QGA GOPNS Point Pair

The QGA 6,3D GOPNS *point pair* 2-vector entity $\mathbf{2}_{\mathcal{Q}}$ is the wedge of two QGA points $\mathbf{P}_{\mathcal{Q}_i}$

$$\begin{aligned}\mathbf{2}_{\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_2}\\ &= (\mathbf{S}_1 \wedge \mathbf{S}_2 \wedge \mathbf{S}_3 \wedge \mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2) \cdot \mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the GIPNS intersection of three GIPNS sphere 1-vector entities $\mathbf{S}_i$ and two GIPNS plane 1-vector entities $\mathbf{\Pi}_i$ that meet in the point pair.

3.3.5 CGA GOPNS Circle

The CGA 6,3D GOPNS *circle* 3-vector entity $\mathbf{C}^{*\mathcal{C}}$ is the wedge of *any three* CGA 6,3D points $\mathbf{P}_{\mathcal{C}_i}$ on the circle

$$\begin{aligned}\mathbf{C}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1}\wedge\mathbf{P}_{\mathcal{C}_2}\wedge\mathbf{P}_{\mathcal{C}_3}\\ &= \mathbf{C}/\mathbf{I}_{\mathcal{C}} = -(\mathbf{S}\wedge\mathbf{\Pi})\cdot\mathbf{I}_{\mathcal{C}}\end{aligned}$$

and is the conformal dual of the GIPNS circle 2-vector entity $\mathbf{C}$. The GIPNS circle $\mathbf{C}$ is the GIPNS intersection of a GIPNS sphere 1-vector $\mathbf{S}$ and GIPNS plane 1-vector $\mathbf{\Pi}$. This is the same GOPNS circle formula as in 4,1D CGA.

The CGA GOPNS circle entity $\mathbf{C}^{*\mathcal{C}}$ can be rotated using standard rotor operations.

It can be verified that $\mathbf{C}^{*\mathcal{C}}$ represents the circle by expanding the GOPNS condition of the set

$$\mathbb{NO}_G(\mathbf{C}^{*\mathcal{C}}) = \left\{ \mathbf{t}\in\mathcal{G}^1_3 \ : \ \mathbf{T}\wedge\mathbf{C}^{*\mathcal{C}} = 0 \right\}$$

where the point $\mathbf{T}$ is the embedding of $\mathbf{t}$. In this case,

$$\mathbf{T} = \mathbf{T}_{\mathcal{C}} = \mathcal{C}_{6,3}(\mathbf{t}).$$

All other GOPNS entities can also be similarly verified by expanding the GOPNS condition of their GOPNS set. The expansions lead to the standard formulas for the geometric surfaces that the entities represent.

3.3.6 QGA GOPNS Circle

The QGA GOPNS *circle* can be constructed as a 4-vector or 5-vector entity.

The QGA 6,3D GOPNS *circle* 4-vector entity $\mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle}$ is the wedge of *any four* QGA points $\mathbf{P}_{\mathcal{Q}_i}$ on the circle

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\\ &= \mathbf{C}_{\langle 3\rangle}\cdot\mathbf{I}_{io} = (\mathbf{S}_1\wedge\mathbf{S}_2\wedge\mathbf{\Pi})\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the GIPNS circle 3-vector entity $\mathbf{C}_{\langle 3\rangle}$. The GIPNS circle $\mathbf{C}_{\langle 3\rangle}$ is the GIPNS intersection of two GIPNS sphere 1-vector entities $\mathbf{S}_i$ and a GIPNS plane 1-vector entity $\mathbf{\Pi}$ that meet in the circle.

The QGA GOPNS circle entity $\mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle}$ generally *cannot* be rotated by standard rotor operations. The QGA points $\mathbf{P}_{\mathcal{Q}_i}$ generally cannot be rotated by standard rotor operations, and therefore the versor outermorphism of $\mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle}$ using standard rotor operations also does not rotate the points or the circle entity.

See also, the QGA GOPNS *ellipse*. The circle is only a special case of the ellipse. Based on the QGA GOPNS ellipse entity, there is also a QGA GOPNS *circle* 5-vector entity $\mathbf{C}^{*\mathcal{Q}}$ that is the wedge of *any four* QGA points $\mathbf{P}_{\mathcal{Q}_i}$ on the circle and the general point at infinity $\mathbf{e}_\infty$

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{e}_\infty\\ &= \mathbf{C}\cdot\mathbf{I}_{io} = (\mathbf{S}\wedge\mathbf{\Pi})\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the GIPNS circle 2-vector entity $\mathbf{C}$. The GIPNS circle $\mathbf{C}$ is the GIPNS intersection of a GIPNS sphere 1-vector $\mathbf{S}$ and GIPNS plane 1-vector $\mathbf{\Pi}$.

The 4-vector entity can be transformed into the 5-vector entity

$$\mathbf{C}^{*\mathcal{Q}} = \mathbf{C}^{*\mathcal{Q}}_{\langle 3 \rangle} \wedge \mathbf{e}_\infty$$

but the inverse transformation is not possible since $\mathbf{e}_\infty$ has no inverse.

3.3.7 CGA GOPNS Line

The CGA GOPNS *line* 3-vector entity $\mathbf{L}^{*\mathcal{C}}$ is a circle entity where one of the three CGA 6,3D points is the point at infinity $\mathbf{e}_\infty$

$$\begin{aligned} \mathbf{L}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{e}_\infty \\ &= \mathbf{L}/\mathbf{I}_\mathcal{C} = -(\mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2) \cdot \mathbf{I}_\mathcal{C} \end{aligned}$$

and represents the line through the two CGA 6,3D points. It is the conformal dual of the GIPNS line 2-vector entity $\mathbf{L}$, which is the GIPNS intersection of two GIPNS plane 1-vector entities $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$.

The line can be described as a circle of infinite radius, touching infinity and through the two points. In the neighborhood of any finite point on the circle, the circle appears linear.

3.3.8 CGA GOPNS Sphere

The CGA 6,3D GOPNS *sphere* 4-vector entity $\mathbf{S}^{*\mathcal{C}}$ is the wedge of any four CGA 6,3D points, not all coplanar, on the surface of the sphere

$$\begin{aligned} \mathbf{S}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3} \wedge \mathbf{P}_{\mathcal{C}_4} \\ &= \mathbf{S}/\mathbf{I}_\mathcal{C} = -\mathbf{S} \cdot \mathbf{I}_\mathcal{C} \end{aligned}$$

and is the conformal dual of the GIPNS sphere 1-vector entity $\mathbf{S}$.

3.3.9 CGA GOPNS Plane

The CGA 6,3D GOPNS *plane* 4-vector entity $\mathbf{\Pi}^{*\mathcal{C}}$ is a sphere entity where one of the three CGA 6,3D points is the point at infinity $\mathbf{e}_\infty$

$$\begin{aligned} \mathbf{\Pi}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3} \wedge \mathbf{e}_\infty \\ &= \mathbf{\Pi}/\mathbf{I}_\mathcal{C} = -\mathbf{\Pi} \cdot \mathbf{I}_\mathcal{C} \end{aligned}$$

and is the conformal dual of the GIPNS plane 1-vector entity $\mathbf{\Pi}$.

The plane can be described as a sphere through the three points with the fourth point at infinity. The sphere is then of infinite radius and appears flat or planar in the neighborhood of any finite point on the sphere.

3.3.10 QGA GOPNS Ellipsoid

The QGA GOPNS *ellipsoid* 6-vector entity $\mathbf{S}^{*\mathcal{Q}}$ with center position

$$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$$

and principal axes semi-diameters r_x, r_y, r_z has the six QGA surface points

$$\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}(\mathbf{p}+r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}(\mathbf{p}-r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}(\mathbf{p}+r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}(\mathbf{p}-r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_5} &= \mathcal{Q}(\mathbf{p}+r_z\mathbf{e}_3)\\
\mathbf{P}_{\mathcal{Q}_6} &= \mathcal{Q}(\mathbf{p}-r_z\mathbf{e}_3)
\end{aligned}$$

and is given by their wedge

$$\begin{aligned}
\mathbf{S}^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2})\wedge(\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4})\wedge(\mathbf{P}_{\mathcal{Q}_5}\wedge\mathbf{P}_{\mathcal{Q}_6})\\
&= \mathbf{S}\cdot\mathbf{I}_{io}
\end{aligned}$$

which is also the QGA dual of the QGA GIPNS ellipsoid 1-vector entity $\mathbf{S}$. The parentheses are used only to emphasize that a point pair can be conceptualized as a pair along an axis.

The six QGA surface points can be any six surface points that span the 2D ellipsoid surface. Therefore, the six points of the surface cannot be all coplanar. At least one point must be out of the plane of the other points. Coplanar points can span only a 1D surface or ellipse.

If the semi-diameters are equal, then this is a QGA GOPNS sphere 6-vector entity. This is a different formula of construction for a sphere entity than the CGA 6,3D GOPNS sphere 4-vector entity which is constructed from just four CGA 6,3D points.

The QGA 2D quadric surface entities, which are based on the ellipsoid entity, can represent only axes-aligned 2D quadric surfaces. The QGA points and QGA 2D quadric surfaces generally cannot be rotated. The surface points used to construct a QGA GOPNS 2D quadric surface entity generally cannot be the surface points of a rotated 2D quadric surface, but must be the surface points of an axes-aligned 2D quadric surface. Attempting to construct a QGA GOPNS 2D quadric surface from rotated surface points which are no longer on an axes-aligned 2D quadric surface will not produce an entity that represents the expected, rotated surface.

However, the QGA 1D surface entities can generally represent rotated conics. QGA GOPNS 1D surface entities can be constructed as the wedge of rotated surface points.

3.3.11 QGA GOPNS Cylinder

Conceptually, the elliptic cylinder is an ellipsoid with one principal axis of infinite length. Compared to the formula for the QGA GOPNS ellipsoid, one point is replaced with a point at infinity. The two point pairs can define the elliptic cross section of the cylinder. In this transformation of an ellipsoid into a cylinder, the fifth point that is paired with a point at infinity must become also a point on the cylinder surface, but it is a rather arbitrary surface point when the other four are coplanar and used to define the cylinder cross section.

The QGA GOPNS *cylinder* 6-vector entities are defined as

$$\begin{aligned}
(\mathbf{H}^{||x})^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{e}_{\infty x}) \wedge (\mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{P}_4) \wedge (\mathbf{P}_{\mathcal{Q}_5} \wedge \mathbf{P}_{\mathcal{Q}_6}) \\
(\mathbf{H}^{||y})^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_2}) \wedge (\mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{e}_{\infty y}) \wedge (\mathbf{P}_{\mathcal{Q}_5} \wedge \mathbf{P}_{\mathcal{Q}_6}) \\
(\mathbf{H}^{||z})^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_2}) \wedge (\mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{P}_4) \wedge (\mathbf{P}_{\mathcal{Q}_5} \wedge \mathbf{e}_{\infty z})
\end{aligned}$$

for cylinders parallel with the x, y, z axis, respectively. These are the QGA duals of the QGA GIPNS cylinder 1-vector entities. The five QGA points in each formula must be any five points, not all coplanar, on the surface of the cylinder.

It is possible to generate an *elliptic*, *parabolic*, *hyperbolic*, or *non-parallel planes pair* cylinder.

For an *elliptic cylinder*, four points can be the sides midpoints of a box that contains an ellipse section perpendicular to the cylinder axis.

For a *parabolic cylinder*, four points can be the corner points of a trapezium contained in a parabola section .

For a *hyperbolic cylinder*, four points can be the two vertices and two other points on one of the branches of a hyperbola section.

The fifth point for the three above cylinder types can be any one of the four points translated by any non-zero distance $d\mathbf{e}_1, d\mathbf{e}_2$ or $d\mathbf{e}_3$ for a cylinder parallel to x, y or z axis, respectively.

For a *non-parallel planes pair* cylinder, the five points can be a cross of points that includes the intersection point, but the points cannot all be coplanar. At least one point must be translated an arbitrary non-zero distance parallel to the cylinder axis. A general rule for surface points that span a 2D surface is that the points cannot be all coplanar.

Many different choices for the five surface points are possible. The surface points must be a set of five points that *span* the 2D surface. Therefore, up to four surface points may be coplanar or in the same cylinder section, and at least one point must be in another parallel section of the cylinder.

The QGA GOPNS 1D surface entities for ellipse, parabola, and hyperbola (conic sections) are very similar to these QGA GOPNS 2D surface cylinder entities. The 1D entities require only the four points on the 1D section, plus the more-general point at infinity $\mathbf{e}_\infty$. The 1D conics are more general since they need not be axes-aligned.

The axis-specific infinity points $\mathbf{e}_{\infty x}$, $\mathbf{e}_{\infty y}$, $\mathbf{e}_{\infty z}$ in the formulas for the QGA GOPNS cylinder 6-vector entities *cannot* be replaced by the more-general $\mathbf{e}_\infty$, and doing so will create an entity that is not the expected cylinder.

3.3.12 QGA GOPNS Parallel Planes Pair

The QGA GOPNS *parallel planes pair* 6-vector entities are defined as

$$\begin{aligned}
(\mathbf{\Pi}^{\perp x})^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_3}) \wedge (\mathbf{e}_{\infty y} \wedge \mathbf{e}_{\infty z}) \wedge (\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_4}) \\
(\mathbf{\Pi}^{\perp y})^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_3}) \wedge (\mathbf{e}_{\infty x} \wedge \mathbf{e}_{\infty z}) \wedge (\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_4}) \\
(\mathbf{\Pi}^{\perp z})^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_3}) \wedge (\mathbf{e}_{\infty x} \wedge \mathbf{e}_{\infty y}) \wedge (\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_4})
\end{aligned}$$

for pairs of axes-aligned parallel planes perpendicular to the x, y, z axes, respectively. These are the QGA duals of the QGA GIPNS parallel planes pair 1-vector entities.

These can be interpreted as axes-aligned ellipsoids that have two infinite principal axes or diameters, and one finite principal axis. The ellipsoid flattens into a pair of parallel planes separated by, and perpendicular to, the finite principal axis. The finite principal axis is sandwiched between the two planes.

The finite principal axis is parallel to the axis which does not have its point at infinity wedged.

The point pair $\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_3}$ must be any two points of a plane perpendicular to the finite principal axis.

The point pair $\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_4}$ must be any two points of *another* parallel plane also perpendicular to the finite principal axis.

The two points $\mathbf{P}_{\mathcal{Q}_1}$ and $\mathbf{P}_{\mathcal{Q}_2}$ may be used to define the finite principal axis or diameter that separates the two parallel planes. If these two points are different positions on any axis-aligned line parallel with the finite principal axis, then the line segment directly between them can be assumed as *the* finite principal axis; the finite principal axis is *any* perpendicular line segment sandwiched between the two parallel planes, or it can be the particular line segment between $\mathbf{P}_{\mathcal{Q}_1}$ and $\mathbf{P}_{\mathcal{Q}_2}$. And then, the points $\mathbf{P}_{\mathcal{Q}_3}$ and $\mathbf{P}_{\mathcal{Q}_4}$ are arbitrary points of the planes through $\mathbf{P}_{\mathcal{Q}_1}$ and $\mathbf{P}_{\mathcal{Q}_2}$, respectively, perpendicular to the finite principal axis between $\mathbf{P}_{\mathcal{Q}_1}$ and $\mathbf{P}_{\mathcal{Q}_2}$.

For example, if

$$
\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}(3\mathbf{e}_1) \\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}(5\mathbf{e}_1) \\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}(3\mathbf{e}_1 + \alpha\mathbf{e}_2 + \beta\mathbf{e}_3) \\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}(5\mathbf{e}_1 + \alpha\mathbf{e}_2 + \beta\mathbf{e}_3)
\end{aligned}
$$

then

$$
(\mathbf{\Pi}^{\perp x})^{*\mathcal{Q}} = (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_3}) \wedge (\mathbf{e}_{\infty y} \wedge \mathbf{e}_{\infty z}) \wedge (\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_4})
$$

gives a representation of the pair of parallel planes $3\mathbf{e}_1$ and $5\mathbf{e}_1$. The arbitrary points $\mathbf{P}_{\mathcal{Q}_3}$ and $\mathbf{P}_{\mathcal{Q}_4}$, with arbitrary scalars α and β (not both zero) for each, are coplanar with $\mathbf{P}_{\mathcal{Q}_1}$ and $\mathbf{P}_{\mathcal{Q}_2}$, respectively. The formula is similar to make other parallel planes pairs.

Same as the QGA GIPNS parallel planes pair entity, the QGA GOPNS parallel planes pair entity represents a standard quadratic equation in one variable x, y or z, such as

$$
(x - x_1)(x - x_2) = 0
$$

where the two solutions x_1 and x_2 represent parallel planes that cut the x-axis through each solution.

3.3.13 QGA GOPNS Plane

The QGA GOPNS *plane* 6-vector entity $\mathbf{\Pi}^{*\mathcal{Q}}$ is given by the wedge of three QGA points on the plane with the three points at infinity

$$
\begin{aligned}
\mathbf{\Pi}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{e}_{\infty x} \wedge \mathbf{e}_{\infty y} \wedge \mathbf{e}_{\infty z} \\
&= \mathbf{\Pi} \cdot \mathbf{I}_{io}
\end{aligned}
$$

and is the QGA dual of the GIPNS plane 1-vector entity $\mathbf{\Pi}$.

This can be interpreted as an ellipsoid or sphere through the three fixed finite points and through three other points that slide to infinity, causing the shape to flatten into a plane through the three fixed finite points.

3.3.14 QGA GOPNS Elliptic Paraboloid

The GOPNS entities for the elliptic paraboloid, hyperbolic paraboloid, and hyperboloids of one and two sheets are constructed similarly to an ellipsoid, as the wedge of any six surface points that span the surface. This section, and the next three, gives details on the construction of these GOPNS 2D surface entities.

The homogeneous equation of a z-axis aligned elliptic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}-\frac{(z-p_z)}{r_z} \;=\; 0.$$

The paraboloid opens up the z-axis for $r_z > 0$, and opens down for $r_z < 0$. The vertex of the paraboloid is (p_x, p_y, p_z).

For a *paraboloid of revolution* or *circular paraboloid* with $r = r_x = r_y$, the focal length is $f = \frac{1}{4}\frac{r^2}{r_z}$, and the focal point is $(p_x, p_y, p_z + f)$.

The QGA GOPNS *z-axis aligned elliptic paraboloid* 6-vector entity $(\mathbf{K}^{||z})^{*\mathcal{Q}}$ with vertex position

$$\mathbf{p} \;=\; p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$$

and principal axes semi-diameters r_x, r_y, r_z has the six QGA surface points

$$\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}(\mathbf{p})\\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}(\mathbf{p}+r_z\mathbf{e}_3 - r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}(\mathbf{p}+r_z\mathbf{e}_3 + r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}(\mathbf{p}+r_z\mathbf{e}_3 - r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_5} &= \mathcal{Q}(\mathbf{p}+r_z\mathbf{e}_3 + r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_6} &= \mathbf{e}_{\infty z}
\end{aligned}$$

and is given by their wedge

$$\begin{aligned}
(\mathbf{K}^{||z})^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1} \wedge (\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{P}_{\mathcal{Q}_4} \wedge \mathbf{P}_{\mathcal{Q}_5}) \wedge \mathbf{e}_{\infty z}\\
&= \mathbf{K}^{||z} \cdot \mathbf{I}_{io}
\end{aligned}$$

which is also the QGA dual of the QGA GIPNS z-axis aligned elliptic paraboloid 1-vector entity $\mathbf{K}^{||z}$.

The z-axis aligned elliptic paraboloid opens up the z-axis for $r_z > 0$, and opens down for $r_z < 0$. The wedge of four points in parentheses represents an ellipse 4-vector entity at distance r_z from the vertex. Conceptually, the sixth point $\mathbf{e}_{\infty z}$ represents the edge of an ellipsoid that is located at infinity.

The conic sections, in planes perpendicular to the z-axis of the z-axis aligned elliptic paraboloid, are ellipses.

The other two very similar x-axis and y-axis aligned elliptic paraboloids can be obtained using transposition operations, but for completeness they are explicity given as follows.

The QGA GOPNS *x-axis aligned elliptic paraboloid* 6-vector entity $(\mathbf{K}^{||x})^{*\mathcal{Q}}$ with vertex position $\mathbf{p}$ and principal axes semi-diameters r_x, r_y, r_z has the six QGA surface points

$$\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}(\mathbf{p})\\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}(\mathbf{p}+r_x\mathbf{e}_1-r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}(\mathbf{p}+r_x\mathbf{e}_1+r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}(\mathbf{p}+r_x\mathbf{e}_1-r_z\mathbf{e}_3)\\
\mathbf{P}_{\mathcal{Q}_5} &= \mathcal{Q}(\mathbf{p}+r_x\mathbf{e}_1+r_z\mathbf{e}_3)\\
\mathbf{P}_{\mathcal{Q}_6} &= \mathbf{e}_{\infty x}
\end{aligned}$$

and is given by their wedge

$$\begin{aligned}
(\mathbf{K}^{||x})^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge(\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{P}_{\mathcal{Q}_5})\wedge\mathbf{e}_{\infty x}\\
&= \mathbf{K}^{||x}\cdot\mathbf{I}_{io}
\end{aligned}$$

which is also the QGA dual of the QGA GIPNS x-axis aligned elliptic paraboloid 1-vector entity $\mathbf{K}^{||x}$.

The x-axis aligned elliptic paraboloid opens up the x-axis for $r_x > 0$, and opens down for $r_x < 0$. The wedge of four points in parentheses represents an ellipse 4-vector entity at distance r_x from the vertex. Conceptually, the sixth point $\mathbf{e}_{\infty x}$ represents the edge of an ellipsoid that is located at infinity.

The QGA GOPNS *y-axis aligned elliptic paraboloid* 6-vector entity $(\mathbf{K}^{||y})^{*\mathcal{Q}}$ with vertex position $\mathbf{p}$ and principal axes semi-diameters r_x, r_y, r_z has the six QGA surface points

$$\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}(\mathbf{p})\\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}(\mathbf{p}+r_y\mathbf{e}_2-r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}(\mathbf{p}+r_y\mathbf{e}_2+r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}(\mathbf{p}+r_y\mathbf{e}_2-r_z\mathbf{e}_3)\\
\mathbf{P}_{\mathcal{Q}_5} &= \mathcal{Q}(\mathbf{p}+r_y\mathbf{e}_2+r_z\mathbf{e}_3)\\
\mathbf{P}_{\mathcal{Q}_6} &= \mathbf{e}_{\infty y}
\end{aligned}$$

and is given by their wedge

$$\begin{aligned}
(\mathbf{K}^{||y})^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge(\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{P}_{\mathcal{Q}_5})\wedge\mathbf{e}_{\infty y}\\
&= \mathbf{K}^{||y}\cdot\mathbf{I}_{io}
\end{aligned}$$

which is also the QGA dual of the QGA GIPNS z-axis aligned elliptic paraboloid 1-vector entity $\mathbf{K}^{||y}$.

The y-axis aligned elliptic paraboloid opens up the y-axis for $r_y > 0$, and opens down for $r_y < 0$. The wedge of four points in parentheses represents an ellipse 4-vector entity at distance r_y from the vertex. Conceptually, the sixth point $\mathbf{e}_{\infty y}$ represents the edge of an ellipsoid that is located at infinity.

3.3.15 QGA GOPNS Hyperbolic Paraboloid

The hyperbolic paraboloid is described as having a saddle-like shape. It can also be visualized as the elliptic paraboloid where one of the elliptic axes has been turned negatively into the opposite direction. This can be visualized as grabbing two edges on the rim of the elliptic parabola and pulling them down while the other two edges remain up; the paraboloid is flexed into a saddle-like shape that is a half-inverted elliptic paraboloid. The GOPNS hyperbolic paraboloid can be formulated very similarly to the GOPNS elliptic paraboloid.

The homogeneous equation of a hyperbolic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)}{r_z} = 0.$$

The QGA GOPNS *z-axis aligned hyperbolic paraboloid* 6-vector entity $(\mathbf{Q}^{||z})^{*\mathcal{Q}}$ with vertex position $\mathbf{p}$ and principal axes semi-diameters r_x, r_y, r_z has the six QGA surface points

$$\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}(\mathbf{p}) \\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}(\mathbf{p} \pm r_z\mathbf{e}_3 - r_x\mathbf{e}_1) \\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}(\mathbf{p} \pm r_z\mathbf{e}_3 + r_x\mathbf{e}_1) \\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}(\mathbf{p} \mp r_z\mathbf{e}_3 - r_y\mathbf{e}_2) \\
\mathbf{P}_{\mathcal{Q}_5} &= \mathcal{Q}(\mathbf{p} \mp r_z\mathbf{e}_3 + r_y\mathbf{e}_2) \\
\mathbf{P}_{\mathcal{Q}_6} &= \mathbf{e}_{\infty z}
\end{aligned}$$

and is given by their wedge

$$\begin{aligned}
(\mathbf{Q}^{||z})^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1} \wedge (\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{P}_{\mathcal{Q}_4} \wedge \mathbf{P}_{\mathcal{Q}_5}) \wedge \mathbf{e}_{\infty z} \\
&= \mathbf{Q}^{||z} \cdot \mathbf{I}_{io}
\end{aligned}$$

which is also the QGA dual of the QGA GIPNS z-axis aligned hyperbolic paraboloid 1-vector entity $\mathbf{Q}^{||z}$.

There is a choice of signs to select which elliptic axis is *up*, and which is down. The elliptic axis that is up, with components $+r_z\mathbf{e}_3$, is the *saddle* axis on which the saddle-like shape is mounted. The elliptic axis that is down, with components $-r_z\mathbf{e}_3$, is the *straddle* axis. For $r_z > 0$, the *up* direction is $\mathbf{e}_3$; and for $r_z < 0$, the *up* direction is $-\mathbf{e}_3$ and the shape is relatively upside down.

The wedge of four points in parentheses represents a curved ellipse 4-vector entity centered at distance r_z from the vertex. Conceptually, the sixth point $\mathbf{e}_{\infty z}$ represents the edge of a half-inverted ellipsoid that is located at infinity.

The conic sections, in planes perpendicular to the z-axis of the z-axis aligned hyperbolic paraboloid, are hyperbolas.

The other two very similar x-axis and y-axis aligned hyperbolic paraboloids can be obtained by similar wedges of any six surface points on the hyperbolic paraboloid, or can be obtained by using transposition operations. The x-axis aligned hyperbolic paraboloid must wedge $\mathbf{e}_{\infty x}$, and the y-axis aligned hyperbolic paraboloid must wedge $\mathbf{e}_{\infty y}$.

3.3.16 QGA GOPNS Hyperboloid of One Sheet

The hyperboloid of one sheet has a shape that is similar to an hourglass that opens up and down the axis around which the shape is aligned.

The GOPNS entity for this surface is the wedge of any six surface points that span the surface. Each principal axis or diameter $2r_x, 2r_y, 2r_z$ must be spanned by a pair of surface points on lines parallel with the x, y, z axes, respectively. Two pairs may be coplanar and represent an ellipse or hyperbola. The third pair, on a line perpendicular to the ellipse or hyperbola, cannot have a point on the ellipse or hyperbola.

The homogeneous equation for a hyperboloid of one sheet is

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)^2}{r_z^2} - 1 \;=\; 0.$$

The QGA GOPNS *z-axis aligned hyperboloid of one sheet* 6-vector entity $(\mathbf{Q}^{||z})^{*\mathcal{Q}}$ with center position

$$\mathbf{p} \;=\; p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$$

and principal axes semi-diameters r_x, r_y, r_z has the six QGA surface points

$$\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}\big(\mathbf{p} + r_z\mathbf{e}_3 + \sqrt{2}r_x\mathbf{e}_1 + \sqrt{2}r_y\mathbf{e}_2\big)\\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}(\mathbf{p} - r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}(\mathbf{p} + r_x\mathbf{e}_1)\\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}(\mathbf{p} - r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_5} &= \mathcal{Q}(\mathbf{p} + r_y\mathbf{e}_2)\\
\mathbf{P}_{\mathcal{Q}_6} &= \mathcal{Q}\big(\mathbf{p} - r_z\mathbf{e}_3 + \sqrt{2}r_x\mathbf{e}_1 + \sqrt{2}r_y\mathbf{e}_2\big)
\end{aligned}$$

and is given by their wedge

$$\begin{aligned}
(\mathbf{Q}^{||z})^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1} \wedge (\mathbf{P}_{\mathcal{Q}_2} \wedge \mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{P}_{\mathcal{Q}_4} \wedge \mathbf{P}_{\mathcal{Q}_5}) \wedge \mathbf{P}_{\mathcal{Q}_6}\\
&= \mathbf{Q}^{||z} \cdot \mathbf{I}_{io}
\end{aligned}$$

which is also the QGA dual of the QGA GIPNS z-axis aligned hyperboloid of one sheet 1-vector entity $\mathbf{Q}^{||z}$.

In the plane $z = p_z$, the conic section is the ellipse with semi-diameters r_x and r_y, represented by the wedge of the four surface points in parentheses. One pair spans the $2r_x$ diameter on a line parallel with the x-axis. Another pair spans the $2r_y$ diameter on a line parallel with the y-axis.

The third pair of surface points spans the $2r_z$ diameter. The pair is taken from the planes $z = p_z \pm r_z$ which spans the $2r_z$ diameter. To span the $2r_z$ diameter, the pair is on a line parallel to the z-axis.

Note that, a pair of surface points that are *not* on a line parallel with the z-axis will *not* span the $2r_z$ diameter; a similar statement is true for the pairs that must span the $2r_x$ and $2r_y$ diameters. Also note that, if five points are coplanar, they must lie on an ellipse or hyperbola, and the wedge of five ellipse or hyperbola points is zero; therefore, a maximum of four coplanar points can be wedged.

The other two very similar x-axis and y-axis aligned hyperboloids of one sheet can be obtained using transposition operations or a similar wedge of six points.

3.3.17 QGA GOPNS Hyperboloid of Two Sheets

The hyperboloid of two sheets has a shape of two dishes that open up and down the axis around which the shape is aligned. It can also be seen as the hyperboloid of one sheet which has been inverted to become upside down and inside out.

The GOPNS entity for this surface is the wedge of any six surface points that span the surface. Each principal axis or diameter $2r_x, 2r_y, 2r_z$ must be spanned by a pair of surface points on lines parallel with the x, y, z axes, respectively. Two pairs may be coplanar and represent an ellipse or hyperbola. The third pair, on a line perpendicular to the ellipse or hyperbola, cannot have a point on the ellipse or hyperbola.

The homogeneous equation for a hyperboloid of two sheets is

$$-\frac{(x-p_x)^2}{r_x^2}-\frac{(y-p_y)^2}{r_y^2}+\frac{(z-p_z)^2}{r_z^2}-1 \;=\; 0.$$

The two dishes or branches open up and down the z-axis. The hyperboloid center point is (p_x, p_y, p_z). The vertices are $(p_x, p_y, p_z \pm r_z)$.

For a *hyperboloid of revolution* or *circular hyperboloid* with $r = r_x = r_y$, the linear eccentricity is $c = \sqrt{r^2 + r_z^2}$, the distance between foci is $2c$, the foci are $(p_x, p_y, p_z \pm c)$, and the focal length is $f = c - |r_z|$.

The QGA GOPNS *z-axis aligned hyperboloid of two sheets* 6-vector entity $(\mathbf{Q}^{||z})^{*\mathcal{Q}}$ with center position

$$\mathbf{p} \;=\; p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$$

and principal axes semi-diameters r_x, r_y, r_z has the six QGA surface points

$$\begin{aligned}
\mathbf{P}_{\mathcal{Q}_1} &= \mathcal{Q}\big(\mathbf{p} + \sqrt{2}r_z\mathbf{e}_3 - r_x\mathbf{e}_1\big)\\
\mathbf{P}_{\mathcal{Q}_2} &= \mathcal{Q}\big(\mathbf{p} + \sqrt{2}r_z\mathbf{e}_3 + r_x\mathbf{e}_1\big)\\
\mathbf{P}_{\mathcal{Q}_3} &= \mathcal{Q}\big(\mathbf{p} + \sqrt{2}r_z\mathbf{e}_3 - r_y\mathbf{e}_2\big)\\
\mathbf{P}_{\mathcal{Q}_4} &= \mathcal{Q}\big(\mathbf{p} + \sqrt{2}r_z\mathbf{e}_3 + r_y\mathbf{e}_2\big)\\
\mathbf{P}_{\mathcal{Q}_5} &= \mathcal{Q}(\mathbf{p} + r_z\mathbf{e}_3)\\
\mathbf{P}_{\mathcal{Q}_6} &= \mathcal{Q}(\mathbf{p} - r_z\mathbf{e}_3)
\end{aligned}$$

and is given by their wedge

$$\begin{aligned}
(\mathbf{Q}^{||z})^{*\mathcal{Q}} &= (\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_2}) \wedge (\mathbf{P}_{\mathcal{Q}_3} \wedge \mathbf{P}_{\mathcal{Q}_4} \wedge \mathbf{P}_{\mathcal{Q}_5} \wedge \mathbf{P}_{\mathcal{Q}_6})\\
&= \mathbf{Q}^{||z} \cdot \mathbf{I}_{io}
\end{aligned}$$

which is also the QGA dual of the QGA GIPNS z-axis aligned hyperboloid of one sheet 1-vector entity $\mathbf{Q}^{||z}$.

In the plane $z = p_z + \sqrt{2}r_z$, the conic section is the ellipse with semi-diameters r_x and r_y, represented by the wedge of the four surface points in parentheses. One pair spans the $2r_x$ diameter on a line parallel with the x-axis. Another pair spans the $2r_y$ diameter on a line parallel with the y-axis.

The third pair of surface points is the vertices which spans the $2r_z$ diameter.

Note that, a pair of surface points that are *not* on a line parallel with the z-axis will *not* span the $2r_z$ diameter; a similar statement is true for the pairs that must span the $2r_x$ and $2r_y$ diameters. Also note that, if five points are coplanar, they must lie on an ellipse or hyperbola, and the wedge of five ellipse or hyperbola points is zero; therefore, a maximum of four coplanar points can be wedged.

The other two very similar x-axis and y-axis aligned hyperboloids of two sheets can be obtained using transposition operations or a similar wedge of six points.

3.3.18 QGA GOPNS Ellipse

The ellipse is the first of the QGA GOPNS 1D planar surfaces, and it serves as the base upon which the other QGA GOPNS 1D planar surfaces are constructed. There are also slightly different, and more-general, QGA GOPNS 1D non-planar curved surfaces.

The homogeneous equation for an ellipse is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}-1 \;=\; 0.$$

The center point of the ellipse is (p_x, p_y). The focal length is $f=\sqrt{|r_x^2-r_y^2|}$. For $r_x \geq r_y$, the foci are $(p_x \pm f, p_y)$; and for $r_y \geq r_x$, the foci are $(p_x, p_y \pm f)$.

The QGA GOPNS *ellipse* 5-vector entity $\mathbf{C}^{*\mathcal{Q}}$ is given by the wedge of four QGA points on the ellipse and the point at infinity

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{e}_\infty\\ &= \mathbf{C}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the standard QGA GIPNS ellipse 2-vector entity $\mathbf{C}$. The four points of an ellipse can be the sides midpoints of the box that contains the ellipse.

The QGA GOPNS *ellipse* 4-vector entity $\mathbf{C}_{\langle 3\rangle}^{*\mathcal{Q}}$ is given by the wedge of four QGA points on the ellipse

$$\begin{aligned}\mathbf{C}_{\langle 3\rangle}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\\ &= \mathbf{C}_{\langle 3\rangle}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the QGA GIPNS ellipse 3-vector entity $\mathbf{C}_{\langle 3\rangle}$, which could be constructed as the QGA GIPNS intersection of three QGA GIPNS 2D surfaces that meet in the ellipse.

The 4-vector entity can be transformed into the 5-vector entity

$$\mathbf{C}^{*\mathcal{Q}} \;=\; \mathbf{C}_{\langle 3\rangle}^{*\mathcal{Q}}\wedge\mathbf{e}_\infty$$

but the inverse transformation is not possible since $\mathbf{e}_\infty$ has no inverse.

In general, the QGA GOPNS ellipse entities and all other QGA entities cannot be rotated by rotors since QGA points do not rotate correctly by rotors. However, it is also possible to represent and rotate conic sections in the $\mathcal{G}_6$ geometric algebra introduced by Perwass. The $\mathcal{G}_6$ geometric algebra has its own type of point embedding

$$\begin{aligned}\mathbf{P}_{\mathcal{D}} &= \mathcal{D}(\mathbf{p})=\mathcal{D}(x\mathbf{e}_1+y\mathbf{e}_2)\\ &= x\mathbf{e}_1+y\mathbf{e}_2+\frac{1}{\sqrt{2}}\mathbf{e}_3+\frac{1}{\sqrt{2}}x^2\mathbf{e}_4+\frac{1}{\sqrt{2}}y^2\mathbf{e}_5+xy\mathbf{e}_6\end{aligned}$$

that is different than CGA and QGA points, and has only entities for 1D conic sections in the xy-plane. This algebra could be called DGA, and it has a rotor $\mathbf{R}$

$$\begin{aligned}\mathbf{R} &= \mathbf{R}_2\mathbf{R}_1\\ \mathbf{R}_1 &= \cos(\theta) - \frac{1}{\sqrt{2}}\sin(\theta)(\mathbf{e}_4 - \mathbf{e}_5)\wedge\mathbf{e}_6\\ \mathbf{R}_2 &= \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right)\mathbf{e}_2\wedge\mathbf{e}_1\end{aligned}$$

for rotations of the DGA GIPNS conic entities. The DGA GIPNS entities are constructed similarly to how the QGA GIPNS entities are constructed, based on the point embedding, standard conic sections formulas, and GIPNS set conditions. The wedge of five points on a conic gives the GOPNS entity. The wedges of four or less points represent the duals of GIPNS intersections of GIPNS conic entities in four or less points.

The QGA GOPNS *circle* entity is simply an ellipse constructed with *four* QGA points on the circle. The CGA 6,3D GOPNS circle uses only *three* CGA 6,3D points on the circle. The QGA GOPNS circle cannot be rotated as an entity using rotor operations, but the CGA 6,3D GOPNS circle can be rotated as an entity.

The QGA GOPNS *curved ellipse* entity $\mathbf{C}^{*\mathcal{Q}}$ is given by the wedge of five QGA points on the non-planar, curved ellipse

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{P}_{\mathcal{Q}_5}\\ &= (\mathbf{S}_1\wedge\mathbf{S}_2)\cdot\mathbf{I}_{io} = \mathbf{C}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the QGA GIPNS intersection of two QGA GIPNS ellipsoids $\mathbf{S}_1$ and $\mathbf{S}_2$ that meet in one curved ellipse.

3.3.19 QGA GOPNS Parabola

The QGA GOPNS parabola entities are essentially the same as the QGA GOPNS ellipse entities but with one edge located at infinity.

The homogeneous equation for a parabola is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)}{r_y} = 0.$$

The parabola opens up the y-axis for $r_y > 0$, and opens down for $r_y < 0$. The vertex of the parabola is (p_x, p_y). The focal point of the parabola is $(p_x, p_y + f)$, where focal length $f = \frac{1}{4}\frac{r_x^2}{r_y}$.

The QGA GOPNS *parabola* 5-vector entity $\mathbf{C}^{*\mathcal{Q}}$ is given by the wedge of four QGA points on the parabola and the point at infinity

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{e}_\infty\\ &= (\mathbf{B}\wedge\mathbf{\Pi})\cdot\mathbf{I}_{io} = \mathbf{C}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the standard QGA GIPNS parabola 2-vector entity $\mathbf{C}$. The four points of a parabola can be the corners of a trapezium contained in the parabola.

The QGA GIPNS parabola 2-vector entity $\mathbf{C}$ could be constructed as the GIPNS intersection of two QGA GIPNS 2D surface 1-vector entities that meet in the parabola. For example, it could be the GIPNS intersection of the QGA GIPNS parabolic cylinder $\mathbf{B}$ and QGA GIPNS plane $\mathbf{\Pi}$.

The QGA GOPNS *parabola* 4-vector entity $\mathbf{C}^*_{\langle 3\rangle}$ is given by the wedge of four QGA points on the parabola

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\\ &= (\mathbf{B}\wedge\mathbf{\Pi}\wedge\mathbf{K})\cdot\mathbf{I}_{io}=\mathbf{C}_{\langle 3\rangle}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the QGA GIPNS parabola 3-vector entity $\mathbf{C}_{\langle 3\rangle}$.

The QGA GIPNS parabola 3-vector entity $\mathbf{C}_{\langle 3\rangle}$ could be constructed as the QGA GIPNS intersection of three QGA GIPNS 2D surfaces that meet in the parabola. For example, it could be the intersection of the QGA GIPNS parabolic cylinder $\mathbf{B}$, GIPNS plane $\mathbf{\Pi}$, and QGA GIPNS elliptic paraboloid $\mathbf{K}$.

The 4-vector entity can be transformed into the 5-vector entity

$$\mathbf{C}^{*\mathcal{Q}} = \mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle}\wedge\mathbf{e}_\infty$$

but the inverse transformation is not possible since $\mathbf{e}_\infty$ has no inverse.

The QGA GOPNS *curved parabola* entity $\mathbf{C}^{*\mathcal{Q}}$ is given by the wedge of five QGA points on the non-planar, curved parabola

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{P}_{\mathcal{Q}_5}\\ &= (\mathbf{K}\wedge\mathbf{B})\cdot\mathbf{I}_{io}=\mathbf{C}\cdot\mathbf{I}_{io}\end{aligned}$$

which could be the QGA dual of the QGA GIPNS intersection of two QGA GIPNS 2D surfaces. For example, it could be the QGA dual of the QGA GIPNS intersection of the QGA GIPNS elliptic paraboloid $\mathbf{K}$ and QGA GIPNS parabolic cylinder $\mathbf{B}$ that meet in one curved parabola.

3.3.20 QGA GOPNS Hyperbola

The QGA GOPNS hyperbola entities are the same forms as the QGA GOPNS ellipse and parabola entities. The wedge of four points on the ellipse, parabola, or hyperbola, and optionally also the point at infinity, gives the QGA GOPNS conic entity.

The homogeneous equation for a hyperbola is

$$\frac{(x-p_x)^2}{r_x^2}-\frac{(y-p_y)^2}{r_y^2}-1 = 0.$$

The origin or center position of the hyperbola is (p_x, p_y). The hyperbola has two branches that open up and down the x-axis. The branch vertices are $(p_x\pm r_x, p_y)$. The linear eccentricity is $c=\sqrt{r_x^2+r_y^2}$, the distance between the foci is $2c$, and the foci are $(p_x\pm c, p_y)$. The focal length is $f=c-|r_x|$ between a vertex and focal point.

The QGA GOPNS *hyperbola* 5-vector entity $\mathbf{C}^{*\mathcal{Q}}$ is given by the wedge of four QGA points on the hyperbola and the point at infinity

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{e}_\infty\\ &= (\mathbf{Q}\wedge\mathbf{\Pi})\cdot\mathbf{I}_{io}=\mathbf{C}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the standard QGA GIPNS hyperbola 2-vector entity $\mathbf{C}$. The four points of a hyperbola can be the two vertices and two other points of one branch of the hyperbola.

The QGA GIPNS hyperbola 2-vector entity $\mathbf{C}$ could be constructed as the GIPNS intersection of two QGA GIPNS 2D surface 1-vector entities that meet in the hyperbola. For example, it could be the GIPNS intersection of the QGA GIPNS hyperboloid $\mathbf{Q}$ and GIPNS plane $\mathbf{\Pi}$.

The QGA GOPNS *hyperbola* 4-vector entity $\mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle}$ is given by the wedge of four QGA points on the hyperbola

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\\ &= (\mathbf{Q}_1\wedge\mathbf{Q}_2\wedge\mathbf{\Pi})\cdot\mathbf{I}_{io}=\mathbf{C}_{\langle 3\rangle}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the QGA GIPNS hyperbola 3-vector entity $\mathbf{C}_{\langle 3\rangle}$.

The QGA GIPNS hyperbola 3-vector entity $\mathbf{C}_{\langle 3\rangle}$ could be constructed as the QGA GIPNS intersection of three QGA GIPNS 2D surfaces that meet in the hyperbola. For example, it could be the GIPNS intersection of two QGA GIPNS hyperboloid entities $\mathbf{Q}_i$ and GIPNS plane $\mathbf{\Pi}$.

The 4-vector entity can be transformed into the 5-vector entity

$$\mathbf{C}^{*\mathcal{Q}} = \mathbf{C}^{*\mathcal{Q}}_{\langle 3\rangle}\wedge\mathbf{e}_\infty$$

but the inverse transformation is not possible since $\mathbf{e}_\infty$ has no inverse.

The QGA GOPNS *curved hyperboloid* entity $\mathbf{C}^{*\mathcal{Q}}$ is given by the wedge of five QGA points on the non-planar, curved hyperboloid

$$\begin{aligned}\mathbf{C}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{P}_{\mathcal{Q}_4}\wedge\mathbf{P}_{\mathcal{Q}_5}\\ &= (\mathbf{Q}\wedge\mathbf{B})\cdot\mathbf{I}_{io}=\mathbf{C}\cdot\mathbf{I}_{io}\end{aligned}$$

which could be the dual of the QGA GIPNS intersection of two QGA GIPNS 2D surfaces. For example, it could be the QGA dual of the QGA GIPNS intersection of the QGA GIPNS hyperboloid $\mathbf{Q}$ and QGA GIPNS parabolic cylinder $\mathbf{B}$ that meet in one curved hyperbola.

3.3.21 QGA GOPNS Line

The QGA GOPNS line 5-vector entity $\mathbf{L}^{*\mathcal{Q}}$ is given by the wedge of two QGA points on the line and the three points at infinity

$$\begin{aligned}\mathbf{L}^{*\mathcal{Q}} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{e}_{\infty x}\wedge\mathbf{e}_{\infty y}\wedge\mathbf{e}_{\infty z}\\ &= \mathbf{L}\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the standard GIPNS line 2-vector entity $\mathbf{L}$. Conceptually, this is an ellipse that has become a line.

The QGA GOPNS line 4-vector entity $\mathbf{L}^{*}_{\langle 3\rangle}$ is given by the wedge of three QGA points on the line and the point at infinity

$$\begin{aligned}\mathbf{L}^{*\mathcal{Q}}_{\langle 3\rangle} &= \mathbf{P}_{\mathcal{Q}_1}\wedge\mathbf{P}_{\mathcal{Q}_2}\wedge\mathbf{P}_{\mathcal{Q}_3}\wedge\mathbf{e}_\infty\\ &= (\mathbf{\Pi}_1\wedge\mathbf{\Pi}_2\wedge\mathbf{V})\cdot\mathbf{I}_{io}\end{aligned}$$

and is the QGA dual of the GIPNS intersection of three GIPNS 2D surface 1-vector entities that meet in the line. Conceptually, this is a parabola that has become a line.

For example, the QGA GIPNS line 3-vector entity $\mathbf{L}_{\langle 3\rangle}$ could be the QGA GIPNS intersection of two standard GIPNS plane entities $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ *and* the QGA GIPNS cone entity $\mathbf{V}$.

3.4 Operations

3.4.1 QGA Components

The following three operations are component extractions that are frequently used to extract the independent components of a QGA entity. Many other operations work by first extracting components and then operating on them individually or independently.

For any QGA entity (point or surface) $\mathbf{Q}$, the components are

$$\begin{aligned}
\mathbf{Q}_x &= (\mathbf{Q}\cdot\mathbf{e}_1)\,\mathbf{e}_1 - (\mathbf{Q}\cdot\mathbf{e}_{ox})\,\mathbf{e}_{\infty x} - (\mathbf{Q}\cdot\mathbf{e}_{\infty x})\mathbf{e}_{ox}\\
\mathbf{Q}_y &= (\mathbf{Q}\cdot\mathbf{e}_2)\,\mathbf{e}_2 - (\mathbf{Q}\cdot\mathbf{e}_{oy})\,\mathbf{e}_{\infty y} - (\mathbf{Q}\cdot\mathbf{e}_{\infty y})\mathbf{e}_{oy}\\
\mathbf{Q}_z &= (\mathbf{Q}\cdot\mathbf{e}_3)\,\mathbf{e}_3 - (\mathbf{Q}\cdot\mathbf{e}_{oz})\,\mathbf{e}_{\infty z} - (\mathbf{Q}\cdot\mathbf{e}_{\infty z})\mathbf{e}_{oz}
\end{aligned}$$

and their sum is

$$\mathbf{Q} = \mathbf{Q}_x + \mathbf{Q}_y + \mathbf{Q}_z.$$

3.4.2 Euclidean 3D Pseudoscalar

The Euclidean 3D pseudoscalar provides the dualization operation on multi-vectors in the $\mathcal{G}_3$ geometric algebra of Euclidean 3D space.

The pseudoscalar of the Euclidean 3D space is

$$\mathbf{I}_3 = \mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$$

and is frequently useful for taking the dual of a 3D vector, or the undual of a bivector.

The dual of a vector $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ is

$$\begin{aligned}
\mathbf{p}^* &= \mathbf{P} = \mathbf{p}/\mathbf{I}_3 = -\mathbf{p}\cdot\mathbf{I}_3\\
&= x(\mathbf{e}_3\wedge\mathbf{e}_2) + y(\mathbf{e}_1\wedge\mathbf{e}_3) + z(\mathbf{e}_2\wedge\mathbf{e}_1)
\end{aligned}$$

and represents $\mathbf{p}$ as a bivector, quaternion, pseudovector, axial vector, or quadrantal rotor around $\mathbf{p}$ if $\mathbf{p}^2 = 1$.

The dual of $\mathbf{p}$ can be symbolized as $\mathbf{p}^*$, but in this text a vector named with a lowercase letter $\mathbf{p}$ has a bivector Euclidean 3D dual that is often named with the uppercase letter $\mathbf{P}$.

An example of undual in the Euclidean 3D algebra $\mathcal{G}_3$ is

$$\mathbf{p} = -\mathbf{P}^* = -(\mathbf{p}^*)^* = -(\mathbf{p}/\mathbf{I}_3)/\mathbf{I}_3 = -\mathbf{p}(\mathbf{I}_3)^2$$

which takes a pseudovector $\mathbf{P}$ back to a vector $\mathbf{p}$.

The dual of $\mathbf{p}$ is symbolized as $\mathbf{p}^*$, but it must be understood in context which pseudoscalar dualization operation produced the dual.

In this text, a postfix $\mathcal{E}, \mathcal{C}$ or $\mathcal{G}$ on the dual

$$\mathbf{O}^* = \begin{cases} \mathbf{O}^{*\mathcal{E}} = \mathbf{O}/\mathbf{I}_3 & , \text{ Euclidean } \mathcal{G}_3 \text{ dual,} \\ \mathbf{O}^{*\mathcal{C}} = \mathbf{O}/\mathbf{I}_{\mathcal{C}} & , \text{ Conformal } \mathcal{G}_{6,3} \text{ dual,} \\ \mathbf{O}^{*\mathcal{Q}} = \mathbf{O}\cdot\mathbf{I}_{io} & , \text{ Quadric } \mathcal{G}_{6,3} \text{ dual,} \end{cases}$$

will be sometimes employed to indicate explicitly which pseudoscalar dualization is used. These other pseudoscalars are introduced in the next sections.

3.4.3 QGA Pseudoscalars

The QGA pseudoscalars provide the dualization operations that take QGA GOPNS entities to and from QGA GIPNS entities.

A 3D Euclidean vector $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is embedded as QGA point $\mathbf{P}_{\mathcal{Q}}$ as

$$\begin{aligned} \mathbf{P}_{\mathcal{Q}} &= \mathcal{Q}(\mathbf{p}) = \mathcal{H}_{Mx}(\mathcal{S}_x(p_x\mathbf{e}_1)) + \mathcal{H}_{My}(\mathcal{S}_y(p_y\mathbf{e}_2)) + \mathcal{H}_{Mz}(\mathcal{S}_z(p_z\mathbf{e}_3)) \\ &= \mathbf{p} + \frac{1}{2}x^2\,\mathbf{e}_{\infty x} + \frac{1}{2}y^2\,\mathbf{e}_{\infty y} + \frac{1}{2}z^2\,\mathbf{e}_{\infty z} + \mathbf{e}_o \end{aligned}$$

and is a 7D vector. The three components of $\mathbf{p}$ are each embedded independently onto a 2D null cone subspace of an independent 3D embedding vector space. The three 2D null cones form a 6D space. The homogeneous component $\mathbf{e}_o$ is the seventh dimension of a QGA point. The wedge of up to six QGA points can represent an QGA GOPNS entity, but the wedge of seven QGA points is proportional to the pseudoscalar

$$\mathbf{I}_{io} = \mathbf{I}_3 \wedge \mathbf{e}_{\infty x} \wedge \mathbf{e}_{\infty y} \wedge \mathbf{e}_{\infty z} \wedge \mathbf{e}_o$$

with respect to QGA points. This pseudoscalar serves as the duality operator that takes a standard QGA GIPNS entity $\mathbf{S}$ into its dual QGA GOPNS entity $\mathbf{S}^*$.

The other pseudoscalar

$$\mathbf{I}_{oi} = \mathbf{I}_3 \wedge \mathbf{e}_{oz} \wedge \mathbf{e}_{oy} \wedge \mathbf{e}_{ox} \wedge \mathbf{e}_{\infty}$$

is also the wedge of seven QGA points and serves as the *un*duality operator that takes a dual QGA GOPNS entity $\mathbf{S}^*$ into its standard or *un*dual QGA GIPNS entity $\mathbf{S}$.

The pseudoscalars $\mathbf{I}_{io}$ and $\mathbf{I}_{oi}$ are defined such that they are multiplicative inverses with the inner product

$$\mathbf{I}_{io}\cdot\mathbf{I}_{oi} = 1.$$

The two QGA pseudoscalars are null 7-blades since their squares are zero. This is different than the CGA pseudoscalar which squares to -1 and is a complex 5-blade.

If $\mathbf{S}$ is a *standard* QGA GIPNS surface entity, and $\mathbf{S}^*$ is the *dual* QGA GOPNS surface entity that represents the same surface, then

$$\begin{aligned} \mathbf{S}^* &= \mathbf{S}\cdot\mathbf{I}_{io} \\ \mathbf{S} &= \mathbf{S}^*\cdot\mathbf{I}_{oi}. \end{aligned}$$

The duality pseudoscalars can also be named

$$\begin{aligned}\mathbf{I}_{sd} &= \mathbf{I}_{io}\\ \mathbf{I}_{ds} &= \mathbf{I}_{oi}\end{aligned}$$

to fit the *standard* and *dual* nomenclature.

3.4.4 CGA 6,3D Pseudoscalar

A 3D Euclidean vector $\mathbf{p}=p_x\mathbf{e}_1+p_y\mathbf{e}_2+p_z\mathbf{e}_3$ is embedded as a CGA 6,3D point $\mathbf{P}_\mathcal{C}$ as

$$\begin{aligned}\mathbf{P}_\mathcal{C} &= \mathcal{C}_{6,3}(\mathbf{p})=\mathcal{H}_M(\mathcal{S}(\mathbf{p}))\\ &= \mathbf{p}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty+\mathbf{e}_o\end{aligned}$$

and is a 5D vector. The vector $\mathbf{p}$ is embedded onto a null 4-cone $\mathbb{K}^4$ in a 4D subspace of the 5D embedding vector space. The homogeneous component $\mathbf{e}_o$ is the fifth dimension of a CGA 6,3D point. The embedding is similar to $\mathcal{G}_{4,1}$ CGA points. The wedge of up to four CGA 6,3D points can represent a CGA 6,3D GOPNS entity, but the wedge of five CGA 6,3D points is proportional to the pseudoscalar

$$\mathbf{I}_\mathcal{C} = \mathbf{I}_3\wedge\mathbf{e}_\infty\wedge\mathbf{e}_o$$

with respect to CGA 6,3D points. Similarly to 4,1D CGA, this one pseudoscalar $\mathbf{I}_\mathcal{C}$ can serve as the duality operator for CGA 6,3D entities.

If $\mathbf{S}_\mathcal{C}$ is a standard CGA 6,3D GIPNS entity, and $\mathbf{S}_\mathcal{C}^*$ is its dual CGA 6,3D GOPNS entity, then

$$\begin{aligned}\mathbf{S}_\mathcal{C}^* &= \mathbf{S}_\mathcal{C}\mathbf{I}_\mathcal{C}^{-1}=-\mathbf{S}_\mathcal{C}\mathbf{I}_\mathcal{C}=-\mathbf{S}_\mathcal{C}\cdot\mathbf{I}_\mathcal{C}\\ \mathbf{S}_\mathcal{C} &= \mathbf{S}_\mathcal{C}^*\mathbf{I}_\mathcal{C}=\mathbf{S}_\mathcal{C}^*\cdot\mathbf{I}_\mathcal{C}.\end{aligned}$$

3.4.5 Euclidean 3D and CGA Rotor

A *rotor* $\mathbf{R}$ is a *rot*ation operat*or* and is isomorphic to a quaternion versor.

It can be verified that Hamilton's quaternion versors map to unit bivectors or pseudovectors as

$$\begin{aligned}\mathbf{i}=\mathbf{k}/\mathbf{j} &\Leftrightarrow \mathbf{e}_3/\mathbf{e}_2=\mathbf{e}_3\mathbf{e}_2=\mathbf{e}_3\wedge\mathbf{e}_2=\mathbf{e}_1/\mathbf{I}_3=\mathbf{e}_1^*\\ \mathbf{j}=\mathbf{i}/\mathbf{k} &\Leftrightarrow \mathbf{e}_1/\mathbf{e}_3=\mathbf{e}_1\mathbf{e}_3=\mathbf{e}_1\wedge\mathbf{e}_3=\mathbf{e}_2/\mathbf{I}_3=\mathbf{e}_2^*\\ \mathbf{k}=\mathbf{j}/\mathbf{i} &\Leftrightarrow \mathbf{e}_2/\mathbf{e}_1=\mathbf{e}_2\mathbf{e}_1=\mathbf{e}_2\wedge\mathbf{e}_1=\mathbf{e}_3/\mathbf{I}_3=\mathbf{e}_3^*.\end{aligned}$$

Given a unit vector $\mathbf{n}$ normal to the plane bivector $\mathbf{N}$ as representing the axis of rotation and the plane of rotation, and given an angle of rotation θ, the rotor $\mathbf{R}$ is

$$\begin{aligned}\mathbf{R} &= \cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\mathbf{n}/\mathbf{I}_3\\ &= \cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\mathbf{N}\\ &= e^{\frac{1}{2}\theta\mathbf{N}}.\end{aligned}$$

The rotation is right-handed counter-clockwise around the axis $\mathbf{n}$ in a standard right-handed axes model. If the axes model is interpreted as left-handed, then the rotation is left-handed clockwise around the axis $\mathbf{n}$.

The rotor $\mathbf{R}$ is valid for rotating Euclidean 3D vectors and multivectors, and for rotating all CGA 6,3D entities. The rotor $\mathbf{R}$ is generally *not valid* for rotating any QGA entities.

The specific rotor $\mathbf{R}$ that rotates vector $\mathbf{a}$ into the line of another vector $\mathbf{b}$ can be constructed as

$$\begin{aligned}\mathbf{R} &= \left(\frac{\mathbf{b}/\mathbf{a}}{|\mathbf{b}/\mathbf{a}|}\right)^{\frac{1}{2}}\\ &= \frac{\left(\frac{\mathbf{b}}{|\mathbf{b}|}+\frac{\mathbf{a}}{|\mathbf{a}|}\right)/\mathbf{a}}{\left|\left(\frac{\mathbf{b}}{|\mathbf{b}|}+\frac{\mathbf{a}}{|\mathbf{a}|}\right)/\mathbf{a}\right|}.\end{aligned}$$

A properly constructed rotor has unit magnitude.

The conical rotation $\mathbf{O}'_{\mathcal{C}}$ of any CGA 6,3D entity $\mathbf{O}_{\mathcal{C}}$ by angle θ around the unit vector axis $\mathbf{n}$ is given by the *versor sandwich* or *rotor operation*

$$\begin{aligned}\mathbf{O}'_{\mathcal{C}} &= \mathbf{R}\mathbf{O}_{\mathcal{C}}\mathbf{R}^{-1}\\ &= \mathbf{R}\mathbf{O}_{\mathcal{C}}\tilde{\mathbf{R}}\end{aligned}$$

where $\mathbf{R}^{-1}$ is the *inverse*, and $\tilde{\mathbf{R}}$ is the *reverse* of $\mathbf{R}$. These are equivalent since

$$\begin{aligned}\mathbf{R}^{-1} &= \cos\left(-\frac{1}{2}\theta\right)+\sin\left(-\frac{1}{2}\theta\right)\mathbf{N}\\ &= \cos\left(\frac{1}{2}\theta\right)-\sin\left(\frac{1}{2}\theta\right)\mathbf{N}\\ &= \cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\tilde{\mathbf{N}}\\ &= \tilde{\mathbf{R}}=e^{-\frac{1}{2}\theta\mathbf{N}}.\end{aligned}$$

The rotation of k-blade

$$\begin{aligned}\mathbf{A}_{\langle k\rangle} &= \bigwedge_{i=1}^{k}\mathbf{a}_i\\ &= \mathbf{A}_{\langle k-1\rangle}\wedge\mathbf{a}_k\\ &= \frac{1}{2}(\mathbf{A}_{\langle k-1\rangle}\mathbf{a}_k+(-1)^{k-1}\mathbf{a}_k\mathbf{A}_{\langle k-1\rangle})\end{aligned}$$

using versor or rotor $\mathbf{R}$ is the *versor outermorphism*

$$\begin{aligned}\mathbf{A}'_{\langle k\rangle} &= \mathbf{R}\mathbf{A}_{\langle k\rangle}\tilde{\mathbf{R}}\\ &= \bigwedge_{i=1}^{k}(\mathbf{R}\mathbf{a}_i\tilde{\mathbf{R}}).\end{aligned}$$

The versor outermorphism formula can be proved or verified by recursively expanding the rotation operation on the geometric product form of $\mathbf{A}_{\langle k\rangle}$ into the rotation of each vector $\mathbf{a}_i$ in the wedge. The identity $\tilde{\mathbf{R}}\mathbf{R}=1$ is inserted to continue the recursive expansion, such as

$$\mathbf{A}_{\langle k\rangle} = \frac{1}{2}(\mathbf{A}_{\langle k-1\rangle}\tilde{\mathbf{R}}\mathbf{R}\mathbf{a}_k+(-1)^{k-1}\mathbf{a}_k\tilde{\mathbf{R}}\mathbf{R}\mathbf{A}_{\langle k-1\rangle}).$$

The rotation of any CGA 6,3D point $\mathbf{P}_{\mathcal{C}}$ using rotor $\mathbf{R}$ is

$$\begin{aligned}
\mathbf{P}'_{\mathcal{C}} &= \mathbf{R}\mathbf{P}_{\mathcal{C}}\tilde{\mathbf{R}} \\
&= e^{\frac{1}{2}\theta\mathbf{N}}\left(\mathbf{p}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty+\mathbf{e}_o\right)e^{-\frac{1}{2}\theta\mathbf{N}} \\
&= e^{\frac{1}{2}\theta\mathbf{N}}(\mathbf{p})e^{-\frac{1}{2}\theta\mathbf{N}}+e^{\frac{1}{2}\theta\mathbf{N}}\left(\frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty\right)e^{-\frac{1}{2}\theta\mathbf{N}}+e^{\frac{1}{2}\theta\mathbf{N}}(\mathbf{e}_o)e^{-\frac{1}{2}\theta\mathbf{N}} \\
&= \mathbf{p}'+\frac{1}{2}(\mathbf{p}')^2\mathbf{e}_\infty+\mathbf{e}_o.
\end{aligned}$$

which is the conformal embedding of the Euclidean 3D vector $\mathbf{p}$ rotated into $\mathbf{p}'$.

The rotation of any CGA 6,3D GOPNS k-vector entity $\mathbf{O}^*_{\mathcal{C}}$ using rotor $\mathbf{R}$ is the versor outermorphism

$$\begin{aligned}
\mathbf{O}^{*\prime}_{\mathcal{C}} &= \mathbf{R}\mathbf{O}^*_{\mathcal{C}}\tilde{\mathbf{R}} \\
&= \mathbf{R}\left(\bigwedge_{i=1}^{k}\mathbf{P}_{\mathcal{C}_i}\right)\tilde{\mathbf{R}} \\
&= \bigwedge_{i=1}^{k}(\mathbf{R}\mathbf{P}_{\mathcal{C}_i}\tilde{\mathbf{R}}) \\
&= \bigwedge_{i=1}^{k}\mathbf{P}'_{\mathcal{C}_i}.
\end{aligned}$$

The rotation of any CGA 6,3D GIPNS entity $\mathbf{O}_{\mathcal{C}}$ using rotor $\mathbf{R}$ is the undual of the rotation of its dual CGA 6,3D GOPNS entity $\mathbf{O}^*_{\mathcal{C}}$ as

$$\begin{aligned}
\mathbf{O}'_{\mathcal{C}} &= -(\mathbf{R}(\mathbf{O}^*_{\mathcal{C}})\tilde{\mathbf{R}})^* \\
&= -(\mathbf{R}(-\mathbf{O}_{\mathcal{C}}\cdot\mathbf{I}_{\mathcal{C}})\tilde{\mathbf{R}})\cdot(-\mathbf{I}_{\mathcal{C}}) \\
&= -(\mathbf{R}(\mathbf{O}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}})\tilde{\mathbf{R}})\mathbf{I}_{\mathcal{C}} \\
&= -(\mathbf{R}\mathbf{O}_{\mathcal{C}}\tilde{\mathbf{R}})\mathbf{I}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} \\
&= \mathbf{R}\mathbf{O}_{\mathcal{C}}\tilde{\mathbf{R}}.
\end{aligned}$$

The pseudoscalar $\mathbf{I}_{\mathcal{C}}$ commutes with the bivector rotor $\mathbf{R}$. This result shows that any CGA 6,3D GIPNS entity $\mathbf{O}_{\mathcal{C}}$ can be rotated using rotor $\mathbf{R}$. Therefore, all CGA 6,3D entities, including points, GOPNS surfaces, and GIPNS surfaces, can be rotated using rotor $\mathbf{R}$.

Attempting to rotate any QGA point $\mathbf{P}_{\mathcal{Q}}$ using rotor $\mathbf{R}$ gives

$$\begin{aligned}
\mathbf{P}'_{\mathcal{Q}} &= e^{\frac{1}{2}\theta\mathbf{N}}\left(\mathbf{p}+\frac{1}{2}x^2\mathbf{e}_{\infty x}+\frac{1}{2}y^2\mathbf{e}_{\infty y}+\frac{1}{2}z^2\mathbf{e}_{\infty z}+\mathbf{e}_o\right)e^{-\frac{1}{2}\theta\mathbf{N}} \\
&= \mathbf{p}'+\frac{1}{2}x^2\mathbf{e}_{\infty x}+\frac{1}{2}y^2\mathbf{e}_{\infty y}+\frac{1}{2}z^2\mathbf{e}_{\infty z}+\mathbf{e}_o.
\end{aligned}$$

However,

$$\begin{aligned}
\mathbf{p}' &= x'\mathbf{e}_1+y'\mathbf{e}_2+z'\mathbf{e}_3 \\
&= e^{\frac{1}{2}\theta\mathbf{N}}(x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3)e^{-\frac{1}{2}\theta\mathbf{N}} \\
&\neq x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3
\end{aligned}$$

and

$$\mathbf{P}'_{\mathcal{Q}} \neq \mathbf{p}' + \frac{1}{2}(x')^2\mathbf{e}_{\infty x} + \frac{1}{2}(y')^2\mathbf{e}_{\infty y} + \frac{1}{2}(z')^2\mathbf{e}_{\infty z} + \mathbf{e}_o$$

which should be the rotated point. The rotation of QGA point $\mathbf{P}_{\mathcal{Q}}$ using rotor $\mathbf{R}$ generally does not give a valid rotated point. Consequently, the versor outermorphism or rotation of any QGA GOPNS entity using $\mathbf{R}$ also generally does not give a valid rotated entity. Using duality, the rotation of any QGA GIPNS entity also generally does not give a valid rotated entity.

3.4.6 CGA Translator

A *translator* $\mathbf{T}$ is a *translat*ion operat*or*.

The successive reflections of a CGA 6,3D point

$$\mathbf{P}_{\mathcal{C}} = \mathbf{p} + \frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty + \mathbf{e}_o$$

in two parallel planes, $\mathbf{\Pi}_1$ and then $\mathbf{\Pi}_2$, is

$$\begin{aligned}
\mathbf{P}'_{\mathcal{C}} &= \mathbf{\Pi}_2\mathbf{\Pi}_1\mathbf{P}_{\mathcal{C}}\tilde{\mathbf{\Pi}}_1\tilde{\mathbf{\Pi}}_2 = \mathbf{\Pi}_2\mathbf{\Pi}_1\mathbf{P}_{\mathcal{C}}\mathbf{\Pi}_1\mathbf{\Pi}_2\\
&= (\mathbf{n} + d_2\mathbf{e}_\infty)(\mathbf{n} + d_1\mathbf{e}_\infty)\mathbf{P}_{\mathcal{C}}(\mathbf{n} + d_1\mathbf{e}_\infty)(\mathbf{n} + d_2\mathbf{e}_\infty)\\
&= (1 + d_1\mathbf{n}\mathbf{e}_\infty - d_2\mathbf{n}\mathbf{e}_\infty)\mathbf{P}_{\mathcal{C}}(1 + d_2\mathbf{n}\mathbf{e}_\infty - d_1\mathbf{n}\mathbf{e}_\infty)\\
&= (1 - (d_2 - d_1)\mathbf{n}\mathbf{e}_\infty)\mathbf{P}_{\mathcal{C}}(1 + (d_2 - d_1)\mathbf{n}\mathbf{e}_\infty)\\
&= \mathbf{P}_{\mathcal{C}} + \mathbf{P}_{\mathcal{C}}(d_2 - d_1)\mathbf{n}\mathbf{e}_\infty - (d_2 - d_1)\mathbf{n}\mathbf{e}_\infty\mathbf{P}_{\mathcal{C}} - (d_2 - d_1)\mathbf{n}\mathbf{e}_\infty\mathbf{P}_{\mathcal{C}}(d_2 - d_1)\mathbf{n}\mathbf{e}_\infty\\
&= \mathbf{P}_{\mathcal{C}} + (d_2 - d_1)(\mathbf{p}\mathbf{n}\mathbf{e}_\infty + \mathbf{e}_o\mathbf{n}\mathbf{e}_\infty - \mathbf{n}\mathbf{e}_\infty\mathbf{p} - \mathbf{n}\mathbf{e}_\infty\mathbf{e}_o) + (d_2 - d_1)^2\mathbf{n}\mathbf{e}_\infty\mathbf{P}_{\mathcal{C}}\mathbf{e}_\infty\mathbf{n}\\
&= \mathbf{P}_{\mathcal{C}} + (d_2 - d_1)(2(\mathbf{p}\cdot\mathbf{n})\mathbf{e}_\infty - 2\mathbf{n}(\mathbf{e}_o\cdot\mathbf{e}_\infty)) + 2(d_2 - d_1)^2\mathbf{e}_\infty\\
&= \mathbf{P}_{\mathcal{C}} + 2(d_2 - d_1)\mathbf{n} + 2(d_2 - d_1)(\mathbf{p}\cdot\mathbf{n} + (d_2 - d_1))\mathbf{e}_\infty\\
&= (\mathbf{p} + 2(d_2 - d_1)\mathbf{n}) + \left(\frac{1}{2}\mathbf{p}^2 + 2(d_2 - d_1)\mathbf{p}\cdot\mathbf{n} + 2(d_2 - d_1)^2\right)\mathbf{e}_\infty + \mathbf{e}_o\\
&= (\mathbf{p} + 2(d_2 - d_1)\mathbf{n}) + \frac{1}{2}(\mathbf{p} + 2(d_2 - d_1)\mathbf{n})^2\mathbf{e}_\infty + \mathbf{e}_o.
\end{aligned}$$

The result is a translation of the point by $2(d_2 - d_1)\mathbf{n}$, twice the distance between the planes.

The CGA translator $\mathbf{T}$ that translates by a distance d in the direction of the unit vector $\mathbf{n}$ is

$$\mathbf{T} = 1 - \frac{1}{2}d\mathbf{n}\mathbf{e}_\infty.$$

Letting $\mathbf{d} = d\mathbf{n}$, the translator that translates by a vector $\mathbf{d}$ is

$$\mathbf{T} = 1 - \frac{1}{2}\mathbf{d}\mathbf{e}_\infty.$$

Using the series expansion formula for the exponential function

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

and the result

$$(\mathbf{n}\mathbf{e}_\infty)^2 = 0$$

the translator $\mathbf{T}$ can be expressed in the exponential form

$$\mathbf{T} = e^{-\frac{1}{2}d\mathbf{n}\mathbf{e}_\infty}.$$

The reverse of $\mathbf{T}$ is

$$\begin{aligned}\tilde{\mathbf{T}} &= 1+\frac{1}{2}d\mathbf{n}\mathbf{e}_\infty\\ &= e^{\frac{1}{2}d\mathbf{n}\mathbf{e}_\infty}\\ &= \mathbf{T}^{-1}.\end{aligned}$$

The translator is a type of versor that can be used in versor outermorphism operations to translate CGA 6,3D GOPNS entities. By duality, it also translates CGA 6,3D GIPNS entities. This is the same construction for a translator as in $\mathcal{G}_{4,1}$ CGA.

The translation of a CGA 6,3D entity $\mathbf{O}_\mathcal{C}$ by a distance d in the direction of the unit vector $\mathbf{n}$ can now be expressed as

$$\mathbf{O}'_\mathcal{C} = \mathbf{T}\mathbf{O}_\mathcal{C}\tilde{\mathbf{T}}.$$

The CGA translator $\mathbf{T}$ is generally *not valid* as a translation operator on QGA entities since it does not translate QGA points properly. However, each *component* of a QGA entity is embedded into an independent $\mathcal{G}_{2,1}$ CGA where the component can be translated by a CGA translator. QGA entities are translated component-wise using a separate CGA translator on each component.

3.4.7 QGA Translation

Each QGA component $\mathbf{Q}_x, \mathbf{Q}_y, \mathbf{Q}_z$ of a QGA GIPNS entity

$$\mathbf{Q} = \mathbf{Q}_x+\mathbf{Q}_y+\mathbf{Q}_z$$

is embedded into an independent $\mathcal{G}_{2,1}$ CGA. Each of the three QGA components can be translated by a CGA translator in its 1D Euclidean vector subspace independently of the other two components.

Let the three QGA component translators be defined as

$$\begin{aligned}\mathbf{T}_x &= 1-\frac{1}{2}d_x\mathbf{e}_1\mathbf{e}_{\infty x}\\ \mathbf{T}_y &= 1-\frac{1}{2}d_y\mathbf{e}_2\mathbf{e}_{\infty y}\\ \mathbf{T}_z &= 1-\frac{1}{2}d_z\mathbf{e}_3\mathbf{e}_{\infty z}.\end{aligned}$$

The translation of the QGA point or GIPNS surface $\mathbf{Q}$ by a displacement vector

$$\mathbf{d} = d_x\mathbf{e}_1+d_y\mathbf{e}_2+d_z\mathbf{e}_3$$

is given by

$$\mathbf{Q}' = \mathbf{T}_x\mathbf{Q}_x\tilde{\mathbf{T}}_x+\mathbf{T}_y\mathbf{Q}_y\tilde{\mathbf{T}}_y+\mathbf{T}_z\mathbf{Q}_z\tilde{\mathbf{T}}_z.$$

This component-wise translation operation is generally *not valid* on QGA GOPNS surfaces. To translate QGA GOPNS surfaces, the QGA pseudoscalar duality operations can be used to transform QGA GOPNS surfaces to QGA GIPNS surfaces.

Each of the three QGA component translators can translate only the matching QGA component and has no effect (identity result) when operating on the other two QGA components. Therefore, it is possible to construct the QGA translator

$$\mathbf{T} = \mathbf{T}_x\mathbf{T}_y\mathbf{T}_z$$

as the product of the QGA component translators. Any QGA entity $\mathbf{Q}$ can be translated as

$$\mathbf{Q}' = \mathbf{T}\mathbf{Q}\tilde{\mathbf{T}}.$$

This QGA translator operation is valid on all QGA entities, including QGA GOPNS entities since the versor outermorphism by the QGA translator is able to translate the QGA points that are within a wedge of points.

The *component-wise translation operation* on GIPNS entities is much faster than the general QGA *translator operation*, even if duals must be taken from a GOPNS to GIPNS entity and then back.

3.4.8 CGA Isotropic Dilator

A *dilator* $\mathbf{D}$ is a *dilat*ion operat*or*. A dilator can isotropically dilate, or uniformly scale, an entity by a dilation factor, or uniform scale, d. The dilator is derived from the successive inversions of a point in two concentric spheres centered on the origin.

The inversion of a CGA 6,3D point

$$\mathbf{P}_{\mathcal{C}} = \mathbf{p}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty+\mathbf{e}_o$$

in a CGA 6,3D GIPNS sphere $\mathbf{S}_1$ centered on the origin is

$$\begin{aligned}
\mathbf{P}'_{\mathcal{C}} &= \mathbf{S}_1\mathbf{P}_{\mathcal{C}}\tilde{\mathbf{S}}_1=\mathbf{S}_1\mathbf{P}_{\mathcal{C}}\mathbf{S}_1\\
&= \left(-\frac{1}{2}r_1^2\mathbf{e}_\infty+\mathbf{e}_o\right)\mathbf{P}_{\mathcal{C}}\left(-\frac{1}{2}r_1^2\mathbf{e}_\infty+\mathbf{e}_o\right)\\
&= \frac{1}{4}r_1^4\mathbf{e}_\infty\mathbf{P}_{\mathcal{C}}\mathbf{e}_\infty-\frac{1}{2}r_1^2\mathbf{e}_\infty\mathbf{P}_{\mathcal{C}}\mathbf{e}_o-\frac{1}{2}r_1^2\mathbf{e}_o\mathbf{P}_{\mathcal{C}}\mathbf{e}_\infty+\mathbf{e}_o\mathbf{P}_{\mathcal{C}}\mathbf{e}_o\\
&= \frac{1}{4}r_1^4\mathbf{e}_\infty\mathbf{e}_o\mathbf{e}_\infty-\frac{1}{2}r_1^2\mathbf{e}_\infty\mathbf{p}\mathbf{e}_o-\frac{1}{2}r_1^2\mathbf{e}_o\mathbf{p}\mathbf{e}_\infty+\frac{1}{2}\mathbf{p}^2\mathbf{e}_o\mathbf{e}_\infty\mathbf{e}_o\\
&= \frac{1}{4}r_1^4(-2\mathbf{e}_\infty)+\frac{1}{2}r_1^2\mathbf{e}_\infty\mathbf{e}_o\mathbf{p}+\frac{1}{2}r_1^2\mathbf{e}_o\mathbf{e}_\infty\mathbf{p}+\frac{1}{2}\mathbf{p}^2(-2\mathbf{e}_o)\\
&= -\frac{1}{2}r_1^4\mathbf{e}_\infty+\frac{1}{2}r_1^2(2\mathbf{e}_\infty\cdot\mathbf{e}_o)\mathbf{p}-\mathbf{p}^2\mathbf{e}_o\\
&= -r_1^2\mathbf{p}-\frac{1}{2}r_1^4\mathbf{e}_\infty-\mathbf{p}^2\mathbf{e}_o\\
&\simeq \frac{r_1^2}{\mathbf{p}^2}\mathbf{p}+\frac{1}{2}\frac{r_1^4}{\mathbf{p}^2}\mathbf{e}_\infty+\mathbf{e}_o.
\end{aligned}$$

The inversion of $\mathbf{P}'_{\mathcal{C}}$ in another sphere $\mathbf{S}_2$ centered on the origin gives

$$\begin{aligned}
\mathbf{P}''_{\mathcal{C}} &= \mathbf{S}_2\mathbf{P}'_{\mathcal{C}}\tilde{\mathbf{S}}_2=\mathbf{S}_2\mathbf{P}'_{\mathcal{C}}\mathbf{S}_2\\
&\simeq \frac{r_2^2}{(r_1^4/\mathbf{p}^2)}\frac{r_1^2}{\mathbf{p}^2}\mathbf{p}+\frac{1}{2}\frac{r_2^4}{(r_1^4/\mathbf{p}^2)}\mathbf{e}_\infty+\mathbf{e}_o\\
&= \frac{r_2^2}{r_1^2}\mathbf{p}+\frac{1}{2}\frac{r_2^4}{r_1^4}\mathbf{p}^2\mathbf{e}_\infty+\mathbf{e}_o
\end{aligned}$$

which is an isotropic dilation of $\mathbf{P}_\mathcal{C}$ by a dilation factor r_2^2/r_1^2.

Let $\mathbf{S}_1$ be the unit sphere with radius $r_1 = 1$, and let $\mathbf{S}_2$ have radius $r_2 = \sqrt{d}$. Then, the CGA 6,3D dilator is

$$\begin{aligned}
\mathbf{D} &= \mathbf{S}_2\mathbf{S}_1 \\
&= \left(-\frac{1}{2}d\mathbf{e}_\infty + \mathbf{e}_o\right)\left(-\frac{1}{2}\mathbf{e}_\infty + \mathbf{e}_o\right) \\
&= -\frac{1}{2}d\mathbf{e}_\infty\mathbf{e}_o - \frac{1}{2}\mathbf{e}_o\mathbf{e}_\infty \\
&= -\frac{1}{2}d(-1 + \mathbf{e}_\infty \wedge \mathbf{e}_o) - \frac{1}{2}(-1 - \mathbf{e}_\infty \wedge \mathbf{e}_o) \\
&= \frac{1}{2}(1+d) + \frac{1}{2}(1-d)\mathbf{e}_\infty \wedge \mathbf{e}_o \\
&\simeq 1 + \frac{1-d}{1+d}\mathbf{e}_\infty \wedge \mathbf{e}_o
\end{aligned}$$

with an isotropic dilation factor of d. The scalar d can be positive or negative. If $d < 0$, the dilator spatially inverts points, or the positions of surfaces, in or through the origin. Dilations relative to translated origins can be achieved by compositions of translators and dilators.

The reverse of $\mathbf{D}$ is

$$\tilde{\mathbf{D}} = \frac{1}{2}(1+d) - \frac{1}{2}(1-d)\mathbf{e}_\infty \wedge \mathbf{e}_o$$

and the dilation of a CGA 6,3D entity $\mathbf{O}_\mathcal{C}$ is

$$\mathbf{O}'_\mathcal{C} = \mathbf{D}\mathbf{O}_\mathcal{C}\tilde{\mathbf{D}}.$$

The CGA 6,3D dilator $\mathbf{D}$ is generally *not valid* as a dilation operator on QGA entities.

3.4.9 QGA Point Anisotropic Dilation

The dilators for QGA points and for QGA GIPNS surfaces are different. The QGA points dilate covariantly, and the QGA GIPNS surfaces dilate contravariantly.

The QGA *point component dilators* are defined as

$$\begin{aligned}
\mathbf{D}^x &= \frac{1}{2}(1+d_x) + \frac{1}{2}(1-d_x)\mathbf{e}_{\infty x} \wedge \mathbf{e}_{ox} \\
\mathbf{D}^y &= \frac{1}{2}(1+d_y) + \frac{1}{2}(1-d_y)\mathbf{e}_{\infty y} \wedge \mathbf{e}_{oy} \\
\mathbf{D}^z &= \frac{1}{2}(1+d_z) + \frac{1}{2}(1-d_z)\mathbf{e}_{\infty z} \wedge \mathbf{e}_{oz}.
\end{aligned}$$

The *anisotropic dilation* of any QGA point $\mathbf{P}$ by factor d_x, d_y, d_z in the x, y, z axis, respectively, is given by

$$\mathbf{P}' = \mathbf{D}^x\mathbf{P}_x\widetilde{\mathbf{D}^x} + \mathbf{D}^y\mathbf{P}_y\widetilde{\mathbf{D}^y} + \mathbf{D}^z\mathbf{P}_z\widetilde{\mathbf{D}^z}.$$

This QGA *point dilation* is generally *not valid* for the dilation of QGA GIPNS surfaces or QGA GOPNS surfaces.

Note that this formula for QGA point dilation is *particular* in its form. It might seem that the dilation factors could be inversed and the dilators reversed, but that change in formula will not dilate a QGA point properly since it would then be the QGA GIPNS surface dilation formula.

3.4.10 QGA GIPNS Surface Anisotropic Dilation

The dilators for QGA points and for QGA GIPNS surfaces are different. The QGA points dilate covariantly, and the QGA GIPNS surfaces dilate contravariantly.

The QGA GIPNS *surface component dilators* are defined as

$$\begin{aligned}
\mathbf{D}_x &= \frac{1}{2}\left(1+\frac{1}{d_x}\right)+\frac{1}{2}\left(1-\frac{1}{d_x}\right)\mathbf{e}_{\infty x}\wedge\mathbf{e}_{ox}\\
\mathbf{D}_y &= \frac{1}{2}\left(1+\frac{1}{d_y}\right)+\frac{1}{2}\left(1-\frac{1}{d_y}\right)\mathbf{e}_{\infty y}\wedge\mathbf{e}_{oy}\\
\mathbf{D}_z &= \frac{1}{2}\left(1+\frac{1}{d_z}\right)+\frac{1}{2}\left(1-\frac{1}{d_z}\right)\mathbf{e}_{\infty z}\wedge\mathbf{e}_{oz}.
\end{aligned}$$

The *anisotropic dilation* of any QGA GIPNS surface $\mathbf{Q}$ by factor d_x, d_y, d_z in the x, y, z axis, respectively, is given by

$$\mathbf{Q}' = \widetilde{\mathbf{D}_x}\mathbf{Q}_x\mathbf{D}_x+\widetilde{\mathbf{D}_y}\mathbf{Q}_y\mathbf{D}_y+\widetilde{\mathbf{D}_z}\mathbf{Q}_z\mathbf{D}_z.$$

This QGA GIPNS *surface dilation* is generally *not valid* for the dilation of QGA points or QGA GOPNS surfaces.

Note that this formula for QGA GIPNS surface dilation is *particular* in its form. It might seem that the dilation factors could be inversed and the dilators reversed, but that change in formula will not dilate a QGA GIPNS surface properly since it would then be the QGA point dilation formula.

3.4.11 QGA GOPNS Surface Anisotropic Dilation

Using the QGA pseudoscalars, a QGA GOPNS surface $\mathbf{Q}$ can be transformed into a QGA GIPNS surface, anisotropically dilated as a QGA GIPNS surface, and then transformed back into a QGA GOPNS surface.

The QGA GIPNS *surface component dilators* are defined as

$$\begin{aligned}
\mathbf{D}_x &= \frac{1}{2}\left(1+\frac{1}{d_x}\right)+\frac{1}{2}\left(1-\frac{1}{d_x}\right)\mathbf{e}_{\infty x}\wedge\mathbf{e}_{ox}\\
\mathbf{D}_y &= \frac{1}{2}\left(1+\frac{1}{d_y}\right)+\frac{1}{2}\left(1-\frac{1}{d_y}\right)\mathbf{e}_{\infty y}\wedge\mathbf{e}_{oy}\\
\mathbf{D}_z &= \frac{1}{2}\left(1+\frac{1}{d_z}\right)+\frac{1}{2}\left(1-\frac{1}{d_z}\right)\mathbf{e}_{\infty z}\wedge\mathbf{e}_{oz}.
\end{aligned}$$

The *anisotropic dilation* of any QGA GOPNS surface $\mathbf{Q}$ by factor d_x, d_y, d_z in the x, y, z axis, respectively, is given by

$$\mathbf{Q}' = (\widetilde{\mathbf{D}_x}(\mathbf{Q}\cdot\mathbf{I}_{oi})_x\mathbf{D}_x+\widetilde{\mathbf{D}_y}(\mathbf{Q}\cdot\mathbf{I}_{oi})_y\mathbf{D}_y+\widetilde{\mathbf{D}_z}(\mathbf{Q}\cdot\mathbf{I}_{oi})_z\mathbf{D}_z)\cdot\mathbf{I}_{io}.$$

This QGA GOPNS *surface dilation* is generally *not valid* for the dilation of QGA points or QGA GIPNS surfaces.

Note that this formula for QGA GOPNS surface dilation is *particular* in its form. It might seem that the dilation factors could be inversed and the dilators reversed, but that change in formula will not dilate a QGA GOPNS surface properly since it would then be the QGA point dilation formula.

3.4.12 QGA Isotropic Dilator

The QGA *isotropic dilator* $\mathbf{D}$ is based on the QGA point component dilators

$$\begin{aligned}\mathbf{D}^x &= \frac{1}{2}(1+d_x)+\frac{1}{2}(1-d_x)\mathbf{e}_{\infty x}\wedge\mathbf{e}_{ox}\\ \mathbf{D}^y &= \frac{1}{2}(1+d_y)+\frac{1}{2}(1-d_y)\mathbf{e}_{\infty y}\wedge\mathbf{e}_{oy}\\ \mathbf{D}^z &= \frac{1}{2}(1+d_z)+\frac{1}{2}(1-d_z)\mathbf{e}_{\infty z}\wedge\mathbf{e}_{oz}\end{aligned}$$

and is defined as

$$\mathbf{D} = \mathbf{D}^x\mathbf{D}^y\mathbf{D}^z$$

and requires the *isotropic dilation factor*

$$d = d_x = d_y = d_z.$$

The reverse of $\mathbf{D}$ is

$$\tilde{\mathbf{D}} = \widetilde{\mathbf{D}^z}\widetilde{\mathbf{D}^y}\widetilde{\mathbf{D}^x}.$$

Any QGA entity (point, GIPNS surface, or GOPNS surface) $\mathbf{Q}$ can be isotropically dilated as

$$\mathbf{Q}' = \mathbf{D}\mathbf{Q}\tilde{\mathbf{D}}.$$

The QGA isotropic dilator $\mathbf{D}$ is generally *not valid* as a dilation operator on CGA 6,3D entities.

3.4.13 QGA and CGA Transposition

The QGA entities generally cannot be rotated using a rotor operation. However, transpositions of the axes is possible for all Euclidean 3D vectors, embedded points, and GIPNS 2D surface 1-vector entities. The transpositions are based on quadrantal rotations and reflections in diagonal planes through the origin. The transpositions can be viewed as planar quadrantal rotations and reflections, or as reflections in diagonal planes, or as conical rotations around diagonal lines through the origin.

Let three standard Euclidean 3D component rotors be defined as

$$\begin{aligned}\mathbf{R}_x &= e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_3\mathbf{e}_2}\\ \mathbf{R}_y &= e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_1\mathbf{e}_3}\\ \mathbf{R}_z &= e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_2\mathbf{e}_1}\end{aligned}$$

for rotations by $\pi\,/\,2$ around the x, y, z axes, respectively. Let six embedded-component rotors be defined as

$$\begin{aligned}
\mathbf{R}_X^+ &= e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_6\mathbf{e}_5}e^{-\frac{1}{2}\frac{\pi}{2}\mathbf{e}_9\mathbf{e}_8}\\
\mathbf{R}_Y^+ &= e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_4\mathbf{e}_6}e^{-\frac{1}{2}\frac{\pi}{2}\mathbf{e}_7\mathbf{e}_9}\\
\mathbf{R}_Z^+ &= e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_5\mathbf{e}_4}e^{-\frac{1}{2}\frac{\pi}{2}\mathbf{e}_8\mathbf{e}_7}\\
\mathbf{R}_X^- &= e^{-\frac{1}{2}\frac{\pi}{2}\mathbf{e}_6\mathbf{e}_5}e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_9\mathbf{e}_8}\\
\mathbf{R}_Y^- &= e^{-\frac{1}{2}\frac{\pi}{2}\mathbf{e}_4\mathbf{e}_6}e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_7\mathbf{e}_9}\\
\mathbf{R}_Z^- &= e^{-\frac{1}{2}\frac{\pi}{2}\mathbf{e}_5\mathbf{e}_4}e^{\frac{1}{2}\frac{\pi}{2}\mathbf{e}_8\mathbf{e}_7}
\end{aligned}$$

for various rotations by $\pi/2$ of the embedded components.

The following six transposition operations are generally *valid* on Euclidean 3D vectors, QGA and CGA 6,3D points, and QGA and CGA 6,3D GIPNS surface 1-vector entities, but they are generally *not valid* on any GOPNS surfaces or on any GIPNS surfaces that are not 1-vector entities. Consequently, the transposition operations are not valid on any GIPNS intersection entities. The GIPNS surface 1-vector entities includes all GIPNS 2D surface entities, but none of the 1D or 0D surfaces.

The transposition of any CGA 6,3D GOPNS surface 4-vector entity requires transforming the CGA 6,3D GOPNS surface 4-vector entity into its dual CGA 6,3D GIPNS surface 1-vector entity by using the CGA 6,3D pseudoscalar duality operations. The CGA 6,3D GOPNS surface 4-vector entities include all of the CGA spherical 2D surfaces, but none of the 1D or 0D surfaces. All CGA 6,3D entities can instead be generally rotated.

The transposition of any QGA GOPNS surface 6-vector entity requires transforming the QGA GOPNS surface 6-vector entity into its dual QGA GIPNS surface 1-vector entity by using the QGA pseudoscalars duality operations. The QGA GOPNS surface 6-vector entities include all of the QGA axes-aligned quadric 2D surfaces, but none of the general 1D or 0D surfaces. The general 1D and 0D surfaces can instead be generally rotated by rotating sets of Euclidean 3D or conformal surface points and then constructing the rotated 1D and 0D GOPNS surfaces using the rotated sets of points embedded as QGA points.

The transposition $\mathrm{T}_{y/x}(\mathbf{Q})$ of point or GIPNS 2D surface 1-vector entity $\mathbf{Q}$ in the plane $y=+x$ is

$$\mathrm{T}_{y/x}(\mathbf{Q}) \;=\; \mathbf{R}_Z^+\mathbf{R}_z(\mathbf{Q}_x-\mathbf{Q}_y)\widetilde{\mathbf{R}_Z^+\mathbf{R}_z}+\mathbf{Q}_z.$$

The transposition $\mathrm{T}_{z/x}(\mathbf{Q})$ of point or GIPNS 2D surface 1-vector entity $\mathbf{Q}$ in the plane $z=+x$ is

$$\mathrm{T}_{z/x}(\mathbf{Q}) \;=\; \mathbf{R}_Y^+\mathbf{R}_y(\mathbf{Q}_z-\mathbf{Q}_x)\widetilde{\mathbf{R}_Y^+\mathbf{R}_y}+\mathbf{Q}_y.$$

The transposition $\mathrm{T}_{z/y}(\mathbf{Q})$ of point or GIPNS 2D surface 1-vector entity $\mathbf{Q}$ in the plane $z=+y$ is

$$\mathrm{T}_{z/y}(\mathbf{Q}) \;=\; \mathbf{R}_X^+\mathbf{R}_x(\mathbf{Q}_y-\mathbf{Q}_z)\widetilde{\mathbf{R}_X^+\mathbf{R}_x}+\mathbf{Q}_x.$$

The transposition $\mathrm{T}_{x\backslash y}(\mathbf{Q})$ of point or GIPNS 2D surface 1-vector entity $\mathbf{Q}$ in the plane $y=-x$ is

$$\mathrm{T}_{x\backslash y}(\mathbf{Q}) \;=\; \mathbf{R}_Z^{-}\mathbf{R}_z(\mathbf{Q}_y-\mathbf{Q}_x)\widetilde{\mathbf{R}_Z^{-}\mathbf{R}_z}+\mathbf{Q}_z.$$

The transposition $\mathrm{T}_{x\backslash z}(\mathbf{Q})$ of point or GIPNS 2D surface 1-vector entity $\mathbf{Q}$ in the plane $z=-x$ is

$$\mathrm{T}_{x\backslash z}(\mathbf{Q}) \;=\; \mathbf{R}_Y^{-}\mathbf{R}_y(\mathbf{Q}_x-\mathbf{Q}_z)\widetilde{\mathbf{R}_Y^{-}\mathbf{R}_y}+\mathbf{Q}_y.$$

The transposition $\mathrm{T}_{y\backslash z}(\mathbf{Q})$ of point or GIPNS 2D surface 1-vector entity $\mathbf{Q}$ in the plane $z=-y$ is

$$\mathrm{T}_{y\backslash z}(\mathbf{Q}) \;=\; \mathbf{R}_X^{-}\mathbf{R}_x(\mathbf{Q}_z-\mathbf{Q}_y)\widetilde{\mathbf{R}_X^{-}\mathbf{R}_x}+\mathbf{Q}_x.$$

3.4.14 Transposition of GIPNS 1D Surfaces

The transpositions $\mathrm{T}_{b/a}$ and $\mathrm{T}_{a\backslash b}$ are generally valid for only points and the axes-aligned GIPNS 2D surface 1-vector entities. However, GIPNS 1D surface 2-vector entities *on axes-aligned planes* can be transposed in the plane using the two functions given in this section.

Let the valid transposition planes be the axes-aligned GIPNS plane 1-vector entities

$$\begin{aligned}
\mathbf{\Pi}_{y/x}=\mathbf{\Pi}_{x\backslash y} &= \mathbf{e}_3+d_z\mathbf{e}_\infty\\
\mathbf{\Pi}_{z/x}=\mathbf{\Pi}_{x\backslash z} &= \mathbf{e}_2+d_y\mathbf{e}_\infty\\
\mathbf{\Pi}_{z/y}=\mathbf{\Pi}_{y\backslash z} &= \mathbf{e}_1+d_x\mathbf{e}_\infty.
\end{aligned}$$

The GIPNS 1D surface 2-vector entity $\mathbf{C}$, on the ab axes-aligned GIPNS plane $\mathbf{\Pi}_{b/a}$, can be transposed two different ways in the ab-plane as

$$\begin{aligned}
\mathbf{C}_{b/a} &= \mathrm{T}_{b/a}(\mathbf{C}\cdot\mathbf{\Pi}_{b/a})\wedge\mathbf{\Pi}_{b/a}\\
\mathbf{C}_{a\backslash b} &= \mathrm{T}_{a\backslash b}(\mathbf{C}\cdot\mathbf{\Pi}_{b/a})\wedge\mathbf{\Pi}_{b/a}
\end{aligned}$$

where ab is either xy, xz, or yz. The conic section $\mathbf{C}_{b/a}$ is transposed in the positive-sloped diagonal, and the conic section $\mathbf{C}_{a\backslash b}$ is transposed in the negative-sloped diagonal line of the plane.

The 1-vector $\mathbf{C}\cdot\mathbf{\Pi}_{b/a}$ represents a GIPNS 2D surface 1-vector entity $\mathbf{Q}$ which can be transposed. The conic section $\mathbf{C}$ is the intersection $\mathbf{C}=\mathbf{S}\wedge\mathbf{\Pi}_{b/a}$.

3.4.15 Intersection of Coplanar GIPNS 1D Surfaces

The GIPNS 1D surface 2-vector entities $\mathbf{C}$, such as lines, circles, ellipses, parabolas, and hyperbolas, are defined as the GIPNS intersections $\mathbf{S}\wedge\mathbf{\Pi}$ of GIPNS 2D surface 1-vector entities $\mathbf{S}$ with a GIPNS plane 1-vector entity $\mathbf{\Pi}$. These 1D surfaces are also known as *conic sections* or *conics*, or as planar *curves*. Non-planar curves are called *strings*.

Coplanar GIPNS 1D surface entities $\mathbf{C}_i$ share a common plane $\mathbf{\Pi}$, and attempting to compute their intersection $\mathbf{X}$ as

$$\mathbf{X} = \bigwedge_{i=1}^{n} \mathbf{C}_i = \bigwedge_{i=1}^{n} (\mathbf{S}_i \wedge \mathbf{\Pi}) = 0$$

always gives a null and invalid result that cannot represent the intersection, if it exists. However, this intersection, if it exists, can be computed as follows.

The intersection $\mathbf{X}$ of n coplanar GIPNS 1D surface entities $\mathbf{C}_i$ on the GIPNS plane $\mathbf{\Pi}$ is

$$\begin{aligned} \mathbf{X} &= \mathbf{\Pi} \wedge \bigwedge_{i=1}^{n} (\mathbf{C}_i \cdot \mathbf{\Pi}) \\ &= \mathbf{\Pi} \wedge \bigwedge_{i=1}^{n} \mathbf{S}_i \end{aligned}$$

In this intersection formula, the plane $\mathbf{\Pi}$ is contracted in each 1D surface 2-vector entity $\mathbf{C}_i$ to give a 2D surface 1-vector entity $\mathbf{S}_i$. The 2D surfaces $\mathbf{S}_i$ and plane $\mathbf{\Pi}$ are intersected as $\mathbf{X}$. If any $\mathbf{C}_i = \mathbf{C}_j$ for $i \neq j$, then $\mathbf{X} = 0$ and is invalid.

The GIPNS intersection entity $\mathbf{X}$ represents a system of homogeneous quadric equations that has real solutions if the intersection exists, and has only imaginary solutions if the intersection does not exist. Tangent points, where two curves touch in a single point, are intersection points.

The QGA point $\mathbf{T}$ is a point of an existing intersection only if

$$\mathbf{T} \cdot \mathbf{X} = 0.$$

For example, consider two coplanar lines

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{\Pi}_1 \wedge \mathbf{\Pi} \\ \mathbf{L}_2 &= \mathbf{\Pi}_2 \wedge \mathbf{\Pi} \end{aligned}$$

on the plane

$$\mathbf{\Pi} = \mathbf{e}_3.$$

Their intersection is

$$\begin{aligned} \mathbf{X} &= \mathbf{\Pi} \wedge (\mathbf{L}_1 \cdot \mathbf{\Pi}) \wedge (\mathbf{L}_2 \cdot \mathbf{\Pi}) \\ &\simeq \mathbf{\Pi} \wedge \mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2 \end{aligned}$$

which represents a QGA or CGA 6,3D GIPNS *point* 3-vector entity. The plane $\mathbf{\Pi}$ could be any plane, but the xy-plane of the plane entity $\mathbf{e}_3$ is well-known.

For another example, consider two coplanar circles

$$\begin{aligned} \mathbf{C}_1 &= \mathbf{S}_1 \wedge \mathbf{\Pi} \\ \mathbf{C}_2 &= \mathbf{S}_2 \wedge \mathbf{\Pi} \\ \mathbf{\Pi} &= \mathbf{e}_3 \end{aligned}$$

on the xy-plane of $\mathbf{e}_3$. Their intersection is

$$\begin{aligned} \mathbf{X} &= \mathbf{\Pi} \wedge (\mathbf{C}_1 \cdot \mathbf{\Pi}) \wedge (\mathbf{C}_2 \cdot \mathbf{\Pi}) \\ &\simeq \mathbf{\Pi} \wedge \mathbf{S}_1 \wedge \mathbf{S}_2 \end{aligned}$$

which represents a QGA or CGA 6,3D GIPNS *point pair* 3-vector entity. The plane $\mathbf{\Pi}$ of the circles could be any plane. The QGA GIPNS point pair $(\mathbf{P}_{\mathcal{Q}_1} \wedge \mathbf{P}_{\mathcal{Q}_2}) \cdot \mathbf{I}_{oi}$ is a 5-vector, and this CGA 6,3D GIPNS point pair is a 3-vector. The QGA GIPNS point pair 5-vector entity is only compatible with QGA points, while this CGA 6,3D GIPNS point pair 3-vector entity is compatible with QGA and CGA 6,3D points.

Chapter 4

Space-Time Algebra

4.1 Postulates of Special Relativity

The two postulates of Special Relativity (SR) are:

The Principle of Relativity. The (non-gravitational) laws of physics hold the same within all frames of reference moving with uniform (constant) velocities relative to each other. The spatial and temporal coordinates in two frames moving with uniform relative velocities are related to each other through linear transformations. A frame with spatial origin fixed at the spatial position of an observer (or particle) is an *inertial frame*, or a local rest frame, of the observer. This principle is often credited to Galileo.

The Principle of Invariant Light Speed. Light waves from a light source propagate spherically in all directions at the same contant speed (c) in an isotropic medium (empty space) for all observers, regardless of the velocity of the observer relative to the light source. This principle is confirmed experimentally (Michelson-Morley, et al.). The measure of time is local to each observer. A single universal time, as conceived in the older theories of Newton and Galileo, is rejected in Einstein's theory of special relativity.

Applying the principle of invariant light speed, a flash of light reflected in a mirror aboard a spacecraft appears to move forward and then backward along a straight line as observed by someone in the spacecraft, while the same flash of light as observed by someone watching the spacecraft from Earth will see the light move through a triangular path.

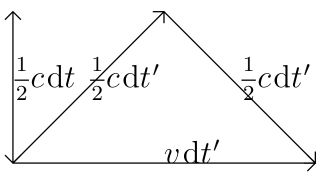


Figure 4.1. Light reflection seen in rest frame t and moving frame t'.

A spacecraft associated with its local time t in its inertial or local rest frame travels away from Earth at constant velocity v. A person on Earth associated with a local time t' observes the spacecraft. Applying the Pythagorean theorem to half of the triangle in the above figure

$$\begin{aligned}
\left(\frac{1}{2}v\,\mathrm{d}t'\right)^2+\left(\frac{1}{2}c\,\mathrm{d}t\right)^2 &= \left(\frac{1}{2}c\,\mathrm{d}t'\right)^2\\
c^2(\mathrm{d}t')^2-v^2(\mathrm{d}t')^2 &= c^2(\mathrm{d}t)^2\\
\mathrm{d}t' &= \sqrt{\frac{c^2(\mathrm{d}t)^2}{c^2-v^2}}=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,\mathrm{d}t=\gamma\,\mathrm{d}t\\
\mathrm{d}t &= \frac{1}{\gamma}\,\mathrm{d}t'
\end{aligned}$$

shows that the time differentials are related by the factor γ, called the *Lorentz factor*. Entity t' experiences a time dilation (increase) by a factor γ relative to entity t. Entity t experiences a time contraction (decrease) by factor $1/\gamma$ relative to entity t'. They also experience relative distance s dilation or contraction:

$$\begin{aligned}
\mathrm{d}s' &= v\,\mathrm{d}t'=v\gamma\,\mathrm{d}t=\gamma\,\mathrm{d}s\\
\mathrm{d}s &= v\,\mathrm{d}t=\frac{1}{\gamma}\,\mathrm{d}s'.
\end{aligned}$$

Both observers see each other moving away at equal, but opposite, velocity v.

Events. In special relativity, an event is something that has a spatial position $\mathbf{v}t$ at a certain time t in the frame of an observer. We will only consider constant velocities such as $\mathbf{v}$ where at $t=t'=0$ the observer and the observed are both at the spacetime origin. In the next sections we will see that an event is associated with a spacetime position of the form $\boldsymbol{v}t=ct\boldsymbol{\gamma}_0+\mathbf{v}t$, with an observer of the form $\boldsymbol{o}=ct\boldsymbol{\gamma}_0$. An event $\boldsymbol{v}$ is observed via light traveling from the event to the eye of an observer $\boldsymbol{o}$. To understand how an event is seen by another observer in another frame, the event must be transformed, using *Lorentz transformations*, into the frame of the other observer.

4.2 Definition of the Space-Time Algebra

The $\mathcal{G}_{1,3}$ Spacetime Algebra (STA) has one vector-unit with positive signature and three vector-units with negative signature or metric

$$\mathbf{e}_i\cdot\mathbf{e}_j = \begin{cases} +1 & \mid\ i=j=1;\\ -1 & \mid\ i=j;\,2\le i\le 4\\ 0 & \mid\ i\ne j.\end{cases}$$

STA was introduced DAVID HESTENES in his book *Space-Time Algebra*[6].

The common convention is to identify the vector-units with their matrix representations, the Dirac gamma matrices, as

$$\boldsymbol{\gamma}_{i-1} = \mathbf{e}_i.$$

Spatial position vectors are written in the form

$$\mathbf{x} = x^1\boldsymbol{\gamma}_1 + x^2\boldsymbol{\gamma}_2 + x^3\boldsymbol{\gamma}_3$$

where $(\boldsymbol{\gamma}_1)^2 = (\boldsymbol{\gamma}_2)^2 = (\boldsymbol{\gamma}_3)^2 = -1$. The coordinates x^1, x^2, x^3 are conventionally associated with x, y, z coordinates, respectively. The vector-unit $\boldsymbol{\gamma}_0$ is the time axis, where $(\boldsymbol{\gamma}_0)^2 = 1$. A temporal position is written in the form $ct\boldsymbol{\gamma}_0$, which is the distance that light travels in the time t.

An *event* is something that happens at a time t and at a spatial position $\mathbf{x}$. An event $\boldsymbol{x}$ is represented by a spacetime vector as

$$\begin{aligned}\boldsymbol{x} &= x^0\boldsymbol{\gamma}_0 + x^1\boldsymbol{\gamma}_1 + x^2\boldsymbol{\gamma}_2 + x^3\boldsymbol{\gamma}_3\\ &= ct\boldsymbol{\gamma}_0 + \mathbf{x}\end{aligned}$$

which is the sum of the event's temporal and spatial positions in spacetime.

The paravector (or "spacetime split") of $\boldsymbol{x}$ is

$$\begin{aligned}\boldsymbol{x}\boldsymbol{\gamma}_0 &= x^0 + x^1\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0 + x^2\boldsymbol{\gamma}_2\boldsymbol{\gamma}_0 + x^3\boldsymbol{\gamma}_3\boldsymbol{\gamma}_0\\ &= x^0 + x^1\boldsymbol{\sigma}_1 + x^2\boldsymbol{\sigma}_2 + x^3\boldsymbol{\sigma}_3\\ &= ct + \mathbf{x}\boldsymbol{\gamma}_0 = ct + \mathbf{x}\wedge\boldsymbol{\gamma}_0\\ &= ct + \mathbf{v}t\boldsymbol{\gamma}_0\\ &= ct + |\mathbf{v}|\hat{\mathbf{v}}t\boldsymbol{\gamma}_0\end{aligned}$$

where $\mathbf{v}$ is the velocity of the event object. The paravector $\boldsymbol{x}\boldsymbol{\gamma}_0$ is an element of an even-grade subalgebra of STA that is isomorphic to $\mathcal{G}_3$ APS. The bivectors $\boldsymbol{\sigma}_k = \boldsymbol{\gamma}_k\boldsymbol{\gamma}_0$ can be identified with the Pauli matrices as their matrix representations. The paravector can also be identified as a form of hyperbolic (or split-complex) number; the bivectors $\boldsymbol{\sigma}_k$ are a geometrical form of the hyperbolic imaginary unit u, where $(\boldsymbol{\sigma}_k)^2 = 1$ and $u^2 = 1$ while $\boldsymbol{\sigma}_k \neq 1$ and $u \neq 1$. The bivectors $\boldsymbol{\sigma}_k$ can also be isomorphically identified with the Euclidean vector-units $\mathbf{e}_k$ in $\mathcal{G}_3$, the algebra of physical space (APS), or as the vector frame of the physical space of an event. Special relativity (SR) can also be fully treated in $\mathcal{G}_3$ APS using just paravectors; however, the $\mathcal{G}_{1,3}$ STA treatment of SR offers a better geometrical interpretation of the Lorentz transformations in spacetime.

The spatial position

$$\mathbf{o} = 0\boldsymbol{\gamma}_1 + 0\boldsymbol{\gamma}_2 + 0\boldsymbol{\gamma}_3 = 0$$

of an observer $\boldsymbol{o}$ (usually a person) is the origin position of a 3D space. In the 3D space extended with $\boldsymbol{\gamma}_0$ into the 4D spacetime, the observer $\boldsymbol{o}$ is assigned the spacetime position

$$\boldsymbol{o} = ct\boldsymbol{\gamma}_0 + \mathbf{o} = ct\boldsymbol{\gamma}_0.$$

The value ct is the distance that light travels in the time t.

An observer $\boldsymbol{o}t = ct\boldsymbol{\gamma}_0$ defines a frame and a time t as measured in the frame. A spacetime position in the frame of $\boldsymbol{o}t$ has the form $\boldsymbol{v}t = \boldsymbol{o}t + \mathbf{v}t$. The local rest frame of $\boldsymbol{v}$, represented as $\boldsymbol{v}t' = ct'\boldsymbol{\gamma}_0$, is a moving frame relative to $\boldsymbol{o}$ that keeps its own local frame time t'.

4.3 Spacetime split

The spacetime position vector $\boldsymbol{x}$ in the frame of an observer $\boldsymbol{o} = ct\boldsymbol{\gamma}_0$ is

$$\begin{aligned}\boldsymbol{x} &= ct\boldsymbol{\gamma}_0 + \mathbf{x}\\ &= x^0\boldsymbol{\gamma}_0 + \mathbf{x}\end{aligned}$$

where $\mathbf{x}$ is the spatial postion vector

$$\begin{aligned}\mathbf{x} &= x^1\boldsymbol{\gamma}_1 + x^2\boldsymbol{\gamma}_2 + x^3\boldsymbol{\gamma}_3\\ &= \mathbf{v}t\end{aligned}$$

and $\mathbf{v}$ is the spatial velocity vector

$$\mathbf{v} = \frac{\mathrm{d}x^1}{\mathrm{d}t}\boldsymbol{\gamma}_1 + \frac{\mathrm{d}x^2}{\mathrm{d}t}\boldsymbol{\gamma}_2 + \frac{\mathrm{d}x^3}{\mathrm{d}t}\boldsymbol{\gamma}_3.$$

The spacetime vector $\boldsymbol{x}$ is also called an *event*. The "spacetime splits"

$$\begin{aligned}x\boldsymbol{\gamma}_0 &= ct + \mathbf{x}\wedge\boldsymbol{\gamma}_0\\ \boldsymbol{\gamma}_0 x &= ct - \mathbf{x}\wedge\boldsymbol{\gamma}_0\end{aligned}$$

are called *paravectors* and are elements of the even sub-algebra of STA, isomorphic to the $\mathcal{G}_3$ Algebra of Physical Space (APS). The algebra of the bivectors $\boldsymbol{\gamma}_i \wedge \boldsymbol{\gamma}_0$ for $i \in \{1, 2, 3\}$ is isomorphic to $\mathcal{G}_3$ and represents the physical space of an event.

4.4 Proper time

Proper time τ is the time read from the clock that is at rest within the *moving frame* of an event $\boldsymbol{x}$. For example, this could be the clock onboard a spaceship that is moving away from Earth. The proper time τ is measured by the length of arc along the event history $\boldsymbol{x}$ through spacetime. In terms of differentials,

$$\begin{aligned}c\mathrm{d}\tau &= |\mathrm{d}\boldsymbol{x}| = \sqrt{(\mathrm{d}\boldsymbol{x})^2} = \sqrt{(\mathrm{d}\boldsymbol{x})\boldsymbol{\gamma}_0\boldsymbol{\gamma}_0(\mathrm{d}\boldsymbol{x})}\\ &= \sqrt{(c\mathrm{d}t + \mathrm{d}(\mathbf{x}\wedge\boldsymbol{\gamma}_0))(c\mathrm{d}t - \mathrm{d}(\mathbf{x}\wedge\boldsymbol{\gamma}_0))}\\ &= \sqrt{c^2\mathrm{d}t^2 - (\mathrm{d}(\mathbf{x}\wedge\boldsymbol{\gamma}_0))^2}\\ \tau &= \int \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - (\mathrm{d}(\mathbf{x}\wedge\boldsymbol{\gamma}_0))^2}\\ &= \int\sqrt{1 - \left(\frac{\mathrm{d}(\mathbf{x}\wedge\boldsymbol{\gamma}_0)}{c\mathrm{d}t}\right)^2}\mathrm{d}t\\ &= \int\sqrt{1 - \frac{(\mathbf{v}\wedge\boldsymbol{\gamma}_0)^2}{c^2}}\mathrm{d}t\end{aligned}$$

and assuming constant velocites in SR, with $\tau=0$ when $t=0$,

$$\begin{aligned}\tau &= \sqrt{1-\frac{(\mathbf{v}\wedge\boldsymbol{\gamma}_0)^2}{c^2}}\,t\\ &= \frac{1}{\gamma}t\end{aligned}$$

so that τ is a contraction of t. Taking derivatives again gives

$$\begin{aligned}\mathrm{d}\tau &= \frac{1}{\gamma}\mathrm{d}t\\ \frac{\mathrm{d}t}{\mathrm{d}\tau} &= \gamma.\end{aligned}$$

Another expression for τ is the dilation form

$$\begin{aligned}\tau &= \gamma\left(t-\frac{|\mathbf{v}\wedge\boldsymbol{\gamma}_0||\mathbf{x}\wedge\boldsymbol{\gamma}_0|}{c^2}\right)\\ &= \gamma\left(t-\frac{|\mathbf{v}\wedge\boldsymbol{\gamma}_0||\mathbf{v}\wedge\boldsymbol{\gamma}_0|t}{c^2}\right)\\ &= \gamma t\left(1-\frac{(\mathbf{v}\wedge\boldsymbol{\gamma}_0)^2}{c^2}\right)=\gamma t\gamma^{-2}=\frac{1}{\gamma}t.\end{aligned}$$

4.5 Velocity and distance dilation

The *spacetime velocity* $\boldsymbol{v}$ can be written

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}=\frac{1}{\gamma}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}$$

then

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \gamma\boldsymbol{v}=\gamma(\boldsymbol{o}+\mathbf{v})$$

showing that velocity and distance are dilated by γ from the perspective of the moving particle $\boldsymbol{v}$ in its local rest frame relative the observer $\boldsymbol{o}$.

4.6 Velocity addition

STA can be used to derive and calculate special relativity velocity addition formulas.

Spacetime velocities, written here in regular lowercase boldtype such as $\boldsymbol{v}$, are associated with particles or observers. An observer (or particle) with a spacetime velocity of the form $\boldsymbol{o}=c\boldsymbol{\gamma}_0$ is considered to be at rest at the spatial origin in the spacetime frame that is local (or tangent) to the observer position $\boldsymbol{o}t$ at local time t. The spacetime velocity variable name, such as $\boldsymbol{o}$ of an observer, is also a name for the observer or particle itself.

A clock that reads a time t (or t') is also associated with the local rest frame of an observer $\boldsymbol{o}$, where at $t=0$ the observer position is at $\boldsymbol{o}t=0$, the spacetime origin of the frame. Generally, the times in two different frames cannot be directly equated, but they can be related by a time transformation, which is a component of a spacetime transformation. Spacetime transformations are also known as Lorentz transformations. The constant speed of light for all observers, and its consequence that each spacetime frame maintains its own time, is the main postulate of the theory of special relativity.

Spatial velocities are written here in upright lowercase boldtype such as $\mathbf{v}=v^i\boldsymbol{\gamma}_i$, $i\in\{1,2,3\}$. A spacetime velocity $\boldsymbol{v}$ has temporal and spatial velocity components and is written in the form $\boldsymbol{v}=c\boldsymbol{\gamma}_0+\mathbf{v}$. A spacetime position is a product of the form $\boldsymbol{v}t$, where t is the clock time of the observer (of $\boldsymbol{v}$) positioned at $\boldsymbol{o}t=ct\boldsymbol{\gamma}_0$. The observer $\boldsymbol{o}t$ is spatially at rest at the spatial origin of the spacetime frame in which $\boldsymbol{v}t$ is measured, but all observers or particles have a moving temporal component $ct\boldsymbol{\gamma}_0$ in time with speed c so that each of the four spacetime dimensions have the unit of velocity or distance.

A spacetime position or point $\boldsymbol{u}t$ in a frame can be transformed into its position as measured in another frame by a passive transformation. A passive transformation does not actively move a point in any ordinary physical sense, but only changes the basis (the spacetime coordinates frame) upon which the point is measured. A spacetime transformation of $\boldsymbol{u}t$ into another frame that maintains a clock time t' takes $\boldsymbol{u}t\rightarrow\boldsymbol{u}'t'$. The spacetime transformation of $\boldsymbol{u}t$ from its current (old) frame into a new frame will transform its spacetime velocity $\boldsymbol{u}\rightarrow\boldsymbol{u}'$ and also its time $t\rightarrow t'$. Unprimed variables $\boldsymbol{u}t$ are the values in the old (original) frame, and primed values $\boldsymbol{u}'t'$ are the new passively transformed values in the new frame. After a transformation is complete and the new frame of reference is understood, the prime marks on the transformed values may be removed to help keep notation simple and avoid multiple-primed variable names or other awkward variable namings or notations.

In spacetime, a Lorentz transformation is a hyperbolic rotation that transforms both the space and time components of a spacetime position or velocity into a different frame that is observed and measured by a different observer at rest at the frame spatial origin. The passive transformations have the form of a *versor operation* through a negative rotation angle as

$$\begin{aligned}(\boldsymbol{u}t)' &= \hat{H}^{-\frac{1}{2}}(\boldsymbol{u}t)\,\hat{H}^{\frac{1}{2}}=\hat{H}^{-\frac{1}{2}}(c\boldsymbol{\gamma}_0+\mathbf{u})\hat{H}^{\frac{1}{2}}t\\ &= \frac{c(\boldsymbol{u}t)'}{(\boldsymbol{u}t)'\cdot\boldsymbol{\gamma}_0}t'=\boldsymbol{u}'t'\end{aligned}$$

where $\boldsymbol{u}t$, in the old frame of an old observer $\boldsymbol{o}t=ct\boldsymbol{\gamma}_0$, is passively transformed into the new frame of another new observer $\boldsymbol{v}t'=ct'\boldsymbol{\gamma}_0$ that is moving as $\boldsymbol{v}t=ct\boldsymbol{\gamma}_0+\mathbf{v}t$ in the frame of old observer $\boldsymbol{o}t$. Each observer, old and new, has its own time variable, t and t' respectively. Both space and time are transformed as time $t\rightarrow t'$ and spatial velocity $\mathbf{u}\rightarrow\mathbf{u}'$. The velocity of light $c\boldsymbol{\gamma}_0$ is constant for all observers, but the time t of one observer is not the same as the time t' of another observer.

The *hyperbolic biradial* H of a particle spacetime velocity $\boldsymbol{v} = c\boldsymbol{\gamma}_0 + \mathbf{v}$ by its observer spacetime velocity $\boldsymbol{o} = c\boldsymbol{\gamma}_0$ is

$$\begin{aligned}
H &= \boldsymbol{v}/\boldsymbol{o} = \boldsymbol{v}\boldsymbol{o}^{-1} = |\boldsymbol{v}/\boldsymbol{o}|\,\widehat{\boldsymbol{v}/\boldsymbol{o}} = |H|\,\hat{H} = \frac{|\boldsymbol{v}|}{|\boldsymbol{o}|}\hat{H}\\
&= \frac{\sqrt{c^2-|\mathbf{v}|^2}}{c}\widehat{\boldsymbol{v}/\boldsymbol{o}} = \sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}\,\widehat{\boldsymbol{v}/\boldsymbol{o}}\\
&= \sqrt{1-\beta_{\mathbf{v}}^2}\,\widehat{\boldsymbol{v}/\boldsymbol{o}} = \frac{1}{\gamma_{\mathbf{v}}}\hat{H}.
\end{aligned}$$

The biradial H is an operator that transforms $\boldsymbol{o}$ into $\boldsymbol{v}$ as $\boldsymbol{v} = H\boldsymbol{o}$. The constant c is the speed of light in a vacuum (empty space). The scalar $\beta_{\mathbf{v}}$ is the *natural speed*

$$\beta_{\mathbf{v}} = \frac{|\mathbf{v}|}{c}.$$

The scalar $\gamma_{\mathbf{v}}$ is the *Lorentz factor*

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}} = \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}.$$

The magnitude $|\boldsymbol{v}|$ of a spacetime velocity $\boldsymbol{v}$

$$|\boldsymbol{v}| = \sqrt{\boldsymbol{v}^2} = \sqrt{c^2-|\mathbf{v}|^2}$$

is a hyperbolic modulus, or *hyperbolic radius*.

The *hyperbolic versor* $\hat{H}$ is the unit hyperbolic biradial

$$\begin{aligned}
\hat{H} &= \widehat{\boldsymbol{v}/\boldsymbol{o}} = \gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o} = \gamma_{\mathbf{v}}\boldsymbol{v}\boldsymbol{o}^{-1} = \frac{\gamma_{\mathbf{v}}}{c}\boldsymbol{v}\boldsymbol{\gamma}_0\\
&= \frac{\gamma_{\mathbf{v}}}{c}(\boldsymbol{v}\cdot\boldsymbol{\gamma}_0 + \boldsymbol{v}\wedge\boldsymbol{\gamma}_0) = \frac{\gamma_{\mathbf{v}}}{c}(c + \mathbf{v}\wedge\boldsymbol{\gamma}_0)\\
&= \gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\frac{|\mathbf{v}|}{c}\hat{\mathbf{v}}\boldsymbol{\gamma}_0 = \gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0\\
&= \cosh(\varphi_{\mathbf{v}}) + \sinh(\varphi_{\mathbf{v}})\hat{\mathbf{v}}\boldsymbol{\gamma}_0\\
&= e^{\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} = e^{\varphi_{\mathbf{v}}\mathbf{M}}.
\end{aligned}$$

The unit bivector $\mathbf{M} = \hat{\mathbf{v}}\boldsymbol{\gamma}_0$ represents a *Minkowski* or spacetime plane. Using the spacetime algebra volume element (pseudoscalar) $\mathbf{I}_4 = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$, the undual of $\mathbf{M}$ is $\mathbf{E} = \mathbf{M}\mathbf{I}_4$. The unit bivector $\mathbf{E}$ represents a spatial plane and it also represents the axis (2-axis) of the hyperbolic versor $\hat{H}$. All spatial vectors in the plane of $\mathbf{E}$ are invariant (eigenvectors) under a hyperbolic rotation or versor operation by $\hat{H}$. The bivector $\mathbf{M}$ has the hyperbolic (split-complex) unit property $\mathbf{M}^2 = 1$, and its dual bivector $\mathbf{E}$ has the complex unit property $\mathbf{E}^2 = -1$.

The hyperbolic angle, or *rapidity*, $\varphi_{\mathbf{v}}$ from $\boldsymbol{o}$ to $\boldsymbol{v}$ is

$$\begin{aligned}
\tanh(\varphi_{\mathbf{v}}) &= \frac{\sinh(\varphi_{\mathbf{v}})}{\cosh(\varphi_{\mathbf{v}})} = \frac{\gamma_{\mathbf{v}}\frac{|\mathbf{v}|}{c}}{\gamma_{\mathbf{v}}} = \frac{\cosh(\varphi_{\mathbf{v}})\frac{|\mathbf{v}|}{c}}{\cosh(\varphi_{\mathbf{v}})} = \frac{|\mathbf{v}|}{c}\\
\varphi_{\mathbf{v}} &= \operatorname{atanh}\!\left(\frac{|\mathbf{v}|}{c}\right) = \operatorname{atanh}(\beta_{\mathbf{v}}).
\end{aligned}$$

Note that, for $|\mathbf{v}| \ll c$ then $\varphi_{\mathbf{v}} \approx \beta_{\mathbf{v}}$, and for $|\mathbf{v}| = c$ then $\varphi_{\mathbf{v}} = \infty$.

Similar to conical rotation using quaternions, the "hyperbolical" rotation or versor operation

$$\begin{aligned} \boldsymbol{u}' &= e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(\boldsymbol{u})e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} \\ &= e^{\frac{1}{2}\varphi_{\mathbf{v}}\mathbf{M}}(\boldsymbol{u}^{\|\mathbf{M}} + \boldsymbol{u}^{\perp\mathbf{M}})e^{-\frac{1}{2}\varphi_{\mathbf{v}}\mathbf{M}} \\ &= e^{\frac{1}{2}\varphi_{\mathbf{v}}\mathbf{M}}(\boldsymbol{u}^{\|\mathbf{M}} + \boldsymbol{u}^{\|\mathbf{E}})e^{-\frac{1}{2}\varphi_{\mathbf{v}}\mathbf{M}} \\ &= e^{\varphi_{\mathbf{v}}\mathbf{M}}\boldsymbol{u}^{\|\mathbf{M}} + \boldsymbol{u}^{\|\mathbf{E}} \end{aligned}$$

turns the direction of the spacetime velocity vector $\boldsymbol{u}$ toward the direction of the spacetime velocity vector $\boldsymbol{v}$, in the same rotational orientation as $\boldsymbol{v}$ by $\boldsymbol{o}$, through the hyperbolic angle φ from the direction of $\boldsymbol{o}$ toward the direction of $\boldsymbol{v}$. The component $\boldsymbol{u}^{\perp\mathbf{M}} = \boldsymbol{u}^{\|\mathbf{E}}$, which is perpendicular to the Minkowski plane of the hyperbolic versor and in the dual spatial plane, is unaffected by the versor operation since it commutes with the versor. The rotation is on the surface of a hyper-hyperboloid that has the two-dimensional spatial axis (2-axis) $\mathbf{E}$. This is an active rotation (motion), not a passive rotation (change of basis).

Usually, we want to use a passive change-of-basis rotation to express a spacetime velocity in one frame in terms of another frame. The particle with spacetime velocity $\boldsymbol{v}$, in the frame of $\boldsymbol{o}$, is at rest in the frame where $(\boldsymbol{v}t)' = ct'\boldsymbol{\gamma}_0$. This change of basis is expressed as a rotation by the negative angle as

$$\begin{aligned} (\boldsymbol{v}t)' &= e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(\boldsymbol{v}t)e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} \\ &= \boldsymbol{v}t\,e^{\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} = \boldsymbol{v}t\left(\frac{\gamma_{\mathbf{v}}}{c}\boldsymbol{v}\boldsymbol{\gamma}_0\right) \\ &= \frac{\gamma_{\mathbf{v}}}{c}\boldsymbol{v}^2 t\boldsymbol{\gamma}_0 = \frac{\gamma_{\mathbf{v}}}{c}\frac{c^2}{\gamma_{\mathbf{v}}^2}t\boldsymbol{\gamma}_0 \\ &= c\frac{t}{\gamma_{\mathbf{v}}}\boldsymbol{\gamma}_0 = ct'\boldsymbol{\gamma}_0. \end{aligned}$$

This change of basis transforms any spacetime point or velocity in the frame of the observer $\boldsymbol{o}$ into the same spacetime point or velocity relative to the local rest frame of the moving particle with spacetime position $\boldsymbol{v}t$ in the frame of $\boldsymbol{o}$.

The spacetime position $\boldsymbol{v}t = (c\boldsymbol{\gamma}_0 + \mathbf{v})t$ as expressed relative to its own local rest frame is

$$\begin{aligned} (\boldsymbol{v}t)' &= e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(\boldsymbol{v}t)e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} = t\boldsymbol{v}e^{\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} \\ &= t\boldsymbol{v}\frac{\gamma_{\mathbf{v}}}{c}\boldsymbol{v}\boldsymbol{\gamma}_0 = t\frac{\gamma_{\mathbf{v}}}{c}\boldsymbol{v}^2\boldsymbol{\gamma}_0 = t\frac{\gamma_{\mathbf{v}}}{c}(c^2 - |\mathbf{v}|^2)\boldsymbol{\gamma}_0 \\ &= t\frac{\gamma_{\mathbf{v}}}{c}c^2\left(1 - \frac{|\mathbf{v}|^2}{c^2}\right)\boldsymbol{\gamma}_0 = ct\gamma_{\mathbf{v}}\gamma_{\mathbf{v}}^{-2}\boldsymbol{\gamma}_0 \\ &= c\frac{t}{\gamma_{\mathbf{v}}}\boldsymbol{\gamma}_0 = ct'\boldsymbol{\gamma}_0 = \boldsymbol{v}'t' \end{aligned}$$

where $t' = t/\gamma_{\mathbf{v}}$ is the *proper time* in the local rest frame of $\boldsymbol{v}t$ where $\boldsymbol{v}t \to \boldsymbol{v}'t'$. The time t', as experienced by the particle $\boldsymbol{v}'t'$ while at rest in its local rest frame, is a contraction of the time t, as experienced by the observer $\boldsymbol{o}t$ while at rest in its own local rest frame and observing the moving particle $\boldsymbol{v}t$.

Consider a particle with spacetime position $\boldsymbol{u}t = ct\boldsymbol{\gamma}_0 + \mathbf{u}t$ in the local rest frame of $\boldsymbol{v}t \to \boldsymbol{v}'t'$, wherein we remove prime marks since the frame is understood. The spacetime position $\boldsymbol{v}t$ is in the local rest frame of the observer $\boldsymbol{o}t$. From the perspective of observer $\boldsymbol{o}$, the spacetime velocity $\boldsymbol{u}$ is added to (or composed with) the spacetime velocity $\boldsymbol{v}$. According to theory, the sum $\mathbf{v}+\mathbf{u}$ of the spatial velocities cannot have a magnitude that exceeds c. To transform $\boldsymbol{u}t$ in the frame of $\boldsymbol{v}$ into the frame $\boldsymbol{o}$, we can use the *reverse* change-of-basis operation with the reversed versor $\hat{H}^{\sim}$

$$\begin{aligned}
(\boldsymbol{u}t)' &= e^{-\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\gamma}_0\hat{\mathbf{v}}}(\boldsymbol{u}t)e^{\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\gamma}_0\hat{\mathbf{v}}} = t\,e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(c\boldsymbol{\gamma}_0+\mathbf{u}^{||\hat{\mathbf{v}}}+\mathbf{u}^{\perp\hat{\mathbf{v}}})e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}\\
&= t\,e^{\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(c\boldsymbol{\gamma}_0+\mathbf{u}^{||\hat{\mathbf{v}}})+t\mathbf{u}^{\perp\hat{\mathbf{v}}}\\
&= t\left(\gamma_{\mathbf{v}}+\gamma_{\mathbf{v}}\frac{|\mathbf{v}|}{c}\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right)(c\boldsymbol{\gamma}_0+|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}})+t(\mathbf{u}-|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}})\\
&= t\gamma_{\mathbf{v}}c\boldsymbol{\gamma}_0+t\gamma_{\mathbf{v}}|\mathbf{v}|\hat{\mathbf{v}}+t\gamma_{\mathbf{v}}|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}}+t\gamma_{\mathbf{v}}\frac{|\mathbf{v}||\mathbf{u}|\cos(\theta_{\mathbf{vu}})}{c}\boldsymbol{\gamma}_0+t(\mathbf{u}-\\
&\quad |\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}})\\
&= t\gamma_{\mathbf{v}}c\left(1+\frac{|\mathbf{v}||\mathbf{u}|\cos(\theta_{\mathbf{vu}})}{c^2}\right)\boldsymbol{\gamma}_0+t\gamma_{\mathbf{v}}(|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}}+\mathbf{v})+t(\mathbf{u}-|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}})\\
&= ct'\boldsymbol{\gamma}_0+\mathbf{u}'t'=\boldsymbol{u}'t'
\end{aligned}$$

where $\boldsymbol{u}t$ is transformed into $\boldsymbol{u}'t'$ in the local rest frame of the observer $\boldsymbol{o}t'=ct'\boldsymbol{\gamma}_0$.

From just above, it is seen that

$$\begin{aligned}
t' &= \gamma_{\mathbf{v}}\left(1+\frac{|\mathbf{v}||\mathbf{u}|\cos(\theta_{\mathbf{vu}})}{c^2}\right)t\\
t &= \frac{1}{\gamma_{\mathbf{v}}\left(1+\frac{|\mathbf{v}||\mathbf{u}|\cos(\theta_{\mathbf{vu}})}{c^2}\right)}t'=\frac{1}{\gamma_{\mathbf{vu}}}t'
\end{aligned}$$

is the time transformation for velocity composition, where the velocities need not be parallel. The time t in the local rest frame of $\boldsymbol{v}t=ct\boldsymbol{\gamma}_0$ is a contraction of the time t' in the local rest frame of $\boldsymbol{o}t'=ct'\boldsymbol{\gamma}_0$. The observer $\boldsymbol{v}t$ experiences only a fraction $\frac{1}{\gamma_{\mathbf{vu}}}$ of the time experienced by the observer $\boldsymbol{o}t'$ while both observe the same particle $\boldsymbol{u}$ traverse a certain distance.

Substituting t with $\frac{1}{\gamma_{\mathbf{vu}}}t'$ gives

$$\begin{aligned}
(\boldsymbol{u}t)' &= ct'\boldsymbol{\gamma}_0+\frac{\gamma_{\mathbf{v}}(|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}}+\mathbf{v})+(\mathbf{u}-|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}})}{\gamma_{\mathbf{v}}\left(1+\frac{|\mathbf{v}||\mathbf{u}|\cos(\theta_{\mathbf{vu}})}{c^2}\right)}t'\\
&= ct'\boldsymbol{\gamma}_0+\frac{(|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}}+\mathbf{v})+\frac{1}{\gamma_{\mathbf{v}}}(\mathbf{u}-|\mathbf{u}|\cos(\theta_{\mathbf{vu}})\hat{\mathbf{v}})}{1+\frac{|\mathbf{v}||\mathbf{u}|\cos(\theta_{\mathbf{vu}})}{c^2}}t'\\
&= ct'\boldsymbol{\gamma}_0+\frac{\mathbf{v}+\mathbf{u}^{||\hat{\mathbf{v}}}+\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}}}{1-\frac{\mathbf{v}\cdot\mathbf{u}}{c^2}}t'\\
&= \boldsymbol{o}t'+\mathbf{u}'t'=\boldsymbol{u}'t'.
\end{aligned}$$

The normalization of the time component to the form $ct'\boldsymbol{\gamma}_0$, treating spacetime as a projective hyperbolic geometry, is the same as a change in the time variables that accounts for time contraction or dilation.

To summarize up to this point, we have the following pair of reciprocal Lorentz transformations for passive changes of spacetime basis between frames:

The transformation of a spacetime point $\boldsymbol{u}t$ in the frame of $\boldsymbol{o}t$ into the frame of $\boldsymbol{v}t'$

$$\begin{aligned}(\boldsymbol{u}t)' &= e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(\boldsymbol{u}t)e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}\\ &= e^{\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\gamma}_0\hat{\mathbf{v}}}(\boldsymbol{u}t)e^{-\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\gamma}_0\hat{\mathbf{v}}}\\ &= ct'\boldsymbol{\gamma}_0+\mathbf{u}'t'=\boldsymbol{u}'t'\end{aligned}$$

transforms any spacetime point $\boldsymbol{u}t$ written in the local rest frame of $\boldsymbol{o}t$ into the same spacetime point written in the local rest frame of $\boldsymbol{v}t'$ by a passive change of basis that is the reverse of the active rotation $\hat{\boldsymbol{v}}$ by $\hat{\boldsymbol{o}}$. Instead of active rotation that takes $\hat{\boldsymbol{o}}\rightarrow\hat{\boldsymbol{v}}$, the passive basis change takes $\hat{\boldsymbol{o}}'\leftarrow\hat{\boldsymbol{o}}\leftarrow\hat{\boldsymbol{v}}$ making $\hat{\boldsymbol{v}}$ the new timelike direction.

The transformation of a spacetime point $\boldsymbol{u}t$ in the frame of $\boldsymbol{v}t$ into the frame of $\boldsymbol{o}t'$

$$\begin{aligned}(\boldsymbol{u}t)' &= e^{-\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\gamma}_0\hat{\mathbf{v}}}(\boldsymbol{u}t)e^{\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\gamma}_0\hat{\mathbf{v}}}\\ &= e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(\boldsymbol{u}t)e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}\\ &= ct'\boldsymbol{\gamma}_0+\mathbf{u}'t'=\boldsymbol{u}'t'\end{aligned}$$

is the reverse of the above transformation. It transforms any spacetime point $\boldsymbol{u}t$ written in the local rest frame of $\boldsymbol{v}t$ into the same spacetime point written in the local rest frame of $\boldsymbol{o}t'$ by a passive change of basis that is the reverse of the active rotation $\hat{\boldsymbol{o}}$ by $\hat{\boldsymbol{v}}$. Instead of active rotation that takes $\hat{\boldsymbol{o}}\leftarrow\hat{\boldsymbol{v}}$, the passive basis change takes back $\hat{\boldsymbol{o}}'\rightarrow\hat{\boldsymbol{o}}\rightarrow\hat{\boldsymbol{v}}$ to restore the original timelike direction.

For both transformations above, the time component can be normalized as

$$(\boldsymbol{u}t)' = \frac{c(\boldsymbol{u}t)'}{(\boldsymbol{u}t)'\cdot\boldsymbol{\gamma}_0}t'=\boldsymbol{u}'t'$$

The normalization effectively substitutes $t=\frac{1}{\gamma_{\mathbf{vu}}}t'$. To avoid notational difficulties with multiple-primed variables, the prime marks may be removed when it is understood which new frame the spacetime point is now written within. Some addition problems follow.

Consider spacetime velocities $\boldsymbol{u}=c\boldsymbol{\gamma}_0+\mathbf{u}$ and $\boldsymbol{v}=c\boldsymbol{\gamma}_0+\mathbf{v}$, both in the frame of the observer $\boldsymbol{o}=c\boldsymbol{\gamma}_0$. What is $\boldsymbol{v}$ in the local rest frame of $\boldsymbol{u}$?

$$\begin{aligned}(\boldsymbol{v}t)' &= e^{-\frac{1}{2}\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0}(\boldsymbol{v}t)e^{\frac{1}{2}\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0}\\ &= ct'\boldsymbol{\gamma}_0+\mathbf{v}'t'\\ &= \frac{c(\boldsymbol{v}t)'}{(\boldsymbol{v}t)'\cdot\boldsymbol{\gamma}_0}t'\\ &= \boldsymbol{v}'t'\end{aligned}$$

Using the same $\boldsymbol{u}$ and $\boldsymbol{v}$ as just above, let spacetime velocity $\boldsymbol{w}=ct\boldsymbol{\gamma}_0+\mathbf{w}$ be in the local rest frame of $\boldsymbol{u}$. What is $\boldsymbol{w}$ in the local rest frame of $\boldsymbol{v}$? First, transform $\boldsymbol{w}$ from the frame of $\boldsymbol{u}$ into the frame of $\boldsymbol{o}$, then transform from the frame of $\boldsymbol{o}$ into the frame of $\boldsymbol{v}$. This may be expressed as the composition of versor operations

$$\begin{aligned}(\boldsymbol{w}t)' &= e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}e^{\frac{1}{2}\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0}(\boldsymbol{w}t)e^{-\frac{1}{2}\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0}e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}\\ &= ct'\boldsymbol{\gamma}_0+\mathbf{w}'t'\\ &= \frac{c(\boldsymbol{w}t)'}{(\boldsymbol{w}t)'\cdot\boldsymbol{\gamma}_0}t'=\boldsymbol{w}'t'.\end{aligned}$$

Instead of using this composition of versor operations, you *might* (but should not) try to form a single versor for this transformation as

$$\begin{aligned}
\hat{H}_{\boldsymbol{vu}} &= \hat{\boldsymbol{v}}/\hat{\boldsymbol{u}} = \hat{\boldsymbol{v}}\hat{\boldsymbol{u}} = \frac{\boldsymbol{vu}}{|\boldsymbol{v}||\boldsymbol{u}|} = \frac{1}{|\boldsymbol{v}||\boldsymbol{u}|}(\boldsymbol{v}\cdot\boldsymbol{u}+\boldsymbol{v}\wedge\boldsymbol{u})\\
&= \cosh(\varphi_{\boldsymbol{vu}}) + \sinh(\varphi_{\boldsymbol{vu}})\frac{\boldsymbol{v}\wedge\boldsymbol{u}}{|\boldsymbol{v}\wedge\boldsymbol{u}|} = e^{\varphi_{\boldsymbol{vu}}\hat{\mathbf{H}}}\\
\hat{\mathbf{H}}_{\boldsymbol{vu}} &= \frac{\boldsymbol{v}\wedge\boldsymbol{u}}{|\boldsymbol{v}\wedge\boldsymbol{u}|}\\
\cosh(\varphi_{\boldsymbol{vu}}) &= \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{|\boldsymbol{v}||\boldsymbol{u}|}\\
\sinh(\varphi_{\boldsymbol{vu}}) &= \cosh(\varphi_{\boldsymbol{vu}})\frac{|\boldsymbol{v}\wedge\boldsymbol{u}|}{\boldsymbol{v}\cdot\boldsymbol{u}} = \frac{|\boldsymbol{v}\wedge\boldsymbol{u}|}{|\boldsymbol{v}||\boldsymbol{u}|}\\
\tanh(\varphi_{\boldsymbol{vu}}) &= \frac{\sinh(\varphi_{\boldsymbol{vu}})}{\cosh(\varphi_{\boldsymbol{vu}})} = \frac{|\boldsymbol{v}\wedge\boldsymbol{u}|}{\boldsymbol{v}\cdot\boldsymbol{u}} = \frac{\sqrt{(\boldsymbol{v}\cdot\boldsymbol{u})(\boldsymbol{v}\cdot\boldsymbol{u})-(\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{u})}}{\boldsymbol{v}\cdot\boldsymbol{u}}\\
&= \sqrt{1-\frac{(\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{u})}{(\boldsymbol{v}\cdot\boldsymbol{u})^2}} = \frac{\sqrt{c^2-\frac{c^2(\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{u})}{(\boldsymbol{v}\cdot\boldsymbol{u})^2}}}{c} = \frac{|\boldsymbol{\varphi}_{\boldsymbol{vu}}|}{c}\\
\varphi_{\boldsymbol{vu}} &= \operatorname{atanh}\left(\frac{|\boldsymbol{v}\wedge\boldsymbol{u}|}{\boldsymbol{v}\cdot\boldsymbol{u}}\right) = \operatorname{atanh}\left(\frac{|\boldsymbol{\varphi}_{\boldsymbol{vu}}|}{c}\right).
\end{aligned}$$

This might *look* promising so far, and so then, you *might* continue to try the transformation of $\boldsymbol{w}$ as

$$\begin{aligned}
(\boldsymbol{w}t)' &= \hat{H}_{\boldsymbol{vu}}^{-\frac{1}{2}}(\boldsymbol{w}t)\hat{H}_{\boldsymbol{vu}}^{\frac{1}{2}}\\
&= e^{-\frac{1}{2}\varphi_{\boldsymbol{vu}}\hat{\mathbf{H}}}(\boldsymbol{w}t)e^{\frac{1}{2}\varphi_{\boldsymbol{vu}}\hat{\mathbf{H}}}\\
&= ct'\boldsymbol{\gamma}_0 + \mathbf{w}'t'\\
&= \frac{c(\boldsymbol{w}t)'}{(\boldsymbol{w}t)'\cdot\boldsymbol{\gamma}_0}t' = \boldsymbol{w}'t'.
\end{aligned}$$

This transformation may sometimes give numerical results that appear to be very close to the expected results. However, this transformation will generally *not* give a good result since while

$$\begin{aligned}
\hat{H}_{\boldsymbol{vu}} &= \hat{\boldsymbol{v}}\boldsymbol{\gamma}_0\boldsymbol{\gamma}_0\hat{\boldsymbol{u}}\\
&= (\hat{\boldsymbol{v}}\cdot\boldsymbol{\gamma}_0 + \hat{\boldsymbol{v}}\wedge\boldsymbol{\gamma}_0)(\hat{\boldsymbol{u}}\cdot\boldsymbol{\gamma}_0 - \hat{\boldsymbol{u}}\wedge\boldsymbol{\gamma}_0)\\
&= \frac{(\boldsymbol{v}\cdot\boldsymbol{\gamma}_0+\boldsymbol{v}\wedge\boldsymbol{\gamma}_0)}{|\boldsymbol{v}|}\frac{(\boldsymbol{u}\cdot\boldsymbol{\gamma}_0-\boldsymbol{u}\wedge\boldsymbol{\gamma}_0)}{|\boldsymbol{u}|}\\
&= \frac{(c+\mathbf{v}\boldsymbol{\gamma}_0)}{\sqrt{c^2-|\mathbf{v}|^2}}\frac{(c-\mathbf{u}\boldsymbol{\gamma}_0)}{\sqrt{c^2-|\mathbf{u}|^2}}\\
&= \frac{\left(1+\frac{|\mathbf{v}|}{c}\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right)}{\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}}\frac{\left(1-\frac{|\mathbf{u}|}{c}\hat{\mathbf{u}}\boldsymbol{\gamma}_0\right)}{\sqrt{1-\frac{|\mathbf{u}|^2}{c^2}}}\\
&= (\gamma_{\mathbf{v}}+\gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0)(\gamma_{\mathbf{u}}-\gamma_{\mathbf{u}}\beta_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0)\\
&= e^{\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}e^{-\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0} = \hat{H}_{\boldsymbol{v}}\hat{H}_{\boldsymbol{u}}^{-1}
\end{aligned}$$

again *appears* promising, you *cannot* take the expected square root as

$$\begin{aligned} \hat{H}_{\boldsymbol{vu}}^{-\frac{1}{2}} &\neq \hat{H}_{\boldsymbol{v}}^{-\frac{1}{2}}\hat{H}_{\boldsymbol{u}}^{\frac{1}{2}} = e^{-\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}e^{\frac{1}{2}\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0} \\ \left(\hat{H}_{\boldsymbol{vu}}^{-\frac{1}{2}}\right)^{\sim} &\neq \hat{H}_{\boldsymbol{u}}^{-\frac{1}{2}}\hat{H}_{\boldsymbol{v}}^{\frac{1}{2}} = e^{-\frac{1}{2}\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0}e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} \end{aligned}$$

since generally $\hat{\mathbf{v}}$ and $\hat{\mathbf{u}}$ are not orthogonal and the versors are not commutative. Versors are commutative only when their bivector exponents are orthogonal (having zero inner product) or commutative.

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