Quantum Special Relativity – Part I

The theory presented in this paper is part of the quantum formulation of Einstein's special theory of relativity. The formulation is based on two postulates which take into account the discrete nature of space and time. Because the Fitzgerald-Lorentz length contraction formula violates the length quantization postulate, this formula is modified to avoid the violation. However, Einstein's time dilation formula does not violate the time quantization postulate. This seems to indicate that we cannot treat time the same way we treat space.

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1. Introduction

In this section I shall briefly describe a relativistic phenomenon known as length contraction. Length contraction is the phenomenon whereby the length of an object with respect to an observer located in a moving reference frame will appear contracted in the direction of motion. The relation between the length of the object \( l_0 \) (proper length) with respect to a frame in which the object is at rest to the length of the object \( l \) (“contracted” length) measured by an observer moving with velocity \( v \) with respect to the stationary observer is given by:

\[
\frac{l}{l_0} = \sqrt{1 - \frac{v^2}{c^2}}
\]  

(1.1)

This equation is sometimes written as

\[
l = l_0 \sqrt{1 - \beta^2}
\]  

(1.2)

where \( \beta \) is the ratio between the velocity of the body, \( v \), to the speed of light, \( c \):

\[
\beta = \frac{v}{c}
\]  

(1.3)

Formula (1.1/1.2) is known as the Fitzgerald-Lorentz length contraction formula or length contraction formula (sometimes also as the Lorentz-Fitzgerald length contraction formula). The formula can be easily derived from the Lorentz transformations. Note that, according to this formula, the length of the body is maximum in the reference frame in which the body is at rest. Thus moving bodies appear contracted or shrink in the direction of motion. However they do not appear contracted in the perpendicular directions to it. An interesting point to observe is that both...
Lorentz's ether theory and Einstein's special relativity produce exactly the same formula for the length contraction of a body. However, in contrast to Einstein's special theory of relativity, Lorentz's theory assumes the existence of an undetectable immobile ether. Einstein's theory, on the other hand, assumes that the speed of light in vacuum is independent of the motion of the light source (a postulate known as the invariance of c). Therefore the speed of light, c, turns out to be a universal constant. In Einstein's own words: “Any ray of light moves in the 'stationary' system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body” [1]. Because no experiment was able to confirm the existence of an immobile ether, Einstein's special theory of relativity is preferred over Lorentz's ether theory.

The purpose of this paper is to modify formula (1.1) to incorporate the discrete nature space. This is done through the Planck length which is defined as:

$$L_p \equiv \sqrt{\frac{hG}{2\pi c^3}}$$

(1.4)

Because the Planck length contains the Planck's constant, h, it is natural to call this theory: the Quantum Gravitational Special Theory of Relativity or simply: Quantum Gravitational Relativity (QGR). The next section outlines the two postulates on which the formulation is based upon. This does not require any sophisticated mathematical tools. Appendix 1 contains the nomenclature used in this paper.

2. Postulates

This formulation assumes that the nature of spacetime is discrete or quantized. This quantization is implemented through the following two postulates:

(1) **Time quantization postulate**

Time is discrete. This means that there is a time, $T_{MIN}$, which is the minimum time with physical meaning. In other words there is no time or time interval smaller than $T_{MIN}$. It is likely that $T_{MIN}$ to be equal to the Planck time, $T_p$.

(2) **Length quantization postulate**

Length is discrete. This means that there is a length, $L_{MIN}$, which is the minimum length with physical meaning. In other words there is no length or distance smaller than $L_{MIN}$. It is likely that $L_{MIN}$ to be equal to the Planck length, $L_p$.

3. The Problem of Fitzgerald-Lorentz's Length Contraction

Now let us consider the problem with the Fitzgerald-Lorentz length contraction formula. This formula is:

$$l = l_0\sqrt{1-\beta^2}$$

(3.1)
If we make \( l_0 \) equal to \( L_{MIN} \) then the value of \( l \) will be

\[
l = L_{MIN} \sqrt{1 - \beta^2}
\]  

(3.2)

Because \( \sqrt{1 - \beta^2} \) is less than 1 for all values of \( v \) in the range: \( 0 < v < c \), we deduce that the length, \( l \), of the body in the direction of the body's movement will appear to be less than \( L_{MIN} \) (except for \( v = 0 \), in which case there is no movement). But this contradicts the second postulate (Length quantization postulate) of this formulation which says that there is no length smaller than \( L_{MIN} \). Thus we draw the conclusion that if postulate 2 is correct then the Fitzgerald-Lorentz's length contraction formula is not. Now one can ask: What should be the value of \( l \) when \( l_0 = L_{MIN} \)? The value of \( l \) for \( L_{MIN} \) should be equal to \( L_{MIN} \) (not equal to \( L_{MIN} \sqrt{1 - \beta^2} \) as predicted by the Fitzgerald-Lorentz formula). This means that we need to modify the Fitzgerald-Lorentz length contraction formula to correct this problem. The corrected formula will be called: The quantum gravitational length contraction (QGLC) formula. This new formula is introduced in the following section.

4. Quantum Fitzgerald-Lorentz Contraction

The formula for the Quantum Gravitational Length Contraction is

\[
l = l_0 \sqrt{1 - \beta^2} + \left(1 - \sqrt{1 - \beta^2}\right)L_{MIN}
\]

(4.1)

This formula guarantees that the length, \( l \), of a body will never appear to be smaller than the minimum length, \( L_{MIN} \), regardless of the proper length, \( l_0 \), of the body and its speed, \( v \), with respect to the observer (of course \( l_0 \) must be greater or equal than \( L_{MIN} \)).

where \( L_{MIN} \) is the minimum length with physical meaning. In other words nature does not have any length or distance smaller than \( L_{MIN} \). So far no experiment could determine the exact value of \( L_{MIN} \). However it is likely that this length to be equal to the Planck length, \( L_P \). The quantum length contraction formula presented in table 1 (Summary section) assumes, precisely, that \( L_{MIN} = L_P \). Having said that we have to keep in mind that \( L_{MIN} \) could be smaller than the Planck length (including a nil value in which case SR will hold). We just don't know.

Formula (4.1) guarantees that the length of a body will never appear to be smaller than the minimum length, \( L_{MIN} \). This is reasonable since it doesn't make any sense that a body appears to be smaller than the minimum length imposed by nature (if there is one). Appendix 2 illustrates the quantitative difference between the Fitzgerald-Lorentz contraction of Special Relativity and its counterpart of Quantum Gravitational Relativity.
5. Summary

The following table (Table 1) shows the formulas for length contraction. For the quantum gravitational formula I have assumed that:

\[ L_{\text{MIN}} = L_p \]  

(5.1)

<table>
<thead>
<tr>
<th>Name of the Formula</th>
<th>Original Relativistic Formula (SR)</th>
<th>Quantum Gravitational Formula (QGR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length contraction</td>
<td>[ l = l_0 \sqrt{1 - \beta^2} ]</td>
<td>[ l = l_0 \sqrt{1 - \beta^2} + \left[ 1 - \sqrt{1 - \beta^2} \right] L_p ]</td>
</tr>
</tbody>
</table>

Table 1: The quantum formulas for the Lorentz length contraction. Note that \( L_{\text{MIN}} \) has been replaced by \( L_P \) (Planck length).

6. Conclusions

In summary, the formula presented in this paper is a straightforward generalization of the Fitzgerald-Lorentz formula of length contraction. Now one can ask: When do we have to use the quantum gravitational formula? When the length involved in the event is of the same order of magnitude as \( L_{\text{MIN}} \) (close enough to this length) we should used the QGLC formula instead of the original Lorentz formula. According to Appendix 2 we draw the conclusion that the faster a body moves with respect to an observer located in an inertial reference frame, the greater the difference between the results given by the length contraction formulas of SR and QGR. Another aspect to take into account is time. It seems that it is not necessary to modify Einstein's time dilation formula:

\[ t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

(6.1)

because if \( t_0 = T_{\text{MIN}} \) then the “dilated” time \( t \) is

\[ t = \frac{T_{\text{MIN}}}{\sqrt{1 - \beta^2}} \]  

(6.2)

and because \( 1/\sqrt{1 - \beta^2} \) is greater than 1 for all values of \( v \) in the range: \( 0 < v < c \), the time \( t \) turns out to be greater than \( T_{\text{MIN}} \). On the other hand for \( v = 0 \), the time \( t \) turns out to be identical to \( T_{\text{MIN}} \). In other words the time \( t \) never gets smaller than \( T_{\text{MIN}} \). This means that the time...
dilation formula does not contradicts postulate 1 (Time quantization postulate) provided that \( t_0 \geq T_{MIN} \). This in turn implies that we have an asymmetry between space and time - we cannot treat time the same way we treat space! Why do we have this awkward asymmetry? To get the answer to this question we need to investigate this topic further. Finally, it is worth to remark that this quantum gravitational framework can be used in future experimental work to determine whether space is quantized or not.

Appendix 1

Nomenclature

The following are the symbols used in this paper

\[ h = \text{Planck's constant} \]
\[ c = \text{speed of light in vacuum} \]
\[ G = \text{Newton's gravitational constant} \]
\[ l_0 = \text{proper length} \]
\[ l = \text{“contracted” length} \]
\[ t_0 = \text{proper time} \]
\[ t = \text{“dilated” time} \]
\[ T_{MIN} = \text{Minimum time with physical meaning} \]
\[ L_{MIN} = \text{Minimum length with physical meaning} \]
\[ T_P = \text{Planck time} \]
\[ L_P = \text{Planck length} \]
\[ v = \text{speed of a massive body with respect to certain observer} \]
\[ \beta = \text{ratio between the speed of a massive body, } v, \text{ and the speed of light, } c. \]

Appendix 2

Calculation of the Fitzgerald-Lorentz's Contraction for a Particle

Let us calculate the Fitzgerald-Lorentz contraction
(a) using Special Relativity, and
(b) using Quantum Special Relativity
for a particle whose proper length, \( l_0 \), is equal to 10 times the Planck length: \( l_0 = 10 L_P \). For the numeric calculations we shall assume that the velocity of the particle with respect to an inertial frame of reference is (i) 0.1\( c \); and then (ii) 0.99\( c \).

(a) Special relativity

The formula in this case is

\[ l = l_0 \sqrt{1 - \beta^2} \]

(Case a-i) \( \nu = 0.1c \)
\[ l(0.1 \, c) = 10 \, L_P \sqrt{1 - 0.1^2} \]
\[ l(0.1 \, c) \approx 1.608 \, 098 \times 10^{-34} \, m \]
The ratio between this length and the Planck length is

\[ \frac{l(0.1 \, c)}{L_P} \approx 9.94988 \]

**Case a-ii** \( v = 0.99 \, c \)

\[ l(0.99 \, c) = 10 \, L_P \sqrt{1 - 0.99^2} \]
\[ l(0.99 \, c) \approx 2.279 \, 930 \times 10^{-34} \, m \]
The ratio between this length and the Planck length is

\[ \frac{l(0.99 \, c)}{L_P} \approx 14.1067 \]

**b) Quantum special relativity**

The formula in this case is

\[ l = l_0 \sqrt{1 - \beta^2} + \left(1 - \sqrt{1 - \beta^2}\right) L_P \]

**Case b-i** \( v = 0.1 \, c \)

\[ l(0.1 \, c) = 10 \, L_P \sqrt{1 - 0.1^2} + \left(1 - \sqrt{1 - 0.1^2}\right) L_P \]
\[ l(0.1 \, c) \approx 1.608 \, 908 \times 10^{-34} \, m \]
The ratio between this length and the Planck length is

\[ \frac{l(0.1 \, c)}{L_P} \approx 9.9549 \]

**Case b-ii** \( v = 0.99 \, c \)

\[ l(0.99 \, c) = 10 \, L_P \sqrt{1 - 0.99^2} + \left(1 - \sqrt{1 - 0.99^2}\right) L_P \]
\[ l(0.99 \, c) \approx 3.668 \, 136 \times 10^{-34} \, m \]
The ratio between this length and the Planck length is

\[ \frac{l(0.99 \, c)}{L_P} \approx 22.6961 \]

Comparing these results we draw the following conclusion: the faster a body moves with respect to
an observer located in an inertial frame of reference, the greater the difference between the results
given by the length contraction formulas of Special Relativity and Quantum Special Relativity.