Equivalent condition of the Generalized Riemann Hypothesis

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We prove next theorem about Dirichlet series $\chi$.

Main theorem

$$\sum_{n=1}^{m} \mu(n)\chi(n) = O(\sqrt{m\log(m)}) \iff \text{G.R.H for } \chi$$

The relation of mobius function and Riemann Hypothesis like this.

Theorem

$$\sum_{n=1}^{m} \mu(n) = O(\sqrt{m\log(m)}) \iff \text{R.H}$$

Proof

Define $M(x)$ like this

$$M(x) := \sum_{n=1}^{x} \mu(n)$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

$$\frac{1}{\zeta(s)} = \int_{x=1}^{\infty} \frac{1}{x^s} d(M(x))$$

$d(M(x))$ is Stieltjes integral.

$$= [M(x)x^{-s}] + s \int_{x=1}^{\infty} M(x)x^{-s-1}$$
\[ M(x) < O(\sqrt{x}) \Rightarrow \text{This integral must converge on } s(\text{Re}(s) = \frac{1}{2}) \]
\[ O(\sqrt{x}) < M(x) < O(\sqrt{x}\log(x)) \Rightarrow \text{This integral may not converge on } s(\text{Re}(s) = \frac{1}{2}) \text{ and must converge not on } s(\text{Re}(s) = \frac{1}{2}) \]

q.e.d

We have got main theorem by rewrite \( M(x) \) to \( M_\chi(x) := \sum \mu(n)\chi(n) \)