On the Ehrenfest paradox and the expansion of the universe

Rommel Nana Dutchou
rommel.nana.dutchou@gmail.com

Abstract

This work presents a formalism of the notions of space and of time which contains that of the special relativity, which is compatible with the quantum theories, and which distinguishes itself from the general relativity by the fact that it allows us to define the possible states of motion between two observers arbitrarily chosen in the nature. Before calculating the advance of the perihelion of an orbit, it is necessary to define the existence of a perihelion and its possible movement. In other words, it is necessary to express the use of a physical space which is a set of spatial positions, a set of world lines constantly at rest according to a unique observer. This document defines all the physical spaces of the nature (some compared with the others) by noting that to choose a temporal variable in one of these spaces, it is enough to choose a particular parametrization along each of its points. If the world lines of a family of observers are not elements of a unique physical space, then even in classical physics, how can they manage to put end to end their rulers to determine the measure of a segment of curve of their reference frame (each will have to ask to his neighbor: a little seriousness please, do not move until the measurement is ended) ? This question is the basis of the solution which will be proposed to paradox of Ehrenfest. A notion of expansion of the universe is established as being a structural reality and a rigorous theoretical formulation of the Hubble's experimental law is proposed. We shall highlight the fact that a relative motion occurs only along specific trajectories and this notion of authorized trajectories is not a novelty in physics as it is stated in the Bohr atomic model. We shall also highlight the fact that a non-uniform rectilinear motion possesses a horizon having the structure of a plan.

1 Introduction

The universe $\mathcal{U}$ is a topological space whose elements are called events and such that each event has a neighborhood homeomorphic to an open subset of $\mathbb{R}^4$. A world line segment can be represented by a continuous function which is defined on a part of $\mathbb{R}$ and which takes its values in $\mathcal{U}$. We can define several parameterizations along a unique world line segment. If this segment represents a trajectory of material body then compared with every experimenter it is or in motion or motionless. For example, a space shuttle can be constantly at rest according to an experimenter on the surface of the earth and be in motion according to an experimenter on the surface of the moon.

Every experimenter possesses a unique physical space which is all the world lines (chosen among those which describe the trajectories of material bodies) which appear to him to be constantly at rest. Thus, when we have arbitrarily defined a local coordinate system, in other words a homeomorphism between an open subset of $\mathcal{U}$ and a bounded part of $\mathbb{R}^4$, every experimenter has a unique explicit formula (which depends on the coordinate system) to characterize each point of his physical space. This formula can be the constancy of a triplet of coordinates which will be considered as spatial by the experimenter or it can be more complex. It is necessary to use a temporal variable of a physical space to define a state vector in quantum mechanics, it is necessary to use this physical space to develop this state vector and write a wave function, it is necessary to use a physical space to define the motion of a body in classical mechanics. In special theory of relativity, we say that there is a Doppler effect when the source moves in the physical space of the receiver.

There are several physical spaces on each open subset of $\mathcal{U}$. That of the experimenter P who is on the platform is different from that of the experimenter P' who is on a boat in movement according to P. If the first one expresses that a bird makes round trips between two fixed points A and B of his physical space, the second can express that the same bird is moving in a zigzag without ever going back at a same point of his physical space. A physical space characterizes the state of motion of an experimenter (whoever owns this physical space) and as we will show at the end of the section 2, even in classical physics, there are coherent families of world lines which are not physical spaces. Every theory has to specify what it considers as being the set of all possible physical spaces of nature. We know that the state of motion of an experimenter has to be defined with respect to another experimenter consequently a physical space has to be defined with respect to another physical space.

Mathematically, a parametrized curve defined on a physical space is not a world line but each point of this curve (which is a space curve) is a world line. A space curve defined on a particular physical space is the path of a particular family of world lines on this physical space. Mathematically, we cannot compare a segment of space curve defined on a physical space $\mathcal{R}$ and a segment of space curve defined on a different physical space $\mathcal{R}'$ (because mathematically a point of $\mathcal{R}$ is not a point of $\mathcal{R}'$ or because these segments of curves are not represented by the same families of segments of world lines) but we can make assumptions to compare their lengths. Each physical theory must provide implicit or explicit assumptions to make this comparison and these assumptions are going to characterize the states of relative motions between physical spaces. Classical physics suggests certain explicit assumptions to make this comparison and the kinematics of the section 4 proposes another hypothesis.

Defining a geometry on a physical space consists in attributing an intrinsic measure to each of its space curve and the geometry (the proper geometry) of a physical space is the one which reports the character superposable or not superposable of its space curves (which can be paths of rays of light) by a simple comparison of their measures. This physical measure of a segment of space curve can be realized by summing the local measures made by a family of experimenters who possess the same physical space and who are arranged along the curve. Although the geometry of each physical space is assumed to be Euclidean in this document [1], one may wonder if in reality it is not Riemannian or more complex and if it does not vary from a physical space to another.The purpose of this work is to highlight the possibilities of construction of the theories which allow to express without ambiguity that every experimenter knows how to characterize mathematically the immobility of a segment of world line.

[1]We can define several inner products on a unique vector space consequently we can define several Euclidean geometries on a unique affine space. We can also define very complex Riemannian geometries on the same space and for each of geometry, the spatial distance between two points is the measure of the smallest of the segments of parametrized curves which connects them.

2 An original approach to classical kinematics: construction of all the three-dimensional relative physical spaces.

We cannot express simply that the four-dimensional universe of the classical physics is an affine space whose difference space is $\mathcal{V}$ because every time we shall evoke the use of a cartesian coordinate system, it will be necessary to specify that this system is on a same and unique decomposition of $\mathcal{V}$ in a direct sum of two subspaces among which one is one-dimensional. It will then be possible to state that one of the cartesian coordinate represents the universal temporal variable and the other three cartesian coordinates are of spatial nature but this can not mean that there is a unique physical space (which is constituted by a special family of world lines) in a physics where the immobility of a body is mathematically a notion relative to the experimenter who notices it: all the physical spaces (each being associated with an experimenter) will be equivalent for the statement of a problem of kinematics but not for the statement of a problem of dynamics. The physical space of an experimenter on the surface of the earth is different from the physical space of an experimenter on the surface of the moon.

An experimenter is always provided with an intrinsically regular clock in the sense where he knows how to appreciate the equality or the difference of time intervals defined by two couples of events arbitrarily chosen on his world line. Establishing the laws of physics means determining the relations which exist between measurable parameters consequently the dating of the events which are not elements of his world line is an inevitably arbitrary choice under the only condition to use an operational definition. This arbitrary character appears as soon as we do not consider the time parameter of classical physics as an inaccessible variable by the experiment but as a variable which must be measured : The fact of expressing the use of dates indicated by a family of clocks synchronized with one of them which is chosen as reference means expressing the use of a dating of the events made by the reference clock through a process which must be explained. An operational process of dating of the events in a physical space will always have to be specified in the formulation of a theory evoking a temporal variable otherwise we can just assume that each physical space must have a privileged temporal variable which will implicitly be used and whose measure (the process to initialize the clocks constantly at rest in the physical space) will be explained subsequently. In this last situation, since choosing a particular parametrization along each of the world lines which constitute a physical space $\mathcal{R}$ is not equivalent to choosing a particular parametrization along each of the world lines which constitute another physical space $\mathcal{R}'$, the relation between the privileged temporal variables of two different physical spaces is not necessarily trivial. Therefore, the following hypothesis is a strong condition:

Hypothesis 1 *There is a universal temporal variable which allows to determine, by a simple subtraction of the values assigned to the events of their trajectories, the elapsed times in all possible regular clocks.*

In a Euclidean space we know how to define the lengths of the segments of the parametrized curves which are straight lines and we can deduct the length of any other segment of parametrized curve by performing Riemann sums of straight line segments which exist between the consecutive points of its subdivisions.

Hypothesis 2 *The proper geometry of each physical space is Euclidean and two experimenters P and P' can always choose their standard of lengths so as to notice the same measures for the straight line segments joining pairs of simultaneous events.*

Proposition 1 *Any transformation between cartesian and rectangular spatial coordinates (x_2, x_3, x_4) and (x'_2, x'_3, x'_4) that P and P' can use to identify the points of their physical spaces is of the shape:*

$$\begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} l_2(t) \\ l_3(t) \\ l_4(t) \end{bmatrix} + \lambda A_t \begin{bmatrix} x'_2 + k'_2 \\ x'_3 + k'_3 \\ x'_4 + k'_4 \end{bmatrix}$$

t being the universal temporal variable and A_t being an isometry.

$<,>_p$ being an inner product on the vector space $\mathcal{V}_p$ associated with the physical space $\mathcal{E}_p$, we can represent this Euclidean physical space by the triplet $(\mathcal{E}_p, \mathcal{V}_p, <,>_p)$ or by the triplet $(\mathcal{E}_p, T_p^O, <,>_p)$ where O is arbitrarily chosen in $\mathcal{E}_p$ and where the affine structure T_p^O is a function defined on $\mathbb{R}^2 \times \mathcal{E}_p^2$ by the relation :

$$\varphi_p(O, T_p^O(a, b, M, N)) = a\varphi_p(O, M) + b\varphi_p(O, N) \tag{1}$$

φ_p is the application with values in $\mathcal{V}_p$ which determines vectors in the physical space of P and T_p^O is an affine structure which allows P to appreciate the character aligned or not aligned of points of his space, and which enables him to recognize that a quadrilateral of $\mathcal{E}_p$ is or is not a parallelogram. The coordinates he uses to parametrize his physical space are cartesians if there is a basis $\mathcal{B}$ of $\mathcal{V}_p$ and an element O of $\mathcal{E}_p$ such as those which are associated with an element M of $\mathcal{E}_p$ are the components in $\mathcal{B}$ of the vector $\varphi_p(O, M)$. To establish a transformation between cartesian spatial coordinates built by P and P' who are at rest on the points O_p and $O_{p'}$ of their spaces, classical physics is going to emit the following assumption :

Hypothesis 3 *Whatever the elements M' and N' of $\mathcal{E}_{p'}$ which describe in $\mathcal{E}_p$ the trajectories $M'(.)$ and $N'(.)$, whatever the date t,*

$$T_{p'}^{O_{p'}}(a, b, M', N')(t) = T_p^{O_{p'}(t)}(a, b, M'(t), N'(t))$$

$(B'_i)_i$ and $(B_i)_i$ are triplets of elements of $\mathcal{E}_{p'}$ and $\mathcal{E}_p$ that define orthonormal bases of euclidean physical spaces by the equalities:

$$\mathcal{B}' = \{(\varphi_{p'}(O_{p'}, B'_i))_i\} \quad \mathcal{B} = \{(\varphi_p(O_p, B_i))_i\}$$

We consider a phenomenon which occurs on the date t, at the points M' and $M'(t)$ of $\mathcal{E}_{p'}$ and $\mathcal{E}_p$. We note (x'_i) and (x_i) its cartesian spatial coordinates which are associated with $\mathcal{B}'$ and $\mathcal{B}$ equipped with the origins O' and O, and we note (k'_i) the coordinates of $\varphi_{p'}(O_{p'}, O')$ in $\mathcal{B}'$. We can write:

$$M' = T_{p'}^{O_{p'}}(x'_4 + k'_4, 1, B'_4, T_{p'}^{O_{p'}}(x'_3 + k'_3, x'_2 + k'_2, B'_3, B'_2))$$

The hypothesis 3 allows to write:

$$M'(t) = T_p^{O_{p'}(t)}(x'_4 + k'_4, 1, B'_4(t), T_p^{O_{p'}(t)}(x'_3 + k'_3, x'_2 + k'_2, B'_3(t), B'_2(t)))$$

We can note:

$$\varphi_p(O, M'(t)) = \varphi_p(O, O_{p'}(t)) + \sum_{i=2}^{4}(x'_i + k'_i)\varphi_p(O_{p'}(t), B'_i(t))$$

The hypothesis 2 allows to assert that the triangle $(O_{p'}(t), B'_i(t), B'_j(t))$ is isosceles right according to P because $(O_{p'}, B'_i, B'_j)$ is isosceles right according to P'. There is thus a scalar λ which describes the ratio of the standards of the lengths and an isometry $\mathcal{A}_t$ of $\mathcal{V}_p$ such as:

$$\varphi_p(O_{p'}(t), B'_i(t)) = \lambda \mathcal{A}_t \varphi_p(O_p, B_i)$$

We obtain:

$$\varphi_p(O, M'(t)) = \varphi_p(O, O_{p'}(t)) + \lambda \mathcal{A}_t \sum_{i=2}^{4} (x'_i + k'_i) \varphi_p(O_p, B_i)$$

Let us look for the Eulerian description of the authorized relative motions in classical kinematics. We can write:

$$\varphi_p(M'(t), N'(t)) = \lambda \mathcal{A}_t \sum_{i=2}^{4} (y'_i - x'_i) \varphi_p(O_p, B_i)$$

$$\varphi_p(M'(t_2), N'(t_2)) - \varphi_p(M'(t_1), N'(t_1)) = (\mathcal{A}_{t_2} \mathcal{A}_{t_1}^{-1} - \mathcal{I}) \varphi_p(M'(t_1), N'(t_1))$$

If the function with value in the whole of the orthogonal matrices $r \mapsto A_r$ is differentiable and if $A_{r_1} = I$, then by writing its first-order Taylor series expansion and by using the characteristic formula $A_r^T A_r = I$ of an orthogonal matrix A_r, we show that the matrix derivative $\left.\frac{d}{dr} A_r \right|_{r=r_1}$ is an antisymmetric matrix.

It results from this proposal that there is a vector function $\vec{w}(t)$ such that, if M' and N' are two points of the space of P' whose paths in the space of P are described by the functions $M'(t)$ and $N'(t)$, then:

$$\frac{d}{dt} \varphi_p(M'(t), N'(t)) = \vec{w}(t) \times \varphi_p(M'(t), N'(t))$$

The operator $\times$ depends on the inner product $<,>_p$. M' being arbitrarily chosen in $\mathcal{E}_{p'}$ and O being arbitrarily chosen in $\mathcal{E}_p$, we can write :

$$\frac{d}{dt} \varphi_p(O, N'(t)) = \vec{w}(t) \times \varphi_p(M'(t), N'(t)) + \vec{v}(t, M') \qquad \forall N' \in \mathcal{E}_{p'}$$

$$\frac{d}{dt} \varphi_p(O, N'(t)) = \vec{w}(t) \times \varphi_p(O, N'(t)) + \vec{v}(t, M', O) \qquad \forall N' \in \mathcal{E}_{p'}$$

$(x_i(t))$ being the cartesian spatial coordinates of $N'(t)$ with respect to an orthonormal basis $\mathcal{B}$ of $\mathcal{V}_p$ equipped with the origin O, we can write:

$$\frac{d}{dt} x_i(t) = \sum_{j=2}^{4} w_{ij}(t) x_j(t) + v_i(t) \quad w_{ji}(t) = -w_{ij}(t) \quad 2 \leq i, j \leq 4$$

S being the cartesian coordinates system of $\mathcal{E}_p$ with respect to $(O, \mathcal{B})$, the Eulerian description of the velocity field $U(\tau, x_2, x_3, x_4) = (U_i)_{2 \leq i \leq 4}$ which represents the motions in S of points of $\mathcal{E}_{p'}$ is :

$$U_i(t, x_2, x_3, x_4) = \sum_{j=2}^{4} w_{ij}(t) x_j + v_i(t) \quad w_{ji}(t) = -w_{ij}(t) \quad 2 \leq i, j \leq 4 \qquad (2)$$

This relation does not depend on coordinate systems that P' can choose to study a phenomenon and will be equivalent to the equations (6), (10), (11) within the frame of the new theory which distinguishes itself by a more realistic modelling since the choice of a process of dating of the events in a physical space by an experimenter should not allow to guess the intrinsic regularities of all the possible clocks. In accordance with (2), declare in classical physics that P' has a uniform rotational motion (respectively a uniform translational motion) with respect to P mean only that any body which is constantly at rest with respect to P' moves with a uniform speed along a circular path (respectively it moves with a uniform speed along a straight path) in the physical space of P when he choose to use the the universal temporal variable and the proper geometry on this physical space.

If $\mathcal{E} = \{M(t), N(t), ...\}$ is a set of Lagrangian trajectories which are described in $\mathcal{E}_p$ by Eulerian formulas (2), and if the basis $\mathcal{B}$ (with respect to which the cartesian spatial coordinates are defined) is orthonormal with respect to an inner product on $\mathcal{V}_p$ which represents the proper geometry of $\mathcal{E}_p$, then we can express the existence of an experimenter P' who notices that all elements of $\mathcal{E}$ are constantly motionless. If the basis $\mathcal{B}$ is orthonormal with respect to an inner product on $\mathcal{V}_p$ which does not represent the proper geometry [2] of $\mathcal{E}_p$, then we cannot express the existence of an experimenter P' who notices that all elements of $\mathcal{E}$ are constantly motionless. In this last situation, $\mathcal{E}$ is not a physical space but we can build several four-dimensional coordinate systems (x_1, x_2, x_3, x_4) in which the equation of each element of $\mathcal{E}$ is the constancy of the triplet (x_2, x_3, x_4): if t_0 is a real number and if R is a coordinate system on $\mathcal{E}_p$, in other words R is a function defined on $\mathcal{E}_p$ and with values in $\mathbb{R}^3$, then an event which belongs to the element $M(t)$ of $\mathcal{E}$ can be described by the coordinates $x_1 = t$, $(x_2, x_3, x_4) = R(M(t_0))$. The concept of a physical space is quite different from that of a coordinate system. The description of the motions of a set of bodies with respect to a four-dimensional coordinate system does not necessarily mean that these motions are described with respect to a certain experimenter, even in classical physics. It is up to each theory to be equipped with tools which allow to specify when a family of world lines of material bodies constitutes a physical space and when this family does not constitutes a physical space. A physical space is inevitably defines with respect to another chosen as reference and these tools can be the use of a special family of coordinates systems (between which transformations have a certain structure), each element of the family being clearly attached to a unique observer. Inspired by the mathematical expression of the Hubble's law, it is necessary to propose specific hypotheses for comparing on the one hand the "proper distance" D (which can change over time) between a particular object and an experimenter P arbitrarily chosen, and on the other hand the "proper distance" D' between the same object and a second experimenter P' which is also arbitrarily selected.

3 On the Ehrenfest paradox

Different approaches to this subject are presented in [3], [4] and [5]. Generally, this problem is introduced to show that we can deduce from special relativity the existence of an observer who finds that his three-dimensional space is not Euclidean and must renounce to an immediate interpretation of some coordinate systems.

Thus, using the assumption that a body $\mathcal{D}$ is described by an inertial reference frame $\mathcal{R}$ as a disc in uniform rotation about an axis perpendicular to the disc plane and passing through its center, it should be concluded :

(i) There exists an observer of $\mathcal{D}$ who can state that "$\mathcal{D}$ has actually and constantly

[2] If $<,>_1$ and $<,>_2$ are two inner products defined on a finite dimensional vector space $\mathcal{V}$, then there exists an automorphism $\mathcal{P}$ of $\mathcal{V}$ such that, for all $(\vec{u}, \vec{v}) \in \mathcal{V}^2$, $< \vec{u}, \vec{v} >_2 = < \mathcal{P}\vec{u}, \mathcal{P}\vec{v} >_1$. If $\mathcal{A}$ is a linear isometry of $(\mathcal{V}, <,>_2)$, then $\mathcal{P}\mathcal{A}\mathcal{P}^{-1}$ is a linear isometry of $(\mathcal{V}, <,>_1)$.

the shape of a disc".
(ii) The observer of $\mathcal{D}$ notices that the relationship which connects the circumference and diameter of $\mathcal{D}$ is not that of Euclidean spaces in other words the ratio between these two quantities is not the number pi.

We will show that (i) is questionable. Indeed, consider two material points which are fixed on $\mathcal{D}$ and such that one is on its center and the other on its circumference. Then:
(a) By assumption, since $\mathcal{D}$ has actually and constantly the shape of a disc with respect to $\mathcal{R}$, this inertial frame can assert that the spatial distance between these two points does not vary over time.
(b) The Lorentz transformation allows to state that there is at least one inertial frame $\mathcal{R}'$ who can say that the spatial distance between these two points varies over time in other words $\mathcal{D}$ is constantly in deformation and has the shape of an ellipse with respect to $\mathcal{R}'$.

Because $\mathcal{R}$ assert that $\mathcal{D}$ does not undergo distortion and has the shape of a disc, and because $\mathcal{R}'$ asserts the opposite, knowing that all inertial reference frames are physically equivalent, it is subjective to assert that there is an observer of $\mathcal{D}$ who notices that $\mathcal{D}$ does not undergo distortion and has the shape of a disc. Thus, (i) is questionable.

To demonstrate (b) it is sufficient to choose $\mathcal{R}'$ as an inertial reference frame whose velocity vector $\vec{v}$ (with respect to $\mathcal{R}$) is in the plane of $\mathcal{D}$ and is therefore orthogonal to the axis of rotation of $\mathcal{D}$. Under these conditions, the transformation of Lorentz teaches that the contraction of the lengths enters $\mathcal{R}$ and $\mathcal{R}'$ is maximal when the radius vector between both material points is colinear to $\vec{v}$ and this contraction of the lengths enters $\mathcal{R}$ and $\mathcal{R}'$ is worthless when the radius vector between both material points is orthogonal to $\vec{v}$. Finally, we know that the radius vector between the two material points occupy alternately each of these two configurations because $\mathcal{D}$ is rotating.

By noticing that even in classical kinematics we can build a coherent [3] family of world lines that are not a set of fixed points with respect to a unique observer (shown at the end of the section 2), we can propose that in a relativist framework:
(a) The family of trajectories (described with respect to an inertial coordinate system by equations that highlight the classical notion of rotational motion) does not constitute a set of fixed points with respect to a unique observer.
(b) It is therefore not surprising that we have difficulty in conceiving that regular digital clocks having these trajectories are synchronisables in the sense of the special relativity.
(c) It is necessary to reinvent the complexity of the equations which have to describe, with respect to an inertial coordinate system, a set of points continuously fixed with respect to an accelerated experimenter: do not plagiarize the equations of classical kinematics.
(d) The geometry of the three-dimensional space of an accelerated observer can remain Euclidean if the fixed points which constitute this three-dimensional space are described (with respect to an inertial coordinate system) by the new complex equations.

[3] Such a family can be artificially used as a set of spatial positions without any ambiguity because there is no intersection between its elements.

4 A relativistic kinematics

In classical physics, because of the strong hypotheses which are postulated to guess the correspondences between the measures of space and time that can perform different experimenters, the relation Doppler-Fizeau depends not only on the motion of the source in the physical space of the receiver, but also depends on motion of the source and the receiver compared with the particular physical space where the theory of the electromagnetism of Maxwell is formulated. This oddity is abnormal according to certain physicists and Woldemar Voigt, in his article on the Doppler effect [1], will establish a linear transformation of coordinates which leaves invariant the wave equation. These transformations constitute the Poincaré group which contains particular subgroups constituted by the Lorentz transformations, formulas which differ from those of Voigt by a change of the standard of lengths during the change of physical space and for the authors it is a question of highlighting practical variables in physical spaces which are in uniform translational motion compared with the ether. Albert Einstein notices in [2] that it is possible to find this special formulas when observers of each physical space chooses to measure the spatial distances and to date the events in a specific way and the new kinematics highlights the symmetry of the Doppler-Fizeau relation.

The intrinsically regular clock of an experimenter can be chosen with a digital display which is cartesian and normally oriented : the duration of proper time elapsed between successive dates t_1 and t_2 is proportional to the positive real number $t_2 - t_1$, the positive constant of proportionality characterizing a choice of the standard. For a digital display which is cartesian and abnormally oriented, this duration is proportional to the positive real number $t_1 - t_2$. As in classical physics, the geometry of each physical space $\mathcal{E}_p$ will be described by a pair $(T_p^O, <,>_p)$ and an experimenter will say that a coordinate system is cartesian if it consists of cartesian spatial coordinates (allowing to identify the points of his physical space) associated with a particular dating of each event. This dating of events, which is a cartesian [4] dating of events, is obtained by emitting at a date t_- of his clock an electromagnetic signal which propagates in the vacuum, by receiving at a date t_+ the signal reflected at the event and by using the formula:

$$t = \frac{1}{2}(t_+ + t_-)$$

It is assumed that the nature is such as every experimenter can make a cartesian dating of events (the dates of transmission and reception have to be uniquely determined) and the following hypothesis is a rigorous formulation of the second postulate of special relativity:

Postulate 1 *Every experimenter P who uses a cartesian dating of events made by himself can define a Euclidean geometry $(T_p^O, <,>_p)$ on his physical space so as to notice that an electromagnetic signal which originates in a given event always propagates in the vacuum in the form of a sphere whose radius increases with a constant speed, the value of this speed characterizing a choice of the standard of lengths.*

It results from this postulate that if P and P' build the cartesian coordinate systems (ct, x_2, x_3, x_4) and $(c't', x'_2, x'_3, x'_4)$ which are relative to orthonormal bases

[4] A cartesian dating of events made by an experimenter P can be mathematically used as a temporal variable in all possible physical spaces and represents the universal temporal variable in classical physics if and only if P is constantly motionless in the particular physical space where the theory of the electromagnetism of Maxwell is formulated.

of their physical spaces (such a coordinate system will be called an observer), c and c' being the values chosen to mathematically represent the speeds of propagation of an electromagnetic signal in each physical space, then whatever the events i and j, we can write:

$$(ct(j)-ct(i))^2 = \sum_{k=2}^{4}(x_k(j)-x_k(i))^2 \Longleftrightarrow (c't'(j)-c't'(i))^2 = \sum_{k=2}^{4}(x'_k(j)-x'_k(i))^2 \quad (3)$$

By definition, the function $(ct, x_2, x_3, x_4) \mapsto (ct', x'_2, x'_3, x'_4)$ is element of the group of the real eligible transformations. The affine solutions form the Poincaré group and we can think that this group contains the Jacobian matrices of the not affine solutions because (3) is valid in a neighborhood of each event. We shall say that a continuously differentiable mapping which is defined on a subset of $\mathbb{R}^4$ is an admissible transformation if there is a pair of real numbers (c, c') such as it transforms any trajectory realized with an instantaneous velocity vector of constant modulus equal to c to a trajectory realized with an instantaneous velocity vector of constant modulus equal to c'.

Theorem 1 *Any real eligible transformation is an admissible transformation.*

S and S' being two observers built by the experimenters P and P', the trajectory in S of a body whose modulus of the velocity vector is constantly equal to c between the events a and b is described by the set:

$$\{(ct, x_2(ct), x_3(ct), x_4(ct)) \ , \ t \in]t_a, t_b[\ , \ \sum_{i=2}^{4}(\frac{dx_i}{dt})^2 = c^2\}$$

We can define the sets:

$$\tilde{S}_1 = \{(u_2, u_3, u_4) \ , \ u_i \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \ , \ u_i = u_i(\xi) \ , \ \sum_{i=2}^{4}(\frac{du_i}{d\xi})^2 = 1\}$$

$$\mathcal{T}(\tilde{S}_1) = \Big\{\{(u_1(\eta), u(u_1(\eta))) \ , \ \eta \in \mathcal{O}\} \ , \ u_1 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \ , \ u \in \tilde{S}_1 \ , \ \mathcal{O} \text{ ouvert de } \mathbb{R}\Big\}$$

We can note:

$$x_1 = ct \quad x'_1 = c't' \quad x'_k = f_k(x_1, x_2, x_3, x_4) \quad f = (f_1, f_2, f_3, f_4)$$

$$\Lambda' = \{(u'_1(\eta), u'_2(u'_1(\eta)), u'_3(u'_1(\eta)), u'_4(u'_1(\eta))) \ , \ \eta \in \mathcal{O}\} = \{f(M), M \in \Lambda\} = f(\Lambda)$$
$$\Lambda \in \mathcal{T}(\tilde{S}_1)$$

The theorem 1 allows to write:

$$\Lambda \in \mathcal{T}(\tilde{S}_1) \Longrightarrow f(\Lambda) \in \mathcal{T}(\tilde{S}_1)$$

By noting $F = (f_{ij})_{ij}$ the Jacobian matrix of f, we get:

$$\begin{pmatrix} \frac{du'_1}{d\eta} \\ \frac{d(u'_2 \circ u'_1)}{d\eta} \\ \frac{d(u'_3 \circ u'_1)}{d\eta} \\ \frac{d(u'_4 \circ u'_1)}{d\eta} \end{pmatrix} = F \begin{pmatrix} \frac{du_1}{d\eta} \\ \frac{d(u_2 \circ u_1)}{d\eta} \\ \frac{d(u_3 \circ u_1)}{d\eta} \\ \frac{d(u_4 \circ u_1)}{d\eta} \end{pmatrix} \qquad \frac{du'_1}{d\eta} \begin{pmatrix} 1 \\ \frac{du'_2}{d\xi'} \\ \frac{du'_3}{d\xi'} \\ \frac{du'_4}{d\xi'} \end{pmatrix} = F \ \frac{du_1}{d\eta} \begin{pmatrix} 1 \\ \frac{du_2}{d\xi} \\ \frac{du_3}{d\xi} \\ \frac{du_4}{d\xi} \end{pmatrix}$$

$$\left(\frac{du_1}{d\eta}\right)^2 - \sum_{i=2}^{4}\left(\frac{d(u_i \circ u_1)}{d\eta}\right)^2 = 0 \implies \left(\frac{du'_1}{d\eta}\right)^2 - \sum_{i=2}^{4}\left(\frac{d(u'_i \circ u'_1)}{d\eta}\right)^2 = 0$$

It results that F is such that:

$$\forall\, h \in \mathbb{R}^4 \;,\; \left\{h_1^2 - \sum_{i=2}^{4} h_i^2 = 0\right\} \implies \left\{h'^2_1 - \sum_{i=2}^{4} h'^2_i = 0\right\} \qquad h' = Fh$$

This last relation allows to demonstrate by calculations of linear algebra that f has the following properties:

i Its Jacobian matrix F is element of the Poincaré group at each event, all its coefficients are expressed using the partial derivatives of f_1, and the square root of the absolute value of Jacobian determinant, which is an eigenvalue of the positive definite symmetric matrix $F^T F$, is given by:

$$\mu_f = f_{11}^2 - \sum_{i=2}^{4} f_{1i}^2 > 0 \tag{4}$$

ii P being equipped with one of his observer S, P' can always choose one of his observer S' such that:

$$f_{ii} = \sqrt{\mu_f} + \frac{f_{1i}^2}{f_{11} + \sqrt{\mu_f}} \qquad f_{ij} = \frac{f_{1i} f_{1j}}{f_{11} + \sqrt{\mu_f}} \qquad f_{i1} = f_{1i} \quad 2 \le i, j \le 4 \quad i \neq j \tag{5}$$

iii By noting $\tau' = f_{11}(\tau, x_2, x_3, x_4)$ and by solving $\dfrac{dx'_i}{d\tau'} = 0$, we get that the components v_i of the velocity vector in S of a point of the physical space of P' are such that:

$$\frac{dx_i}{d\tau}(\tau) = \frac{v_i}{c}(\tau) = -\frac{f_{1i}}{f_{11}}(\tau, x_2(\tau), x_3(\tau), x_4(\tau)) \quad 2 \le i \le 4 \tag{6}$$

iv If f is continuously differentiable to second order then there are three matrices A_i^f such that:

$$\frac{\partial}{\partial x_j}\begin{pmatrix} f_{i2} \\ f_{i3} \\ f_{i4} \end{pmatrix} = A_j^f \begin{pmatrix} f_{i2} \\ f_{i3} \\ f_{i4} \end{pmatrix} \qquad 2 \le i, j \le 4 \tag{7}$$

We can note $h(\tau)$ the function which is defined along the trajectory of P' and which realizes the correspondence between the temporal coordinate τ in S and the temporal coordinate in S'. If $(x'_i(O_{p'}))_i$ represents the coordinates of P' in S' and if $\tau_-(\alpha)$ and $\tau_+(\alpha)$ are the temporal coordinates in S of emission and reception by P' of signals which allow him to date the event α, then the postulate 1 gives the relations :

$$\sqrt{\sum_{i=2}^{4} (f_i(.) - x'_i(O_{p'}))^2} = \frac{|h(\tau_+(.)) - h(\tau_-(.))|}{2}$$

$$f_1(.) = \frac{h(\tau_+(.) + h(\tau_-(.))}{2}$$

Let us specify f_1 which is the generative function of f because it allows to build its Jacobian matrix. $M'(.) = (\sigma, y_2(\sigma), y_3(\sigma), y_4(\sigma))$ and $N'(.) = (\tau, x_2(\tau), x_3(\tau), x_4(\tau))$ being two trajectories of body in motion in S, we note $\sigma(\tau)$ the function which to the date τ of emission of an electromagnetic signal by N' associates the date σ of interception of the signal by M'. If the cartesian digital clock used by P to build S is normally oriented, then :

$$\sigma(\tau) - \tau = \sqrt{\sum_{i=2}^{4}(y_i(\sigma(\tau)) - x_i(\tau))^2}$$

The calculation gives :

$$\left(\sigma(\tau) - \tau - \sum_{i=2}^{2}(y_i(\sigma(\tau)) - x_i(\tau))\frac{dy_i}{d\sigma}\right)\frac{d\sigma}{d\tau} = \sigma(\tau) - \tau - \sum_{i=2}^{2}(y_i(\sigma(\tau)) - x_i(\tau))\frac{dx_i}{d\tau}$$

The duration in S' which separates the emission of the signal by N' and its reception by M' is the absolute value of the number :

$$f_1(\sigma(\tau), y_2(\sigma(\tau)), y_3(\sigma(\tau)), y_4(\sigma(\tau))) - f_1(\tau, x_2(\tau), x_3(\tau), x_4(\tau))$$

If the admissible transformation f is a real eligible transformation then when M' and N' are constantly at rest according to P', $\frac{dy_i}{d\sigma}$ and $\frac{dx_i}{d\tau}$ being determined by (6), this duration does not depend on the variable τ. We obtain:

$$\frac{d\sigma}{d\tau}(f_{11} + \sum_{i=2}^{2} f_{1i}\frac{dy_i}{d\sigma})(\sigma(\tau), M'(\sigma(\tau))) = (f_{11} + \sum_{i=2}^{2} f_{1i}\frac{dx_i}{d\tau})(\tau, N'(\tau))$$

$$\frac{d\sigma}{d\tau}\frac{\mu_f}{f_{11}}(\sigma(\tau), M'(\sigma(\tau))) = \frac{\mu_f}{f_{11}}(\tau, N'(\tau))$$

By noting :

$$l_i(\tau) = \frac{y_i(\sigma(\tau)) - x_i(\tau)}{\sigma(\tau) - \tau}$$

We get :

$$\frac{1}{\mu_f}(f_{11} + \sum_{i=2}^{4} f_{1i}l_i)(\sigma(\tau), M'(\sigma(\tau))) = \frac{1}{\mu_f}(f_{11} + \sum_{i=2}^{4} f_{1i}l_i)(\tau, N'(\tau))$$

Thus, $\forall(\tau, x_2, x_3, x_4) \in \mathbb{R}^4$, $\forall(l_2, l_3, l_4) \in \mathbb{R}^3$, $l_2^2 + l_3^2 + l_4^2 = 1$, the function :

$$\frac{1}{\mu_f}(f_{11} + \sum_{i=2}^{4} f_{1i}l_i)(\tau + \epsilon, x_2 + \epsilon l_2, x_3 + \epsilon l_3, x_4 + \epsilon l_4)$$

does not depend on the the positive variable ϵ. It results :

$$\frac{d}{d\tau}(\frac{f_{11}}{\mu_f}) + \sum_{i=2}^{4}\left(\frac{d}{d\tau}(\frac{f_{1i}}{\mu_f}) + \frac{d}{dx_i}(\frac{f_{11}}{\mu_f})\right) l_i + \sum_{j=2}^{4}\sum_{i=2}^{4}\left(\frac{d}{dx_j}(\frac{f_{1i}}{\mu_f})\right) l_i l_j = 0 \qquad (8)$$

We have to associate with this equation the fact that (6) gives the velocity vector known along the trajectory of P' and as in classical physics (according to (2) he can move with a rotational motion relative to P), his world line is not sufficient to specify his state of motion. All the solutions of (8) and (4) are physically acceptable.

Any affine function f_1 which is solution of (4) is acceptable and the associated transformation f is an element of the Poincaré group.

In a coordinate system, a clock that revolves around itself (possibly without translational motion) is physically distinguishable from a clock that does not revolve around itself therefore it would not be a conceptual absurdity if the relationship between "the proper time of a clock H in motion in an inertial coordinate system S" and "the temporal variable of S" depended on the movement of H around itself (with respect to S). This is the case in the new model. Rigorously, a unique world line segment which describes a trajectory of material body can have a certain cartesian proper time (which is simply a parametrization recognized as affine) when it is considered as a point of a physical space $\mathcal{R}$ and it may have a different cartesian proper time when it is considered as a point of a different physical space $\mathcal{R}'$. This is forbidden in general relativity which supposes, by definition, that a cartesian proper time comes from the metric tensor: it is not a consequence of special relativity and it is not a consequence of the equivalence principle.

By noting $g_p = \dfrac{f_{1p}}{\mu_f}$, $g_{pq} = \dfrac{\partial}{\partial x_q} g_p$, $1 \leq p, q \leq 4$, we can write:

$$g_{11} + \sum_{i=2}^{4} (g_{i1} + g_{1i}) l_i + \sum_{j=2}^{4} \sum_{i=2}^{4} g_{ij} l_i l_j = 0$$

By noting

$$\alpha^2 = -1 \quad 0 \leq \theta \leq \pi \quad 0 \leq \psi < 2\pi$$

$$l_2 = \cos\theta \quad l_3 = \frac{1}{2}\sin\theta(e^{\alpha\psi} + e^{-\alpha\psi}) \quad l_4 = \frac{1}{2\alpha}\sin\theta(e^{\alpha\psi} - e^{-\alpha\psi})$$

This equation can be noted:

$$\sum_{n=-2}^{2} s_n e^{\alpha n \psi} = 0$$

It is a trigonometric polynomial. The explicit shape of s_{-2} and s_2 gives :

$$g_{34} + g_{43} = 0 \quad g_{33} = g_{44}$$

The symmetries of the equation (8) or the permutations of the indices of the transformations $l_i \longmapsto (\theta, \psi)$ allow to write:

$$g_{ij} + g_{ji} = 0 \quad g_{ii} = g_{jj} \quad 2 \leq i, j \leq 4 \quad i \neq j$$

By replacing these relations in the explicit shape of s_{-1}, s_0 and s_1, we obtain :

$$g_{1i} + g_{i1} = 0 \quad g_{ii} = -g_{11} \quad 2 \leq i \leq 4$$

We can write:

$$g_{pq} = -g_{qp} \quad g_{ii} = -g_{11} \quad 2 \leq i \leq 4 \quad 1 \leq p, q \leq 4 \quad p \neq q \tag{9}$$

By using alternately the antisymmetric relation of the equation (9) and Schwarz' theorem we can write:

$$g_{pqn} = -g_{qpn} = -g_{qnp} = g_{nqp} = g_{npq} = -g_{pnq} = -g_{pqn} \quad 1 \leq p, q, n \leq 4 \quad p \neq q \neq n$$

We get:

$$g_p = \sum_{q \neq p} G^{pq}(x_p, x_q) \qquad 1 \leq p, q \leq 4$$

Let us exploit the relation $g_{22} = g_{33}$. By noting $G_k^{ij} = \dfrac{\partial}{\partial x_k} G^{ij}$ we can write:

$$G_2^{21}(x_2, x_1) + G_2^{23}(x_2, x_3) + G_2^{24}(x_2, x_4) = G_3^{31}(x_3, x_1) + G_3^{32}(x_3, x_2) + G_3^{34}(x_3, x_4)$$

By performing a derivation of both sides of this equality with respect to a same variable we can write:

$$\frac{\partial}{\partial x_1} G_2^{21}(x_2, x_1) = \frac{\partial}{\partial x_1} G_3^{31}(x_3, x_1) \quad \frac{\partial}{\partial x_4} G_2^{24}(x_2, x_4) = \frac{\partial}{\partial x_4} G_3^{34}(x_3, x_4)$$

On one side we have a function of (x_2, x_1) and of the other one we have a function of (x_3, x_1). We can write, using Schwarz' theorem :

$$\frac{\partial}{\partial x_2} G_1^{21}(x_2, x_1) = \frac{\partial}{\partial x_3} G_1^{31}(x_3, x_1) = \tilde{H}(x_1)$$

$$\frac{\partial}{\partial x_2} G_4^{24}(x_2, x_4) = \frac{\partial}{\partial x_3} G_4^{34}(x_3, x_4) = \tilde{R}_4(x_4)$$

We can write:

$$G_1^{21}(x_2, x_1) = \tilde{H}(x_1)x_2 + \tilde{L}_2(x_1) \quad G_4^{24}(x_2, x_4) = \tilde{R}_4(x_4)x_2 + \tilde{w}_{24}(x_4)$$

$$G_1^{31}(x_3, x_1) = \tilde{H}(x_1)x_3 + \tilde{L}_3(x_1) \quad G_4^{34}(x_3, x_4) = \tilde{R}_4(x_4)x_3 + \tilde{w}_{34}(x_4)$$

Since $g_{22} = g_{33} = g_{44}$ we can write, $2 \leq i, j \leq 4$, $i \neq j$:

$$G^{i1}(x_i, x_1) = H(x_1)x_i + L_i(x_1) + D_{ii}(x_i) \quad G^{ij}(x_i, x_j) = R_j(x_j)x_i + w_{ij}(x_j) + D_{ij}(x_i)$$

We get, $2 \leq i, j, k \leq 4$, $i \neq j \neq k$, $D_i = D_{ii} + D_{ij} + D_{ik}$:

$$g_i = [H(x_1) + R_j(x_j) + R_k(x_k)]x_i + D_i(x_i) + L_i(x_1) + w_{ij}(x_j) + w_{ik}(x_k)$$

Since $g_{ii} = g_{jj}$ we can write, $2 \leq i, j, k \leq 4$, $i \neq j \neq k$:

$$H(x_1) + R_j(x_j) + R_k(x_k) + \frac{dD_i}{dx_i}(x_i) = H(x_1) + R_i(x_i) + R_k(x_k) + \frac{dD_j}{dx_j}(x_j)$$

$$R_i(x_i) - \frac{dD_i}{dx_i}(x_i) = R_j(x_j) - \frac{dD_j}{dx_j}(x_j)$$

On one side we have a function of x_i and of the other one we have a function of x_j then we can write, $2 \leq i \leq 4$, $h \in \mathbb{R}$:

$$R_i(x_i) - \frac{dD_i}{dx_i}(x_i) = h$$

By redefining the function H we can write, $2 \leq i, j, k \leq 4$, $i \neq j \neq k$:

$$g_i = [H(x_1) + \frac{dD_j}{dx_j}(x_j) + \frac{dD_k}{dx_k}(x_k)]x_i + D_i(x_i) + L_i(x_1) + w_{ij}(x_j) + w_{ik}(x_k)$$

Since $g_{ij} = -g_{ji}$ we can write, $2 \leq i, j \leq 4$, $i \neq j$:

$$\frac{d^2 D_j}{dx_j^2}(x_j)x_i + \frac{dw_{ij}}{dx_j}(x_j) = -\frac{d^2 D_i}{dx_i^2}(x_i)x_j - \frac{dw_{ji}}{dx_i}(x_i)$$

We get, $2 \leq i, j, k \leq 4$, $i \neq j \neq k$:

$$\frac{d^3 D_j}{dx_j^3}(x_j)x_i + \frac{d^2 w_{ij}}{dx_j^2}(x_j) = -\frac{d^2 D_i}{dx_i^2}(x_i) \qquad \frac{d^3 D_j}{dx_j^3}(x_j) = -\frac{d^3 D_i}{dx_i^3}(x_i)$$

$$\frac{d^3D_j}{dx_j^3}(x_j) = -\frac{d^3D_i}{dx_i^3}(x_i) = -(-\frac{d^3D_k}{dx_k^3}(x_k)) = \frac{d^3D_k}{dx_k^3}(x_k) = -\frac{d^3D_j}{dx_j^3}(x_j)$$

We get, $i \neq 1$, a_i, α_i, c_i, $w_{ij} \in \mathbb{R}$:

$$\frac{d^3D_i}{dx_i^3}(x_i) = 0 \quad D_i(x_i) = \frac{a_i}{2}x_i^2 + \alpha_i x_i + c_i \quad \frac{d^2w_{ij}}{dx_j^2}(x_j) = -a_i \quad \frac{dw_{ij}}{dx_j}(x_j) = w_{ij} - a_i x_j$$

We can write :

$$a_j x_i + (w_{ij} - a_i x_j) = -a_i x_j - (w_{ji} - a_j x_i)$$

We get, $2 \leq i, j \leq 4$, $i \neq j$, $b_{ij} \in \mathbb{R}$:

$$w_{ji} = -w_{ij} \qquad w_{ij}(x_j) = -\frac{a_i}{2}x_j^2 + w_{ij}x_j + b_{ij}$$

By noting $\tilde{b}_i = b_{ij} + b_{ik} + c_i$ and by redefining the function H we can write, $2 \leq i, j, k \leq 4$, $i \neq j \neq k$:

$$g_i = H(x_1)x_i + L_i(x_1) + \frac{a_i}{2}(x_i^2 - x_j^2 - x_k^2) + a_j x_j x_i + a_k x_k x_i + w_{ij}x_j + w_{ik}x_k + \tilde{b}_i$$

Since $g_{11} = -g_{ii}$, $i \neq 1$, we can write :

$$\sum_{n=2}^{4} G_1^{1n}(x_1, x_n) = -[H(x_1) + \sum_{n=2}^{4} a_n x_n] \qquad \frac{\partial}{\partial x_i}G_1^{1i}(x_1, x_i) = -a_i$$

$$G^{1i}(x_1, x_i) = -a_i x_i x_1 - T_i(x_1) - P_i(x_i) \qquad \frac{\partial}{\partial x_1}(\sum_{n=2}^{4} T_n(x_1)) = H(x_1)$$

Since $g_{1i} = -g_{i1}$, $i \neq 1$, we can write :

$$-a_i x_1 - \frac{dP_i}{dx_i}(x_i) = -\frac{dH}{dx_1}(x_1)x_i - \frac{dL_i}{dx_1}(x_1)$$

We get, $i \neq 1$:

$$\frac{d^2H}{dx_1^2}(x_1)x_i + \frac{d^2L_i}{dx_1^2}(x_1) = a_i \quad \frac{d^2H}{dx_1^2}(x_1) = 0 \quad \frac{d^2L_i}{dx_1^2}(x_1) = a_i$$

We get, $i \neq 1$, L_1, L_i, H, $\tilde{c}_i$, $d_i \in \mathbb{R}$:

$$H(x_1) = Hx_1 + L_1 \quad L_i(x_1) = \frac{a_i}{2}x_1^2 + L_i x_1 + \tilde{c}_i \quad \frac{dP_i}{dx_i}(x_i) = Hx_i + L_i$$

$$P_i(x_i) = \frac{H}{2}x_i^2 + L_i x_i + d_i \qquad \sum_{n=2}^{4} T_n(x_1) = \frac{H}{2}x_1^2 + L_1 x_1 + d_1$$

By noting $b_i = \tilde{b}_i + \tilde{c}_i$, $b_1 = d_1 + d_2 + d_3 + d_4$, $i \neq 1$, We get :

$$g_1 = -\frac{H}{2}(\sum_{p=1}^{4} x_p^2) - (\sum_{n=2}^{4} a_n x_n)x_1 - (\sum_{p=1}^{4} L_p x_p) - b_1 \tag{10}$$

$$g_i = Hx_i x_1 + \frac{a_i}{2}(x_1^2 - \sum_{n=2}^{4} x_n^2) + (\sum_{n=2}^{4} a_n x_n)x_i + L_i x_1 + L_1 x_i + (\sum_{n=2}^{4} w_{in}x_n) + b_i \tag{11}$$

$$w_{ji} = -w_{ij} \qquad 2 \leq i, j \leq 4 \tag{12}$$

Let us look for compatibility equations.

By noting $\mu_g = g_1^2 - g_2^2 - g_3^2 - g_4^2 = \dfrac{1}{\mu_f}$, the generative function f_1 being supposed continuously differentiable to second order, we can write:

$$\frac{d}{dx_i}(\frac{g_j}{\mu_g}) = \frac{d}{dx_j}(\frac{g_i}{\mu_g}) \quad \frac{d}{dx_i}(\frac{g_1}{\mu_g}) = \frac{d}{dx_1}(\frac{g_i}{\mu_g}) \quad 2 \le i, j \le 4 \quad i \ne j$$

By noting $\mu_{gl} = \dfrac{d}{dx_l}\mu_g$, we can write:

$$(g_{ij} - g_{ji})\mu_g = g_i \mu_{gj} - g_j \mu_{gi} \quad (g_{i1} - g_{1i})\mu_g = g_i \mu_{g1} - g_1 \mu_{gi} \quad 2 \le i, j \le 4 \quad i \ne j \quad (13)$$

These are equalities between polynomials of degree at most equal to five. In this document we are only going to be interested in the solutions for which $(H, a_2, a_3, a_4) = (0, 0, 0, 0)$. We do not ask if there are others solutions.

When Q is a polynomial of finite degree, one can note $Z_l[Q]$ the sum of monomials of degree l. We can write:

$$Z_2[g_i \mu_{g1} - g_1 \mu_{gi}] = Z_1[g_i] \times Z_1[\mu_{g1}] - Z_1[g_1] \times Z_1[\mu_{gi}]$$

$$Z_2[(g_{i1} - g_{1i})\mu_g] = 2L_i Z_2[\mu_g]$$

The equality $\dfrac{\partial^2}{\partial x_1 \partial x_i} Z_2[(g_{i1} - g_{1i})\mu_g] = \dfrac{\partial^2}{\partial x_1 \partial x_i} Z_2[g_i \mu_{g1} - g_1 \mu_{gi}]$ gives:

$$L_1(L_j^2 + L_k^2 + w_{ij}^2 + w_{ik}^2) = 0 \quad 2 \le i, j, k \le 4 \quad i \ne j \ne k$$

We must distinguish the solutions for which $L_1 = 0$ and the solutions for which $L_1 \ne 0$.

4.1 The Big Bang observational reference frame: $\mathcal{R}^0$

The solutions for which $L_1 \ne 0$ are formed between an observer S' of one particular physical space $\mathcal{R}^0$ and an observer S of a physical space $\mathcal{R}$ which is different from $\mathcal{R}^0$. I do not preach the existence of several Big Bang physical spaces because of an analysis of the effects of the composition of a Lorentz transformation and the solution that I will get.We can write :

$$\mu_g = (L_1 x_1 + b_1)^2 - \sum_{n=2}^{4} (L_1 x_n + b_n)^2$$

$$L_i = w_{ij} = 0 \quad g_1 = -L_1 x_1 - b_1 \quad g_i = L_1 x_i + b_i \qquad 2 \le i, j \le 4 \quad i \ne j$$

The Eulerian description of the velocity field $U(\tau, x_2, x_3, x_4) = (U_i)_{2 \le i \le 4}$ which represents the motions in S of points of $\mathcal{R}^0$ is given by the equation (6) and one can write:

$$U_i(\tau, x_2, x_3, x_4) = -\frac{g_i}{g_1}(\tau, x_2, x_3, x_4) = \frac{L_1 x_i + b_i}{L_1 \tau + b_1} \quad 2 \le i \le 4$$

In a description of Lagrangian trajectories in S of points of $\mathcal{R}^0$, we must write:

$$\frac{dx_i}{d\tau}(\tau) = \frac{L_1 x_i(\tau) + b_i}{L_1 \tau + b_1} \quad 2 \le i \le 4$$

We get

$$\frac{d^2 x_i}{d\tau^2}(\tau) = 0 \quad 2 \le i \le 4$$

Thus, each point of $\mathcal{R}^0$ moves with a constant speed in S. Furthermore, there exists a unique point of $\mathcal{R}^0$ which is continuously motionless in $\mathcal{R}$: its spatial coordinates in S are $x_i = -\frac{b_i}{L_1}$, $2 \leq i \leq 4$. However, this world line possesses no peculiarity in $\mathcal{R}^0$ because it depends on the choice of the physical space $\mathcal{R}$ with which S is associated. Indeed, if $\mathcal{E}$ is a physical space which is such that a transformation between one of its observers and S is element of the Poincaré group, then $\mathcal{R}$ and $\mathcal{E}$ has no common point if they are distinct.

Besides, for each value of L_1, each of the three numbers b_2, b_3, and b_4 may be expressed as a function of b_1. In fact, using the expression of $\frac{dx_i}{d\tau}(\tau)$, it is sufficient to know, at a time τ_0 in S, the position and the speed of a particular point which is arbitrarily chosen in $\mathcal{R}^0$.

Furthermore, equation (4) is used to write:

$$\mu_g = \frac{1}{\mu_f} > 0$$

Thus, for each value of the couple (L_1, b_1), the cartesian temporal variable τ (which is assumed to be normally oriented) exists only in the real interval $]\frac{-b_1}{L_1}; +\infty[$. At each time τ of this real interval of cartesian dating of the events, the physical space $\mathcal{R}$ exists only in the open ball with center $(x_i = -\frac{b_i}{L_1})$, $2 \leq i \leq 4$ and radius $\mid L_1\tau + b_1 \mid$.

Besides, we can write:

$$\frac{d}{d\tau}(x_i(\tau) + \frac{b_i}{L_1}) = \frac{L_1}{L_1\tau + b_1}(x_i(\tau) + \frac{b_i}{L_1}) \quad 2 \leq i \leq 4$$

$O_{\mathcal{R}}$ being this unique world line which is both a point of $\mathcal{R}$ and a point of $\mathcal{R}^0$, if M' is a point of $\mathcal{R}^0$ whose spatial coordinates in S are represented by $(x_i(\tau))_{2\leq i\leq 4}$, and if $d_\tau(M', O_{\mathcal{R}})$ is the spatial distance which separates M' and O_R in $\mathcal{R}$ at the cartesian time τ, we can write:

$$d_\tau(M', O_{\mathcal{R}}) = \sqrt{\sum_{i=2}^{4}(x_i(\tau) + \frac{b_i}{L_1})^2}$$

$$\frac{d}{d\tau}d_\tau(M', O_{\mathcal{R}}) = \frac{L_1}{L_1\tau + b_1}d_\tau(M', O_{\mathcal{R}}) \tag{14}$$

Thus, for each value of τ, the speed of estrangement between M' and $O_{\mathcal{R}}$ is proportional to the distance $d_\tau(M', O_{\mathcal{R}})$: This is the description of the structural expansion of the universe. We can write the solution:

$$d_\tau(M', O_{\mathcal{R}}) = (\tau + \frac{b_1}{L_1})\sqrt{\sum_{i=2}^{4}(\frac{L_1 x_i(\tau_0) + b_i}{L_1\tau_0 + b_1})^2} \quad , \quad \tau_0 \in]-\frac{b_1}{L_1}; +\infty[$$

The world lines M' and $O_{\mathcal{R}}$ are constantly at rest in $\mathcal{R}^0$. We can write a description of the structural expansion of the universe with regard to each of the points of $\mathcal{R}$. In fact, if O is a point arbitrarily chosen in $\mathcal{R}$, and if β is the angle which is formed in $\mathcal{R}$ by the vector space $\varphi_{\mathcal{R}}(O, O_{\mathcal{R}})$ and $\varphi_{\mathcal{R}}(O_{\mathcal{R}}, M')$, and if $d(O, O_{\mathcal{R}})$ is the constant spatial distance between O and $O_{\mathcal{R}}$ in $\mathcal{R}$, then we can write:

$$d_\tau(M', O) = \sqrt{d_\tau^2(M', O_{\mathcal{R}}) + 2d_\tau(M', O_{\mathcal{R}})d(O, O_{\mathcal{R}})\cos\beta + d^2(O, O_{\mathcal{R}})}$$

This suggests a characteristic equation of the structural expansion of the universe.
Besides, we can write:

$$f_{1l} = \frac{g_l}{\mu_g} \qquad 1 \leq l \leq 4$$

We get:

$$f_1(\tau, x_2, x_3, x_4) = C_{f_1} - \frac{1}{2L_1} \ln[(\tau + \frac{b_1}{L_1})^2 - \sum_{i=2}^{4} (x_i + b_i)^2]$$

All the generatives functions of the real eligible transformations between two designated physical spaces can be deduced from each other by a mathematical affine transformation. This is because a generative function is a cartesian dating of events and therefore it depends on the arbitrary choice of the unity of durations, on the arbitrary choice of the numerical value assigned to a reference event, and the arbitrary choice of one of the two orientation of the mathematical real number line in order to characterize the succession of events. So we need to specify the value of the couple (L_1, C_{f_1}) in specific physical conditions. On the unique world line that is both constantly at rest in $\mathcal{R}$ and $\mathcal{R}^0$, we can write:

$$f_1(\tau, -\frac{b_2}{L_1}, -\frac{b_3}{L_1}, -\frac{b_4}{L_1}) = C_{f_1} - \frac{1}{L_1} \ln(\tau + \frac{b_1}{L_1})$$

If the cartesian temporal variables of S and S' are both normally oriented then the constant sign of the function f_{11} is positive consequently $L_1 < 0$. If c is the positive number which is chosen to represent the intensity in the vaccum of an electromagnetic signal in $\mathcal{R}$ then we can write:

$$f_1(ct, -\frac{b_2}{L_1}, -\frac{b_3}{L_1}, -\frac{b_4}{L_1}) = C_{f_1} - \frac{1}{L_1} \ln(ct + \frac{b_1}{L_1})$$

The following theorem is based on a purely mathematical argument of a symmetry:

Theorem 2 *The cartesian datings of the events being normally oriented in $\mathcal{R}$ and $\mathcal{R}^0$, the standarts of lengths of these separate physical spaces are identical when $L_1 = -1$. If furthermore $\mathcal{R}$ chosen to make a cartesian dating of the events which takes values in the real interval $]0; +\infty[$, then $b_1 = 0$ and we shall say that the cartesian clocks of $\mathcal{R}$ and $\mathcal{R}^0$ are synchronized if $C_{f_1} = 0$.*

Even in a situation where the physical spaces $\mathcal{R}$ and $\mathcal{R}^0$ choose the same real number c to represent the intensity of an electromagnetic signal in vacuum, the explicit relation that will exist between the temporal variable $t' = \frac{f_1}{c}$ of $\mathcal{R}^0$ and the temporal variable t of $\mathcal{R}$ will depend on the numerical value of c which characterizes (in each physical space) the particular relation which exists between the natural period which is arbitrarily chosen to be the unity of the durations and the natural measure which is arbitrarily chosen to be the unity of the lengthes. This importance of the numerical value of c can already be noticed in a transformation of Lorentz.

Let us look for more information on the structure of the transformation f. If the observers S and S' are chosen so that the particular conditions of the theorem 2 are satisfied, and also so that the unique world line which is a point of $\mathcal{R}$ and $\mathcal{R}^0$ has the spatial coordinates $(0, 0, 0)$ in S, then we can write:

$$f_1(x_1, x_2, x_3, x_4) = \ln[\sqrt{x_1^2 - x_2^2 - x_3^2 - x_4^2}]$$

By noting $\delta = x_1^2 - x_2^2 - x_3^2 - x_4^2$, the equations (5) can be written:

$$f_{i1} = -\frac{x_i}{\delta} \quad f_{ii} = \frac{x_1\sqrt{\delta} + \delta + x_i^2}{x_1\delta + \delta^{\frac{3}{2}}} \quad f_{ij} = \frac{x_i x_j}{x_1\delta + \delta^{\frac{3}{2}}} \quad 2 \leq i,j \leq 4 \quad i \neq j$$

By noting $f_{lpq} = \dfrac{\partial^2 f_l}{\partial x_p \partial x_q}$, $2 \leq i,j,k \leq 4$, $i \neq j \neq k$, we get:

$$f_{ijk} - f_{ikj} = 0 \quad f_{iji} - f_{iij} = -\frac{x_1 x_j}{x_1\delta^{\frac{3}{2}} + \delta^2} \quad f_{ij1} - f_{i1j} = \frac{x_i x_j}{x_1\delta^{\frac{3}{2}} + \delta^2}$$

Thus, the conclusions of the Schwarz theorem are not completely satisfied and this indicates that the real eligible transformation f which is defined from $\mathcal{R}$ to $\mathcal{R}^0$, as well as its inverse which is defined from $\mathcal{R}^0$ to $\mathcal{R}$, are not continuously differentiable to second order.

4.2 Transformations between two different physical spaces of $\mathcal{R}^0$

We are interested in the solutions for which $L_1 = 0$. Because we come to a contradiction when it is not satisfied, all the relations stemming from the equality $Z_2[g_i\mu_{g1} - g_1\mu_{gi}] = Z_2[(g_{i1} - g_{1i})\mu_g]$ produce the equation:

$$L_i w_{jk} + L_j w_{ki} + L_k w_{ij} = 0 \qquad 2 \leq i,j,k \leq 4 \quad i \neq j \neq k \tag{15}$$

The equality $Z_1[g_i\mu_{g1} - g_1\mu_{gi}] = Z_1[(g_{i1} - g_{1i})\mu_g]$ gives the equations, $2 \leq i,j,k \leq 4$, $i \neq j \neq k$:

$$L_j(b_j L_i - b_i L_j) + L_k(b_k L_i - b_i L_k) = b_1(L_j w_{ji} + L_k w_{ki})$$

$$w_{ji}(b_j L_i - b_i L_j) + w_{ki}(b_k L_i - b_i L_k) = b_1(w_{ji}^2 + w_{ki}^2)$$

$$w_{jk}(b_j L_i - b_i L_j) = b_1 w_{jk} w_{ji}$$

The equality $Z_0[g_i\mu_{g1} - g_1\mu_{gi}] = Z_0[(g_{i1} - g_{1i})\mu_g]$ gives the equations, $2 \leq i,j,k \leq 4$, $i \neq j \neq k$:

$$b_j(b_j L_i - b_i L_j) + b_k(b_k L_i - b_i L_k) = b_1(b_j w_{ji} + b_k w_{ki})$$

The equality $Z_2[g_i\mu_{gj} - g_j\mu_{gi}] = Z_2[(g_{ij} - g_{ji})\mu_g]$ gives the equation (15). The equality $Z_1[g_i\mu_{gj} - g_j\mu_{gi}] = Z_1[(g_{ij} - g_{ji})\mu_g]$ gives the equations, $2 \leq i,j,k \leq 4$, $i \neq j \neq k$:

$$L_k(b_i w_{jk} + b_j w_{ki} + b_k w_{ij}) = 0$$

$$w_{ki}(b_i w_{jk} + b_j w_{ki} + b_k w_{ij}) - L_i(b_j L_i - b_i L_j) = b_1 L_i w_{ij}$$

$$-L_k(b_j L_i - b_i L_j) = b_1 L_k w_{ij}$$

We can write:

$$b_i w_{jk} + b_j w_{ki} + b_k w_{ij} = 0 \qquad 2 \leq i,j,k \leq 4 \quad i \neq j \neq k \tag{16}$$

Furthermore :

$$(L_2, L_3, L_4) = (0,0,0) \quad or \quad b_i L_j - b_j L_i = b_1 w_{ij} \quad 2 \leq i,j \leq 4 \quad i \neq j$$

The equality $Z_0[g_i\mu_{gj} - g_j\mu_{gi}] = Z_0[(g_{ij} - g_{ji})\mu_g]$ gives the equations, $2 \leq i,j,k \leq 4$, $i \neq j \neq k$:

$$b_1(b_i L_j - b_j L_i) + b_k(b_i w_{jk} + b_j w_{ki} + b_k w_{ij}) = b_1^2 w_{ij}$$

We can write:
$$b_1(b_iL_j - b_jL_i) = b_1^2 w_{ij} \quad 2 \le i,j \le 4 \quad i \ne j$$
Since $\mu_g > 0$, the function g_1 never vanishes and we can write:
$$(L_2, L_3, L_4) = (0,0,0) \quad \Longrightarrow \quad (g_1, w_{23}, w_{24}, w_{34}) = (b_1, 0, 0, 0)$$
We get:
$$b_iL_j - b_jL_i = b_1 w_{ij} \quad 2 \le i,j \le 4 \quad i \ne j \tag{17}$$
The equations (15) and (16) are consequences of the equations (17).The Eulerian description of the velocity field $U(\tau, x_2, x_3, x_4) = (U_i)_{2\le i\le 4}$ which represents the motions in S of points of $\mathcal{R}'$ is given by the equation (6) and one can write:
$$U_i(\tau, x_2, x_3, x_4) = \frac{L_i\tau + w_{ij}x_j + w_{ik}x_k + b_i}{L_2x_2 + L_3x_3 + L_4x_4 + b_1} \quad 2 \le i,j,k \le 4 \quad i \ne j \ne k \tag{18}$$

4.3 The admissible rectilinear motions

One can study a rectilinear motion by imposing relations:
$$U_3(\tau, x_2, x_3, x_4) = U_4(\tau, x_2, x_3, x_4) = 0 \quad \forall(\tau, x_2, x_3, x_4)$$
Wet get:
$$L_3 = L_4 = b_3 = b_4 = w_{23} = w_{24} = w_{34} = 0$$
If $L_2 = 0$ then f is an element of the Poincaré group. Necessarily, the constant f_{11} depends only on the intensity of the velocity $\frac{v_i}{c}(f) = -\frac{f_{1i}}{f_{11}}$, $2 \le i \le 4$, which results from the equation (6). By calculating the inverse of the Jacobian matrix of f which is given by the equations (5), we obtain [5], $i \ne 1$:
$$(f^{-1})_{11} = \frac{f_{11}}{\mu_f} \qquad (f^{-1})_{1i} = -\frac{f_{1i}}{\mu_f}$$
We can write:
$$\frac{v_i}{c}(f^{-1}) = \frac{f_{1i}}{f_{11}} \qquad \sum_{i=2}^{4}[\frac{v_i}{c}(f^{-1})]^2 = \sum_{i=2}^{4}[\frac{v_i}{c}(f)]^2 = \frac{v^2}{c^2}$$
The following theorem is based on a purely mathematical argument of a symmetry:

Theorem 3 *The standarts of lengths of the physical spaces $\mathcal{R}$ and $\mathcal{R}'$ are identical when $(f^{-1})_{11} = f_{11}$.*

[5] Since it was specified that the equations (5) describe a situation where the observer S' is chosen in a particular way with regard to S, in generality there is a constant orthogonal matrix M_f such that $f_{11}^{-1} = \frac{f_{11}}{\mu_f}$ and $\begin{pmatrix} f_{12}^{-1} \\ f_{13}^{-1} \\ f_{14}^{-1} \end{pmatrix} = -\frac{1}{\mu_f} M_f \begin{pmatrix} f_{12} \\ f_{13} \\ f_{14} \end{pmatrix}$ and we always get
$$\sum_{i=2}^{4}[\frac{v_i}{c}(f^{-1})]^2 = \sum_{i=2}^{4}[\frac{v_i}{c}(f)]^2$$

We can write:

$$\begin{aligned}(f^{-1})_{11} = f_{11} &\implies \frac{f_{11}}{\mu_f} = f_{11} \\ &\implies \mu_f = 1 \\ &\implies f_{11}^2 - \sum_{i=2}^{4} f_{1i}^2 = 1 \\ &\implies f_{11}^2(1 - \frac{v^2}{c^2}) = 1 \\ &\implies |f_{11}| = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\end{aligned}$$

Alternatively, if $L_2 \neq 0$ then there is a number ν such that $b_2 = \nu L_2$. We get:

$$g_1(x_1, x_2, x_3, x_4) = -L_2(x_2 + \frac{b_1}{L_2}) \quad g_2(x_1, x_2, x_3, x_4) = L_2(x_1 + \nu) \quad g_3 = g_4 = 0$$

We can write:

$$\frac{1}{\mu_f} = \mu_g = L_2^2[(x_2 + \frac{b_1}{L_2})^2 - (x_1 + \nu)^2] > 0$$

Thus, for each value of x_1, the transformation f is defined for values of x_2 which are such as $(x_2 + \frac{b_1}{L_2})^2 - (x_1 + \nu)^2 > 0$. In particular, f is not defined on the plane of equation $x_2 = -\frac{b_1}{L_2}$. We already know that in a physical space which is different of $\mathcal{R}^0$, each cartesian time τ which is normally oriented is always greater than some number whose numerical value is derived from an arbitrary choice. The coordinate system S can be chosen so that x_1 take its values in real interval $]-\nu; +\infty[$. In a description of Lagrangian trajectories in S of points of $\mathcal{R}'$, we must write:

$$\frac{d}{d\tau}x_4(\tau) = \frac{d}{d\tau}x_3(\tau) = 0 \quad \frac{d}{d\tau}x_2(\tau) = \frac{\tau + \nu}{x_2(\tau) + \frac{b_1}{L_2}} \qquad 2 \leq i \leq 4 \quad i \neq j \neq k$$

We get:

$$x_2(\tau) = -\frac{b_1}{L_2} + \iota\sqrt{(\tau + \nu)^2 + (x_2(\tau_0) + \frac{b_1}{L_2})^2 - (\tau_0 + \nu)^2} \quad \tau_0 \in]-\nu; +\infty[\quad \iota \in \{-1, 1\}$$

The generative function $f_1(x_1, x_2, x_3, x_4) = f_1(x_1, x_2)$ is determined by the relations:

$$f_{11} = \frac{-1}{L_2}\frac{x_2 + \frac{b_1}{L_2}}{(x_2 + \frac{b_1}{L_2})^2 - (x_1 + \nu)^2} \qquad f_{12} = \frac{1}{L_2}\frac{x_1 + \nu}{(x_2 + \frac{b_1}{L_2})^2 - (x_1 + \nu)^2} \tag{19}$$

5 Conclusion

We can distinguish two practices in physics. The first is the proposal of formulae (Newton's first law of motion, Maxwell's equations, Lorentz force, Newton's law of gravitation, Doppler effect, Schrödinger's equation...) to describe the evolution of the elements of a system according to their intrinsic natures and their states of motion. The second is the precision of the observers who can notice the accuracy of these formulae (an observer is needed to notice the states of motions of the elements of a system). This second aspect is practicable only if we define beforehand all the possible observers by indicating how determining the states of motion of some with regard to the others.

When we express that we must associate to every experimenter a particular family of world lines which follow him in his motion and represents his physical space, it is a definition which offers mathematical possibilities. The classical kinematics operates a possibility to define all the physical spaces of the nature (some compared with the others) and this document operates another possibility. If a spatial position is defined as being a world line and not simply an element of $\mathcal{U}$, it is because an experimenter can notice in corpuscular model that a body makes round trips between two fixed points or is moving a zigzag without return by a same point, and in a quantum model he can be interested in the variation over time of the probability of finding a particle system in a bounded region of his physical space which is the set of fixed points.

We can define several temporal variables in each physical space (because it is sufficient to choose a particular parametrization along each point of this space) and some of them, by example the cartesian datings of events made by different experimenters who are in motion the ones compared with the others, are directly measurable. We showed at the end of the section 2 that we can propose mathematically consistent but physically false answers to the following question: if $\mathcal{R}$ and $\mathcal{R}'$ are two physical spaces, what is the structure of the states of motion that an experimenter of $\mathcal{R}$ notices for each world line which constitute $\mathcal{R}'$? One is free to choose any geometry and any temporal variable on $\mathcal{R}$ to write equations.

In classical physics we express that every experimenter can define a Euclidean geometry on his physical space so that all notice the same spatial distances between pairs of simultaneous events, this simultaneity resulting from a privileged temporal variable the existence of which is supposed. We establish then the states of relative motion between two arbitrarily chosen physical spaces in the nature and we obtain the equation (2) which highlights the existence of rotational motions, possibly coupled with translational motions.

A cartesian dating of events made by an experimenter on the surface of the earth is a temporal variable t, and a cartesian dating of events made by an experimenter on the surface of the moon is another temporal variable t'. In a relativistic theory, by definition, none of the measurable temporal variables (and more generally none of the relations of simultaneity on $\mathcal{U}$) is recognized as privileged by all the possible experimenters consequently we cannot resume the assumption of the classical physics on the conservation of lengths of certain segments of parametrized curves defined on the physical spaces. We propose then the postulate 1 which results from special relativity and experiments of Michelson and Morley, and we obtain the equations (6) and (8), and we propose the equation (14) as a rigorous theoretical formulation of the Hubble's experimental law.

References

[1] Woldemar Voigt, Ueber das Doppler'sche Princip (On the Principle of Doppler), Gttinger Nachrichten (7): 4151 (1887).

[2] Albert Einstein , Zur Elektrodynamik bewegter Körper, Annalen der Physik 322(10): 891-921 (1905).

[3] Øyvind Grøn, Space Geometry in a Rotating Reference Frame: A Historical Appraisal In G. Rizzi and M. Ruggiero (Kluwer, 2004), eds., Relativity in Rotating Frames (2004).

[4] Nathaniel I. Reden, Life on a Merry-go-round: An Examination of Relativistic Rotating Reference Frames, Amherst College (2005).

[5] Angelo Tartaglia, Lengths on Rotating Platforms, Foundations of Physics Letters 12 (1), 17-28 (1999).