Nagual Numbers:
A Critique of the Transfinite in Five Acts

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Abstract:

We are transported to the infinite hotel via processes unknown and find a way to discuss with Georg Cantor himself the cardinality of infinite sets. Using Cantor's first theorem we enumerate the numbers between zero and one, and discover that Cantor's second theorem has not been proven with the rigor we expected and the diagonalization proof fails spectacularly for certain representations. However Cantor has the last laugh. Later we visit the large but finite hotel and discover that transcendental numbers of certain classes are in fact countable, and that uncountable infinites are only created by the addition of a class of numbers or objects we describe as nagual numbers.

Prelude

To set the stage for later discussions of infinities we consider first what some might consider merely a curiosity of nomenclature of rational numbers, that some require infinite decimal representations, e.g.:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{10^i}=\frac{1}{3}$$

or in more concise notation, $1/3=.3\overline{3}$ (repeating). The difficulty here lies in our physical inability to actually carry out the infinite sum, thus making the equality seem somehow impossible to obtain. Despite this problem, it can easily be seen that carrying out the long division 1/3 will in fact yield the decimal expansion $.3\overline{3}$. While this example may seem trivial to those familiar with these notations, the same problem becomes more interesting in evaluating more complex expressions. In fact this question is important for our very definition of what it means for two expressions to be equal. One way to define such an equality is found in [Rudin, 1964]:

Infinite limit equality definition: An infinite limit of some function f: $\lim_{x\to\infty} f(x)$ is said to be equal to a value g if for every positive real number ε there exists a finite y such that $(g-\varepsilon) < f(y) < (g+\varepsilon)$, and if y exists for every ε we can write $\lim_{x\to\infty} f(x) = g$.

In other less rigorous words, we say that if an infinite sum can get as close to a number as we would like, we can say it is equal to that number. This might seem counterintuitive because if I give you an infinite list of numbers:

0.3
0.33
0.333
0.3333
0.33333
…

you might correctly point out that not one of these numbers on its own is exactly equal to one third. However there is no question that they converge to one third. So is it okay to say that in the limit as *n* goes to infinity, the series of *n* 3s after the decimal is equal to one third? It turns out that the whole of transfinite analysis relies on this important bit of semantics:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{10^i} \equiv \frac{1}{3} \qquad \text{however} \qquad \frac{1}{3} \notin \{0.3, 0.33, 0.333\ldots\}$$

Act 1

In a fog one day, we awake in the infinite hotel. A lively conversation is taking place at the hotel bar and it quickly becomes apparent that it is a very special occasion. An infinite supply of scotch has been delivered to celebrate. As the bottles of scotch were delivered they were opened one at a time and each bottle was divided equally between all the guests. After going through several bottles like that the total of scotch for any individual was still zero! Finally somebody pointed out that each guest should be able to take 10 bottles each because the bottles were numbered. Each guest should just multiply his room number by 10 and take the 10 bottles which are numbered beginning with that number. After much laughter over the error, the conversation begins.

We meet Georg Cantor then, in the infinite hotel lobby. He looks fabulous. A hush seems to move through the crowd as he approaches.

“So I understand you have a critique of the transfinite.” He says. “Would you care to make a wager?”

Quickly we agree to a bet, the loser of whom will pay for the scotch. Prices can run quite high in this hotel, but we are confident and agree to the deal amidst a cry of approval from nearby mathematicians. After the drinks arrive we get right down to business and I present him with a numbered list of all real numbers between zero and one. After all it's only fair. It is he who first told us how to number them! Traditionally it can be done with an ordering of the rational numbers using a diagonal covering or some other covering of the plane of ordered pairs needed to construct a rational number. In this case however we go with a slightly different numbering scheme, presenting them in binary notation as an ordered list:

0.1000000...	=½
0.0100000...	=1/4
0.1100000...	=3/4
0.0010000...	=1/8
0.1010000...	=5/8
0.0110000...	=3/8
0.1110000...	=7/8
0.0001000...	=1/16
0.1001000...	=9/16
0.0101000...	=5/16
0.1101000...	=13/16
0.0011000...	=3/16
0.1011000...	=11/16
0.0111000...	=7/16
0.1111000...	=15/16
0.0000100...	=1/32
…	

Fig. 1. The start of a list of all numbers between 0 and 1 in binary.

Of course we don't give him the full list but rather a formula for the ith binary digit of the jth number on the list. To see the ordering algorithm you can view the numbers from the rightmost 1 reading to the left and see we are just counting upwards in binary, effectively covering the interval from zero to one with a sparse binary tree.

Cantor quiets down the audience and then begins his diagonalization argument to show that there is a number between zero which is not on my list. He doesn't need to say much with this crowd, they are all familiar with his famous proof. He begins to select his number C which is not on our list. He picks each digit C_i to disagree with the i^{th} number on our list, along the diagonal. He picks it very quickly by looking at the table of numbers in Fig. 1. The number C is

C= 0.00111111111111111111111111111....

We can see from inspection this disagrees with the first number on our list on 1st binary digit, it disagrees from the second with it's second binary digit, and so forth. So has Georg really given us a new number not on our list? Not at all. In fact this number is ¼ and it is the 2nd number on our list! :

$1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots = 1/4$

Georg is surprised at first but then is somewhat nonplussed by this discovery that his famous diagonally constructed number is in fact on our list despite the difference from every digit of the base representation. He reaches for his wallet as though to pay the tab but then stops and says:

"In the Deutsche Mathematiker-Vereinigung (1890) I also used a binary system (using the symbols *m* and *w* instead of 0 and 1) to present my proof of the existence of uncountable infinity, however what I considered as a target set was not the numbers between zero and one but the set of infinite ordered sequences of digits."

"Georg, I thought the bet was about the numbers between zero and one."

"But is not the cardinality of the set of numbers between 0 and 1 equal to the cardinality of the set of all series of digits?"

Interlude

Most people have been introduced to Cantor's proof of the uncountability of the numbers between 0 and 1 using the proof using decimal notation. On the one hand, we might not even bother to go down this road because something as unimportant as the symbolic representation of a number could not possibly be crucial in the determination of a fundamental property of sets such as cardinality. On the other hand, if the representation of the numbers doesn't matter we should be able to do it in decimal as well. So lets leave Georg deep in thought and take a look at some numbers in decimal representation.

The first thing to note at this point about the famous proof that the real numbers are uncountable is that it only applies immediately to a certain set of orderings which it purports to disprove: those in decimal form (or other sequence form such that it makes sense to construct a diagonal proof). Because it is a proof of contradiction it proves that we cannot give a list in decimal form of all numbers between 0 and 1. It therefore does not prove that no list of the numbers exists. That would require also proving that every defined list of numbers could be written as a list of numbers in decimal form. Surely, every number can be written in decimal form, but does that mean that every countable list of numbers can also be written as a countable list of decimal numbers? That is harder to prove.

We return to our list R of reals in one to one correspondence with the natural numbers, so that the real numbers are ordered R_1, R_2, R_3, etc. These numbers are taken from the list and expanded in decimal form, i.e. $R_i = \lim_{n\to\infty}\sum_{m=1}^{n}\frac{R_{im}}{10^m}$, where R_{in} is the n^{th} digit of the i^{th} real number in the list. A real number is then constructed by going down the diagonal of this list, and picking an n^{th} digit different from the n^{th} digit of the n^{th} number. Thus we have constructed a number $Q = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{Q_m}{10^m}$ such that $Q_m \neq R_{mm}$. It is then argued that the list R_i does not contain this real number and hence is incomplete.

As we can tell from our binary example, this proof also relies on the theory that every number has a *unique representation* as an infinite sum of integral multiples of inverse powers of ten, i.e. decimal representations of numbers are unique. In fact this is not always the case! To correct this error, many authors say that the numbers Q_m must be chosen “avoiding infinite sequences of the number nine” while forgetting to also to avoid the equally important ambiguity of infinite sequences of the number zero. However, these minor variations in the popular account are not so important. The important thing is that because each item on the list is effectively a limit of an infinite sum, it can gain an equality with another number by virtue of it's being “as close as we need it to be” to that other number. In this case the uniqueness of the representation is more difficult to prove, though it appears at first avoiding the digits 9 and 0 when constructing our number Q should remove ambiguities. There are plenty of other digits, and any other two would seem to do the trick handily.

Interesting as this might be, we can move right past it at this stage because of what Georg just told us, namely that we needn't use the real numbers between 0 and 1 as our target set. If we instead consider as a target set the set of all infinite ordered numberings of digits. In this case the uniqueness is guaranteed and we can proceed toward a proof of uncountably infinite set.

Act 2

At the register.

“Well”, said Georg, “... “.
 “Well what? “
“Well I don't think I'm going to pay.”
“No? But we agreed. And you haven't pointed to a number not on my list”.

Georg Smiles.

“Well that's the point really. I don't have to believe in the existence of the uncountables for them to exist. But just for fun, here's one anyway”

Immediately he gives us:

C = 1/3 = 0.010101010101010101.....

Now it is our turn to be shocked. How is this nice rational number not on our list?   A quick look confirms that it is not. Our list only has numbers which at some point terminate with an infinite number of zeroes or an infinite number of ones. This one is a repeating pattern zero one zero one.

The audience is excited and money changes hands amongst the betting crowd. It looks like it's time to pay the bar bill.

However before we pay we can try to weasel our way out as we are armed with the infinite limit equality definition. We tell Georg that in the limit as the number of terms on our list goes to infinity, his number is equal to a number on our list. He replies that he wants to see this exact number:

$$C = 1/3 = \sum_{n=1}^{\infty} \frac{1}{2^{2n}}$$

on our list. Our reply is that insofar as we can consider his number to be well defined, it exists on our list:

“Listen Geog. We have given you a list of numbers Rn. For any given small number epsilon, we can find a number n on our list such that Rn is within epsilon of your number. Thus, in the limit that the amount of numbers on my list goes to infinity, my list contains your number.”

A murmur fills the room as some arguments break out. Opinions seem mixed. Georg collects himself and prepares a response.

Interlude

You aren't going to get through this adventure without hearing about power sets. These rather mundane sounding creations, the “set of all subsets of a set”, somehow manage to have wedged themselves into a backbone of the literature on cardinalities of infinite sets. The definition of a power set of a set is rather straightforward if a set is finite, and the elements of the power set can be listed. It is clear that the cardinality of such a power set is larger than the cardinality of the original set. The trouble of course comes when the original set is infinite, like say, the set of all integers.

Some interesting arguments have been made extending the analysis to when the original set is countably inifinite, and they conclude that the power set is uncountable. Usually they rely on contradiction or paradox after the assumption that the power set is countable. In fact the difficulty arises in determining the exact meaning of “all subsets”.

Lets tackle this problem, the power set of the integers and its cardinality, as though we were very naïve new arrivals at the infinite hotel. We would start to make a list of subsets. We can make a list of all subsets of size 1, and put them in the first column. We can also make a list of all subsets of size two and put them in the second column. If you doubt that this list can be made, just refer to any proof that the rational numbers are countable. We can further make a list of all subsets of size three and put them in the third column, and so on. Now we can begin to make our list of all subsets.

We begin of course with the empty set {}, then we add our first set of size 1: {0}, then our first set of size 2: {0,0} then the second set of size 1: {1}... as you can see we have covered an infinite plane with all the subsets of the integers in columns arranged by size and so we can snake down through them collecting them all. Right?

Well it turns out we are going to miss a few subsets. For example, the set of all even numbers. Is that on the list? How about the set of all numbers that represent computer programs that halt on any input, is that on the list? The set of “all subsets” of an infinite set can contain some pretty odd members. Infinite subsets are very difficult to order. Hence the term “uncountable”. And herein lies exactly what torments us about transfinite numbers.

Act III

“Nonsense” says Georg.

If you recall, we were just explaining to him that in the limit that my list is infinitely long, it contains his number 1/3.

“Your list contains nothing but sums of inverse powers of two. While your list might contain convergent series that approach 1/3, and yes to any required resolution, the exact number 1/3 is not on your list!”

Georg is definitely in charge of the situtation here, and enjoying his scotch tremendously. We had agreed from the start after all, that while equal to an infinite limit of decimal sums, the number 1/3 will never appear on any list of terminating decimal numbers. It makes sense it won't appear on this list of numbers in binary either.

“Do you really want me to tell you more numbers that aren't on your list?” Georg asks this with such mystique, and the hush in the room is so great, that I simply cannot let him continue. I concede the bet.

“The hard part is finding a number between 0 and 1 that *is* on his list” remarks a nearby mathematician to much general amusement. Glasses are raised and a toast is made to “the numbers between 0 and 1 that are on his list, though it is such a miniscule subset of them all”. “Of measure zero” yells another.

Thankfully the bartenders are not using bitcoin, and I am able to settle the tab after brief arithmetic and taking out loans from a subset of the financiers in the room. The mathematicians begin to relax and enjoy conversation again.

“So Georg”, I say. “Why don't you grab a small subset of your friends and come visit me sometime in the computer science hotel sometime? I think you would enjoy the quiet. There are 2^{1024} rooms available.”

“Not a chance,” says Georg. “will you ever find me leaving this place!”. He laughs heartily, and I laugh with him. I can't say I blame him. Can you?

Act 4.

Somehow I arrive at the large but finite hotel, and things look similar at first. An applied mathematics professor and friend called Finite Finney are enjoying a tea in the lounge.

“How was your trip?” he asks.
“Good” I say. “I learned a lot about infinity. Lost a lot of money but it was worth it.”
“Humbug” he says. “Infinity doesn't exist and you know it.”
“What about the integers?”
“Which set is that?”.
“Uh, what about 32 bit integers?”
“Of course those exist. They go all the way to maxint”.
“What about bigintegers?”
“They exist also. At any moment there is a largest possible integer in the set, but the set definition adjusts itself to a higher number if needed.”

We sit and stare for a few moments. Eventually I get up and go for a stroll. The finitist approach is hard to argue with, and certainly avoids any transfinite numbers, but something appears not quite right to me about it.

I return to question him further. What about the computer program:

```
bigint n = 0;
while(true) {
      print n;
      n++;
}
```

This program will try to list the integers. It will try to list all of them. In so much as we can talk about infinite sets, it appears that this finite program refers to and in fact defines an infinte set, namely the integers. Of course Finite Fin will point out that this computer will run out of power, memory, and cosmic rays will fry it's cpu, the sun will go supernova, etc. He might be right but it is hard to argue that the concept of the integers doesn't exist. Most people understand the concept, and use the word.

Me: “So, how about we just agree that this program represents the integers.”
Finny: “For sake of argument, lets say I'm fine with that”
Me: “Well this is quite a bit of progress if you are admitting the existence of infinite sets”
Finny: “Only because you redefined infinite sets as representations. Which do by the way form an ordered list”.
Me: “Wait What?”
Finny: “Programs are numbers. Each one can be tagged with a unique bigint. Some of these bigints might wind up creating the same thing but they're still a list. For example take the text file of the source code as the number.”
Me: “So there's also a finite program that will represent PI for example, in a similar way to the way we just identified the integers.”.
Finney: “In fact, a countably infinite number of programs that do that”
Me: “Wow. Hey you are doing that a lot today”
Finney: “What?”
Me: “Saying infinite”

So you see, numbers like *pi* are countable. *Pi* is the root of an infinite polynomial equation with deterministic coefficients. It can be represented as the output of a finite program. So is *e* and of course $\sqrt{2}$. They all appear on the list which Finite Fin has proposed! What is going on here? Are the real numbers countable while we are in the finite hotel?

Finny: “Yeah you're right, I didn't mean that. Only for your new definition. These are just representations of numbers here, not the numbers themselves so don't get too ahead of yourself.”
Me: “What's the difference between a number and its representation”?

Finny: “Hmm. Good point. Well, the integers don't really exist.   Pi doesn't really exist. These are just programs that represent them”.

Me: “But this still implies that there is an ordered list of the things these represent”

Finny: “A list of the representations, yes”.

Me: “So, the set of numbers that can be represented as a finite computer progam is countably infinite”.

Finny: “Countably infinite? No such thing. You guys need to fix your language problems. ”.

Finney settles the tab. As I walk to my room, it looks like I had made my travellers salesman choice incorrectly. Armed with this new list I would have done much better in the infinite hotel and saved myself a lifetime of debt. Or would I have? Lets take a look at the diagonal argument Cantor might have given us on the set of all finite computer programs. The program we generate by taking a different digit from each number is no longer a finite program, and therefore can't be a member of our set of finite programs. So, infinite programs are off the list. An infinite program is one for which the number of lines of source code, so I guess it had better be an interpreted language. But are there numbers represented by such infinite programs? And if so, therefore are not the numbers between 0 and 1 uncountable? If so, I still would have lost my bet.

Act 5

Over breakfast the next day I tell Finney about my thoughts from the previous evening.

Me: “So I can't help think that we have arrived at an interesting set of numbers last night”
Finney: “How so?“
Me: “Consider the set of numbers which cannot be represented as a finite program”.
Finney: “Hmm. In my mind that's not a whole lot more interesting than your supposed set of integers. Neither really exist”
Me: “But the set of the integers can be represented as a finite program!”
Finney: “OK, I'll listen. So what of your set of numbers? What are you going to call them?”
Me: “Nagual numbers”.
Finney: “What's nagual?”
Me: “It means a few different things, but one use has been to declare as nagual those things which are atonal, which is to say they cannot be symbolically represented”
Finney: “OK fine with me, but if you want to go sort through the list of things that don't actually exist you might have to do so somewhere else. We're full tonight”.
Me: “Aw really? Ok Ok. Well my infinite thanks for what you have given me in this finite time!”
Finney: “Hehe, No problem. And a word of advice? Stop hanging out with those weirdos.”

As I left the hotel I was deep in thought. One of these nagual numbers could represent the “halting program”, it might take as its input any other test program and outputs a boolean for whether the test program halts. No finite program could do this but what about an infinite one? I cringed to think of what Finney would say about infinite programs.

Epilogue

I made my way onto the underground. “Mind the gap”. I though of the gaps between the tonal, the nameable numbers. After all, this set still has measure zero. Something else must fill up the continuity and give us extension and measure. What is it?

I am back in London, on my way home. A shiver runs up my spine as I think about these numbers and I realize that I might have to visit a third hotel. What could the indescribable nagual hotel possibly look like? The fear is difficult to conquer and I collapse on the floor of the subway. As I hold on desperately to a handle, the subway shifts out of focus. As I try to identify symbolic objects around me and regain my composure, I realize that to do so is a lossy process. I am approximating. I am modeling the world around me with my feeble brain. I seem to have recovered and the next stop is mine so I get up and slide out the door. Now it is no longer fear but merely curiosity which has taken a hold of me. Is home the nagual hotel? If so that's fine with me, I didn't feel like traveling anymore anyway.