Random ODE and application to statistics.

Michail Zak
Senior Research Scientist (Emeritus)
Jet Propulsion Laboratory California Institute of Technology
Pasadena, CA 91109

Abstract.

It is demonstrated that any statistics can be represented by an attractor of the solution to the corresponding system of ODE coupled with its Liouville equation. Such a non-Newtonian representation allows one to reduce foundations of statistics to better-established foundations of ODE. In addition to that, evolution to the attractor reveals possible micro-mechanisms driving random events to the final distribution of the corresponding statistical law. Special attention is concentrated upon the power law and its dynamical interpretation: it is demonstrated that the underlying dynamics supports a "violent reputation" of the power-law statistics. As preliminary information, a review of origin of randomness in physics including new class of random ODE is presented.

1. Introduction.

This paper presents an attempt of dynamical interpretation of statistics laws. Unlike a law in science that is expressed in the form of an analytic statement with some constants determined empirically, a law of statistics represents a function all values of which are determined empirically; in addition to that, the micro-mechanisms driving the underlying sequence of random events to their final probability distribution remained beyond any description.

The objective of this paper is to create a system of ODE that has a prescribed statistics as a chaotic attractor. Such representation allows one to reduce foundations of statistics to better-established foundations of ODE, as well as to find a virtual model of micro-mechanisms that drive a sequence of random events to the final distribution of the corresponding statistical law.

It is clear that the ODE to be found cannot be based upon Newtonian dynamics since their solutions must be random. The Langeven equations must also be disqualified since their randomness is generated by random inputs. The only ODE that have intrinsically random solution will be considered in the section 3 of this paper. The distinguished property of these equations is that they are coupled with their Liouville equation by mean of a specially selected feedback. This feedback violates the Lipschits condition and creates instability that leads to randomness. The proposed model is represented by a modified Madelung equation (that represent a hydrodynamics version of the Schrodinger equation) in which the quantum potential is replaced by different, specially chosen "information potential". As a result, the dynamics attains both quantum and classical properties.

Based on the Madelung version of the Schrödinger equation, the origin of randomness in quantum mechanics has been traced down to instability *generated by quantum potential* at the point of departure from a deterministic state. The instability triggered by failure of the Lipchitz condition splits the solution into a continuous set of random samples representing a hidden statistics of Schrödinger equation, i.e., the transitional stochastic process as a "bridge" to quantum world. The proposed model has similar properties, but it is not conservative, and therefore, it can have attractors.

2. Origin of randomness in dynamics.

The concept of randomness entered Newtonian and quantum physics in different ways, but approximately at the same time. In 1926, Synge, J.L. introduced a new type of instability - orbital instability- in classical mechanics, [1], that can be considered as a precursor of chaos discovered a couple of decades later, [2]. In 1927, Heisenberg, W., [3] ***postulated*** randomness in quantum physics via the uncertainty principle. Since then it was a well-established opinion in the scientific community that there is a "deterministic " randomness, attributed to chaos, and a "true" randomness postulated by quantum physics. In this paper we contest this sub-division by providing a proof that the randomness in quantum physics does not have to be postulated: it follows from dynamics instability as randomness in Newtonian physics does. However in Newtonian physics this instability is measured by positive ***finite*** Liapunov exponents averaged over infinite time period, while in quantum physics the instability is accompanied by a loss of the Lipchitz condition and represented by an infinite divergence of trajectories at a singular point. Although from a mathematical viewpoint such a difference is significant, from physical viewpoint it does not justify division of randomness into "deterministic "(chaos) and "true" (quantum physics).

We start this section with revisiting mathematical formalism of chaos in a non-traditional way that is based upon the concept of orbital instability. After that, turning to the Madelung version of the Schrödinger equation, we describe a transition from determinism to randomness in quantum mechanics. In the last part we discuss randomness in non-Lipchitz version of Newtonian dynamics and its application to foundation of statistics.

A. Randomness in Newtonian physics.

a. Orbital instability as a precursor of chaos.

Chaos is a special type of instability when the system does not have an alternative stable state and displays an irregular aperiodic motion. Obviously this kind of instability can be associated only with ignorable variables, i.e. with such variables that do not contribute into energy of the system. In order to demonstrate this kind of instability, consider an inertial motion of a particle M of unit mass on a smooth pseudosphere *S* having a constant negative curvature G_0, Fig. 1.

$$G_0 = const > 0 \qquad (1)$$

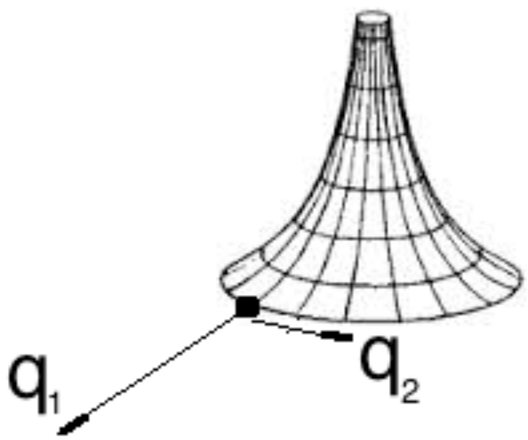


Figure 1. Inertial motion on a smooth pseudosphere.

Remembering that trajectories of inertial motions must be geodesics on *S*, compare two different trajectories assuming that initially they are parallel, and the distance ε_0 between them, are small (but not infinitesimal!),

$$0 < \varepsilon_0 << 1 \qquad (2)$$

As shown in differential geometry, the distance between these geodesics increases exponentially

$$\varepsilon = \varepsilon_0 e^{\sqrt{-G_0}t}, \quad G_0 < 0, \qquad (3)$$

Hence no matter how small the initial distance ε_0, the current distance ε tends to infinity.

Let us assume now that accuracy to which the initial conditions are known is characterized by the scale *L*. This means that any two trajectories cannot be distinguished if the distance between them is less than *L* i.e. if

$$\varepsilon < L \qquad (4)$$

The period during which the inequality (4) holds has the order

$$\Delta t \approx \frac{1}{\sqrt{|-G_0|}} \ln \frac{L}{\varepsilon_0} \qquad (5)$$

However for

$$t >> \Delta t \tag{6}$$

these two trajectories diverge such that they can be easily distinguished and must be considered as two different trajectories. Moreover the distance between them tends to infinity no matter how small is ε_0. That is why the motion once recorded cannot be reproduced again (unless the initial condition are known exactly), and consequently it attains stochastic features. The Liapunov exponent for this motion is positive and constant

$$\sigma = \lim_{\substack{t\to\infty \\ \varepsilon_0 \to 0}} [\frac{1}{t} \ln \frac{\varepsilon_0 e^{\sqrt{-G_0}t}}{\varepsilon_0}] = \sqrt{-G_0} = const > 0 \tag{7}$$

Remark. In theory of chaos, the Liapunov exponent measures divergence of initially close trajectories averaged over infinite period of time. But in this particular case, even "instantaneous" Liapunov exponent taken at a fixed time has the same value (7).

Let us introduce a system of coordinates on the surface *S*: the coordinate q_1 along the geodesic meridians and the coordinate q_2 along the parallels. In differential geometry such a system is called semigeodesic. The square distance between adjacent points on the pseudosphere is

$$ds = g_{11} dq_1^2 + 2q_{12} dq_1 dq_2 + g_{22} dq_2^2 \tag{8}$$

where

$$g_{11} = 1, \quad q_{12} = 0, \quad g_{22} = -\frac{1}{G_0} e^{(-2\sqrt{-G_0} q_1)} \tag{9}$$

The Lagrangian for the inertial motion of the particle *M* on the pseudosphere is expressed via the coordinates and their temporal derivatives as

$$L = g_{ij} \dot{q}_i \dot{q}_j = \dot{q}_1^2 - \frac{1}{G_0} e^{(-2\sqrt{-G_0} q_1)} \dot{q}_2^2 \tag{10}$$

and consequently,

$$\frac{\partial L}{\partial q_2} = 0 \tag{11}$$

$$\frac{\partial L}{\partial q_1} \neq 0 \quad if \quad \dot{q}_2 \neq 0 \tag{12}$$

Hence q_1 and q_2 play the roles of position and ignorable coordinates, respectively, and therefore, the inertial motion of a particle on a smooth pseudosphere is unstable with respect to the ***ignorable*** coordinate, [1]. This instability known as orbital instability is not bounded by energy and it can persist indefinitely. As shown in [2], eventually orbital instability leads to stochasticity. Later on such motions were identified as chaotic.

b. Randomness in chaotic systems.

In this sub-section we present a sketch of general theory of chaos in context of origin of randomness starting with the flow generated by an autonomous ODE

$$\frac{dx_i}{dt} = V_i(\mathbf{x}), \quad i = 1,2,...m \tag{13}$$

and compare two neighboring trajectories in *m*-dimensional phase space with initial conditions x_0 and $x_0 + \Delta x_0$ denoting $\Delta x_0 = w$. These evolve with time yielding the tangent vector $\Delta x(x_0, t)$ with its Euclidian norm

$$d(x_0,t) = \left\| \Delta x(x_0,t) \right\| \tag{14}$$

Now the Liapunov exponent can be introduced as the mean exponential rate of divergence of two initially close trajectories

$$\tilde{\lambda}(x_0,w) = \lim_{\substack{t\to\infty \\ d(0)\to 0}} \left(\frac{1}{t}\right) \ln \frac{d(x_0,t)}{d(x_0,0)} \tag{15}$$

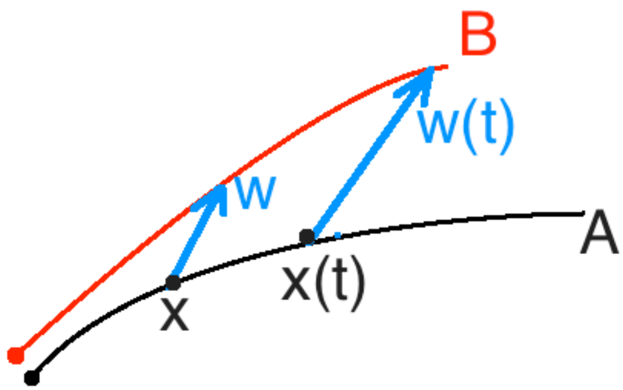
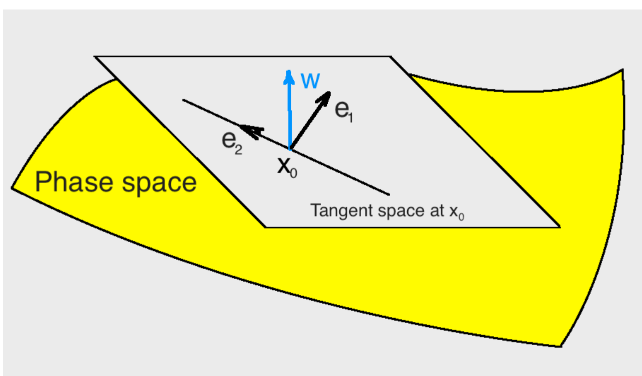


Figure 2. Two nearby trajectories that separate as time evolves.



Ficure 3. Tangent space for the Liapunov exponents.

Therefore in general the Lyapunov exponent cannot be analytically expressed via the parameters of the underlying dynamical system (as it can be done in case of inertial motion on a pseudosphere), and that makes prediction of chaos a hard task. However some properties of the Liapunov exponents can be expressed in an analytical form. Firstly, it can be shown that in an *m*-dimensional space, there exist *m* Liapunov exponents

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \ldots \geq \tilde{\lambda}_m \tag{16}$$

while at least one of them must vanish. Indeed, as follows from Eqs. (13) and (14), **w** grows only linearly in the direction of the flow, and the corresponding Liapunov exponent is zero. Secondly it has been proven that the sum of the Liapunov exponents is equal to the average phase space volume contraction

$$\sum_{i=1}^{m} \tilde{\lambda}_i = \Lambda_0 \tag{17}$$

where the instantaneous phase space volume contraction

$$\Lambda = \nabla \cdot \mathbf{V} \tag{18}$$

But

$$\Lambda_0 = \Lambda \tag{19}$$

when

$$\nabla \cdot \mathbf{V} = const \tag{20}$$

Therefore in case (20), the sum of the Liapunov exponents is expressed analytically

$$\sum_{i=1}^{m} \tilde{\lambda}_i = \nabla \cdot \mathbf{V} \tag{21}$$

Thus the result we extracted from the theory of chaos, which can be used for comparison to quantum randomness is the following: the origin of randomness in Newtonian mechanics is instability of ignorable variables that leads to exponential divergence of initially adjacent trajectories; this divergence is measured by Liapunov exponents, which form a discrete spectrum of numbers that must include positive ones.

B. Randomness in quantum mechanics.

a. Background.

Quantum mechanics has introduced randomness into the basic description of physics via the uncertainty principle. In the Schrödinger equation, randomness is included in the wave function. But the Schrödinger equation does not *simulate* randomness: it rather describes its evolution from the prescribed initial (random) value, and this evolution is fully deterministic. The main purpose of this section is to trace down the mathematical origin of randomness in quantum mechanics, i.e. to find or build a "bridge" between the deterministic and random states. In order to do that, we will turn to the Madelung equation, [4]. For a particle mass *m* in a potential *F,* the Madelung equation takes the following form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\frac{\rho}{m} \nabla S\right) = 0 \tag{22}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + F - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0 \tag{23}$$

Here ρ and S are the components of the wave function $\psi = \sqrt{\rho} e^{iS/\hbar}$, and $\hbar$ is the Planck constant divided by 2π. The last term in Eq. (23) is known as quantum potential. From the viewpoint of Newtonian mechanics, Eq. (22) expresses continuity of the flow of probability density, and Eq. (23) is the Hamilton-Jacobi equation for the action *S* of the particle. Actually the quantum potential in Eq. (23), as a feedback from Eq. (22) to Eq. (23), represents the difference between the Newtonian and quantum mechanics, and therefore, it is solely responsible for fundamental quantum properties.

The Madelung equations (22), and (23) can be converted to the Schrödinger equations using the ansatz

$$\sqrt{\rho} = \Psi \exp(-iS / h) \tag{24}$$

where ρ and S being real function.

Reversely, Eqs. (22), and (23) can be derived from the Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 \Psi - F\Psi = 0 \tag{25}$$

using the ansatz , which is reversed to (24)

$$\Psi = \sqrt{\rho}\exp(iS/\hbar) \tag{26}$$

So there is one-to-one correspondence between the solutions of the Madelung and the Schrödinger equations. From the stochastic mechanics perspective, the transformation of nonlinear Madelung equation into the linear Schrödinger equation is just a suitable mathematical technique that provides an easy way of finding their solutions.

b. Search for transition from determinism to randomness.

In this sub-section we will address the problem of correspondence between quantum and Newtonian randomness. Turning to Eq. (23), we start with some simplification assuming that $F = 0$. Rewriting Eq. (23) for the one-dimensional motion of a particle, and differentiating it with respect to x, one obtains

$$m\frac{\partial^2 x(X,t)}{\partial t^2} - \frac{\hbar^2}{2m}\frac{\partial}{\partial X}\left[\frac{1}{\sqrt{\rho(X)}}\frac{\partial^2\sqrt{\rho(X)}}{\partial X^2}\right] = 0 \tag{27}$$

where $\rho(X)$ is the probability distribution of x over its possible values X.

Let us choose the following initial conditions for the deterministic state of the system:

$$x = 0,\ \ \rho = \delta(|x| \to 0), \qquad \dot{\rho} = 0 \quad at \qquad t = 0 \tag{28}$$

We intentionally did not specify the initial velocity $\dot{x}$ expecting that the solution will comply with the uncertainty principle.

Now let us rewrite the one-dimensional version of Eqs. (22) and (23) as

$$\frac{\partial^2\rho}{\partial t^2} + \frac{\hbar^2}{2m^2}\frac{\partial^4\rho}{\partial X^4} + \xi = 0 \qquad at \qquad t \to 0 \tag{29}$$

where ξ includes only lower order derivatives of ρ. For the first approximation, we ignore ξ (later that will be justified,) and solve the equation

$$\frac{\partial^2\rho}{\partial t^2} + a^2\frac{\partial^4\rho}{\partial X^4} = 0 \qquad at \qquad t \to 0 \qquad\qquad a^2 = \frac{\hbar^2 T^2}{2m^2 L^4} \tag{30}$$

subject to the initial conditions (28). The closed form solution of this problem is known from the theory of nonlinear waves, [5]

$$\rho = \frac{1}{\sqrt{4\pi t\dfrac{\hbar}{2m}}}\cos\left(\frac{x^2}{4t\dfrac{\hbar}{2m}} - \frac{\pi}{4}\right) \qquad at \qquad t \to 0 \tag{31}$$

Based upon this solution, one can verify that $\xi \to 0$ at $t \to 0$, and that justifies the approximation (30) (for the proofs see the sub-section *b**). It is important to remember that the solution (31) is valid only for small times, and only during this period it is supposed to be positive and normalized.

Rewriting Eq. (27) in dimensionless form

$$\ddot{x} - a^2\frac{\partial}{\partial X}\left[\frac{1}{\sqrt{\rho(X)}}\frac{\partial^2\sqrt{\rho(X)}}{\partial X^2}\right] = 0 \tag{32}$$

and substituting Eq. (31) into Eq. (32) at $\boldsymbol{X = x}$, after Taylor series expansion, simple differentiations and appropriate approximations, one arrives at the following differential equation instead of (32)

$$\ddot{x} = c\frac{x}{t^2}, \qquad c = -\frac{3}{8\pi^2 a^2} \tag{33}$$

This is the Euler equation, and it has the following solution, [7]

$$x = C_1 t^{\frac{1}{2}+s} + C_2 t^{\frac{1}{2}-s} \qquad at \qquad 4c+1>0 \tag{34}$$

$$x = C_1\sqrt{t} + C_2\sqrt{t}\ln t \qquad at \qquad 4c+1=0 \tag{35}$$

$$x = C_1\sqrt{t}\cos(s\ln t) + C_2\sqrt{t}\sin(s\ln t) \qquad at \qquad 4c+1<0 \tag{36}$$

where $2s = \sqrt{|4c+1|}$ (37)

Thus, the qualitative structure of the solution is uniquely defined by the dimensionless constant a^2 via the constants *c* and *s*, (see Eqs. (33) and (37). But the cases (35) and (36) should be disqualified at once since they are in a conflict with the approximations used for derivation of Eq. (33), (see sub-section *b**).
Hence, we have to stay with the case (34). This gives us the limits

$$0 < |c| < 0.25, \tag{38}$$

In addition to that, we have to drop the second summand in Eq. (34) since it is in a conflict with the approximation used for derivation of Eq. (30) (see sub-section *b**). Therefore, instead of Eq. (34)) we now have

$$x = C_1 t^{\frac{1}{2}+s} \qquad at \qquad 4c+1>0 \tag{39}$$

For illustration, let us evaluate the constant *c* based upon the following data:

$$\hbar = 10^{-34} m^2 kg/\sec, m = 10^{-30} kg, \ L = 2.8\times 10^{-15} m, \quad L/T = \tilde{C} = 3\times 10 m/\sec$$

where *m*- mass of electron, and $\tilde{C}$ -speed of light. Then,

$$c = -1.5\times 10^{-4}, \qquad i.e. \qquad |c| < 0.25$$

Hence, the value of *c* is within the limit (38). Thus, for the particular case under consideration, the solution (39) is

$$x = C_1 t^{0.9998} \tag{40}$$

In the next sub-section, prior to analysis of the solution (39), we will present the proofs justifying the solution (31).

b. Proofs***.**

1. Let us first justify the statement that $\xi \to 0 \ at \qquad t \to 0$ (see Eq. (29)).

For that purpose, consider the solution (31)

$$\rho = \frac{1}{\sqrt{4\pi a t}}\cos\left(\frac{X^2}{4at} - \frac{\pi}{4}\right) \quad at \qquad t \to 0 \tag{1*}$$

As follows from the solution (39),

$$\frac{x}{t} \approx o(t^{s-1/2}) \to \infty, \quad \frac{x^2}{t} \approx o(t^{2s}) \to 0 \qquad at \qquad t \to 0 \text{ since } 0 < s < 1/2 \tag{2*}$$

Then, finding the derivatives from Eq. (1') yields

$$|\frac{\partial^n \rho}{\partial X^n}| / |\frac{\partial^{n-1}\rho}{\partial X^{n-1}}| \approx o(t^{-1}) \to \infty \qquad at \qquad t \to 0 \tag{3*}$$

and that justifies the inequalities

$$|\frac{\partial^4 \rho}{\partial X^4}| >> |\frac{\partial^3 \rho}{\partial X^3}|, |\frac{\partial^2 \rho}{\partial X^2}|, |\frac{\partial \rho}{\partial X}|, \rho \tag{4*}$$

Similarly,

$$|\frac{\partial^n \rho}{\partial t^n}| / |\frac{\partial^{n-1}\rho}{\partial t^{n-1}}| \approx o(t^{-1}) \to \infty \qquad at \qquad t \to 0 \tag{5*}$$

and that justifies the inequalities

$$|\frac{\partial^2 \rho}{\partial t^2}| >> |\frac{\partial \rho}{\partial t}|, |\frac{\partial \rho}{\partial X}|^2$$

Also as follows from the solution (18)

$$\begin{aligned} &|\frac{\partial S}{\partial x}| \approx o(t^{S-0.5}), \qquad |\frac{\partial^2 S}{\partial x^2}| \approx o(t^{-1}), \\ &|\frac{\partial S}{\partial x}| / |\frac{\partial^2 S}{\partial x^2}| \approx o(t^{S+0.5}) \to 0 \qquad at \qquad t \to 0 \end{aligned} \tag{6*}$$

It should be noticed that for Eq. (34), the evaluations (6*) do not go through, and that was the reason for dropping the second summand.

Finally, the inequalities (4*), (5*) and (6*) justify the transition from Eq. (29) to Eq. (31).

2. Next let us first prove the positivity of ρ in Eq. (31) for small times. Turning to the evaluation (2*)

$\frac{x^2}{t} \approx o(t^{2s}) \to 0 \qquad at \qquad t \to 0$, one obtains for small times

$$\rho = \frac{1}{\sqrt{4\pi at}} \cos(-\frac{\pi}{4}) > 0 \qquad at \qquad t \to 0 \tag{7*}$$

In order to prove that ρ is normalized for small times, turn to Eq.(30) and integrate it over X

$$\int_{-\infty}^{\infty} \frac{\partial^2 \rho}{\partial t^2} dX + a^2 \int_{-\infty}^{\infty} \frac{\partial^4 \rho}{\partial X^4} dX = 0 \tag{8*}$$

Taking into account the initial conditions (28) and requiring that ρ and all its space derivatives vanish at infinity, one obtains

$$\frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \rho \, dX = 0 \tag{9*}$$

But as follows from the initial conditions (28)

$$\int_{-\infty}^{\infty} \rho \, dX = 0, \quad \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho \, dX = 0 \qquad at \qquad t = 0 \tag{10*}$$

Combining Eqs. (9*) and (10*), one concludes that the normalization constraint is preserved during small times.

3. The solutions (34), (35) and (36) have been derived under assumption that $\frac{x^2}{t} \to 0 \qquad at \qquad t \to 0$

(11*)

since this assumption was exploited for expansion of ρ in Eq. (31) in Taylor series. However, in the cases (35) and (36),

$$\frac{x^2}{t} \approx o(1) \qquad at \qquad t \to 0,$$

and that disqualify their derivation. Actually these cases require an additional analysis that is out of scope of this paper. For the same reason, Eq. (13) has been truncated to the form (39).

4. Presence of the potential *F* that was ignored in the approximation Eq. (27) does not change Eq. (33) since *F* is bounded while x / t^2 is unbounded at $t \to 0$.

c. Analysis of solution.

Turning to the solution (39), we notice that it satisfies the initial condition (28) i.e. *x =0* at *t =0* for *any* values of C_1 : all these solutions co-exist in a superimposed fashion; it is also consistent with the sharp initial condition for the solution (31) of the corresponding equation (22). The solution (31) describes the simplest *irreversible* motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of the Lipchitz condition at *t=0,* Fig.5); then the solution splits into a continuous set of random samples representing a stochastic process with the probability density ρ controlled by Eq. (31). The irreversibility of the process follows from the fact that the backward motion obtained by replacement of *t* with *(-t)* in Eqs. (31) and (39) leads to imaginary values. Actually Fig.4 illustrates a jump from determinism to a coherent state of superimposed solutions that is ***lost in solutions of the Schrödinger equation.***

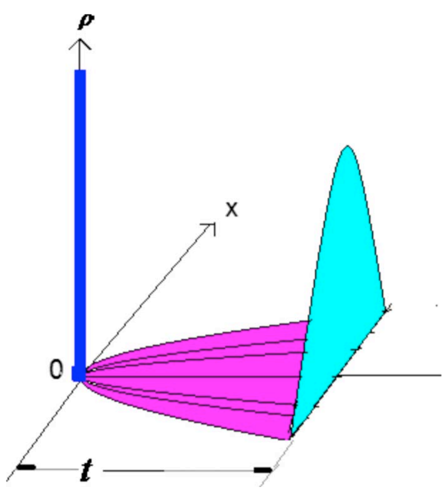


Figure 4. Hidden statistics of transition from determinism to randomness.

Let us show that this jump is triggered by instability of the deterministic state. Indeed, turning to the solution represented by Eq. (39) with $|C_1| \leq 0.25$, we observe that for fixed values of C_1, the solution (39) is ***unstable*** since

$$\frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}} \qquad > 0 \tag{41}$$

and therefore, an initial error always grows generating ***randomness***. Initially, at *t=0*, that growth is of ***infinite rate*** since the Lipchitz condition at this point is violated (such a point represents a ***terminal*** repeller)

$$\frac{d\dot{x}}{dx} \to \infty \qquad at \qquad t \to 0 \tag{42}$$

This means that an *infinitesimal* initial error becomes finite in a bounded time interval. That kind of instability (similar to blow-up, or Hadamard, instability) has been introduced and analyzed in [6,8].

Considering first Eq.(39) at fixed C_1 as a sample of the underlying stochastic process (54), and then varying C_1, one arrives at the whole ensemble of one-parametrical random solutions characterizing that process, (see Fig.5). It should be stressed again that this solution is valid only during a small initial period representing a "bridge" between deterministic and random states, and that was essential for the derivation of the solutions (39), and (31).

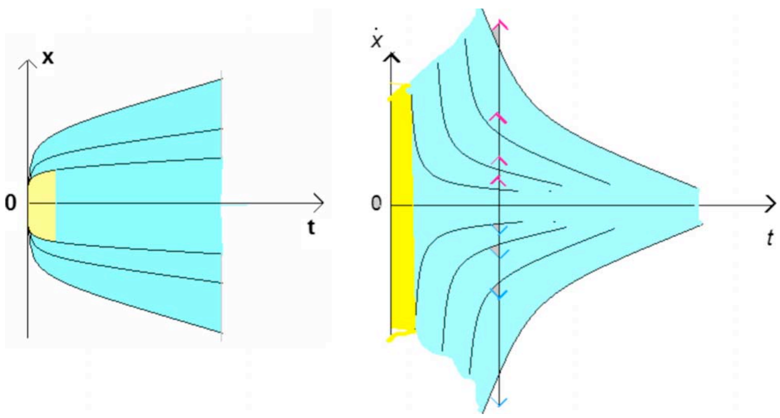


Figure 5. Family of random trajectories and particle velocities

Returning to the quantum interpretation of Eqs. (22) and (23), one notice that during this transitional period, the quantum postulates are preserved. Indeed, as follows from Eq. (41),

$$\dot{x} \to \infty \quad at \quad t \to 0 \tag{43}$$

i.e. the initial velocity is *not defined*, (see the flat area in Fig. 5), and ***that confirms the uncertainty principle.*** It is interesting to note that an enforcement of the initial velocity would "blow-up" the solution (39); at the same time, the qualitative picture of the solution is not changed if the initial velocity is not enforced: the solution is composed of superposition of a family of random trajectories with the singularity (43) at the origin. Next, the solution (39) justifies the *belief* sheared by the most physicists that particle trajectories do not exist, although, to be more precise, as follows from Eq. (39), ***deterministic*** trajectories do not exist: each run of the solution (39) produces different trajectory that occurs with probability governed by Eq. (31). It is easily verifiable that the transition of motion from one trajectory to another is very sensitive to errors in initial conditions in the neighborhood of the deterministic state. Indeed, as follows from Eq. (39),

$$C_1 = x_0 t_0^{-(s+0.5)}, \qquad \frac{\partial C_1}{\partial x_0} = t_0^{-(s+0.5)} \to \infty \quad as \qquad t_0 \to 0 \tag{44}$$

where x_0 and t_0 are small errors in initial conditions.

Actually Eq. (39) represents a hidden statistics of the underlying Schredinger equation. As pointed out above, ***the cause of the randomness is non-Lipchitz instability*** of Eq. (39) at *t=0*. Therefore, trajectories of quantum particles have the same "status" as trajectories of classical particles in a chaotic motion with the only difference that the random "choice" of the trajectory is made only at $t_0 \to 0$. It should be emphasized again that the transition (39) is *irreversible*. However, as soon as the difference between the current probability density and its initial sharp value becomes finite, one arrives at the conventional quantum formalism described by the Schrödinger, as well as the Madelung equations. Thus, in the conventional quantum formalism, the *transition* from the classical to the quantum state has been ***lost*,** and that created a major obstacle to interpretation of quantum mechanics as an extension of the Newtonian mechanics. However, as demonstrated above, the quantum and classical worlds can be reconciled via the more subtle mathematical treatment of the ***same equations***. This result is generalizable to multi-dimensional case as well as to case with external potentials.

d. Comments on equivalence of Schrödinger and Madelung equations.

Equivalence of Schrödinger and Madelung equations was questioned by some quantum physicists on the ground that to recover the Schrödinger equation from the Madelung equation, one must add by hand a quantization condition, as in the old quantum theory. However, this argument has been challenged by other physicists. We will not go into details of this discussion since this paper is focused on mathematical rather than physical equivalence of Schrödinger and Madelung equations. Firstly we have to notice that the Schrödinger equation is more attractive for computations due to its linearity, while the Madelung equations have a methodological advantage: they allow one to trace down the Newtonian origin of the quantum physics. Indeed, if one drops the Planck's constant, the Madelung equations degenerate into the Hamilton-Jacobi equation supplemented by the Liouville equation. However despite the fact that these two forms of the same governing equations of quantum physics can be obtained from one another without a violation of any of mathematical rules, there is more significant difference between them, and this difference is associated with the concept of stability. Indeed, as demonstrated above, the solution of the Madelung equations with deterministic initial condition (28) is unstable, and it describes the jump from the determinism to randomness. This illuminates the origin of randomness in quantum physics. However the Schrödinger equation does not have such a solution; moreover, it does not "allow" posing such a problem and that is why the randomness in quantum mechanics had to be postulated. So what happens with mathematical equivalence of Schrödinger and Madelung equations? In order to answer this question, let us turn to the concept of stability. It should be recalled that stability is not an invariant of a physical model. It is an attribute of mathematical description: it depends upon the frame of reference, upon the class of functions in which the motion is presented, upon the metrics of configuration space, and in particular, upon the way in which the distance between the basic and perturbed solutions is defined, [1,8]. As an example, consider an inviscid stationary flow with a *smooth* velocity field, [2]

$$v_x = A\sin z + C\cos y,\ v_y = B\sin x + A\cos z,\ v_z = C\sin y + b\cos x \qquad (45)$$

Surprisingly, the trajectories of individual particles of this flow are unstable (Lagrangian turbulence). It means that this flow is stable in the Eulerian representation, but unstable in the Lagrangian one. The same happens with stability in Hilbert space (Schrödinger equalion), and stability in physical space (Madelung equations). One should recall that stability analysis is based upon a departure from the basic state into a perturbed state, and such departure requires an expansion of the basic space. However, Schrödinger and Madelung equations in the ***expanded*** spaces are not necessarily equivalent any more, and that explains the difference in the concept of stability of the same solution as well as the interpretation of randomness in quantum mechanics.

There is another "mystery" in quantum mechanics that can be clarified by transfer to the Madelung space: a ***belief*** that a particle trajectory does not exist. Indeed, let us turn to Eq. (39). For any particular value of the arbitrary constant C_1, it presents the corresponding particle's trajectory. However as a result of non-Lipchitz instability at $t = 0$, this constant is supersensitive to infinitesimal disturbances, and actually it becomes random at $t=0$. That makes random the choice of the whole trajectory, while the randomness is controlled by Eq. (30). Actually this provides a justification for the ***belief*** that a particle can occupy any place at any time: it is due to randomness of its trajectory. However it should be emphasized that the particle makes random choice only once: at $t = 0$. After that it stays on the chosen trajectory. Therefore in our interpretation, this belief does not mean that a trajectory does not exist: it means only that the trajectory

exists, but it is unstable. Based upon that, we can extract some deterministic information about the particle trajectory by posing the following question: find such a trajectory that has the highest probability to appear. The solution of this problem is straight forward: in the process of collecting statistics for the arbitrary constant C_1 find such its value that has the highest frequency to appear. Then the corresponding trajectory will have the highest probability to appear as well. Actually this could, in principle, allow one to transmit intentional massage with high probability using quantum entanglement.
Thus, strictly speaking, the Schrödinger and Madelung equations are equivalent only in the open time interval

$$t > 0 \qquad (46),$$

since the Schrödinger equation does not include the infinitesimal area around the singularity at

$$t = 0 \qquad (47)$$

while the Madelung equation exists in the closed interval

$$t \geq 0 \qquad (48)$$

But all the "machinery" of randomness emerges precisely in the area around the singularity (47). That is why the source of randomness is missed in the Schrödinger equation, and the randomness had to be postulated.
Remark. An example of fundamental difference between stability in open and closed intervals is given in [8].
Hence although historically the Schrödinger equation was proposed first, and only after a couple of months, Madelung introduced its hydrodynamic version that bears his name, strictly speaking, the foundations of quantum mechanics would be saved of many paradoxes had it be based upon the Madelung equations.

C. Randomness in non-Lipchitz Newtonian dynamics.
The discovery of the origin of randomness in quantum mechanics opens up a strong support to the correspondence principle: the randomness in physics (both quantum and Newtonian) is caused by dynamical instability. However this support comes with some complications: the types of dynamical instability in Newtonian and quantum physics are qualitatively different.

Indeed in Newtonian physics it is Liapunov instability of ignorable variables, i.e. such variables that do not contribute into energy of the system. For an *m*-dimensional system, this instability manifests itself in appearance of positive Liapunov exponents in the spectrum of *m* Liapunov exponents, while each of these exponents measures the averaged divergence of adjacent trajectories.

In quantum physics, the instability has a different nature: it is caused by the loss of uniqueness of the solution in a singular point due to failure of Lipchitz condition at this point. In context of terminal dynamics, [6], this point represents a terminal repeller that is characterized by infinite divergence of trajectories. As a result of that, quantum system makes a random choice of the trajectory only once – at the beginning of the transition from determinism to randomness, while a Newtonian system may change trajectories continuously during its chaotic motion.

a. Terminal repeller.

In order to capture the fundamental properties of the effects associated with failure of the Lipchitz condition, let us turn to a simple ODE, [6]

$$m\dot{v} = \alpha v^k, \quad k = \frac{N}{N+2} < 1, \quad m, \alpha > 0 \qquad (49)$$

where N is a natural number.

One can verify that that for Eq. (49), the equilibrium point $v = 0$ becomes a terminal repeller, and since

$$\frac{d\dot{v}}{dv} = k\frac{\alpha}{m}v^{k-1} \to \infty \quad at \quad v \to 0 \qquad (50)$$

it is infinitely unstable. If the initial condition is infinitely close to this repeller, the transient solution will escape it during a finite time period

$$t_0 = \int_{v_0}^{0} \frac{mdv}{\alpha v^k} = \frac{mv_0^{1-k}}{\alpha(1-k)} < \infty \qquad (51)$$

while for a regular repeller the time period would be infinite. Here the motion is irreversible since the inversion of time in the solution of Eq. (50)

$$v = \pm[\frac{\alpha}{m}(1-k)t]^{1/1-k} \qquad (52)$$

leads to imaginary values of v since $k < 1$.

But in addition to that, terminal repellers possess even more surprising characteristics: the solution (52) becomes totally unpredictable. Indeed, two different motions described by Eq. (52) are possible for "almost the same" initial conditions:

$$v_0 = +\varepsilon \to 0 \quad or \quad v_0 = -\varepsilon \to 0 \quad at \quad t = 0 \qquad (53)$$

The most essential property of this result in that the divergence of these two solutions is characterized by an unbounded rate

$$\sigma = \lim_{t \to t_0} (\frac{1}{t} \ln \frac{\alpha t^{1/(1-k)}}{m|v_0|}) \to \infty \quad at \quad |v_0| \to 0 \qquad (54)$$

In contrast to the classical case where $t_0 \to \infty$, here σ can be defined within an arbitrarily small time interval t_0 since during this interval the initial infinitesimal distance between the solutions becomes finite. Thus a terminal repeller represents a vanishingly small, but infinitely powerful "pulse of randomness" that is pumped into the system via terminal repeller, Figs.6,7. Obviously, failure of the uniqueness of the solution here results from the violation of the Lipchitz condition(50) at $v = 0$.

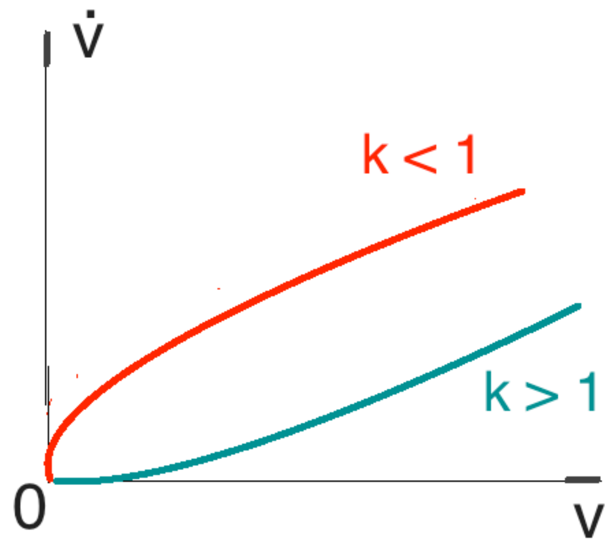


Figure 6.Terminal repeller in phase space (*k* < *1*), classical repeller in phase space (*k* > *1*).

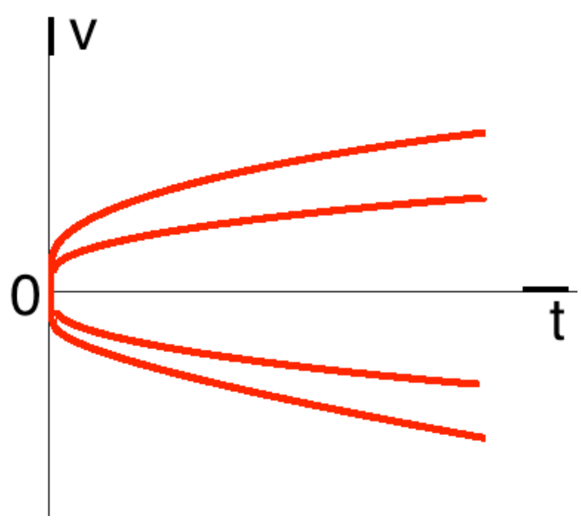


Figure 7. Terminal repeller in physical space.

Now one can verify that the solution (39) that describes transition from determinism to randomness in quantum physics belong to the same class as the solution (52) that starts with the terminal repeller , (compare Fig. 7 with Fig. 5 that present qualitative description of solutions).

3. General model of random ODE.

In this section we will introduce a general model of random ODE as a quantum-classical hybrid. Actually our approach is based upon a modification of the Madelung equation, and in particular, upon replacing the quantum potential with a different Liouville feedback, Fig.8

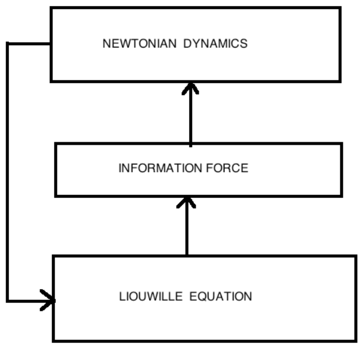


Figure 8. Quantum-classical hybrid.

A. Mathematical formulation.
In Newtonian physics, the concept of probability ρ is introduced via the Liouville equation

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{F}) = 0 \qquad (55)$$

generated by the system of ODE

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}[\mathbf{v}_1(t), \ldots \mathbf{v}_n(t), t] \qquad (56)$$

where **v** is velocity vector.
It describes the continuity of the probability density flow originated by the error distribution

$$\rho_0 = \rho(t = 0) \qquad (57)$$

in the initial condition of ODE (57).

Let us rewrite Eq. (56) in the following form

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}[\rho(\mathbf{v})] \qquad (58)$$

where **v** is a velocity of a hypothetical particle.

This is a fundamental step in our approach: in Newtonian dynamics, the probability never explicitly enters the equation of motion. In addition to that, the Liouville equation generated by Eq. (58) could be nonlinear with respect to the probability density ρ

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \{\rho \mathbf{F}[\rho(\mathbf{V})]\} = 0 \qquad (59)$$

and therefore, the system (58),(59) departs from Newtonian dynamics. However although it has the same topology as quantum mechanics (since now the equation of motion is coupled with the equation of continuity of probability density), it does not belong to it either. Indeed Eq. (58) is more general than the Hamilton-Jacoby equation (23): it is not necessarily conservative, and **F** is not necessarily the quantum potential although further we will impose some restriction upon it that links **F** to the concept of information. The topology of the system (58), (59) is illustrated in Fig.8.

Remark. Here and below we make distinction between the random *variable v(t)* and its *values V* in probability space.

Prior to considering a specific form of the force **F,** we will make a comment concerning the normalization constrain satisfaction

$$\int_V \rho \, dV = 1 \qquad (60)$$

in which V is the volume where Eqs. (58) and (59) are defined.
Turning to Eq. (59) and integrating it over the volume V

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_V dV \nabla \bullet \{\rho \mathbf{F}[\rho(\mathbf{V})]\} = -\oint_\Phi d\Phi \nabla \bullet (\rho \mathbf{F}) = 0 \qquad (61)$$

if

$$\rho = 0, \quad |\mathbf{F}| < \infty \quad at \quad \Phi \qquad (62)$$

where Φ is the surface bounding the volume V.
Therefore, if the normalization constraint (58) is satisfied at $t = 0$, it is satisfied for all the times.

B. Information force instead of quantum potential.

In this sub-section, we propose the structure of the force **F** that plays the role of a feedback from the Liouville equation (59) to the equation of motion (58). Turning to one-dimensional case, let us specify this feedback as

$$F = c_0 + \frac{1}{2}c_1\rho - \frac{c_2}{\rho}\frac{\partial \rho}{\partial v} + \frac{c_3}{\rho}\frac{\partial^2 \rho}{\partial v^2} \qquad (63)$$

$$c_0 > 0, c_1 > 0, c_3 > 0 \qquad (64)$$

Then Eq.(58) can be reduced to the following:

$$\dot{v} = c_0 + \frac{1}{2}c_1\rho - \frac{c_2}{\rho}\frac{\partial \rho}{\partial v} + \frac{c_3}{\rho}\frac{\partial^2 \rho}{\partial v^2} \qquad (65)$$

and the corresponding Liouville equation will turn into the following PDE

$$\frac{\partial \rho}{\partial t} + (c_0 + c_1\rho)\frac{\partial \rho}{\partial V} - c_2\frac{\partial^2 \rho}{\partial v^2} + c_3\frac{\partial^3 \rho}{\partial V^3} = 0 \qquad (66)$$

This equation is known as the KdV-Bergers' PDE. The mathematical theory behind the KdV equation became rich and interesting, and, in the broad sense, it is a topic of active mathematical research. A homogeneous version of this equation that illustrates its distinguished properties is nonlinear PDE of parabolic type. But a fundamental difference between the standard KdV-Bergers equation and Eq. (66) is that Eq. (66) ***dwells in the probability space***, and therefore, it must satisfy the normalization constraint

$$\int_{-\infty}^{\infty} \rho \, dV = 1 \qquad (67)$$

However as shown in [5], this constraint is satisfied: in physical space it expresses conservation of mass, and it can be easily scale-down to the constraint (67) in probability space. That allows one to apply all the known results directly to Eq. (66). However it should be noticed that all the conservation invariants have different physical meaning: they are not related to conservation of momentum and energy, but rather impose constraints upon the Shannon information.
In physical space, Eq. (66) has many applications from shallow waves to shock waves and solitons. However, application of solutions of the same equations in probability space is fundamentally different. In the next sections we present a phenomenon that exists neither in Newtonian nor in quantum physics.

C. Emergence of randomness.

In this sub-section we introduce a fundamentally new phenomenon: transition from determinism to randomness in ODE that coupled with their Liouville PDE. We demonstrate that emergence of randomness in the proposed dynamics has the same mathematical origin as that in quantum mechanics.
In order to complete the solution of the system (65), (66), one has to substitute the solution of Eq. (66):

$$\rho = \rho(V, t) \quad at \quad V = v \tag{68}$$

into Eq.(65). Since the transition from determinism to randomness occurs at $t \to 0$, let us turn to Eq. (66) with sharp initial condition

$$\rho_0(V) = \delta(V)\, at \qquad t = 0, \tag{69}$$

Then applying one of the standard analytical approximations of the delta-function, one obtains the asymptotic solution

$$\rho = \frac{1}{t\sqrt{\pi}} e^{-\frac{V^2}{t^2}} \quad at \quad t \to 0 \tag{70}$$

Substitution this solution into Eq. (63) shows that

$$O(c_0 + \frac{1}{2}c_1\rho) = \frac{1}{t}, \quad O(\frac{c_2}{\rho}\frac{\partial\rho}{\partial v}) = \frac{1}{t^2},$$
$$and \quad O(\frac{c_3}{\rho}\frac{\partial^2\rho}{\partial v^2}) = \frac{1}{t^4} \quad at \quad t \to 0, \quad v \neq 0 \tag{81}$$

i.e.

$$c_0 + \frac{1}{2}c_1\rho << \frac{c_2}{\rho}\frac{\partial\rho}{\partial v} << \frac{c_3}{\rho}\frac{\partial^2\rho}{\partial v^2} \quad at \quad t \to 0, \quad v \neq 0 \tag{82}$$

and therefore, the first three terms in Eq. (65) can be ignored

$$\dot{v} = \frac{c_3}{\rho}\frac{\partial^2\rho}{\partial v^2} \quad at \quad t \to 0, \quad v \neq 0 \tag{83}$$

or after substitution of Eq. (70)

$$\dot{v} = \frac{4c_3 v^2}{t^4} \quad at \quad t \to 0, \quad v \neq 0 \tag{84}$$

Eq. (84) has the following solution (see Fig. 9)

$$v = \frac{t^3}{4c_3 + Ct^3} \quad at \quad t \to 0, \quad v \neq 0 \tag{85}$$

where C is an arbitrary constant.
This solution has the following property: the Lipchitz condition at $t \to 0$ fails

$$\frac{\partial\dot{v}}{\partial v} = \frac{8c_3 v}{t^4} = \frac{8c_3 t^3}{t^4(4c_3 + Ct^3)} \to \infty \quad at \quad t \to 0, \quad v \neq 0 \tag{86}$$

and as a result of that, the uniqueness of the solution is lost. Indeed, as follows from Eq. (85), for any value of the arbitrary constant C, the solutions are different, but they satisfy the same initial condition

$$v \to 0 \quad at \quad t \to 0 \tag{87}$$

Due to violation of the Lipchitz condition (86), the solution becomes unstable. That kind of instability when infinitesimal errors lead to finite deviations from basic motion (the Lipchitz instability) has been discussed in sub-section 2C. This instability leads to unpredictable shift of solution from one value of *C* to another. It means that appearance of any specified solution out of the whole family is random, and ***that randomness is controlled by the feedback (63) from the Liouville equation (66).*** Indeed if the solution (85) runs independently many times with the same initial conditions, and the statistics is collected, the probability density will satisfy the Liouville equation (66), Fig.10.

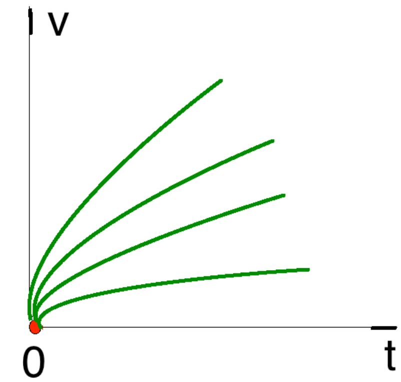


Figure 9. Family of random solutions describing transition from determinism to stochastisity.

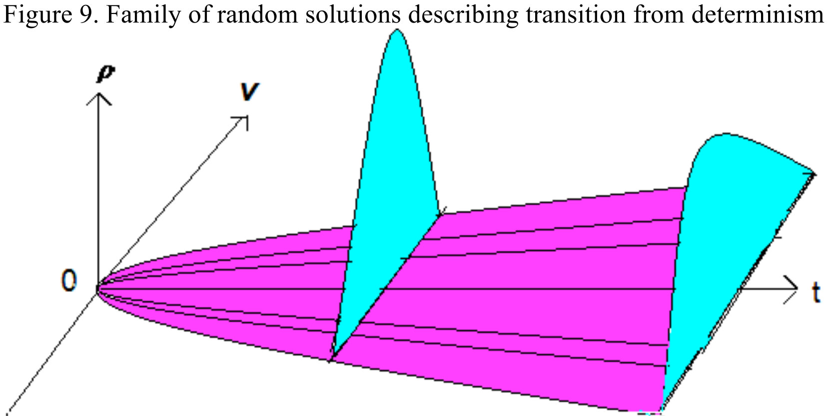


Figure 10. Stochastic process and probability density.

Remark. It should be emphasized that with the probability density defined by Eq. (69), the point $v = 0$ must be excluded from consideration since at this point Eq. (65) is meaningless.

D. Departure from Newtonian and quantum physics.

In this sub-section we will derive a distinguished property of the system (65),(66) that is associated with violation of the second law of thermodynamics i.e. with the capability of moving from disorder to order without help from outside. That property can be predicted qualitatively even prior to analytical proof: due to the nonlinear term in Eq. (66), the solution form shock waves and solitons in probability space, and that can be interpreted as "concentrations" of probability density, i.e. departure from disorder and decrease of entropy, see Fig.11.

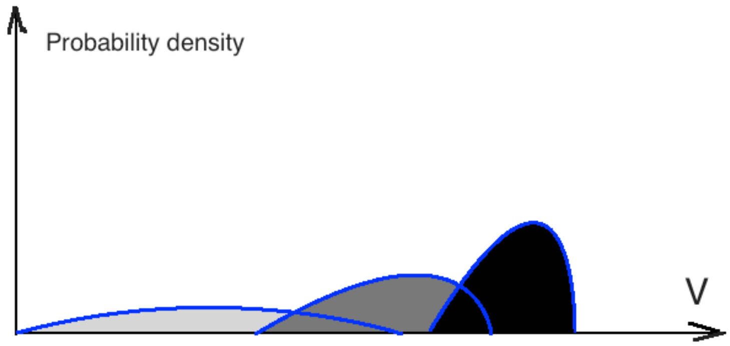


Figure 11. Formation of shock waves in probability space.

In order to demonstrate it analytically, let us turn to Eq. (66) at

$$c_1 >> |c_2|, c_3 \qquad (88)$$

and find the change of entropy H

$$\frac{\partial H}{\partial t} = -\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \rho \ln \rho \, dV = -\int_{-\infty}^{\infty} \frac{1}{c_1} \dot{\rho}(\ln \rho + 1) dV = \int_{-\infty}^{\infty} \frac{1}{c_1} \frac{\partial}{\partial V}(\rho^2) \ln(\rho + 1) dV$$
$$= \frac{1}{c_1}[\, \Big|_{-\infty}^{\infty} \rho^2 (\ln \rho + 1) - \int_{-\infty}^{\infty} \rho \, dV] = -\frac{1}{c_1} < 0 \qquad (89)$$

At the same time, the original system (65), (66) is isolated: it has no external interactions. Indeed the information force Eq. (63) is generated by the Liouville equation that, in turn, is generated by the equation of motion (65). In addition to that, the particle described by ODE (65) is in equilibrium $\dot{v} = 0$ prior to activation of the feedback (63). Therefore the solution of Eqs. (65), and (66) can violate the second law of thermodynamics, and that means that this class of dynamical systems does not belong to physics as we know it. This conclusion triggers the following question: are there any phenomena in Nature that can be linked to dynamical systems (65), (66)? The answer will be discussed bellow.

Thus despite the mathematical similarity between Eq.(66) and the KdV-Bergers equation, the physical interpretation of Eq.(66) is fundamentally different: it is a part of the dynamical system (65),(66) in which Eq. (66) plays the role of the Liouville equation generated by Eq. (65). As follows from Eq. (89), this system being isolated has a capability to decrease entropy, i.e. to move from disorder to order without external resources. In addition to that, the system displays transition from deterministic state to randomness (see Eq. (86)).

This property represents departure from classical and quantum physics, and, as shown in [9,10,11], it provides a link to behavior of livings. That suggests that this kind of dynamics requires extension of modern physics to include physics of life.

4. Random ODE with chaotic attractor.

A. Mathematical model.

In this section we consider a special feedback

$$f = \frac{\zeta}{\rho(v,t)} \int_{-\infty}^{v} [\rho(\eta,t) - \rho^*(\eta)] d\eta \tag{90}$$

that does not belong to the family of feedbacks described by Eq. (63).
Here $\rho^*(v)$ is a preset probability density satisfying the constraints (60), and ζ is a positive constant with dimensionality [1/sec]. As follows from Eq. (90), *f* has dimensionality of a force per unit mass that depends upon the probability density ρ, and therefore, it can be associated with the concept of information, so we will call it the *information force*. In this context, the coefficient ζ can be associated with the Planck constant that relates Newtonian and *information forces*. But since we are planning to deal with objects that belong to the macro-world, ζ must be of order of a viscose friction coefficient.
With the feedback (90), Eqs. (65) and (66) take the form, respectively

$$\dot{v} = \frac{\zeta}{\rho(v,t)} \int_{-\infty}^{v} [\rho(\eta,t) - \rho^*(\eta)] d\eta \tag{91}$$

$$\frac{\partial \rho}{\partial t} + \zeta[\rho(t) - \rho^*] = 0 \tag{92}$$

The last equation has the analytical solution

$$\rho = [(\rho_0 - \rho^*)e^{-\zeta t} + \rho^*] \tag{93}$$

Subject to the initial condition

$$\rho(t=0) = \rho_0 \tag{94}$$

that satisfy the constraints (60).
This solution converges to a preset stationary distribution $\rho^*(V)$. Obviously the normalization condition for ρ is satisfied if it is satisfied for ρ_0 and $\rho*$. Indeed,

$$\int_{-\infty}^{\infty} \rho V dV = [\int (\rho_0 - \rho^*) V dV] e^{-\zeta t} + \int_{-\infty}^{\infty} \rho^* V dV = 1 \tag{95}$$

Rewriting Eq. (93) in the form

$$\rho = \rho_0 + \rho^*(1 - e^{-\zeta t}) \tag{96}$$

one observes that $\rho \geq 0$ at all $t \geq 0$ and $-\infty > V > \infty$.
As follows from Eq. (93), the solution of Eq. (92) has an attractor that is represented by the preset probability density $\rho^*(v)$. Substituting the solution (93) in to Eq. (91), one arrives at the ODE that describes the stochastic process with the probability distribution (93)

$$\dot{v} = \frac{\zeta e^{-\zeta t}}{[\rho_0(v) - \rho^*(v)]e^{-\zeta t} + \rho^*(v)} \int_{-\infty}^{v} [\rho_0(\eta) - \rho^*(\eta)]\ d\eta \tag{97}$$

As will be shown below, the randomness of the solution of Eq. (96) is caused by instability that is controlled by the corresponding Liouville equation.
It is reasonable to assume that the solution (93) starts with a sharp initial condition

$$\rho_0(V) = \delta(V) \tag{98}$$

As a result of that assumption, all the randomness is supposed to be generated *only* by the controlled instability of Eq. (96). Substitution of Eq. (97) into Eq. (96) leads to two different domains of *v*: $v \neq 0$ and *v*=0 where the solution has two different forms, respectively

$$\int_{-\infty}^{v} \rho^*(\zeta) d\zeta = \left(\frac{C}{e^{-\zeta t} - 1}\right)^{1/\zeta}, \qquad v \neq 0 \tag{99}$$

$$v \equiv 0 \tag{100}$$

Indeed, $\dot{v}=\dfrac{\zeta e^{-\zeta t}}{\rho^*(v)(e^{-\zeta t}-1)}\int_{-\infty}^{v}\rho^*(\eta))]d\eta$

whence $\dfrac{\rho^*(v)}{\int_{-\infty}^{v}\rho^*(\eta)d\eta}dv=\dfrac{\zeta e^{-\zeta t}}{e^{-\zeta t}-1}dt$. Therefore, $\ln\int_{-\infty}^{v}\rho^*(\eta)d\eta=\ln(\dfrac{C}{e^{-\zeta t}-1})^{1/\zeta}$ and that leads to Eq. (99) that presents an implicit expression for v as a function of time since ρ^* is the known function. Eq. (100) represents a singular solution, while Eq. (99) is a regular solution that include arbitrary constant C . The regular solutions is unstable at $t=0$, $|v|\to 0$ where the Lipschitz condition is violated

$$\frac{d\dot{v}}{dv}\to\infty \; at \qquad t\to 0 \quad ,|v|\to 0 \tag{101}$$

and therefore, an initial error always grows generating *randomness*.

Let us analyze the behavior of the solution (99) in more details. As follows from this solution, all the particular solutions for different values of C intersect at the same point v=0 at t=0, and that leads to non-uniqueness of the solution due to violation of the Lipcshitz condition. Therefore, the same initial condition v=0 at t=0 yields infinite number of different solutions forming a family (99); each solution of this family appears with a certain probability guided by the corresponding Liouville equation (92). For instance, in cases plotted in Fig.12 a and Fig.12 b, the "winner" solution is, respectively,

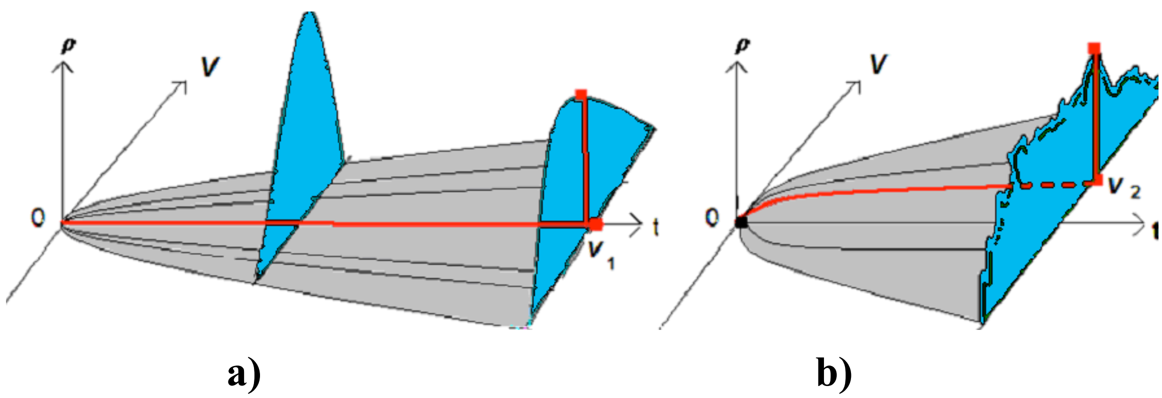


Figure 12. Stochastic processes and their attractors.

$$v_1=\varepsilon\to 0, \qquad \rho(v_1)=\rho_{\max}, \qquad and \qquad v=v_2, \quad \rho(v_2)=\sup\{\rho\} \tag{102}$$

since it passes through the maximum of the probability density . However, with lower probabilities, other solutions of the same family can appear as well. Obviously, this is a non-classical effect. Qualitatively, this property is similar to those of quantum mechanics: the system keeps all the solutions simultaneously and displays each of them "by a chance", while that chance is controlled by the evolution of probability density (93).

Let us emphasize the connections between solutions of Eqs. (91) and (92): the solution of Eq. (91) is an one-parametrical family of trajectories (99), and each trajectory occurs with the probability described by the solution (96) of Eq. (92). The scenario of transition from determinism to randomness here similar to that of quantum mechanics considered in sub-section 2Bc (see Eqs. (41) and (42)): ***combination of failure of the Lipchitz condition and emergence the Hadamard instability at the same point t=0 leads to disturbance that can take any trajectory of the multivalued family with the probability controlled by the Liouville equation (92).*** However unlike the classical chaos, here (as well as in quantum mechanics) the source of randomness is concentrated in one point $t=0$; beyond of this point, the solution evolves along the "chosen" trajectory.

The approach is generalized to n-dimensional case simply by replacing v with a vector $v=v_1,v_2,...v_n$ since Eq. (92) does not include space derivatives

$$\dot{v}_i = \frac{\zeta}{\rho(\{v\},t)} \int_{-\infty}^{v_i} [\rho(\{\eta\},t) - \rho^*(\{\eta\})] d\eta_i \tag{103}$$

$$\frac{\partial \rho(\{V\},t)}{\partial t} + n\zeta\rho(\{V\}t) - \rho^*(\{V\}) = 0 \tag{104}$$

B. Examples.

Let us start with the following normal distribution

$$\rho^*(V) = \frac{1}{\sqrt{2\pi}} e^{-\frac{V^2}{2}} \tag{105}$$

Substituting the expression (105) and (98) into Eq. (99) at V=v, and $\zeta = 1$ one obtains

$$v = erf^{-1}(\frac{C_1}{e^{-t} - 1}), \qquad v \neq 0 \tag{106}$$

As another example, let us choose the target density ρ^* as the Student's distribution, or so-called power law distribution

$$\rho^*(V) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})} (1 + \frac{V^2}{\nu})^{-(\nu+1)/2} \tag{107}$$

Substituting the expressions (107) and (98) into Eq. (99) at V=v, ν=1, and $\zeta = 1$ one obtains

$$v = \cot(\frac{C}{e^{-t} - 1}) \; for \qquad v \neq 0 \tag{108}$$

The 3D plot of the solutions of Eqs.(106) and (108), are presented in Figures 13, and 14,respectively.

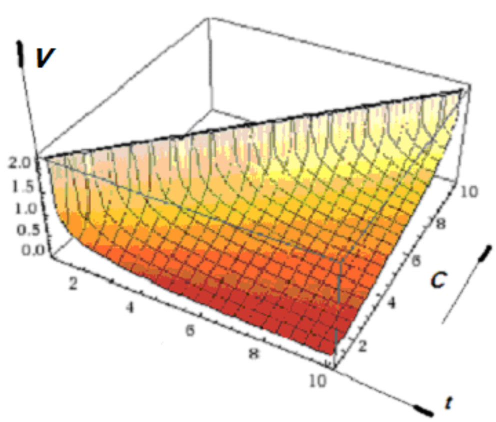


Figure 13. Dynamics driving random events to normal distribution.

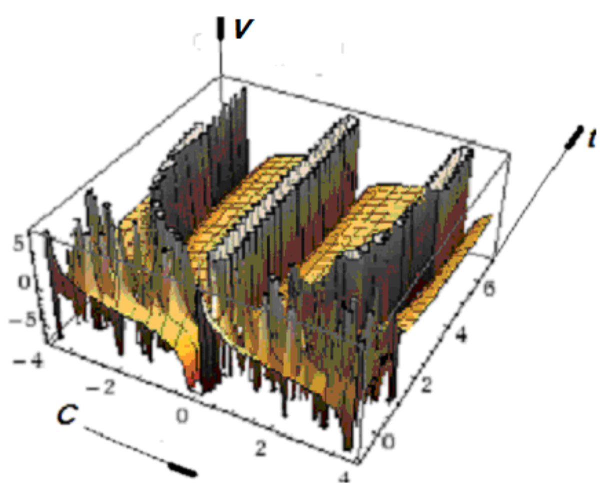


Figure 14. Dynamics driving random events

Since our further analysis will be concentrated on the power law statistics, we take a closer look at the solution (108) and consider the trajectories of the underlying particle. The evolution of a particular solutions at *C=1, 10,* and *100* is plotted in Fig 15. Here *C* is the parameter representing a particular random sample of the solution. This solution describes (in an implicit form) a one-parametric family of dynamical processes $v = v(t, C)$. Each scenario (for a fixed *C*) occurs with the current probability (93) that asymptotically approaches the power-law distribution (107) at critical points. Those critical points that occur at large *v* can be associated with catastrophes.

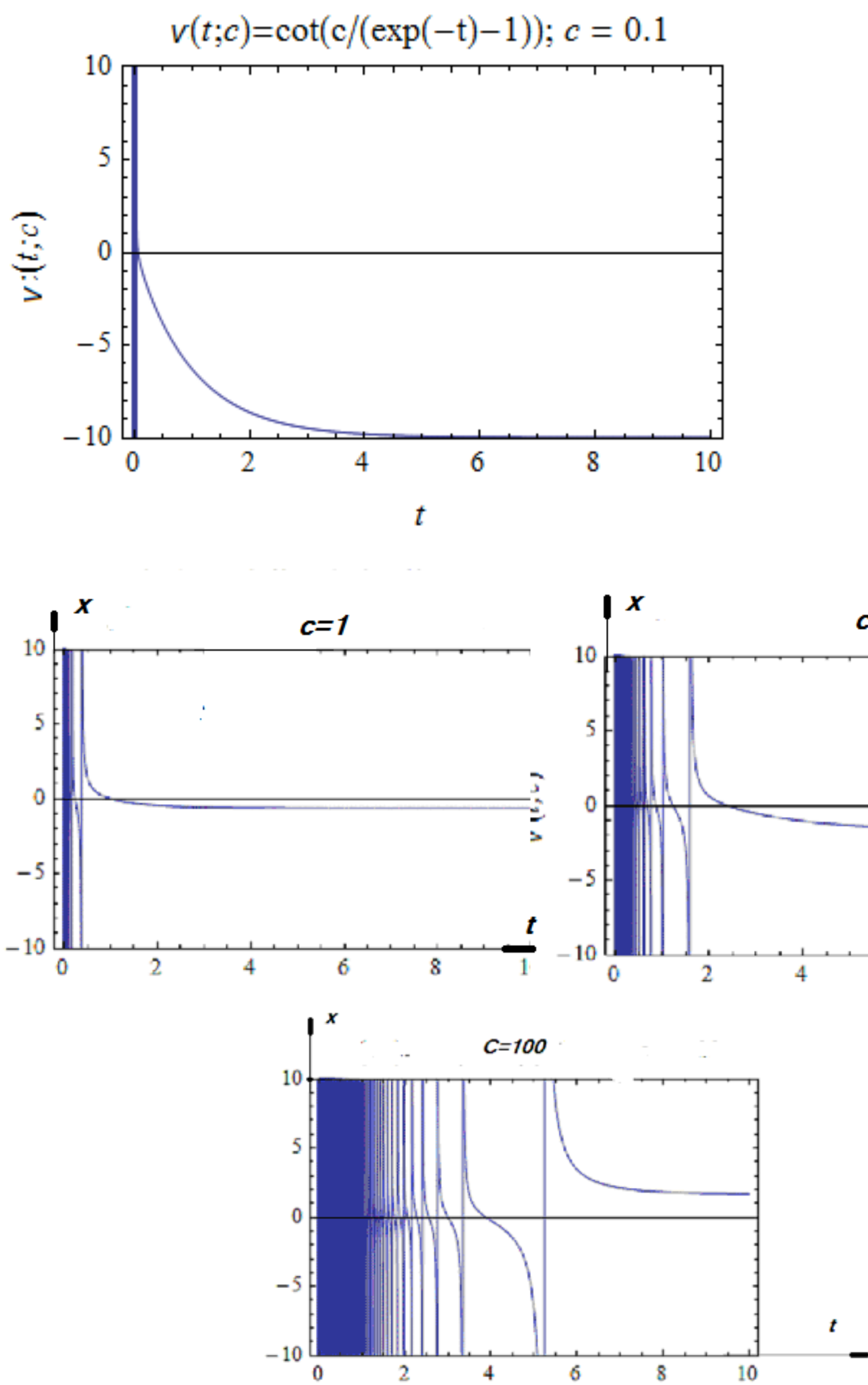


Figure 15. The evolution of a particular solutions at C=1, 10, 100.

C. Relations to chaos.

In this sub-section we outline similarities and differences between the dynamical attractor described above and classical chaos in Newtonian dynamics. Turning to Eqs. (1) through (21), we observe that the origin of randomness in Newtonian mechanics is instability of ignorable variables, i.e., such variables that do not contribute to energy of the system, and that leads to exponential divergence of initially adjacent trajectories; this divergence is measured by Liapunov exponents, which form a discrete spectrum of numbers that must include positive ones.

In dynamics of ODE (65), (66) and (91),(92) described in the previous sections, as well as in quantum systems (see Eqs.(30) and (39)) the instability has a different nature: it is caused by the loss of uniqueness of the solution at a singular point due to failure of Lipchitz condition at this point. In context of terminal dynamics, [6], this point represents a terminal repeller that is characterized by infinite divergence of trajectories. In other words, it is characterized by an unbounded positive Liapunov exponent, (see Eq. (54)), and the power of the non-Lipschitz-originated instability is equivalent to the Hadamard, or blown-up instability. As a result of that, this system makes a random choice of the trajectory only once – at the beginning of the transition from determinism to randomness, while a Newtonian system may change trajectories continuously during its chaotic motion. However as follows from Eq. (89), the random ODE considered above belongs neither to Newtonian, nor to quantum mechanics since they violate the second law of thermodynamics due to a capability to decrease entropy without external interaction. Actually these systems can be considered as quantum-classical hybrids. However regardless of some differences, the attractors of the system (91),(92) have to be considered as a special type of chaos since randomness there is originated from dynamical instability rather than from an external random input.

5. A mystery of power-law statistics.

The objective of this section was inspired by a mysterious power-law statistics that predicts social catastrophes: wars, terrorist attacks, market crushes est. Resent interest in literature is concentrated on half-a-century finding that the severity of interstate wars is power-law distributed, and that belongs to the most striking empirical regularities in world politics. Surprisingly, similar catastrophes were identified in physics (Ising systems, avalanches, earthquakes), and even in geometry (percolation). Although all these catastrophes have different origins, their similarity is based upon the power law statistics, and as a consequence, on scale invariance, self-similarity and fractal dimensionality, [12]. According to the theory of self-organized criticality, that explains the origin of this kind of catastrophes, each underlying dynamical system is attracted to a critical point separating two qualitatively different states (phases). This attraction is represented by a relaxation process of slowly driven system. Transitions from one phase to another are accompanied by sudden release of energy that can be associated with a catastrophe, and the severity of the catastrophe is power law distributed. However, in order to overcome the critical point and enter a new phase, a slow input of *external* energy is required. The origin of this energy is well understood in physical systems, but not in social ones, since there are no well-established models of social dynamics. For that reason, we turn to the previous section and start with comparison the underlying dynamics of normal and power law distribution, (see Figs. 13,14, and 16). Let us recall that the normal distribution is commonly encountered in practice, and is used throughout statistics, natural sciences, and social sciences as a simple model for complex phenomena. For example, the observational error in an experiment is usually assumed to follow a normal distribution, and the propagation of uncertainty is computed using this assumption. But statistical inference using a normal distribution is not robust to the presence of outliers (data that is unexpectedly far from the mean, due to exceptional circumstances, observational error, etc.). When outliers are expected, data may be better described using a heavy-tailed distribution such as the power-law distribution. As demonstrated in Fig. 15, normal and power law distributions have very close configurations excluding the tails. However despite of that, the types of the random events described by these statistics are of fundamental difference. Indeed, processes described by normal distributions are usually coming from physics, chemistry, biology, etc., and they are characterized by a smooth evolution of underlying dynamical events. On the contrary, the processes described by power laws are originated from events driven by human decisions (wars, terrorist acts, market crushes), and therefore, they are associated with catastrophes. Surprisingly, the 3D plots of Eqs.(106) and (108) (see Figs.13 and 14) describing dynamics that drives random events to the normal and the power law distributions, respectively, demonstrate the same striking difference between these distributions, that is: a smooth evolution to normal distribution, and “violent”, full of densely distributed discontinuities (see Fig. 15) transition to power law distribution. In Fig.15, C is the parameter representing a particular random sample of the solution. This solution describes (in an implicit

form) a one-parametric family of dynamical processes $v = v(t, C)$. Each scenario (for a fixed *C*) occurs with the current probability (93) that asymptotically approaches the power-law distribution (107) at critical points. Those critical points that occur at large *v* can be associated with catastrophes.
Is this a coincidence? Indeed, the proposed random dynamics is based upon global assumptions, and it does not bear any specific information about a particular statistics as an attractor. However the last statement should be slightly modified:

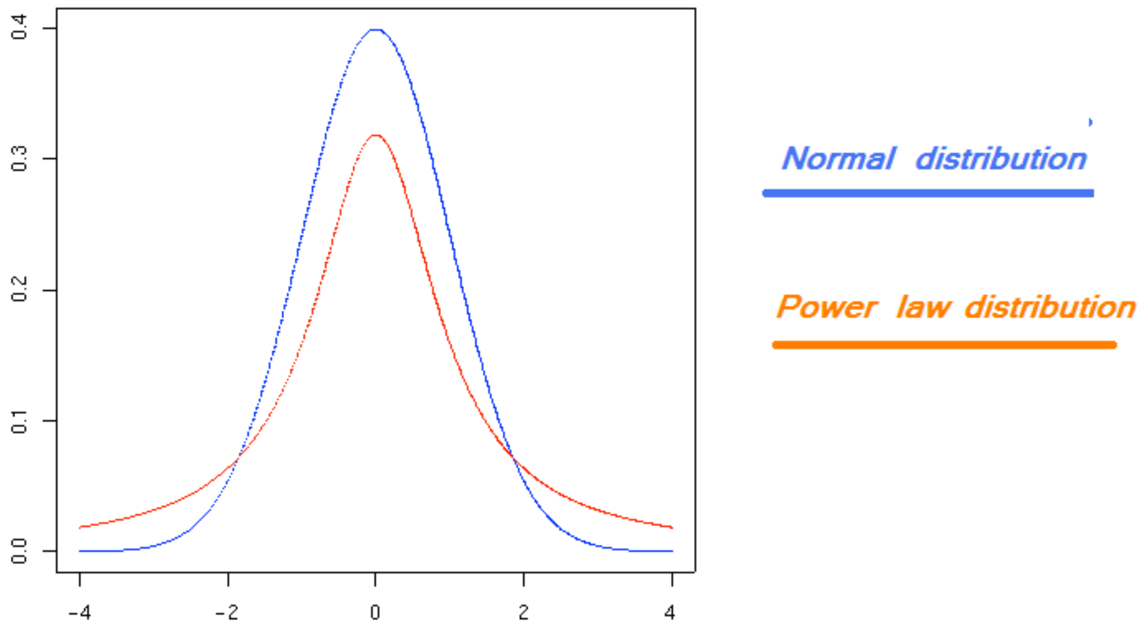


Figure 16. Normal and power law distributions.

actually the model of random dynamics is tailored to describe Livings' behavior, and in particular, decision making process,[9,10,11]. Is that why the random dynamics captures "violent" properties of power law statistics that is associated with human touch? We will discuss possible answers to this question below.
First of all we have to consider the problem of uniqueness of a dynamical system which solutions approach a preset attractor. In classical dynamics the answer to this problem is clear: there is infinite number of different dynamical systems that approach the same attractor, and that is true for static, periodic and chaotic attractors. Although in random dynamics (91),(92) we are dealing with a special type of chaotic attractor that represents a preset statistics, the answer is similar. Indeed, let us turn to Eq. (91) and add an arbitrary terms as following

$$\dot{v} = \frac{1}{\rho(v,t)}\{\int_{-\infty}^{v}[\rho(\zeta,t) - \rho^*(\zeta)]d\zeta + A + B(t)\} \qquad (109)$$

where *A* is an arbitrary constant, and *B(t)* is arbitrary function of time.
It easily verifiable that this modification does not change Eq.(92),

$$\frac{\partial\rho}{\partial t} + \rho(t) - \rho^* = 0 \qquad (110)$$

and therefore, the attractor ρ^* remains the same, despite of different solution of Eq.(109).

That excludes the possibility of a rigorous proof that the random dynamics necessarily describes the same random events that constitute the corresponding statistics. But in order to explain, at least, some correlations between them, we will turn to a less rigorous approach known as an Occam Razor. After several modifications, a "scientific" version of this approach can be formulated as following: one should proceed to simpler theories until simplicity can be traded for greater explanatory power. Obviously application of this principle does not guarantee a success, however, there is known some encouraging results of its applications: In science, Occam's razor is used as a heuristic (rule of thumb) to guide scientists in the development of theoretical models rather than as an arbiter between published models. In physics, parsimony was an important heuristic in the formulation of special relativity by Albert Einstein, the development and application of the principle of least action by Pierre Louis Maupertuis and Leonhard Euler, and the development of quantum mechanics by Ludwig Boltzmann, Max Planck, Werner Heisenberg and Louis de Broglie. There have been attempts to derive Occam's Razor from probability theory, notable attempts made by Harold Jeffreys and E. T. Jaynes. Using Bayesian reasoning, a simple theory is preferred to a complicated one because of a higher prior probability. Hence, application of the Occam Razor requires a definition of the concept of complexity/simplicity. However in our case the simplest dynamical system is obvious: it follows from Eq. (109) when

$$A \equiv 0,\ B \equiv 0 \tag{111}$$

i.e. the ***most likely*** scenario describing dynamics that drives random events to power law distribution is still a dynamical system Eq.(91) and (92) that was discussed above. However if additional information about the random events dynamics is available, it should be incorporated to the enlarged model Eq. (109) through a best-fit adjustment of the constant *A* and the parameterized function *B(t)* thereby trading simplicity for greater explanatory power. We have to emphasize that the enlarged dynamical system still has the same power-law statistics as an attractor.

6. General case.

Based upon the proposed model, a simple algorithm for finding a dynamical system that is attracted to a preset *n*-dimensional statistics can be formulated. The idea of the proposed algorithm is very simple: based upon the system (103), and (104), introduce the probability density $\rho^*(v_1, v_2, \ldots v_n)$ representing the statistics to which the solution of Eq. (104) is to be attracted, and insert it in Eqs. (103) and (104). The solution of Eq. (103) will eventually approach a chaotic attractor which probability density coincides with the preset statistics. As in the case of the one-dimensional power-law statistics, here the solution of Eq. (103) will present the ***most likely*** scenario describing dynamics that drives random events to preset statistics. Moreover if additional information about the random events dynamics is available, it should be incorporated to the enlarged model of Eq. (109) that is

$$\dot{v}_i = \frac{1}{n\rho(v_{1,} \ldots v_n, t)}\Big[\int_{-\infty}^{v_i}[\rho(\zeta_i, v_{j\neq i}, t) - \rho^*(\zeta_i, v_{j\neq i})]d\zeta_i + \\ A_i + B_i(t) + \sum_{j\neq i}^{n} T_{ij}\tanh v_j\Big] \tag{112}$$

through the best-fit adjustment of the constants A_i, the parameterized functions $B_i(t)$*),* as well as the constants T_{ij} that introduce zero-divergence terms for *n >1*, thereby trading simplicity for greater explanatory power and without a change of the preset statistics as is the one-dimensional case.

Remark. It is easily verifiable that the augmented terms in Eqs.(112) do not effect the corresponding Liouville equation (104), and therefore, they do not change the static attractor in the probability space (that corresponds to the chaotic attractor in physical space). However, they may significantly change the configuration of the random trajectories in physical space making the dynamics more sophisticated.

It should be noticed that the proposed approach imposes a weak restriction upon the *space structure* of the function $\rho(\{v\})$: it should be only integrable since there is no space derivatives included in Eq. (104). This means that $\rho(\{v\})$ is not necessarily to be differentiable. For instance, it can be represented by a

Weierstrass-like function $f(v) = \sum_{0}^{\infty} a^n \cos(b^n \pi v)$, where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + 1.5\pi$.

7. Discussion and conclusion.

The main result being proved in this paper is that ***for any statistics it can be found a set of dynamical systems having a chaotic attractor whose probability density is identical to the underlying statistics, and therefore, the statistics becomes a static attractor in probability space.*** It has been hypothesized that the simplest system in each set is likely to present a first approximation to the scenario simulating dynamics that drives random events to the corresponding stochastic attractor. It was demonstrated how to find these dynamical systems given the underlying statistics. Special attention was concentrated on power-law statistics, and its interpretation, with help of the dynamical system, is proposed.

Let us discuss now possible contribution of the presented result to the foundation of statistics and probability in general. There are two broad categories of probability interpretations that can be called "physical" and "evidential" probabilities.

Physical probabilities, which are also called objective or frequency probabilities, are associated with random physical systems such as roulette wheels, rolling dice and radioactive atoms. In such systems, a given type of event (such as a dice) tends to occur at a persistent rate, or "relative frequency", in a long run of trials. Physical probabilities either explain, or are invoked to explain, these stable frequencies. Thus talk about physical probability makes sense only when dealing with well defined random experiments.

Evidential probability, also called Bayesian probability (or subjectivist probability), can be assigned to any statement whatsoever, even when no random process is involved, as a way to represent its subjective plausibility, or the degree to which the statement is supported by the available evidence. On most accounts, evidential probabilities are considered to be degrees of belief, defined in terms of dispositions to gamble at certain odds.

Comparing these alternative interpretations in view of the random dynamics approach introduced above, one concludes that the latter partially unites these two extremes. Indeed, the randomness there is of dynamical origin: it is generated by the Liouville equation along with non- Lipchitz instability caused by the feedback to the equations of motion. However the dynamical origin does not mean a physical one since the random dynamics is not Newtonian and not Quantum: it is rather a quantum-classical hybrid that describes behavior of Livings, [9,10,11],Fig.17.

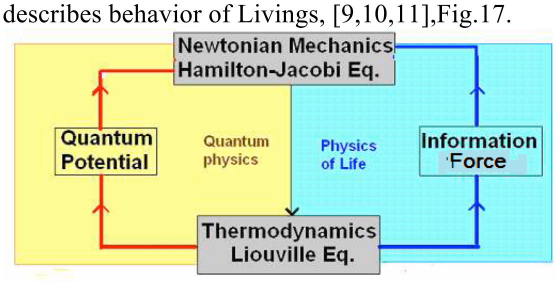


Figure 17. Classic Physics, Quantum Physics and Physics of Life.

That is why this dynamics includes "human factor": self-image, self-awareness, collective mind, etc., and that place it in between of physical and Bayesian interpretations. But in addition to that, the dynamical interpretation introduced above has a solid mathematical foundation in the form of ordinary differential equations.

It should be emphasized that the proposed approach is based upon a new class of random ODE in dynamics.

References.

1. J. Synge, 1926, On the geometry of dynamics, Phil. Trans. R. Sos. Lond., Ser.A 226, 31-106.
2. V. Arnold, 1988, Mathematical methods of classical mechanics, Springer, New York

3. Heisenberg, W. (1927), "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik", *Zeitschrift für Physik* **43** (3–4): 172–198, Bibcode:1927ZPhy...43..172H
4. Madelung, E. (1927). "Quantentheorie in hydrodynamischer Form". *Z. Phys.* **40**: 322–326.
5. G. Whitham, *Linear and Nonlinear Waves* Wiley-Interscience, New York, 1974

6. Zak, M., "Terminal Model of Newtonian Dynamics," Int. J. of Theoretical Physics, No.32, 159-190, 1992.

7. E. Kamke, *Differentialgleichungen: Losungsmethoden und Losungen,* Gewohnliche Differentialgleichungen Vol. I, edited by B. G. Teubner Leipzig, 1977.

8. Zak, M., 1994, *Postinstability models in Dynam*ics, Int. J. of Theor. Phys.vol. 33, No. 11, 2215-2280

9. Zak, M., 2007, Physics of Life from First Principles, EJTP **4**, No. 16(II), 11–96

10.Zak,M., 2014, Interference of probabilities in dynamics, AIP Advances 4, 087130.
11. Zak, M., 2008, Quantum-inspired maximizer, JOURNAL OF MATHEMATICAL PHYSICS **49**, 042702.
12. Christensen, K., Moloney, N., 2005, Complexity and Criticality, Imperial College Press