Essential Eigenvalues and Eigenvectors

Robert Mereau

May 7, 2015

The output probability between two vertices v_S and v_R for an adjacency matrix A at time t is given by:

$$P(t) = |\langle v_S|e^{-iAt}|v_R\rangle|^2 \tag{1}$$

We know the time evolution operator can be expressed as

$$e^{-iAt} = \sum_{n=1}^{D} e^{-i\lambda_n t}|v_n\rangle\langle v_n| \tag{2}$$

where λ_n and v_n are eigenvalues and eigenvectors of A respectively and D is the dimension of the matrix. So we can rewrite the equation for output probability as

$$P(t) = |\sum_{n=1}^{D} e^{-i\lambda_n t}\langle v_S|v_n\rangle\langle v_n|v_R\rangle|^2 \tag{3}$$

We will call the product of the inner products $\langle v_S|v_n\rangle\langle v_n|v_R\rangle = E_n$. So our equation for output probability now reads:

$$P(t) = |\sum_{n=1}^{D} e^{-i\lambda_n t}E_n|^2 \tag{4}$$

So the eigenvalues are responsible for the phase of E_n at time t. It turns out that for the system of APST networks I determined previously, only a small subsection of the summation is responsible for high output probability. For instance, lets consider the output probability through time for a modified path of diameter 740, with attached stars of size 100 (Thanos).

We can see that the output probability through time is negligibly different if we use all 941 terms of the summation (Figure 1), or only 6 (Figure 2).

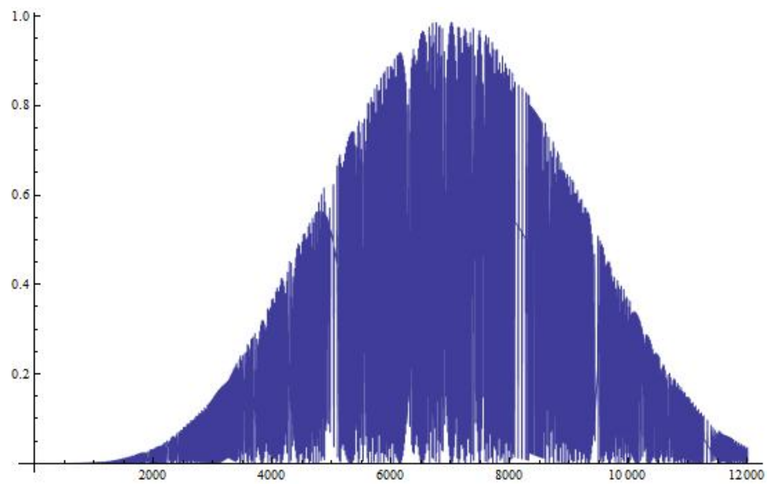

Figure 1: Here we can see the output probability $P(t)$ across Thanos using the entire summation of the evolution operator, for a total of 941 terms.

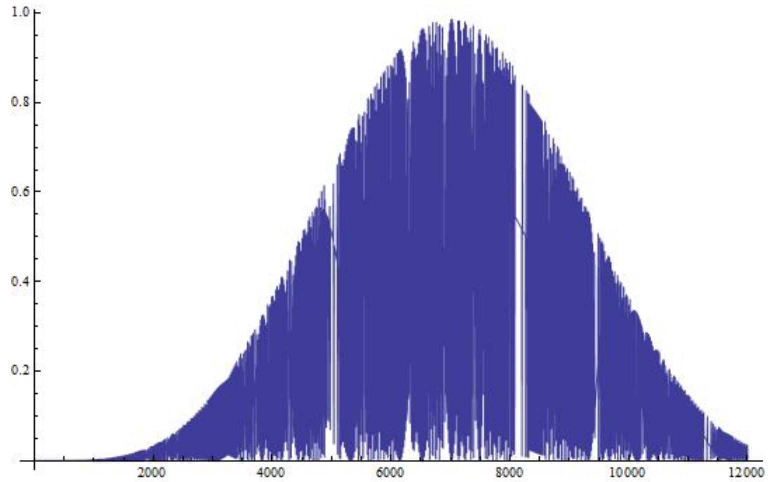

Figure 2: Here we can see the output probability $P(t)$ across Thanos using only 6 essential terms of the summation. We can see that it is almost indistinguishable from the entire summation. If a single term out of the essential 6 is not included, $P(t)$ effectivly vanishes.

Before going into the reason this is occurring, lets consider a smaller example of another modified path with a diameter of 86 and attached stars of size 10 (Tony). Again looking at $P(t)$ across the diameter, we can see the effect of the essential 6 terms in the summation (Figures 3 and 4).

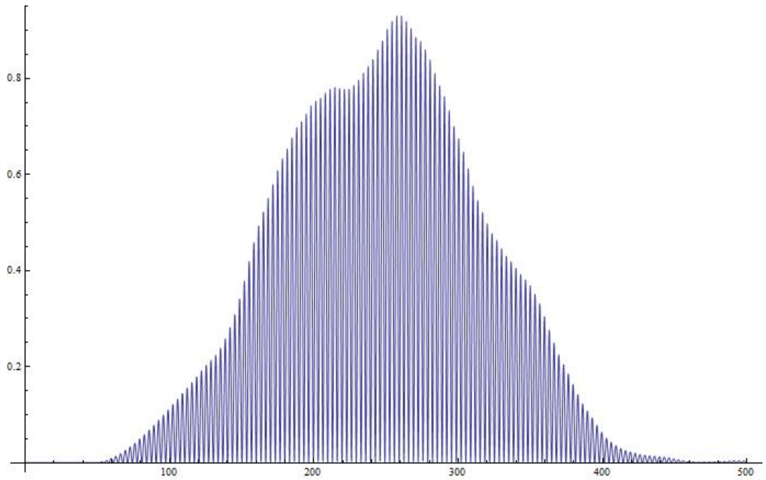

Figure 3: Here we can see the output probability $P(t)$ across Tony using the entire summation of the evolution operator, for a total of 107 terms.

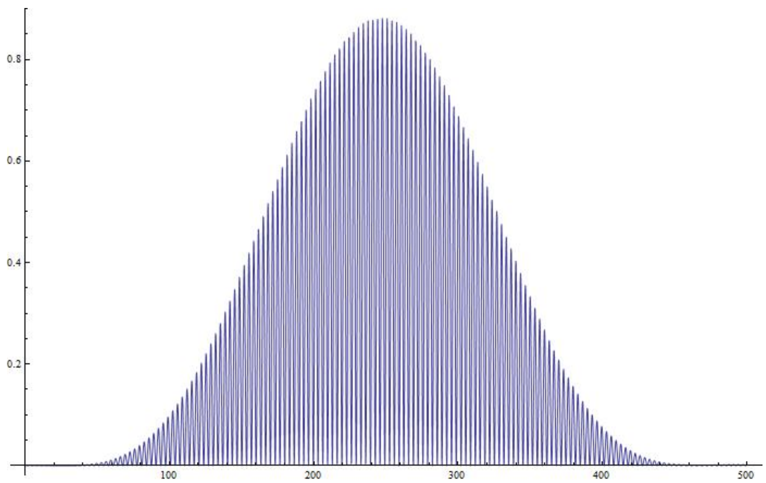

Figure 4: Here we can see the output probability $P(t)$ across Tony using only the essential 6 terms. Again, removing a single term causes $P(t)$ to vanish.

Of course, we can see that by using the summation of the entire system, Figure 3 is actually more distorted and does not behave as geometrically as using only the essential terms. The point is that there are six terms in each case of these networks, which push $P(t)$ to APST values. The reason is due to the large value of E_n for the essential six terms, in combination of the fact that the phase of these terms (Eigenvalues) are close. For example, consider a graphical representation of the different E_n terms for Tony.

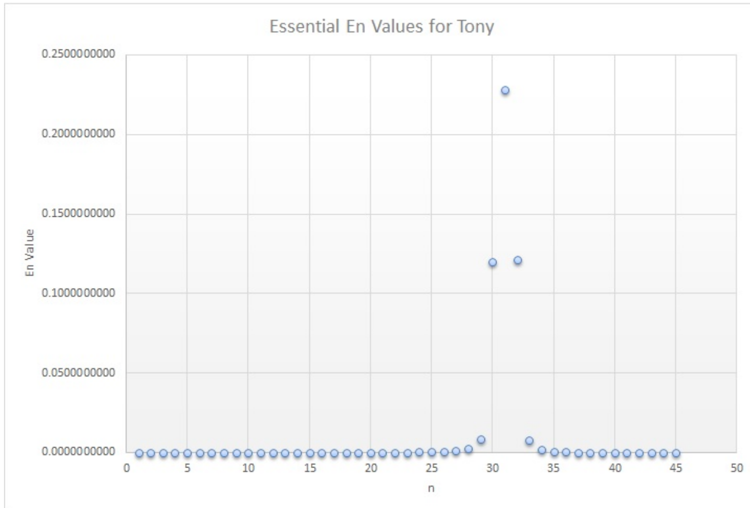

Figure 5: Here we can see the different values of E_n for Tony. Due to a convienent numbering system, we can see a sharp peak where three values are much larger than anywhere else. These are the essential values. Of course, close inspection can beg the question if there are actually five essential values (two additional points at the bottom of the peak), but we will see that this is not the case.

We can see that there are three values of n for which E_n is disproportionately large. (These values have multiplicity 2 for a total of 6 essential E_n terms). So in terms of the summation given by Equation 4, these values pack the punch. Furthermore, it is also imperative than the phases (Eigenvalues) of the essential E_n are close or they can cancel each other out. Of course, this is just the case:

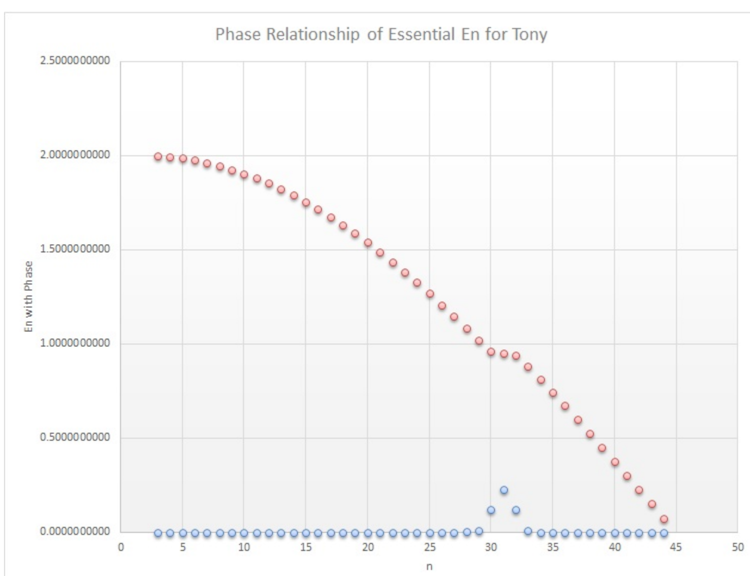

Figure 6: This blue points in this figure show the identicle representation of the different E_n values for Tony as above. Additionally, each red point directly above a blue point represents the corresponding λ_n which controls the phase of the E_n below.

From Figure 6, it becomes abundantly clear why these network are able to achieve APST. The eigenvalues plateau for the essential E_n values, meaning that the phases can add constructively! (Note the interesting exponential trend before the plateau and the linear trend afterwards). APST is fundamentally achieved because the eigenvalues of the essential E_n are close values. This hold true for all cases of these networks.

I have previously shown that the peak output probability increases as the size of a network increases. Furthermore, I have also shown the eigenvalues corresponding to essential E_n converge as the networks grow as well. Now we can see that these two results walk hand in hand. As the eigenvalues of the essential E_n converge to a single value, complete constructive phase addition is made possible.

Lastly, it turns out that the essential E_n and λ_n for different networks are actually very close in value. That is, the essential values of E_n and λ_n for one network are very close to that of another. The is illustrated on the last page.

At first, it seems that a potential setback could be that though the essential eigenvalues converge, leading to higher peak probability, they also seem to be increasing. This would imply that the frequency of oscillation also increases making the time window to retrieve a state smaller. However, through the analysis of much larger networks, I have found that essential eigenvalues are converging to ± 1.

Diameter = 86, Star = 10

$-0.120258\, e^{(0.+0.964652\, i)\, t}$

$-0.120258\, e^{(0.-0.964652\, i)\, t}$

$0.228153\, e^{(0.+0.951752\, i)\, t}$

$0.228153\, e^{(0.-0.951752\, i)\, t}$

$-0.121276\, e^{(0.+0.939162\, i)\, t}$

$-0.121276\, e^{(0.-0.939162\, i)\, t}$

Diameter = 158, Star = 20

$-0.122871\, e^{(0.+0.980199\, i)\, t}$

$-0.122871\, e^{(0.-0.980199\, i)\, t}$

$0.23883\, e^{(0.+0.975379\, i)\, t}$

$0.23883\, e^{(0.-0.975379\, i)\, t}$

$-0.123084\, e^{(0.+0.970594\, i)\, t}$

$-0.123084\, e^{(0.-0.970594\, i)\, t}$

Diameter = 233, Star = 30

$0.119259\, e^{(0.+0.986245\, i)\, t}$

$-0.119259\, e^{(0.-0.986245\, i)\, t}$

$-0.242426\, e^{(0.+0.983498\, i)\, t}$

$0.242426\, e^{(0.-0.983498\, i)\, t}$

$0.128069\, e^{(0.+0.98094\, i)\, t}$

$-0.128069\, e^{(0.-0.98094\, i)\, t}$

Diameter = 452, Star = 60

$-0.119401\, e^{(0.+0.9927\, i)\, t}$

$-0.119401\, e^{(0.-0.9927\, i)\, t}$

$0.246202\, e^{(0.+0.991705\, i)\, t}$

$0.246202\, e^{(0.-0.991705\, i)\, t}$

$-0.129295\, e^{(0.+0.990786\, i)\, t}$

$-0.129295\, e^{(0.-0.990786\, i)\, t}$