A modified circle-cutting strategy for conceptualizing $n!$ and its application to derive yet another well-known mathematical result: the approximate sum of a convergent series involving factorials equals unity

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Abstract: $n!$ is defined as the product $1.2.3........n$ and it popularly represents the number of ways of seating $n$ people on $n$ chairs. In a previous paper we conceptualized a new way of describing $n!$, using sequential cuts to an imaginary circle and derived a well known result. In this paper we use the same intuitive approach but reverse the cutting strategy by starting with $n$-cuts to the circle. We observe that this method leads us to estimate the approximate sum of an infinite convergent series involving factorials as unity.

Results:

$n!$ is defined as the product of $n$ consecutive positive integers from 1 to $n$, each considered exactly once. It has been popularly described as the number of ways in which $n$ people can be seated in $n$ chairs (1).

We use yet another way to visualize the factorial function (modification of the method in reference 2).

$n!=(n)(n-1)(n-2)............4.3.2.1=(n)(n-1)(n-2)............4.3.2$

We can consider cutting a circle into $n!$ pieces as a sequential step wise process.

Let us consider an intact circle.

In the first step let us cut it at $n$ positions. This will give rise to $n$ pieces. (In this step we have used $n$ cuts and end up with $n$ arcs or pieces of the circle. Note that until this step we have used a total of $n$ cuts).

In second step let us cut each of the $n$ pieces/arcs from the first step at $\{(n-1)-1\}=(n-2)$ positions. In the second step we use $n(n-2)$ cuts and we end up with $n(n-1)$ pieces. Until now we have used $n+n(n-2)$ cuts to end up with $n(n-1)$ pieces or arcs.

In the third step let us cut each $n(n-1)$ pieces from the second step at $\{(n-2)-1\}=(n-3)$ positions. In the third step we use $n(n-1)(n-3)$ cuts and end up with $n(n-1)(n-2)$ pieces. Until now we have used $n+n(n-2)+n(n-1)(n-3)$ cuts to end up with $n(n-1)(n-2)$ pieces or arcs.

In the fourth step let us cut each of the $n(n-1)(n-2)$ pieces from the third step at $\{(n-3)-1\}=(n-4)$ positions. In the fourth step we use $n(n-1)(n-2)(n-4)$ cuts and we end
up with \( n(n-1)(n-2)(n-3) \) pieces. Until now we have used \( n+n(n-2)+ n(n-1)(n-3) + n(n-1)(n-2)(n-4) \) cuts and end up with \( n(n-1)(n-2)(n-3) \) pieces or arcs.

Proceeding in this manner........

In the \((n-1)\)th step we will cut each of the \( n(n-1)(n-2) \ldots 5.4.3 \) pieces from previous step at \( \{n-(n-2)\}-1 \) position. Thus in the \((n-1)\)th step we use \( n(n-1)(n-2) \ldots 5.4.3.1 \) cuts and we end up with \( n(n-1)(n-2) \ldots 5.4.3.2 \) pieces. Until now we have used a total of

\[
\begin{align*}
&n+n(n-2) + n(n-1)(n-3) + n(n-1)(n-2)(n-4) + \ldots \ldots + n(n-1)(n-2) \ldots 5.4.3.1 \text{ cuts} \\
&\text{end up with } n(n-1)(n-2) \ldots 5.4.3.2 = n! \text{ pieces or arcs.}
\end{align*}
\]

Since a total of \( x \)-cuts to circle would result in \( x \)-pieces or arcs therefore

\[
\begin{align*}
&n+n(n-2) + n(n-1)(n-3) + n(n-1)(n-2)(n-4) + \ldots \ldots + n(n-1)(n-2) \ldots 5.4.2 + \\
&\{n(n-1)(n-2)\ldots 5.4.3.1\}
\end{align*}
\]

\[= n! \]

or

\[
\begin{align*}
&n+n!\left\{(n-2)/(n-1)!\right\} + \{n-3\}/(n-2)! + \{n-4\}/(n-3)! + \ldots \ldots 4/5! + 3/4! \\
&+2/(3!) + 1/(2!)
\end{align*}
\]

\[= n! \]

dividing both sides by \( n! \) we obtain

\[
\begin{align*}
&\{1/(n-1)!\} + \{(n-2)/(n-1)!\} + \{(n-3)/(n-2)!\} + \{(n-4)/(n-3)!\} + \ldots \ldots 4/5! + 3/4! \\
&+2/(3!) + 1/(2!)
\end{align*}
\]

\[= 1 \]

As \( n \) becomes large and approaches \( \infty \), \( \{1/(n-1)!\} \) approaches 0.

Then

\[
\begin{align*}
&\{(n-2)/(n-1)!\} + \{(n-3)/(n-2)!\} + \{(n-4)/(n-3)!\} + \ldots \ldots 4/5! + 3/4! \\
&+2/(3!) + 1/(2!)
\end{align*}
\]

\[\approx 1 \]
Rewriting and replacing (n-2) by N we derive that the sum of the infinite series 
\(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{N}{(N+1)!}\) ≈ 1

This is a well documented convergent series whose approximate sum we have derived using a novel strategy and the sum equals unity. Its sum may represent the sum of probabilities of selecting any one of (N-1) permissible arrangements of possible N! arrangements for an infinite large number of objects that have been grouped into infinite sets with increasing number of objects from 2 to N and likely to have applications in fields such as chemistry, astronomy, etc.

Reference:

1. The Factorial function and Generalization 
   (Manjul Bhargava)

2. An Intuitive Conceptualization of n! and Its Application to Derive a Well Known Result 
   (Prashanth R. Rao)
   vixra.org>Number theory>1503 (March 2015)