An Elementary Proof of Gilbreath's Conjecture

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Abstract

Given the fact that the Gilbreath's Conjecture has been a major topic of research in Aritmatic progression for well over a Century,and as bellow:

2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61
1 2 2 4 2 4 2 4 6 2 6 4 2 4 6 6 2
1 0 2 2 2 2 2 2 4 4 2 2 2 2 0 4
1 2 0 0 0 0 0 2 0 2 0 0 0 2 4
1 2 0 0 0 0 2 2 2 2 0 0 2 2
1 2 0 0 0 2 0 0 0 2 0 2 0
1 2 0 0 2 2 0 0 2 2 2 2
1 2 0 2 0 2 0 2 0 0 0
1 2 2 2 2 2 2 2 0 0
1 0 0 0 0 0 0 2 0
1 0 0 0 0 0 2 2
1 0 0 0 0 2 0
1 0 0 0 2 2
1 0 0 2 0
1 0 2 2
1 2 0
1 2
1

The Gilbreath's conjecture in a way as easy and comprehensive as possible.
He proposed that these differences, when calculated repetitively and left as bsolute values, would always result in a row of numbers beginning with 1,In this paper we bring elementary proof for this conjecture.

Introduction:

Given the fact that the Gilbreath's Conjecture has been a major topic of research in aritmatic progression for well over a century, the Gilbreath conjecture in a way as easy and comprehensive as possible.
Hopefully it will help the right person take this conjecture out of the unsolved

list and into the list of accomplishments of mathematics.

To begin the story, the anecdote goes that an undergraduate student named norman Gilbreath was doodling on a napkin one day in a cafe and found a very interesting characteristic of the list of sequential prime numbers and the diferences between them. He proposed that these diferences, when calculated repetitively and left as absolute values, would always result in a row of numbers beginning with 1 (after the first row). No one has been able to prove it.

In 1878, eighty years before Gilbreath's discovery, François Proth had, however, published the same observations along with an attempted proof, which was later shown to be false.

Andrew Odlyzko_verified that d_1^k is 1 for $k \leq n = 3.4 \times 10^{11}$ in 1993, but the conjecture remains an open problem. Instead of evaluating *n* rows, Odlyzko evaluated 635 rows and established that the 635th row started with a 1 and continued with only 0's and 2's for the next *n* numbers. This implies that the next *n* rows begin with a 1,see[15]

Notation

We define d_n^k is K th row, n th, Number, in $d_n^k = \left|d_{n+1}^{k-1} - d_n^{k-1}\right|$

We should prove that $d_1^k = 1$, for any k

Theorem: $d_1^k = 1$,for any k

Proof: Assume that the Gilbreath's Conjecture is correct until p_m ,that is m- th prime in first row by induction, we prove that this Conjecture is correct for p_{m+1} ,hence below table is correct by induction

2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61…….p_{m-2} p_{m-1} p_m
1 2 2 4 2 4 2 4 6 2 6 4 2 4 6 6 2
1 0 2 2 2 2 2 2 4 4 2 2 2 2 0 4
1 2 0 0 0 0 0 2 0 2 0 0 0 2 4

1 2 0 0 0 0 2 2 2 2 0 0 2 2
1 2 0 0 0 2 0 0 0 2 0 2 0
1 2 0 0 2 2 0 0 2 2 2 2
1 2 0 2 0 2 0 2 0 0 0
1 2 2 2 2 2 2 2 0 0
1 0 0 0 0 0 0 2 0
1 0 0 0 0 0 2 2
1 0 0 0 0 2 0
1 0 0 0 2 2
1 0 0 2 0
1 0 2 2
1 2 0
1 2
1
.
.
.
1

Notice that in above table for K- th row, n- th, number,we have $d_n^k = \left|d_{n+1}^{k-1} - d_n^{k-1}\right| < 2^n \leq 2^m$,Now we prove that this table is correct for p_{m+1},

For simplicity this conjecture we state some Lemmas as below:

Lemma 1: if p_m to be m-th prime ,so $p_m < 2^m$ for $m > 1$

Proof: According to [1] ,this is Correct obviously

Lemma 2: the Second row is correct ,i.e $d_m^2 = |p_{m+1} - p_m| < p_m \leq 2^m$

Proof, this is obviously correct by refer to [1]

Lemma 3: the third row is correct ,i.e $d_{m-1}^3 = \left|\, |p_{m+1} - p_m| - |p_m - p_{m-1}| \right| < p_{m-1} \leq 2^{m-1}$,

Proof :this is obviously correct by refer to [1]

Lemma 4: k th row is correct, $4 \leq k \leq m+1$,i.e $d_{m-(k-2)}^k = \left|d_{m-(k-3)}^{k-1} - d_{m-(k-2)}^{k-1}\right| < 2^{m-(k-2)}$

Proof: we assume that this is not hold for $k \geq 4$,notice that from $k = 4$ to $k = m + 1$, we have $d_{m-(k-2)}^{k} = \left|d_{m-(k-3)}^{k-1} - d_{m-(k-2)}^{k-1}\right| \geq 2^{m-(k-2)}$

So for simplicity we write abbreviation as below:

$a_1 = a - b > 2^{m-2}$

$a_2 = a_1 - b_1 > 2^{m-3}$

.

.

.

$a_{m-2} = a_{m-3} - b_{m-3} > 2^{(m-1)-(m-2)}$

We add above formula ,hence: $a_1 + a_2 + \cdots + a_{m-2} > 2 + 2^2 + \cdots 2^{m-2}$

But each item is smaller than a, and $a < p_{m-1}$

So $(m-2)p_{m-1} > 2^{m-1} - 2$

Or $p_{m-1} > \frac{2^{m-1}-2}{m-2}$, according to [1], there are constants numbers c_1&c_2 such that $c_1(m-1)\log(m-1) < p_{m-1} < c_2(m-1)\log(m-1)$

So this is Contradiction for $m \geq 8$

Lemma 5: suppose the s-th row is correct $s \geq 4$,so $s + 1 \leq k \leq m + 1$, we prove that $d_{m-(k-2)}^{k} = \left|d_{m-(k-3)}^{k-1} - d_{m-(k-2)}^{k-1}\right| < 2^{m-(k-2)}$

Proof : this proof is similar to Lemma 4, by substitute m instead $m - s$

So $p_{m-s-1} > \frac{2^{m-s-1}-2}{m-s-2}$, this is Contradiction for $m - s \geq 8$

Theorem: $d_1^k = 1$,for any k

Proof:According to above Lemmas 3 &4 this theorem is hold, for $k = m + 1$

$d_1^{m+1} \leq 2^5 - 1$,

i.e 5 row from the bottom this lemma is correct, s for $k = m + 1$, $d_1^{m+1} \leq 2^5 - 1$,if $b = b_1 = b_2 = \cdots = b_{m-3} = 0$,we have $p_6 > \frac{5\times2\hat{}5}{4}$ or $13 > 40$

and this is contradiction, for 4,3,2 rows from the bottom we reach to contradiction, so the second row from the bottom i.e $d_2^m \leq 2$ then we have for $k = m + 1$, $d_1^{m+1} < 2$,so $d_1^{m+1} = 1$,therfore we proof the Theorem.

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