On the geometry of Space

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Abstract

In this paper, we explore the possibility that black holes and Space could be the geometrically Compactified Transverse Slices ("CTS"s) of their higher (+1) dimensional space. Our hypothesis is that we might live somewhere in between partially compressed regions of space, namely $4\mathrm{d}_{L+R}$ hyperspace compactified to its 3d[1] transverse slice, and fully compressed dark regions, i.e. black holes, still containing all $_L$d432-1-234d$_R$ dimensional fields. This places the DGP, ADD, Kaluza-Klein, Randall-Sundrum, Holographic and Vanishing Dimensions theories in a different perspective.

We first postulate that a black hole could be the result of the compactification (fibration) of a 3d burned up S^2 star to its 2d transverse slice; the 2d dimensional discus itself further spiralling down into a bundle of one-dimensional fibres.

Similarly, Space could be the compactified transverse slice (fibration) of its higher $4\mathrm{d}_{L+R}$ S^3 hyper-sphere to its 3d transverse slice, the latter adopting the topology of a closed and flat left+right handed trefoil knot. By further extending these two ideas, we might consider that the Universe in its initial state was a "Matroska" $4\mathrm{d}_{L+R}$ hyperspace compactified, in cascading order, to a bundle of one-dimensional fibres. The Big Bang could be an explosion from within that broke the cascadingly compressed Universe open.

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[1]As we shall only consider spatial dimensions and leave the time dimension out, ordinary 3d+1t spacetime will be reduced to '3d' space; 4d+1t spacetime will be denoted '4d' space, etc.

1 Introduction

Bekenstein-Hawking entropy indicates that a black hole could be the 2d geometrically Compactified Transverse Slice ("CTS") of its higher 3d dimension.

Existing papers and recent data from the WMAP and Planck satellites, on the other hand, might indicate that 'ordinary' 3d space could itself be the geometrically Compactified Transverse Slice of its higher $4d_{L+R}$ hyperspace.

By further extending these two ideas, we might consider that Space in its initial state was a "Matroska" $4d_{L+R}$ hyperspace compactified, in cascading order, to a collection of one-dimensional fibres.

The postulate is made that the Big Bang could be an explosion from within that broke the cascadingly compactified Space open.

From the initial compactified or *"zipped"* point of view, we say that the Universe started as a $4d_{L+R}$ hypersphere compactified into (3d+1) Minkowski space which, as a "Matroska", was further compactified (collapsed compressed) into a 2d brane finally resulting in a bundle of 1d dimensional fibres in which time behaved as another direction in space.

Uncompressing or *"unzipping"*, we then say that the Big Bang is the unfolding of the compactified 1d bundle of fibres within a less compactified 2d sheet that immediately thereafter inflated to the 3d space we live in. Our 3d space is itself the Compactified Transverse Slice of its higher $4d_{L+R}$ hypersphere.

Black holes might well be reminiscences of the original compactified state that still include all dimensional fields, from $4d_{L+R}$ to 1d included. Black holes are places where the compactified Space did not open, or opened when it was hot, but, when cooling down, collapsed again to their original state. Formulated in a different way, a black hole might be a $4d_{L+R}$(x,y,z,w) coordinate system collapsed and compactified to its 3d(x,y,z) transverse slice coordinate system, spiralling down into a 2d(x,y) transverse slice coordinate system, spiralling down into a bundle of 1d fibres. To be consistent with our approach, it would be better to describe black holes, apart from transverse slices, as Bose-Einstein condensates, but that is a different story for later[1].

We might thus live in between partially compactified regions of 3d space embedded into $4d_{L+R}$ and dark regions or black holes still containing all compactified dimensions (i.e. $4d_{L+R}$, 3d, 2d and 1d). This places the Dvali-Gabadadze-Porrati (DGP)[2], Arkani-Hamed, Dimopoulos and Dvali (ADD)[3], Kaluza-Klein[4], Randall-Sundrum[5], Holographic and Vanishing Dimensions[6] theories in a different perspective.

2 Black holes as the geometrically Compactified Transverse Slice of a 2-sphere S^2

2.1 The Bekenstein-Hawking entropy of a Schwarzschild black hole

Between 1916 and 1917, K. Schwarzschild[7, 8], but also J. Droste[9], H. Weyl[10] and D. Hilbert[11], found, based on Einstein's 1915 publication of his General Relativity Theory, what was later called the *"Schwarzschild radius"*[12].

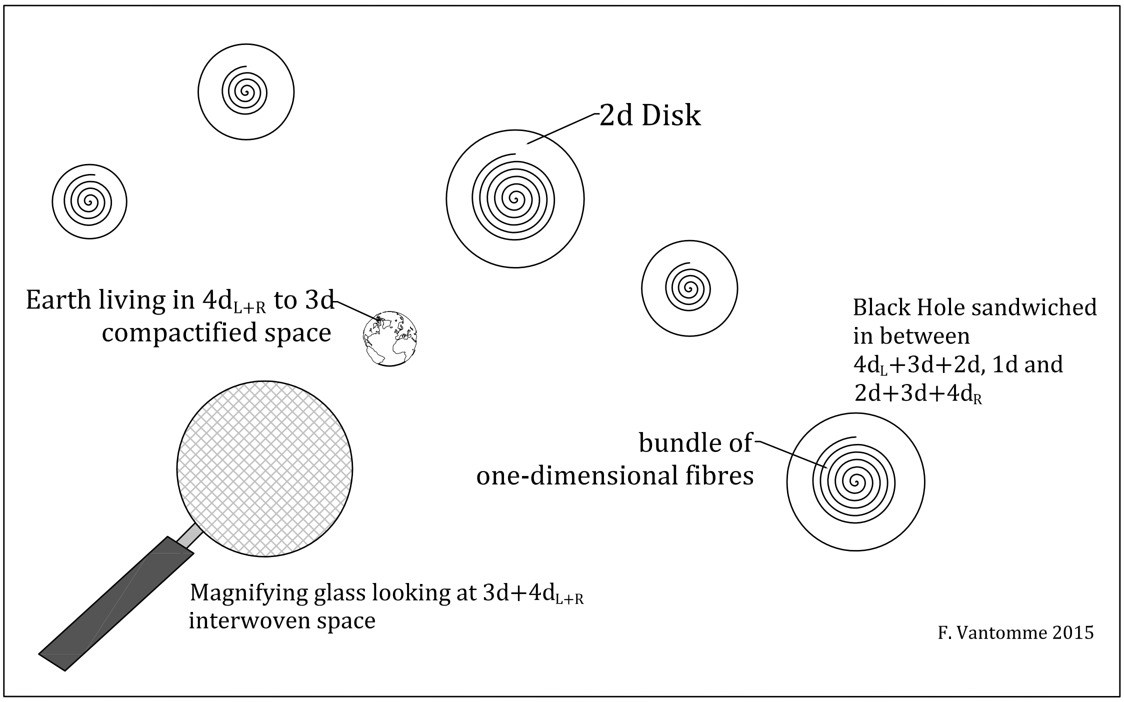


Figure 1: Top 2D view from within. We might live in between original dark regions or black holes still containing all compactified dimensions $4d_L$32-1-23$_R$4d and partially compactified - or partially opened - regions containing 34d_{L+R} dimensions. Thus, not only Space gets contracted but - more importantly - *the dimensions themselves are being contracted.*

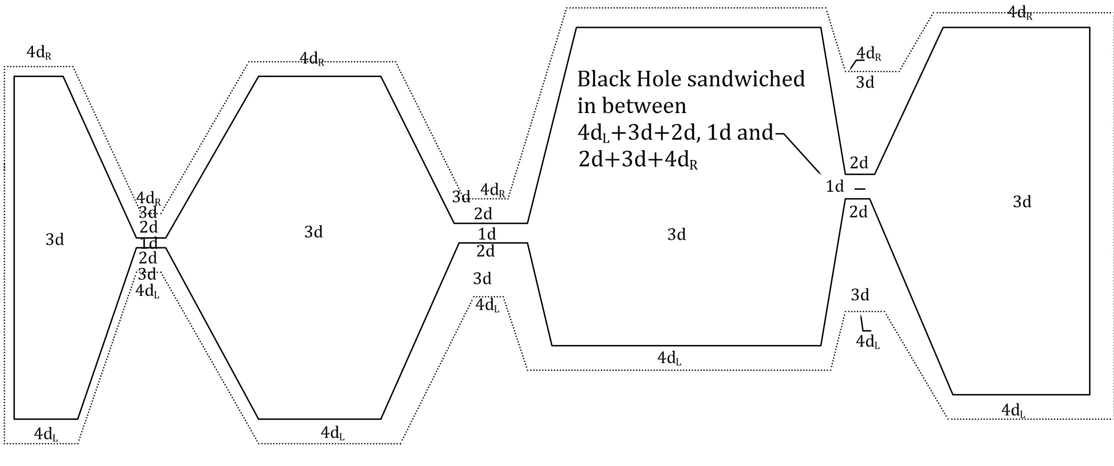


Figure 2: Schematic front 2D view. $4d_{L+R}$ hyperspace is everywhere compactified to its 3d tranverse slice and in punctual black hole regions even further compactified to $4d_L$32-1-23$_R$4d space. These are singularities with a surrounding 2d discus embedded in 34d_{L+R} space.

An object with a mass M that under its own gravitational forces is compactified to fit into a certain critical radius, shall form a black hole within which the Newtonian escape velocity equals the speed of light. This critical radius is called the Schwarzschild radius and is given by the well-known equation:

$$r_{Sch} = \frac{2GM}{c^2} \tag{1}$$

The minimum mass M for a stellar black hole to form is now set at about 2 to 3 solar masses. Let us, by way of example, take 3 solar masses and mould them together to create one bigger sun with a mass of 3 x 1.988435.10^{30} kg. At the end of this $3M_\odot$ star's life, when almost all nuclear fuel has burned up and gravity overcomes the nuclear forces, the dying star will undergo a supernova explosion and collapse to form a black hole. The above equation states that this black hole shall have a radius of about 8,859 km.

In 1972, J. Bekenstein[13], later complemented by S. Hawking[14], discovered that it is possible to calculate the entropy of the *interior* region of such a Schwarzschild black hole. The Bekenstein-Hawking entropy is given by:

$$S_{BH} = \frac{k_B A}{4 l_p^2} \tag{2}$$

whereby $A = 4\pi r^2$ and $l_p^2 = \frac{G\hbar}{c^3}$

$$S_{BH} = \frac{k_B (4\pi r^2) c^3}{4\hbar G} \tag{3}$$

In our example, this results in a S_{BH} of about 1.303 x 10^{55} J/K (note[2]).

Strangely, the Bekenstein-Hawking entropy of the black hole horizon's interior region is related to the *surface area* $4\pi r^2$ of our previous $3M_\odot$ stellar 2-sphere S^2 (this is a normal 3-dimensional sphere) and thus not to the *volume* $\frac{4}{3}\pi r^3$ of our collapsed $3M_\odot$ star.

Even more remarkably, the Bekenstein-Hawking entropy, measured in Planck areas, is reduced to $\frac{1}{4}$ of the 2-sphere's surface, which is the same as the two dimensional transverse slice area of S^2, namely (πr^2). This points out that a black hole horizon's interior region could well be a genuine two-dimensional discus.

2.2 Dimensional compactification

Actually, various authors have already pointed out that a black hole could well be a *de facto* two-dimensional discus. Others have called our attention to the fact that the 2d discus might be further reducible by spiralling it down into a collection of 1d fibres. It is important to note that this, as far as we can see, differs from the current mainstream

[2]For the sake of clarity, we consider hereby only the black hole's interior Bekenstein-Hawking entropy, i.e. within the event horizon, *not* its exterior entropy outside the event horizon.

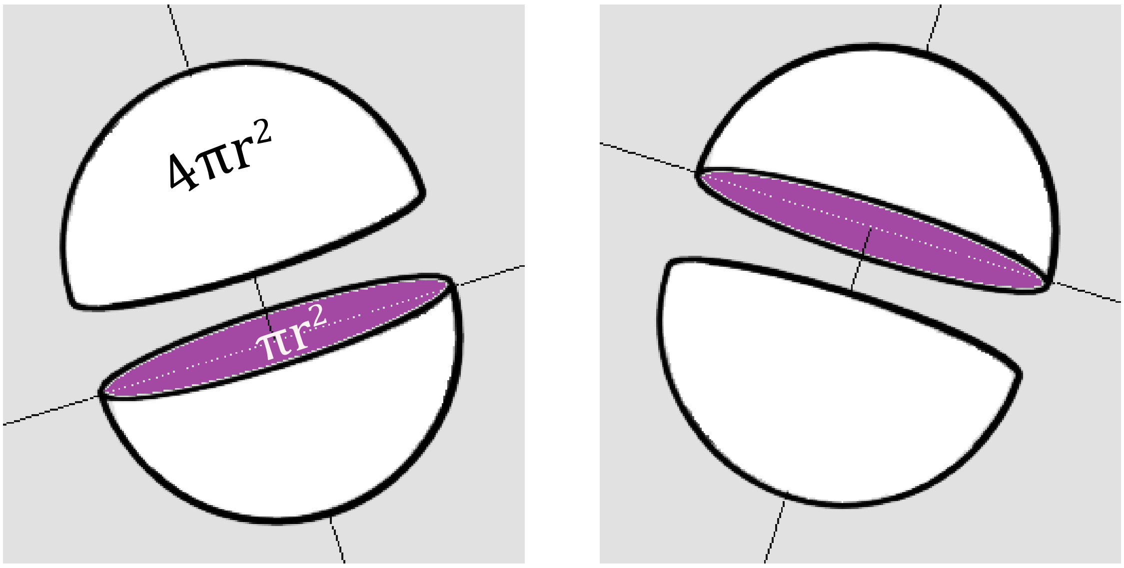


Figure 3: The surface area of a 2-sphere S^2 equals $4\pi r^2$. The transverse slice area equals πr^2. You notice that the horizontal slice of S^2 are two D^2 disks glued together, which correspond in S^3 to two solid tori glued together along their boundaries.

thought that a black hole collapses directly from 3d to a single 1d singularity, i.e. without first passing through the 2d discus.

G. 't Hooft[15, *1999, The Holographic principle: Opening lecture*] proposed the idea that particles close to a black hole's horizon can be accurately described by a 2d function instead of a 3d function (we will call this the 'limited holographic principle'). He then added that this holographic description of particles could well have a universal validity and might apply for more than only particles entering into a black hole. This extended idea that the entire Universe might be two-dimensional was further expanded by, amongst others, L. Susskind[16, *1994, The world as a hologram*] (we will call this the 'extended holographic principle').

The limited holographic principle is, in our view, a correct but partial description of reality, while, on the other hand, the extended holographic principle needs to be reconsidered and put into a different perspective.

In a paper from 1992, Bañados, Teitelboim and Zanelli (BTZ)[17] showed that a black hole solution can be given that consists of two-dimensional fields. In 1995, S. Carlip[18, *1995, The (2+1)-dimensional black hole*] pointed out that such a BTZ black hole has different characteristics than a 'normal' Schwarzschild or Kerr black hole: (1) the BTZ metric has constant negative curvature, any point in the black hole space time has a neighbourhood isometric to Anti-de Sitter space and *(2) it has no curvature singularity at the origin.* But the BTZ black hole still has two interesting characteristics that are similar to an 'ordinary' black hole: it appears as the final state of collapsing matter and it has thermodynamic properties much like those of a 3d black hole.

Continuing in this line of reasoning, there are recent hints that the dimensional fields near the black hole's horizon can be even further reduced to a set of two-dimensional fields (sheets or branes) spiralling down into a bundle of one-dimensional fibres[19, 20, *J.D. Bekenstein and A.E. Mayo (2001), Black holes are one-dimensional*]. The dimensional fields (3d+1), (2d+1) and (1d+1) seem thus to converge near the black hole's horizon[21, *S. Carlip (2012), Spontaneous dimensional reduction?*]. Thus, not only Space itself gets contracted but - more importantly - *the dimensions themselves are being contracted.*

We will now argue that Space we live in might be, in an equivalent way, the geometrically Compactified Transverse Slice ("CTS") of its higher (+1) dimensional space, being the 4-dimensional$_{L+R}$ hypersphere.

3 Space as the geometrically Compactified Transverse Slice of a 3-sphere S^3

Proceeding one step further, we now pose the question how Space would look like if Space itself was a Schwarzschild black hole or, in line with our hypothesis, better expressed as the geometrically Compactified Transverse Slice of a 3-sphere S^3.

3.1 The Transverse Slice of a 3-sphere S^3: the Clifford torus

As already mentioned above, the transverse slice that is formed once our almost burned out $3M_{\odot}$ stellar 2-sphere S^2 collapses under its own gravity, is equal to the *area* of a simple circle in the plane (i.e. a 1-sphere S^1) with area πr^2.

This is $\frac{1}{4}$ of the surface area of S^2 ($4\pi r^2$). S^2 is a two-dimensional manifold folded into a 3d ball. The unit 2-sphere S^2 is given by the following equation:

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \tag{4}$$

Per analogy, S^3 is a 3-dimensional manifold folded into a 4-dimensional hypersphere. The unit 3-sphere S^3 is given by one additional dimension (w):

$$S^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\} \tag{5}$$

Equally per analogy, the (complex) transverse slice of S^3 is equal to $\frac{4}{4}$ of its surface area, or, the hyper-surface area itself.

Now, the equation for the hyper-surface area of the hypersphere is not so easily found by analogy, as H.S.M. Coxeter nicely described in his book "Regular Polytopes"[22]: *"For instance, seeing that the circumference of a circle is $2\pi r$, while the surface of a sphere is $4\pi r^2$, we might be tempted to expect the hyper-surface of a hyper-sphere to be $6\pi r^3$ or $8\pi r^3$. It is unlikely that the use of analogy, unaided by computation, would* *ever lead us to the correct expression, $2\pi^2r^3$"*. Contrary to the hyper-surface area, the hyper-volume of S^3 is given by $\frac{1}{2}\pi^2r^4$.

Note that the equation $2\pi^2r^3$ for the hyper-surface area of the hypersphere is exactly the same as the equation for the interior volume of a torus, which is $2\pi^2Rr^2$, always on condition that R = r, i.e. this torus must be formed by the product of circles with the same radii. Such a torus formed by the product of constantly equal radii is a flat and equatorial Clifford torus.

The 3-sphere S^3 is given by:

$$S^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\} \tag{6}$$

The Clifford torus is given by:

$$\left\{(x, y, z, w) \in S^3 : x^2 + y^2 = z^2 + w^2 = \frac{1}{2}\right\} \tag{7}$$

One Clifford torus within S^3 is:

$$x^2 + y^2 = \frac{1}{2} \tag{8}$$

The other Clifford torus within S^3 is:

$$z^2 + w^2 = \frac{1}{2} \tag{9}$$

The Clifford torus thus slices S^3 into two completely identical solid (Clifford) tori glued together along their boundaries with the latitudes of one pasted to the longitudes of the other[23, 24].

M. Dehn[25], looking at this from a different view angle, noted that the middle torus laying at the equatorial transverse slice of S^3 could be cut out, removed from S^3 and sewn in back differently (later named *'Dehn surgery'*)[26]. Now, such a torus removed from the equatorial transverse slice is in itself a Clifford torus. The same was observed by P. Heegaard. He too saw, from yet another viewpoint, that the standard genus 1 splitting (or *Heegaard splitting*) of S^3 similarly results in a Clifford torus. Otherwise said, the boundary where the two solid Clifford tori meet, this is the (hyper-)equatorial transverse slice of S^3, is equally another, third, middle Clifford torus. It is hereby to be noted that the Lawson Conjecture, proven in 2012 by S. Brendle[27], states that nothing but the Clifford torus is the embedded minimal surface in S^3 of genus 1[28].

3.2 The Compactification of a 3-sphere S^3 through the Hopf Fibration

We shall now explain how we could approach the *compactification* of S^3 to its transverse slice. In 1931, H. Hopf invented a way of mapping - in mathematics known as a *submersion* - the higher dimensional space S^3 to a lower dimensional space S^2, the so-called Hopf Fibration or Hopf Map[29]. We shall use this continuous map (or continuous function): $\pi : S^1 \to S^3 \to S^2$ to describe the compression of S^3 to S^2.

The Hopf Fibration has already been broadly described in mathematical and physics literature[30]. We shall therefore only give a recapitulation which we will then complement with some distinct insights. It is hereby worthy to note that during recent years, the idea has re-emerged that the purely mathematical idea of the (Hopf) Fibration might correspond to a true physical reality, a.o. as a possible technique to describe the quantum states of a single qubit and the entanglement between two qubits[31, 32].

A Fibration, in general, consists of three components: a base space (B), a fibre (F) and a total space (E), all together completed with a continuous map $\pi : F \to E \to B$. The base space (B) is made of points $b \in B$. Every point b has an open neighbourhood (U). The inverse pre-image of all points b and U, notated, respectively, as $\pi^{-1}(b)$ and $\pi^{-1}(U)$, is a fibre (F). The Fibration tool lets us represent the higher dimensional total space (E) on a lower dimensional space (B) with each point $b \in B$ having a one-to-one corresponding fibre (F) attached to it, for $b \in B$, $F_b := \pi^{-1}(b)$ is called the fibre over b.

A Fibration can be trivial or non-trivial. A trivial Fibration exists when the total space E = B x F. Euclidean space $\mathbb{R}^3$ is an example of a trivial Fibration because the total space $\mathbb{R}^3_{(E)} = \mathbb{R}^1_{(B)}$ x $\mathbb{R}^2_{(F)}$ or $\mathbb{R}^2_{(B)}$ x $\mathbb{R}^1_{(F)}$ whereby, in the first case, the fibres (F) are 2-dimensional parallel planes and, in the second case, the fibres are one-dimensional parallel lines.

The Hopf Fibration is an example of a non-trivial Fibration because $S^3 \neq S^2$ x S^1 and, being a specific type of Fibration, it can similarly be written as
$\pi : S^1_{(F)} \to S^3_{(E)} \to S^2_{(B)}$. The base space S^2 is made of particle-points $b \in B$. The inverse pre-image of all compressed particle-points $b \in S^2$, which is obtained by uncompressing these points b from S^2 to $\mathbb{R}^3$ and then by doing an inverse stereographic projection to S^3, is a one-dimensional Hopf circle that in itself is made of points $p \in S^1_{(F)}$.

We can consider the link between S^3 and S^2 therefore even better as a many-to-one relationship because many points p on the fibre circle S^1 embedded in S^3 correspond to one single point $b \in S^2$. N. Johnson published a nice video on the internet[33] (around minute 39 and further) that helps to visualize this concept: every *point* in S^2 corresponds to a Hopf circle embedded in S^3. *Two points* in S^2 correspond to a Hopf link in S^3. Moving on, an *arc* (this is a part of a circle) corresponds to a Hopf band. A complete *circle* corresponds to an empty torus (T^2) and a *disk* D^2 in base space S^2 corresponds to a solid torus (D^2 x S^1). The transverse slice of S^2 are two D^2 disks glued together, which correspond in S^3 to two solid tori glued together along their boundaries.

We visualize the Bloch sphere[34] - or, better said, the Bloch "hyper-" sphere - as an imagery shell in S^3, its surface through which the S^1 Hopf circle freely rotates. The possible states of a single qubit can be represented by any point p on the S^1 Hopf circle (F). Given that point p is only allowed to move along the one-dimensional surface of S^1 and S^1 is in itself confined to a two-dimensional plane, this plane embedded within the Bloch S^3 hyper-sphere, point p must be described by 4 coordinates $x^2+y^2+z^2+w^2 = 1$ or as a pair of two complex numbers $z = x + iy$ and $w = z + iw$ whereby $S^3 =$

$\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. The Hopf Fibration gives us a technique to represent points $p \in S^1 \in S^3$ (4 coordinates) as points $b \in S^2$ (3 coordinates):

$$
\begin{aligned}
&2(xy + zw) = x \\
&2(xw - yz) = y \\
&(x^2 + z^2) - (y^2 + w^2) = z
\end{aligned}
$$

Formulated in the more familiar linear superposition of the basis states $|0\rangle$ and $|1\rangle$, the possible states where one can find point p within the quantum qubit space in S^3 is given by: $|\psi\rangle : \alpha|0\rangle + \beta|1\rangle$ whereby α and β are complex probability amplitudes that are trapped within the equation $\{(\alpha,\beta) \in \mathbb{C}^2 \mid |\alpha|^2 + |\beta|^2 = 1\}$ which is equal to $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. The below table gives a schematic résumé:

Total space *(E)*		**Base space *(B)***
***S^1* embedded in *S^3* (4d)**		***S^2* (3d)**
Pre-image in *S^3*		**Image in *S^2***
S^1 Hopf circle rotating within the Bloch hyper-sphere		1 point-like particle
S^1 x *S^1* Hopf link	Compactified to	2 point-like particles
Locations are defined by *x, y, z, w* 4-tuples (or quadruples) or as a pair of two complex numbers $z = x + iy$ and $w = z + iw$		Locations are defined by *x, y, z* triples

Just out of curiosity, we would like to mention the remarkable achievement made in 2013 by a research team from the Technischen Universität Darmstadt[35] in Germany. The researchers were able to trap photons inside a crystal atom that was previously cooled down with lasers to form a Bose-Einstein condensate, thus extracting all motion-energy from the crystal atoms, only leaving their rest mass m_0. In terms of the foregoing, a possible interpretation of this phenomenon could be as follows. At room temperature, the S^1 Hopf circles rotate very fast along the surfaces of their corresponding Bloch hyper-spheres. The fundamental particles that make up the crystal atom (quarks, gluons, electrons, etc.) can all be considered points $b \in S^2$ with to each of them a S^1 Hopf circle attached. When the temperature of the crystal atoms reach absolute zero (kelvin) and form a Bose-Einstein condensate, the S^1 Hopf circles freeze to a standstill. At that moment, the photons, which are also Hopf circles in S^3, are beamed into the outer "open" crystal atom fundamental particles' Hopf circles. When the laser cooling is taken away and the crystal atom's Hopf circles start rotating again, the heavily bouncing photons are trapped inside the outer-shell S^1 Hopf circles and the photon's motion-energy is given of throughout the chain of crystal atoms.

Another curiosity is that for every point p on the Hopf circle $S^1 = \{(x,y) : x^2 + y^2 = 1\}$, there exists a corresponding fibre over p that can be written as another type of Hopf Fibration $\pi : S^0 \to S^1 \to S^1$, whereby the fibre $S^0 = \{(x) \in \mathbb{R}^1 : x^2 = 1\}$. The basis states of a qubit might then be represented as the collapse of S^1 to its transverse slice S^0. The points x on S^0 can only have the value of x^2, namely -1 or 1, laying at the "West pole" and "East Pole" of the X axis, or at the "North pole" and "South pole" of the Z axis.

3.3 The 120-cell meets both criteria

We will now look for a geometric shape that fits both criteria 3.1 (Transverse Slice) and 3.2 (Compactification).

3.3.1 Criterium 3.1: Clifford Tori

Section 3.1 requires that the figure sought must be formed by two solid Clifford tori and that the boundary where these two tori meet, this is at the (hyper-)equatorial transverse slice of S^3, must equally be another, third, middle Clifford torus. Section 3.2 requires that the figure be a collapse-compression and, ideally, a fibration to represent the dimensional reduction.

At first view, the best candidate for this position seems to be the 120-cell. In the interest of brevity and to avoid repeating what already exists in literature, we refer to Chapter 9 of H.S.M. Coxeter's book *Twisted Honeycombs*[36] and we found an even more visually appealing description in the book of G. Toth *Glimpses of Algebra and Geometry, Second Edition*[37], on page 386:

"The explicit construction of the 120-cell is technical. There is an easy way, however, to see how these 120 dodecahedra fit together in S^3, and it is based on the fact that, up to adjustment by an isometry, the centroids of the dodecahedral cells can be considered as the 120 elements of the binary icosahedral group $\boldsymbol{I}^*$ in S^3 discussed in Section 23. Recall that in terms of the Clifford decomposition of S^3, $\boldsymbol{I}^*$ is made up of the vertices of two regular decagons inscribed in the orthogonal circles $C_{\pm 1}$, and the rest appear (in two groups of 50) in the Clifford tori $C_{\pm 1/\sqrt{5}}$ (Figure 23.5). In view of this, the 120-cell can be constructed as follows. First make a necklace of 10 (spherical) dodecahedra such that the centroids of the dodecahedra are the 10 elements of $\boldsymbol{I}^*$ on C_1. It turns out that these dodecahedra have dihedral angle 120° as above. A pair of consecutive dodecahedra in the necklace are pasted together at a common pentagonal face. Each of the 5 edges of this common face is the shared edge of two other pentagonal faces, one from each of the consecutive dodecahedra. These two faces meet at a dihedral angle of 120°, so that another dodecahedron can be pasted in. Since we have five edges (of the common pentagonal face), we can paste in 5 extra dodecahedra around the two consecutive dodecahedra in the necklace. This cluster of 5 dodecahedra makes a bulge in the necklace. Since the necklace has 10 places (of consecutive dodecahedra) for this construction, we can add 10 bulges to the necklace, a total of 10 5 dodecahedra. These, along with the original 10 dodecahedra, use up 60 dodecahedra, and give a bumpy polyhedral Clifford torus in S^3. As computation shows, the centroids of the dodecahedra in the bulges make up the 50 elements of $\boldsymbol{I}^*$ in $C_{1/\sqrt{5}}$. Finally, the entire construction can be repeated for C_{-1} and $C_{-1/\sqrt{5}}$, and the two

bumpy Clifford tori fit together to form the 120-cell. (This visualization of the 120-cell also reveals that the faces of each dodecahedral cell are contained in the perpendicular bisectors of the line segments connecting the centroid (in $\boldsymbol{I}^*$) of the cell and the 2+5+5 nearby elements in $\boldsymbol{I}^*$.)"

and further on page 313:

"Removing the middle torus C_0 from S^3, we see that S^3 falls into the disjoint union of two solid tori. Going backward, we reach the inevitable conclusion: The 3-sphere is obtained from two solid tori by pasting them together along their boundaries!"

Thus, one ring-shaped complex of 60 dodecahedra forms a Clifford torus, as do the other 60 dodecahedra. The two interlocking 60 dodecahedra (Clifford tori) form a 120-cell which in itself is a Clifford torus. What is more, the 120-cell comes with a bonus because just as 12 pentagons tile the surface of a 2-sphere S^2, 120 dodecahedra tile the hyper-surface of a 3-sphere S^3. However, to do the trick, one must inflate the dihedral angles of each of the 120 dodecahedral cells by nearly 4° degrees. In ordinary 3d Euclidean space, the dihedral angles of a dodecahedron are about $116°,34'$, while in 4d hyperspace the angles are bended to 120°. These 120 "round balloon cells" tessellate the hyper-surface $2\pi^2 r^3$ of the hyper-sphere S^3. As we shall see in Section 3.4 below, this inflation exists for a well-deserved reason.

J.P. Luminet[38, 39, 40], J. Stillwell[41] and J. Weeks[42, 43] are amongst the researchers[44, 45, 46] that are testing this Poincaré Dodecahedral Space model against the data obtained from the NASA WMAP (Wilkinson Microwave Anisotropy Probe) and ESA Planck satellites that image the extremely tiny anisotropies in the Cosmic Background Radiation Field or CMB (Cosmic Microwave Background). Given that the 120-cell is a complex geometric figure, in his paper "Plato, Poincaré and the Enchanted Dodecahedron", the mathematician L. Brenton[47] naturally asks the question whether we are visiting many different (dodecahedral-cell) rooms or the same room over and over again? We shall now see that the 120-cell is not the end of the story, which favours the "many different rooms" hypothesis.

3.3.2 Criterium 3.2: Compactification through fibration

Next in line is the compactification through fibration. Although it is not the same as the Hopf Fibration described under Section 3.2, we say that the 120-cell forms a Hopf-like Fibration because the 120-cell can be partitioned into 12 intertwined rings of 10-cell rings. Furthermore, the dual of the 600-cell is the 120-cell. A dual polytope means that the vertices of one polytope touch the faces of the other at one single coordinate in space. The centres of the 120 dodecahedral cells are the 120 vertices of its dual, the 600-cell, as can bee seen from their Schläfi symbols (120-cell: $\{5, 5, 3\}$ and for the 600-cell $\{3, 3, 5\}$).

In any case, the puzzle of the compactification through fibration of the 120° 120-cell falls better in place when we now come to the conclusion that the Poincaré Dodecahedral Space is nothing more than the expression of the underlying closed left+right-handed trefoil knot.

3.4 The underlying concept: the closed left+right-handed trefoil knot as the transverse slice of S^3

There exists an intimate relationship between Poincaré Dodecahedral Space and the trefoil knot. To present things in a more visually understandable way, let us represent S^3 as an apricot that we will cut in two halfs (see Figure 4 on the next page).

As we need to cut 4d space, we will use a knife with a special surface: the complex surface $z^2 + w^3 = 0$ (z and w being complex numbers) which follows the pattern of a trefoil knot but - behold - is different from the trefoil knot. The trefoil knot is the inner-stone or kernel that is obtained by slicing the S^3 apricot open. For an apricot to be able to be sliced, you must have a left- and a right-half. As seen above, S^3 can be written as a pair of complex numbers $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. If we now intersect our apricot S^3 with the complex surface knife ($z^2 + w^3 = 0$) we obtain a left-handed trefoil laying on the "left" Clifford torus $x^2 + y^2 = \frac{1}{2}$ (see (8) above) and a, mirrored, right-handed trefoil laying on the "right" Clifford torus $z^2 + w^2 = \frac{1}{2}$ (see (9) above) within S^3 that, glued together, form the total kernel.

To put it differently, when the left-handed trefoil, laying on the $x^2 + y^2 = \frac{1}{2}$ torus, and the right-handed trefoil, laying on the $z^2 + w^2 = \frac{1}{2}$ torus, are merged (compactified) into a third middle torus, then there will be coordinates in space where the two trefoils will cross and touch (see Figure 5 below). If, after the merger, the middle-torus is further compressed and finally taken away, the "left-over" will be a left+right handed trefoil knot glued together - now not only at their cross-touching coordinates, but - along their entire boundaries. In other words, with the help of the complex surface knife, we have been able to transverse slice or cut a closed left+right-handed trefoil out of S^3.

Dodecahedral space, given by the equation $z^2 + w^3 + p^5 = 0$, and thus very closely related to our complex surface knife ($z^2 + w^3 = 0$), can be seen in Figure 4 as it were the outer-stone that falls of when S^3 is sliced in two by the complex surface knife leaving the left+right-handed trefoil kernel visible. If you punch a hole in the center of two opposite pentagonal faces of each of the 120 dodecahedral cells and connect this 5-fold axis all the way long, the axis will become the inner-axis of a trefoil knot. If you then reflect many points on symmetrically opposite sides of this 5-fold axis mirror, that is the trefoil knot kernel, you will obtain the Poincaré Dodecahedral Space as "outer-stone", the latter being merely a *reflection* of the underlying inner trefoil kernel.

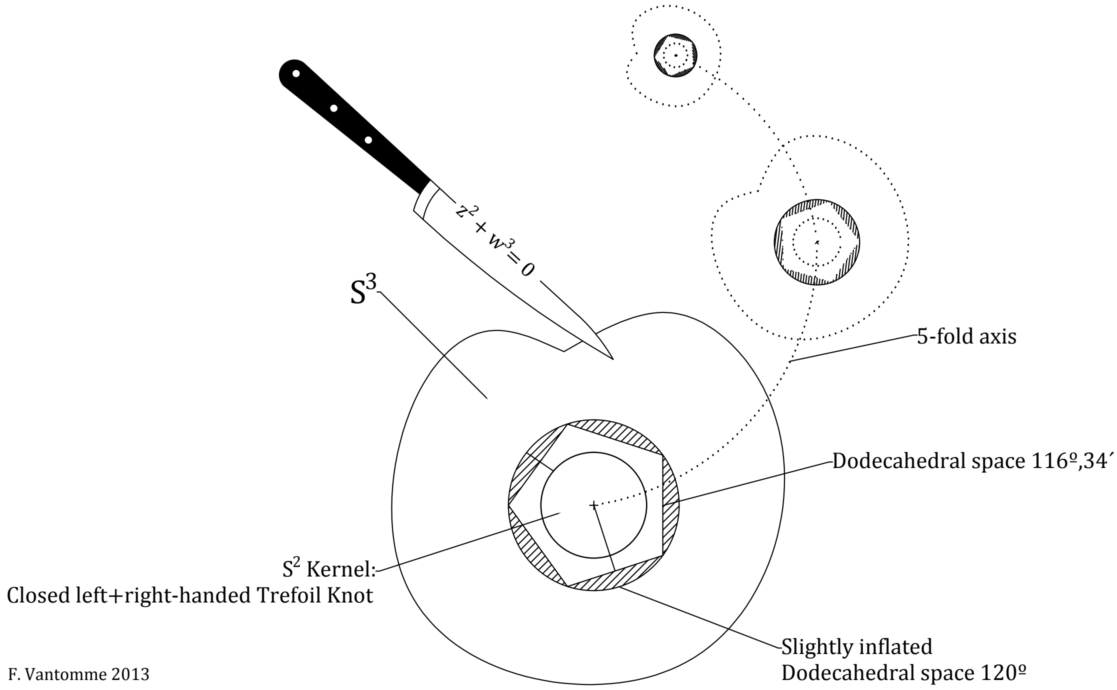


Figure 4: To visualize that Dodecahedral space is merely the symmetrical reflection of the underlying closed left+right-handed trefoil S^2 kernel, we have drawn the 120° Dodecahedral space a little removed and in parallel to the kernel but, in reality, the Dodecahedral space must be removed. Also, the outer S^3 is visible but is actually compressed into the S^2 kernel.

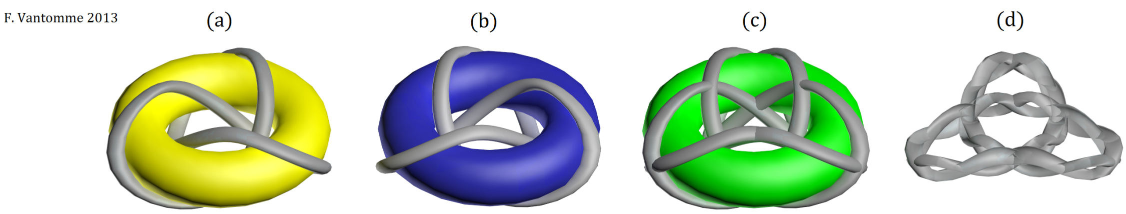


Figure 5: The (a) left-handed trefoil - (3,2) torus knot - laying on the left Clifford torus C_{-1} and (b) the right-handed trefoil laying on the, mirrored, right Clifford torus C_1, are merged into one (c) third middle Clifford torus C_0 that (d) further compresses and finally disappears. What is left is a closed left+right handed trefoil knot loop.

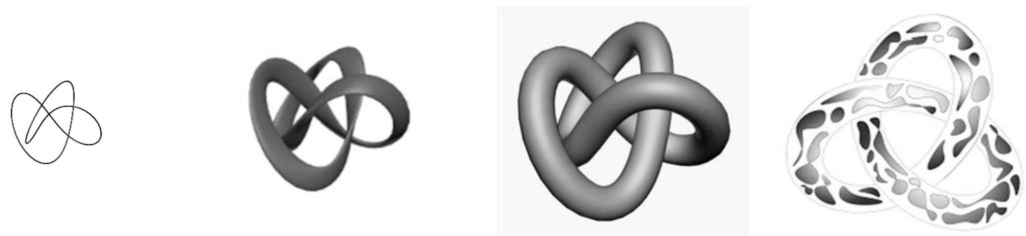
We can now finally answer the question of how Space might look like if Space itself was a Schwarzschild black hole or, in line with our hypothesis, the Compactified Transverse Slice of S^3: a left- and right-handed trefoil glued together along their boundaries forming one closed trefoil. This answers why we favour the "many different rooms" hypothesis: the trefoil forms a closed loop in which part of the light emitted in front of you will come back and hit you in the back. Lastly, contrary to the 120-cell, the trefoil *is* a fibred knot that is susceptible to fibration (read "dimensional compactification") via the Milnor map.

4 Further Reflections

The following is a series of thoughts and reflections:

Cosmic exponential inflation and homogeneity. The CTS "Matroska" model of Space may solve both the exponential inflation and homogeneity problem. We postulate that the Big Bang could be an explosion from within that broke the cascadingly compactified or "zipped" Space open. Although Figure 6 here below is not perfect because it presents a one-handed trefoil only, viewed from the outside, it gives a feeling of the explosion from within.

Looking at phases 2 and 3, you see immediately that the inflation from 234d to 34d is exponential. Similarly, you notice that the homogeneity on large scales can be better explained because the matter of Space, everything and all already present at the moment of the Big Bang, must have been in a state of closest contact and therefore highest possible order.



Phase 1: all dimensions and existing matter collapsed-compressed to a bundle of one-dimensional fibre(s) in which time behaves as another direction in Space.

Phase 2: explosion from within that breaks one-dimensional (1234d+1) space-time open to (234d+1). This appears as a closed 2-dimensional trefoil knot.

Phase 3: exponential inflation from (234d+1) to (34d+1) space. This appears as a closed 3-dimensional trefoil knot.

Phase 4: cooling down. Some regions collapse back to their original fully compressed state

Figure 6: The above pictures are an approximation of the cosmic exponential inflation which corresponds to the pattern of an Evolving Trefoil. For a better understanding, the outside observer should imagine himself at the inside of the trefoil. It can be helpful to have a look at the Evolving Trefoil sculpture from C. Séquin and B. Collins at the Missouri Western State University[48]. However, in order to obtain a more accurate image, one needs to replace the empty spaces with fully compressed $1234\mathrm{d}_{L+R}$ space and most of the intertwining filaments should be blown up to $34\mathrm{d}_{L+R}$ space.

Kaluza-Klein. Our model seems to revindicate Kaluza-Klein theory[49, 50]. Each particle point $b \in S^2$ living in 3d space has as pre-image a corresponding Hopf circle S^1 embedded in S^3 4d$_{L+R}$ attached to it.

AdS/CFT and Chern-Simons. The hypothesis that we live in between partially compactified regions of space and dark regions still containing all compactified dimensions is in consonance with the AdS/CFT correspondence[51, 52, 53] or Maldacena duality[54] as well as, and at the same time, with Chern-Simons theory. Gravitational equations in n+1 dimensions can be completely equivalently written in non-gravitational equations in n dimensions.

Beyond the Standard Model. The proposed CTS model may lead us to yet another possible but still very speculative route to unification. There exists an intriguing parallelism between the proposed geometry of Space as a closed left+right handed trefoil knot (transverse surface slice) and a variation on the $SU(2)_L$ x $SU(2)_R$ x $SU(4)$ Pati-Salam model.

As shown in Figure 7 below, each point-like particle in ordinary 3d Base Space has as pre-image a compactified S^1 Hopf circle which symmetries are represented by the $U(1)$ gauge group and which corresponds with the interactions of the electromagnetic force.

The fast rotating one-dimensional Hopf circle S^1, confined to a two-dimensional complex (because it is compactified and therefore partially imaginary) plane is itself embedded within an imaginary compactified S^3 4d hypersphere which symmetries are represented by the $SU(2)$ gauge group and which corresponds with the interactions of the weak force[55].

S^3 is then compactified to S^2 which equals $SO(3)$ or the rotational symmetry of the ordinary sphere. The fact that $SU(2)$ is compactified to $SO(3)$ makes that their symmetries are very similar but not exactly equal.

It might be important to note that the CTS model is left-right symmetric and thus doubled. As a first impression, the following variation on Pati-Salam emerges: [($U(1)_L$ x $SU(2)_L$) + ($U(1)_R$ x $SU(2)_R$)] x $SU(4)$. Note that $U(1)_L$ is unbroken (or still compactified) while $SU(2)_L$) is broken (or opened up) . To preserve the beauty of the symmetry, the right-hand trefoil could also be partially broken but just in the opposite way namely $U(1)_R$ broken (or opened up) and $SU(2)_R$) still compactified. Recall then that according to the Goldstone-Nambu theorem, to each broken symmetry generator corresponds one Goldstone boson (massless field).

The Standard Model Brout-Englert-Higgs (BEH) sector should then, for the sake of symmetry, likewise be extended with a duplicate BEH sector (these are the so called 2HDM or Two Higgs Doublet Models).

At first sight, it might be tempting to consider the 4d$_L$ and 4d$_R$ as a hypersphere in 8 dimensions but it is perhaps safer to consider the left-handedness and right-handedness as two 'sub-dimensions' of a single 4d hypersphere.

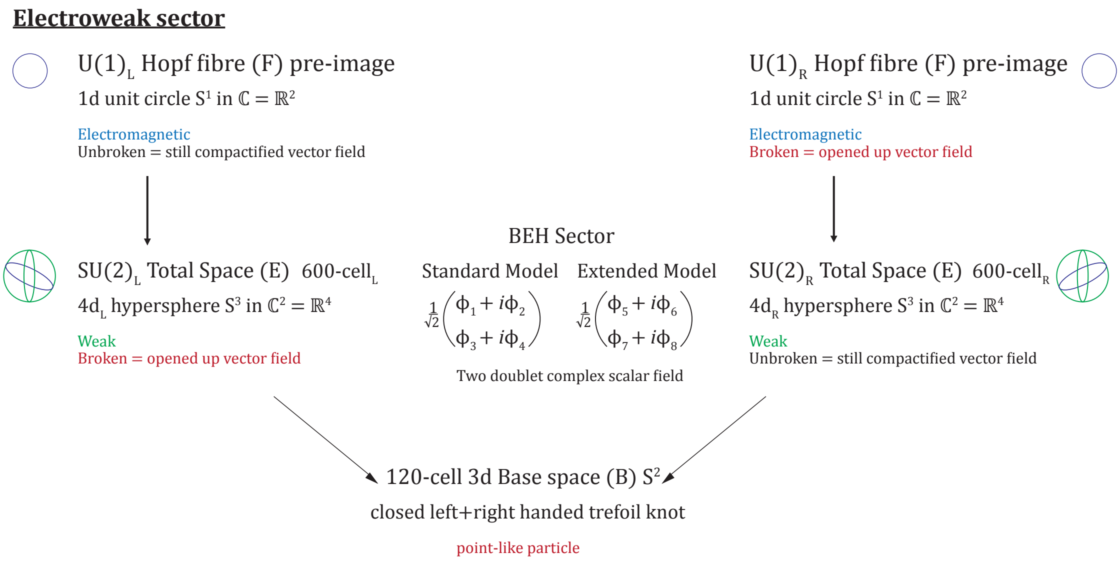


Figure 7: The closed left+right handed trefoil knot might appear as a variation on Pati-Salam: [$(U(1)_L$ x $SU(2)_L)$ + $(U(1)_R$ x $SU(2)_R)$] x $SU(4)$.

5 Testing

In conclusion, we will explore a couple of ways in which the theory could be tested (verified or falsified) against empirical evidence.

Neutrinos. It is an odd story that a neutrino kicked out with a certain energy (E) begins its journey through space as a particular weak-type neutrino (ν_α), e.g. a (ν_μ) emitted from a supernova, the (ν_μ) being a superposition of the mass-type neutrinos $(\nu_1,\ \nu_2$ and/or $\nu_3)$, and may during its journey through interstellar space oscillate and transform into a different weak-type neutrino $(\nu_e$ or $\nu_\tau)$ which are themselves different superposition-compositions of the mass-type neutrinos. The longer the distance (L), the higher the probability that oscillation will occur. Furthermore, it is bizarre that the oscillations seem to affect all neutrinos synchronously because they arrive at the detector at nearly the same time after having travelled enormous distances. One can wonder how that can be possible.

As already considered by some physicists[56, 57], neutrinos could well be travelling through different dimensional fields with different particles associated to them. Going back to our example and keeping the Matroska model at the back of our minds, we see emerge an alternative way to explain neutrino oscillations. The muon-neutrinos emitted from the 234d$_{L+R}$ supernova field are blown into ordinary 34d$_{L+R}$ space, turning them all together into ν_e, then bump into a 1234d$_{L+R}$ black hole converting them instantaneously to tau-neutrinos, leave the black hole after them and change back into ν_e, etc.

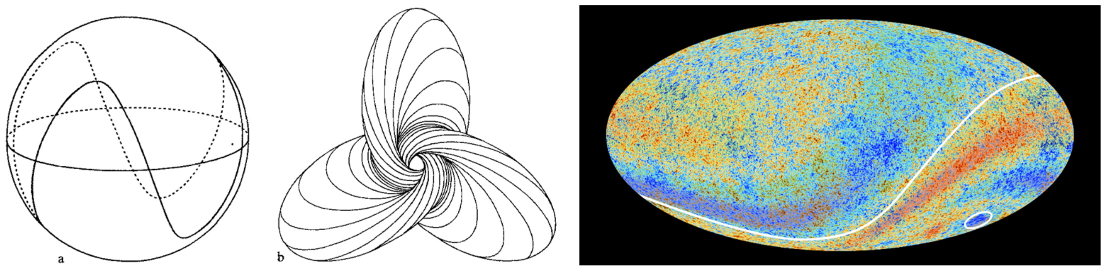
We support the hypothesis that the neutrino is a single particle that is affected by different superposition-compositions of dimensional fields it travels through. The superposition-composition of a $234d_{L+R}$ field has a different mass mixture than a $34d_{L+R}$ dimensional field with only (ν_1, ν_2) mixture, as will have a single $4d_{L+R}$ field that will make that the affected neutrino obtains a unique ν_1 mass. If a neutrino passes from $34d_{L+R}$ to $234d_{L+R}$ space, its mass will increase because it is affected by more superposed dimensional fields. Conversely, if a neutrino passes from $34d_{L+R}$ to $4d_{L+R}$ space, its mass should decrease. If its initial energy is high, the neutrino will be able to cross $234d_{L+R}$ compactified fields. If, however, its initial energy is lower, the neutrino will not be able to penetrate the high density compactified fields and will be confined to the $34d_{L+R}$ lower density compactified fields.

The Orbital Angular Momentum (OAM) of light. Since around 1990, various research groups have been exploring the OAM of electromagnetic waves, the possible applications for data transfer and the effects Kerr Black Holes (KBH) may have on the OAM of light waves passing nearby. The $1234d_{L+R}$ dimensional field lines converging in the region of a KBH could have a measurable impact on the shape of the wavefront of the EM waves in the vicinity of such KBH but also on the quantity of photon superposition states that should increase with the electromagnetic waves approaching the KBH. The authors welcome any suggestions for accurate testing.

The flat Universe, by Planck. The 2013 Planck results[58, 59, 60] give a total density parameter Ω_{tot} of 0.9995 $\pm$0.0034, which is close to the Ω_{tot} 1.0027 $\pm$0.0039 previously obtained by WMAP-9. These data are consistent with a geometrically flat Universe. H.B. Lawson, Jr. proved in 1969[61] that the only flat minimal torus in S^3 is the Clifford torus. Then, in 2012, S. Brendle proved the Lawson conjecture[27]. The Lawson conjecture states that any embedded minimal torus in S^3 is congruent to the Clifford torus; the latter - as thus already proven in 1969 by Lawson Jr. - being a flat torus embedded in S^3. The middle Clifford torus C_0 that spans the transverse slice of S^3 is geometrically flat, also because this torus is formed by the product of two unit circles $S^1(\frac{1}{\sqrt{2}})$ x $S^1(\frac{1}{\sqrt{2}})$[28]. As can be seen in Figure 5d above, our hypothesis is that the left-handed trefoil knot (Fig. 5a) and the right-handed trefoil knot (Fig. 5b) are glued along their boundaries and compactified within the span of the flat middle Clifford torus C_0 (Fig. 5c) that further squeezes together and finally 'dissolves' (for the sake of clarity, these Clifford tori are imaginary shells) leaving a (nearly) flat closed left+right handed trefoil knot (Fig. 5d) as the remainder. What is more, the mirrored trefoils tend to collapse[62] within a flat area.

Aside from a flat geometry, Planck also confirmed the existence of a cold spot and an asymmetry in the average temperature fluctuations of the CMB radiation emitted roughly 380,000 years after the Big Bang, with a slightly colder northern hemisphere and a slightly warmer southern hemisphere. Finishing this paper, we stumbled over a picture drawn back in 1985 by U. Pinkall (Figure 8, a and b)[63] which shows the inverse stereographic projection π^{-1} of a curve γ (or Villarceau circle), laying over a S^2 Clifford torus, onto $\mathbb{R}^3$.

The similarity between the wave pattern found in this drawing and in the Planck data seems purely accidental but it would be interesting to see how this model behaves in a high performance computer simulation, the results of which could then be tested against the topological models examined by Planck[60], in particular the T3 Cubic Torus and the Dodecahedral space.



Source and copyright: Ulrich Pinkall, "Hopf tori in S^3," *Inventiones mathematicae* **81** (1985) no. 2, page 383

Source and copyright: ESA and the Planck Collaboration, released 21/03/2013 12:00 pm, Planck enhanced anomalies.

Figure 8: At the left, U. Pinkall's drawing from 1985. At the right, the ESA Planck enhanced anomalies data image dated 2013.

6 Conclusion

The Matroska Compactified Transverse Slice (MCTS) model that we propose is based on already existing ideas but it is the link between them that is totally new. If found to be generally true, it would add a new mathematical and geometrical layer to the apparent chaotic Universe. We have postulated that (1) a black hole could be the result of the collapse-compactification (fibration) of a 3d burned up S^2 star to its two-dimensional transverse slice; the two-dimensional discus itself further spiralling down into a bundle of one-dimensional fibres. Similarly, (2) Space might be the collapsed-compactified (fibration) of its higher $4\mathrm{d}_{L+R}$ S^3 hyper-sphere to its 3d transverse slice, this surface adopting the topology of a closed and flat left+right handed trefoil knot. By further extending these two ideas, we might consider that Space in its initial state was a "Matroska" $4\mathrm{d}_{L+R}$ hyperspace collapsed compactified, in cascading order, to a bundle of one-dimensional fibres. The Big Bang could be an explosion from within that broke the cascadingly compressed Space open. This idea turns our conception of the cosmos somehow upside down for the reason that we might be living in between partially compactified $34\mathrm{d}_{L+R}$ regions of space and fully compactified dark regions still containing all $1234\mathrm{d}_{L+R}$ dimensional fields. We welcome any suggestions for testing to allow the proposed topology to be tested (verified or falsified) against empirical data and mathematical reasoning.

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This paper was first presented on the 8^{th} of November 2013 to the Intellectual Property Register of the Community of Madrid with solicitation reference M-8654/2013 and was accepted for registration on the 20^{th} of December 2013 with number 16/2013/8704 and references Exp.: 09-RTPI-08956.1/2013, Doc.: 09/200171.5/13.

Recent findings in Cosmology and Astrophysics (a.o [64], [65], [66]) seem to gently corroborate the idea and this led to a renewed interest in the subject and the partial rewriting of this paper. See IP Register of the Community of Madrid: accepted with number 16/2016/1790, Exp.09-RTPI-00708.0/2016, Doc.: 09/033121.4/16.

I thank my children for being so wonderful and making me laugh.

References

[1] G. Dvali and C. Gomez, "Black Holes as Critical Point of Quantum Phase Transition," `arXiv:1207.4059 [hep-th]`.

[2] G. Dvali, G. Gabadadze, and M. Porrati, "4D gravity on a brane in 5D Minkowski space," *Physics Letters B* **485** (2000) no. 1, 208–214. `http://www.sciencedirect.com/science/article/pii/S0370269300006699`.

[3] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, "The Hierarchy problem and new dimensions at a millimeter," *Phys.Lett.* **B429** (1998) 263–272, `arXiv:hep-ph/9803315 [hep-ph]`.

[4] A. Salam and J. Strathdee, "On kaluza-klein theory," *Annals of Physics* **141** (1982) no. 2, 316–352. `http://www.sciencedirect.com/science/article/pii/0003491682902913`.

[5] L. Randall and R. Sundrum, "Large mass hierarchy from a small extra dimension," *Physical Review Letters* **83** (1999) no. 17, 3370. `http://prl.aps.org/abstract/PRL/v83/i17/p3370_1`.

[6] L. Anchordoqui, D. C. Dai, M. Fairbairn, G. Landsberg, and D. Stojkovic, "Vanishing dimensions and planar events at the LHC," *Modern Physics Letters A* **27** (2012) no. 04, . `http://www.worldscientific.com/doi/abs/10.1142/S0217732312500216`.

[7] K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der EINSTEINschen Theorie," *Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.)* **1916** (1916) 189–196, `arXiv:physics/9905030 [physics]`.

[8] K. Schwarzschild, "Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der EINSTEINschen Theorie," *Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.)* **1916** (1916) 424–434, `arXiv:physics/9912033 [physics.hist-ph]`.

[9] J. Droste, "The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field," *Ned. Acad. Wet., SA* **19** (1917) 197.

[10] H. Weyl, "On the Theory of Gravitation," *Annalen Phys.* **54** (1917) 117–145.

[11] S. Antoci, "David Hilbert and the origin of the Schwarzschild solution," `arXiv:physics/0310104 [physics]`.

[12] C. Corda, "A clarification on the debate on 'the original Schwarzschild solution'," *Electron.J.Theor.Phys.* **8** (2011) no. 25, 65–82, `arXiv:1010.6031 [gr-qc]`.

[13] J. D. Bekenstein, "Black holes and entropy," *Phys.Rev.* **D7** (1973) 2333–2346.

[14] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Communications in Mathematical Physics* **31** (1973) no. 2, 161–170. `http://link.springer.com/article/10.1007/BF01645742`.

[15] G. 't Hooft, "The Holographic principle: Opening lecture," `arXiv:hep-th/0003004 [hep-th]`.

[16] L. Susskind, "The world as a hologram," *arXiv preprint hep-th/9409089* (1994) . `http://arxiv.org/abs/hep--th/9409089`.

[17] M. Bañados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys.Rev.Lett.* **69** (1992) 1849–1851, `arXiv:hep-th/9204099 [hep-th]`.

[18] S. Carlip, "The (2+ 1)-dimensional black hole," *Classical and Quantum Gravity* **12** (1995) no. 12, 2853. `http://iopscience.iop.org/0264-9381/12/12/005`.

[19] J. D. Bekenstein and A. E. Mayo, "Black holes are one-dimensional," *General Relativity and Gravitation* **33** (2001) no. 12, 2095–2099. `http://link.springer.com/article/10.1023/A:1015278813573`.

[20] S. P. Robinson and F. Wilczek, "Relationship between Hawking radiation and gravitational anomalies," *arXiv preprint gr-qc/0502074* (2005) . `http://arxiv.org/abs/gr-qc/0502074`.

[21] S. Carlip, "Spontaneous dimensional reduction?," *arXiv preprint arXiv:1207.4503* (2012) . `http://arxiv.org/abs/1207.4503`.

[22] H. S. M. Coxeter, *Regular Popytopes.* DoverPublications. com, 1973.

[23] M. C. Sullivan, "Trefoil Surgery,". `http://galileo.math.siu.edu/mikesullivan/TTLects/ts.pdf`.

[24] N. Kapouleas and S.-D. Yang, "Minimal surfaces in the three-sphere by doubling the Clifford torus," *American journal of mathematics* **132** (2010) no. 2, 257–295. http://muse.jhu.edu/journals/ajm/summary/v132/132.2.kapouleas.html.

[25] M. Dehn and J. Stillwell, *Papers on group theory and topology.* Paper 6 : The two trefoil knots (Die beiden Kleeblattschlingen. Math. Ann. 75 (1914), 402-413). Springer New York, 1987.

[26] J. Stillwell, "Poincaré and the early history of 3-manifolds," *Bulletin of the American Mathematical Society* **49** (2012) no. 4, 555–576. http://www.ams.org/journals/bull/0000-000-00/S0273-0979-2012-01385-X/S0273-0979-2012-01385-X.pdf.

[27] S. Brendle, "Embedded minimal tori in S^3 and the Lawson conjecture," *arXiv preprint arXiv:1203.6597* (2012) .

[28] P. Norbury, "The 12th problem," *Gazette of the Australian Mathematical Society* **32** (2005) 244–246. http://www.austms.org.au/Publ/Gazette/2005/Sep05/millennium.pdf.

[29] Z. Treisman, "A Young Person's Guide to the Hopf Fibration," *arXiv preprint arXiv:0908.1205* (2009) .

[30] D. W. Lyons, "An elementary introduction to the Hopf fibration," *Mathematics magazine* **76** (2003) no. 2, 87–98. http://www.jstor.org/stable/10.2307/3219300.

[31] T. Kitagawa, "Physical Interpretation of Hopf Fibration." http://www.learningace.com/doc/1620000/df1ae1fee6122d02ff5ef50b469105e1/physical_hopf.

[32] H. Liu, "Hopf Fibration,". http://www.nilesjohnson.net/hopf-articles/Hongwan_Liu-Hopf_fibration.pdf.

[33] N. Johnson, "Visualizations of the Hopf fibration," 2011. http://www.youtube.com/watch?v=QXDQsmL-8Us. As from minute 39.

[34] R. Mosseri and R. Dandoloff, "Geometry of entangled states, Bloch spheres and Hopf fibrations," *Journal of Physics A: Mathematical and General* **34** (2001) no. 47, 10243. http://iopscience.iop.org/0305-4470/34/47/324.

[35] G. Heinze, C. Hubrich, and T. Halfmann, "Stopped Light and Image Storage by Electromagnetically Induced Transparency up to the Regime of One Minute," *Physical review letters* **111** (2013) no. 3, 033601. http://prl.aps.org/abstract/PRL/v111/i3/e033601.

[36] H. H. S. M. Coxeter, *Twisted honeycombs*, vol. 4. AMS Bookstore, 1970.

[37] G. Toth, *Glimpses of algebra and geometry.* Springer, 2002.

[38] J.-P. Luminet, "A cosmic hall of mirrors," *arXiv preprint physics/0509171* (2005) . http://arxiv.org/abs/physics/0509171.

[39] J.-P. Luminet, "Geometry and topology in relativistic cosmology," *arXiv preprint arXiv:0704.3374* (2007) . http://arxiv.org/abs/0704.3374.

[40] J.-P. Luminet, "The shape and topology of the universe," *arXiv preprint arXiv:0802.2236* (2008) . http://arxiv.org/abs/0802.2236.

[41] J. Stillwell, "The story of the 120-cell," *Notices of the AMS* **48** (2001) no. 1, . http://www.ams.org/notices/200101/fea-stillwell.pdf.

[42] J. Weeks, "The Poincaré dodecahedral space and the mystery of the missing fluctuations," *Notices of the AMS* **51** (2004) no. 6, 610–619. https://e-math2.ams.org/notices/200406/fea-weeks.pdf.

[43] J.-P. Luminet, J. Weeks, A. Riazuelo, R. Lehoucq, and J.-P. Uzan, "Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background," *Nature* **425** (2003) 593, arXiv:astro-ph/0310253 [astro-ph].

[44] R. Kirby and M. Scharlemann, "Eight faces of the Poincaré homology 3-sphere," *Uspekhi Matematicheskikh Nauk* **37** (1982) no. 5, 139–159.

[45] S. Caillerie, M. Lachieze-Rey, J.-P. Luminet, R. Lehoucq, A. Riazuelo, and J. Weeks, "A new analysis of Poincaré dodecahedral space model," *arXiv preprint arXiv:0705.0217* (2007) . http://arxiv.org/abs/0705.0217.

[46] T. Kazimierczak, "The shape and the size of the Universe according to the Poincaré dodecahedral space hypothesis," *Advances in Astronomy and Space Physics* **2** (2012) 60–62.

[47] L. Brenton, "Plato, Poincaré, and the Enchanted Dodecahedron: Is the Universe Shaped Like the Poincare Homology Sphere?," *Math Horizons* **20** (2013) no. 3, 12–17. http://www.jstor.org/stable/10.4169/mathhorizons.20.3.12.

[48] C. H. a. Séquin, "Sculpture Designs and Math Models," February, 2013. http://www.cs.berkeley.edu/~sequin/SCULPTS/CHS_bronzes/EvolvingTrefoil/Evolving-Trefoil.jpg.

[49] T. Kaluza, "Zum Unitätsproblem der Physik," *Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.)* **1921** (1921) 966–972.

[50] O. Klein, "Quantentheorie und fünfdimensionale Relativitatstheorie," *Z.Phys.* **37** (1926) 895–906.

[51] D. R. Brill, "Multi-black-holes in 3D and 4D anti-de Sitter spacetimes," *arXiv preprint gr-qc/9608010* (1996) . http://arxiv.org/abs/gr-qc/9608010.

[52] E. Witten, "Anti de Sitter space and holography," *arXiv preprint hep-th/9802150* (1998) . `http://arxiv.org/abs/hepth/9802150`.

[53] J. de Boer, "Introduction to the AdS/CFT correspondence," *University of Amsterdam Institute for Theoretical Physics (ITFA) Technical Report* (2003) 03–02.

[54] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *AIP Conf.Proc.* **484** (1999) 51–63.

[55] J. Maldacena, "The symmetry and simplicity of the laws of physics and the Higgs boson," `arXiv:1410.6753 [physics.pop-ph]`.

[56] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, and J. March-Russell, "Neutrino masses from large extra dimensions," *Physical Review D* **65** (2001) no. 2, 024032. `http://prd.aps.org/abstract/PRD/v65/i2/e024032`.

[57] T. Hällgren, *Aspects of Dimensional Deconstruction and Neutrino Physics*. PhD thesis, KTH, 2007. `http://kth.diva-portal.org/smash/record.jsf?pid=diva2:12470`.

[58] **Planck Collaboration** Collaboration, P. Ade *et al.*, "Planck 2013 results. I. Overview of products and scientific results," `arXiv:1303.5062 [astro-ph.CO]`.

[59] **Planck Collaboration** Collaboration, P. Ade *et al.*, "Planck 2013 results. XVI. Cosmological parameters," `arXiv:1303.5076 [astro-ph.CO]`.

[60] **Planck Collaboration** Collaboration, P. Ade *et al.*, "Planck 2013 results. XXVI. Background geometry and topology of the Universe," `arXiv:1303.5086 [astro-ph.CO]`.

[61] H. B. Lawson, "Local rigidity theorems for minimal hypersurfaces," *The Annals of Mathematics* **89** (1969) no. 1, 187–197. `http://www.jstor.org/stable/10.2307/1970816`.

[62] D. Kozlov, "Knots and links as form-generating structures," in *Mathematics and Modern Art*, pp. 105–115. Springer, 2012. `http://link.springer.com/chapter/10.1007/978-3-642-24497-1_10`.

[63] U. Pinkall, "Hopf tori in S^3," *Inventiones mathematicae* **81** (1985) no. 2, 379–386. `http://www.springerlink.com/index/Q75502X6K61V0456.pdf`.

[64] V. Vikram, C. Chang, B. Jain, D. Bacon, A. Amara, M. Becker, G. Bernstein, C. Bonnett, S. Bridle, D. Brout, M. Busha, J. Frieman, E. Gaztanaga, W. Hartley, M. Jarvis, T. Kacprzak, O. Lahav, B. Leistedt, H. Lin, P. Melchior, H. Peiris, E. Rozo, E. Rykoff, C. Sanchez, E. Sheldon, M. Troxel, R. Wechsler, J. Zuntz, T. Abbott, F. B. Abdalla, R. Armstrong, M. Banerji, A. H. Bauer, A. Benoit-Levy, E. Bertin, D. Brooks, E. Buckley-Geer, D. L. Burke, D. Capozzi,

A. Carnero Rosell, M. Carrasco Kind, F. J. Castander, M. Crocce, C. B. D'Andrea, L. N. da Costa, D. L. DePoy, S. Desai, H. T. Diehl, J. P. Dietrich, C. E. Cunha, J. Estrada, A. E. Evrard, A. Fausti Neto, E. Fernandez, B. Flaugher, P. Fosalba, D. Gerdes, D. Gruen, R. A. Gruendl, K. Honscheid, D. James, S. Kent, K. Kuehn, N. Kuropatkin, T. S. Li, M. A. G. Maia, M. Makler, M. March, J. Marshall, P. Martini, K. W. Merritt, C. J. Miller, R. Miquel, E. Neilsen, R. C. Nichol, B. Nord, R. Ogando, A. A. Plazas, A. K. Romer, A. Roodman, E. Sanchez, V. Scarpine, I. Sevilla, R. C. Smith, M. Soares-Santos, F. Sobreira, E. Suchyta, M. E. C. Swanson, G. Tarle, J. Thaler, D. Thomas, A. R. Walker, and J. Weller, "Wide-field lensing mass maps from des science verification data," *ArXiv e-prints* (Apr., 2015) , `arXiv:1504.03002`.

[65] R. Massey, L. Williams, R. Smit, M. Swinbank, T. D. Kitching, *et al.*, "The behaviour of dark matter associated with 4 bright cluster galaxies in the 10kpc core of Abell 3827," *Mon.Not.Roy.Astron.Soc.* **449** (2015) 3393, `arXiv:1504.03388 [astro-ph.CO]`.

[66] I. Szapudi, A. Kovcs, B. R. Granett, Z. Frei, J. Silk, *et al.*, "Detection of a Supervoid Aligned with the Cold Spot of the Cosmic Microwave Background," `arXiv:1405.1566 [astro-ph.CO]`.

[67] W. Israel, "Aspects of Black Hole Entropy,". `http://www.cbpf.br/~cosmogra/Escolas/9_escola/IX_ESCOLA_275_291.pdf`.

[68] S. Hawking, "Particle Creation by Black Holes," *Commun.Math.Phys.* **43** (1975) 199–220.

[69] J. D. Bekenstein, "Do we understand black hole entropy?," `arXiv:gr-qc/9409015 [gr-qc]`.

[70] A. Krause, "Dual branes, discrete chain states and the entropy of the Schwarzschild black hole," `arXiv:hep-th/0212339 [hep-th]`.

[71] J. J. van Wijk and A. M. Cohen, "Visualization of the Genus of Knots," in *Visualization, 2005. VIS 05. IEEE*, pp. 567–574, IEEE. 2005. `http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1532843`.

[72] U. Moschella, "The de Sitter and anti-de Sitter sightseeing tour," in *Einstein, 1905–2005*, pp. 120–133. Springer, 2006. `http://link.springer.com/content/pdf/10.1007/3-7643-7436-5_4.pdf`.

[73] J. D. Bekenstein and A. T. Kamajian, "Information in the holographic universe," *Scientific American* **15** (2006) 74–81. `http://www.nature.com/scientificamerican/journal/v15/n3s/full/scientificamerican0206-74sp.html`.

[74] S. Klaus, "The solid trefoil knot as an algebraic surface," *International Center for Mathematics* (2010) 2.

[75] S. Schleimer and H. Segerman, "Sculptures in S^3," *arXiv preprint arXiv:1204.4952* (2012) . http://arxiv.org/abs/1204.4952.