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Theorem of the Keplerian kinematics

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Abstract As described in the literature the velocity of a Keplerian orbiter on a fixed orbit is always the sum of a uniform rotation velocity and a uniform translation velocity, both coplanar. This property is stated here as a theorem and demonstrated as true. The consequences are investigated among which the Newton's law of gravitation appears as its derivative with respect to time, the classical mechanical energy is deduced, the Galileo's equivalence principle is respected, an alternative to the orbit determination emerges. However the kinematics gives a less restrictive interpretation of the Newton's factor GM, and show that the Einstein's equivalence principle can not be correct. Furthermore they provide an explanation of the galaxy rotation by extending the Plank-Einstein relation to the macroscopic scale.

Keywords Kinematics · Laws of Kepler · Gravitation · Orbital dynamics

1 Introduction

Although the three laws of Kepler are widely known [1], there exists a special property of the Keplerian motion for a fixed conic that is too often forgotten : the velocity is simply the addition of a uniform circular and a uniform translation velocities. This kinematics aspect of the motion, fully referenced in the literature [2–8], is generally quoted by the means of the hodograph plane representation, although some authors as R.H. Battin state it in a different way [9]. In all cases this special property is presented as a consequence of the Newton's law of gravitation. From this point of view it appears somehow difficult and complex to use, although it could be a powerful tool when presented in an other way.

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Our aim in this article is to inverse the usual vision of this property by stating it as a theorem, and then showing its consequences, among which the Newton's law of gravitation appears, as a its trivial derivation with respect to time, the classical expression of the mechanical energy emerges as well as an alternative to the orbit determination.

In no way at all we will pretend that such a vision of the Keplerian motion could be a satisfactory gravitation theory that could compete with the Newton's or Einstein's ones. Our purpose is only to perform a pure kinematics study without any postulate, nor assumption, nor theory.

We will first state and proof the theorem, and second we will investigate some of its consequences.

2 Stating and proving the theorem

2.1 Statement

Accordingly to what is demonstrated in the literature [2–8] about the mathematical structure of the Keplerian motion, we deduce the possibility of writing the following theorem :

Theorem 1 *The velocity of a Keplerian orbiter on a fixed orbit is always the sum of a uniform rotation velocity and a uniform translation velocity, both coplanar.*

We do not know why nature chose to exhibit this behavior, and this is not the purpose of this paper, but the works of the literature show that it applies to all Keplerian motions. Our work here will consist to prove that it is mathematically true, i.e. that it fully describes the Keplerian motion.

At a kinematics point of view such a velocity will be written as follows :

$$\mathbf{v} = \mathbf{v}_R + \mathbf{v}_T \tag{1}$$

where $\mathbf{v}_R = \boldsymbol{\omega} \times \mathbf{r}$ is the uniform rotation velocity (its norm is constant), with $\boldsymbol{\omega}$ being the frequency of rotation, $\mathbf{r}$ being the radius vector from the focus of the orbit to the orbiter, and $\mathbf{v}_T$ is the uniform translation velocity (its norm and direction are constant).

It is important to remember that the indice R does not stand for radial, but for rotation, while the indice T does not stand for tangential but for translation. Figure 1 exhibits the different velocities at 4 steps of a conic trajectory. Note that the translation velocity is always perpendicular to the main axis of the conic.

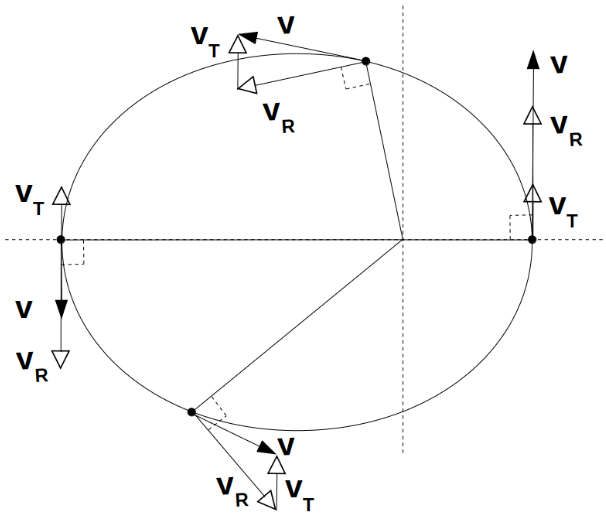

Fig. 1 The velocity of a Keplerian orbiter $\mathbf{v}$ on a fixed orbit is always the sum of a uniform rotation velocity $\mathbf{v}_R$, perpendicular to the radius vector, and a uniform translation velocity $\mathbf{v}_T$, which direction is always perpendicular to the main axis of the conic. Both are coplanar and have a constant norm all along the trajectory.

Of course the rotation velocity being uniform by statement, and the frequency of rotation being perpendicular to the velocity/radius plane, we must verify :

$$v_R = \|\mathbf{v}_R\| = \|\boldsymbol{\omega} \times \mathbf{r}\| = \omega r = constant \tag{2}$$

Derivating this last relationship with report to time we get a trivial but important expression :

$$\dot{\omega} r + \omega \dot{r} = 0 \tag{3}$$

The scalar $\dot{\omega}$ shall correspond to a vector $\dot{\boldsymbol{\omega}}$ which is collinear to the vector $\boldsymbol{\omega}$. Finally because the translation velocity is also uniform by statement we can write

$$v_T = \|\mathbf{v}_T\| = constant \tag{4}$$

This being stated, we are now going to give the proof of this theorem.

2.2 Proof

2.2.1 The angular momentum and its constancy

We define the angular momentum $\mathbf{L}$ as follows :

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} \tag{5}$$

This angular momentum does not refer to the mass as it is only a kinematics vector. R.H. Battin called it *the massless angular momentum* [10]. It is trivial to see that its derivation with respect to time, by including the relation 3, is null, thus the angular momentum is constant as expected for a central field motion.

2.2.2 First law of Kepler

The vector cross product of the rotation velocity by the momentum leads to

$$\mathbf{v}_R \times \mathbf{L} = v_R^2 \left(1 + \frac{\mathbf{v}_R . \mathbf{v}_T}{v_R^2}\right) \mathbf{r} \tag{6}$$

Therefore the scalar version of this expression is

$$\frac{L}{v_R} = \left(1 + \frac{v_T}{v_R} \cos\theta\right) r \quad \text{or} \quad p = (1 + e\cos\theta)\, r \tag{7}$$

This last equation is the one of a conic where $p = L/v_R$ is the semilatus rectum, $e = v_T/v_R$ is the eccentricity and θ is the angle between the directions of the rotation and the translation velocity, i.e. the true anomaly. We see that both p and e are constant and therefore the equation 7 is nothing else but the first law of Kepler [1].

The theorem 1 also provides an elegant way to describe the eccentricity vector, thus the direction of the periapsis, by the means of the translation velocity :

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{L}}{k} - \frac{\mathbf{r}}{r} = \frac{\mathbf{v}_T \times \mathbf{L}}{k} \text{ with } k = L v_R \tag{8}$$

2.2.3 Second law of Kepler

The second Kepler's law derives simply from the constancy of the angular momentum, demonstrated above. As explained by L. Landau and E. Lifchitz [1], the momentum can also be written as a function of the position and the derivative of the true anomaly with respect to time :

$$L = r^2 \dot{\theta} \tag{9}$$

In this expression we see that the angular momentum is twice the areal velocity. The first being constant, the second will also be, which is the second law of Kepler.

From the equation 7 it is trivial to relate the angular frequency of rotation to the derivative of the true anomaly with respect to time :

$$\dot{\theta} = \omega(1 + e\cos\theta) = \omega\frac{p}{r} \tag{10}$$

2.2.4 Third law of Kepler

The third Kepler's law also derives simply from the constancy of the angular momentum [1]. Indeed the integration with respect to time of the relation 9 , over a complete period T of revolution, gives

$$LT = \int_0^{2\pi} r^2 \, \mathrm{d}\theta \tag{11}$$

For the case where the trajectory is an ellipse, the right side of this equation is worth $2\pi a\, b$, where a is the semi-major axis and b the semi-minor one. Knowing that $a = p/(1-e^2)$ and $b = 1/\sqrt{1-e^2}$, and remembering the definition of the semilatus rectum p given by the equations 7, it is easy to finally get the following relation :

$$L\ v_R = 4\pi^2 a^3/T^2 \tag{12}$$

Because L and v_R are constants, this last expression is nothing else but the third law of Kepler stating that the square of the period of revolution is proportional to the cube of the semi-major axis [1].

3 Consequences

3.1 Newton's law of gravitation

When we derive the equation 1 with respect to time, we get the acceleration $\mathbf{a}$ of a Keplerian orbiter : $\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}$. Now including the equation 3 we can write $\mathbf{a} = -(\boldsymbol{\omega}/r^2) \times [\mathbf{r} \times (\mathbf{r} \times \mathbf{v})]$, and finally

$$\mathbf{a} = -\frac{L\, v_R}{r^3}\mathbf{r} \tag{13}$$

This is the expression of the Newton's gravitational acceleration if

$$L\, v_R = G\, M \tag{14}$$

where G is the constant of gravitation and M is the attracting mass. We can also notice that the equation 14 is consistent with the expression 12 of the third Kepler's law when compared to the one of the literature [1].

As we see here the kinematics does agree with the mathematical structure of the Newton's acceleration, but this last is not any more a prior to the existence of the velocity. As explained by the theorem 1, it only becomes a trivial consequence, the centripetal acceleration due to the rotation velocity. In a sense we can say that, from a kinematics point of view, the gravitation law is not a law of attraction, but a law of rotation.

The Newton's postulate proposing that $G\, M$ should be the numerator of expression 13 can not be reached, nor discussed, by the kinematics that describe it rather as $L\, v_R$, both factors being however accordingly constant.

3.2 Galileo's equivalence principle

Galileo has shown in the early 17th century that the motion in a gravitational field is mass independent. This is quite consistent with the kinematics structure of the Keplerian motion as the equation 1 is also mass independent.

A body falling, for instance from the top of the tower of Pisa, must follow the equation 1. If the starting velocity of the body is null, we have $\mathbf{v}_T = -\mathbf{v}_R$, which means $e = 1$ in equation 7. If we drop the object, $\mathbf{v}_T$ will slightly decrease, and therefore the mobile will enter a conic with an eccentricity close but lower than 1. The body will fall on a sharp ellipse which focus is at the center of gravity of the two implied bodies, the falling object and the Earth, so nearly at the center of the Earth. Of course at such a distance from the Earth's center, and locally from the top of the tower to the ground, such a conic trajectory could be confused with a straight line at a first approximation.

3.3 Mechanical energy

If we develop the square of the equation 1, and include the result 7, it is trivial to define the massless mechanical energy E_M as follows :

$$E_M = \frac{1}{2}v^2 - \frac{L\,v_R}{r} = \frac{1}{2}v_R^2(e^2-1) \tag{15}$$

This expression is interesting because it describes the classical mechanical energy (divided by the mass of the orbiter) made of the addition of the usual kinetic and potential parts. It also shows, with its right member, that this energy is a constant for a fixed conic. Therefore the kinematics does agree with the classical physics of the gravitation [1], as far as, once again, the relation 14 is true.

3.4 Orbit determination alternative

If we know $\mathbf{r}$ and $\mathbf{v}$ at a single time t, we can trivially calculate the angular momentum $\mathbf{L}$, and thus get directly the rotation velocity v_R by the means of the equation 14. Now the direction of the rotation velocity must be perpendicular to the radius vector, then we can calculate the vector $\mathbf{v}_R$, so the translation velocity from equation 1 with $\mathbf{v}_T = \mathbf{v} - \mathbf{v}_R$. At this point we are able to calculate the eccentricity, the semilatus rectum, and the true anomaly by the means of the equation 7, i.e. $e = v_T/v_R$, $p = L/v_R$ and $\theta = \arccos[(1/e)(p/r-1)]$. We then get all the characteristics needed to draw the complete conic of the orbiter, including the direction of the periapsis given by the equation 8.

Of course once we have the former information it is easy to calculate the velocity of the mobile at any other position on the conic, and this is useful to resolve other problems as space rendezvous, which are usually handled by the means of the Lambert's problem [17]. This subject deserves a complete article to be fully discussed from the present kinematics point of view.

3.5 Accelerating an orbiter with a mechanical thrust

If an orbiter has a null translation velocity, it will only possess the uniform rotation velocity that is well described in the literature [1], i.e. $v = v_R = \sqrt{GM/r}$, which is consistent with the equations 7 and 14, and the eccentricity of the conic is of course null. This velocity is due to the gravitation and we can not get rid of it, as far as the attracting mass exists.

Let us now apply a very tiny thrust during a very short time, so we provide an impulsional momentum to the body, thus a translation velocity, in a specific direction. The consequence is a change of eccentricity (see the definition of the eccentricity in equation 7), and the orbiter can not stay on its initial circular orbit, whatever the direction of $\mathbf{v}_T$, thus of the thrust, is. This result is consistent with what is described in the literature when either a tangential or a radial thrust is applied to an orbiter [11,12].

Generalizing this, we know that the the acceleration equation for any thrusted trajectory of an orbiter is [13–15] :

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r} + \frac{\mathbf{F}}{m} \tag{16}$$

where $\mathbf{F}$ is the thrusting force applied to the orbiter, and m its mass. Straight forward integrating this expression with respect to time, regarding paragraph 3.1, we get :

$$\mathbf{v} = \mathbf{v}_R + \int_{t_0}^{t} \frac{\mathbf{F}}{m} dt \tag{17}$$

While $\mathbf{v}_R$ is the rotation velocity due to the gravitation, the integral in the right hand of this equation is of course nothing else but $\mathbf{v}_T$, and we find back the equation 1. This expression is a new way to represent the velocity of a thrusted orbiter.

A very interesting question here is can we produce an acceleration with a mechanical thrust that would cause a rotation around the gravitational axis of rotation. A force must have a physical connection to the axis of rotation to cause a rotation, but the force of a mechanical thrust applied to an orbiter has no physical connection to the central body of the Kepleran motion. Therefore a mechanical thrust can not cause any rotation around the gravitational center, but only a translation. At the contrary the gravitatinal force on the orbiter has a physical connection to the central body, the gravitation itself, and therefore can cause the rotation.

A series of micro-thrusts set to simulate a perfect rotation will only be a succession of rectilinear thrusts, changing the translation velocity $\mathbf{v}_T$ of the equation 1, therefore changing the eccentricity, and the direction of the main axis, at each micro-thrust, as short as it could be. If the direction of the thrust changes with the time, $\mathbf{v}_T$ will consequently change, and thus the characteristics of the conic. We then typically get a low-thrust trajectory structure, well known by the space engineers placing satellites into orbit [13–15]. The experiment shows therefore that a mechanical acceleration is not equivalent to a gravitational acceleration. The gravitation provides the pure rotation $\mathbf{v}_R$ while the mechanical acceleration can only provide the translation $\mathbf{v}_T$, accordingly to the equations 1 and 17.

4 More consequences

4.1 Dark matter

The experimental measurements have shown that the rotation velocities of galaxies are not consistent with the Newton's postulate of gravitation [18]. Indeed, considering the trajectories of the stars as nearly circular in the galactic disk, the kepler's third law indicates that the rotation velocity of a star must be proportional to the inverse of the square root of the distance to the central body (see equations 12 and 14). Mathematically we must verify : $v = \sqrt{k/r}$, where k is the newton's factor GM, G being the gravitation constant and M the mass of the central body (approximately the central black hole plus the bulb). Obvioulsy k can not depend upon the distance r to the center of the galaxy, because whatever this distance is, G and M will not vary, and therefore the velocity should decrease with the distance. However this is not what is observed [18], the velocity tending to be a constant for all the stars of the disk, whatever their distance. It looks like k is proportional to the distance r. Such an observation leads trivially to consider two possible solutions to this problem : either the Newton's postulate is fully right, and then we must add a new postulate (existence of a dark matter, existence of a new force at long range, ...) to explain the observations, either Newton is not fully right.

Let us now look at what the kinematics say about this. For them $k = Lv_R$ (see equation 12), so the rotation velocity verifies $v = \sqrt{Lv_R/r} = \sqrt{L\omega}$, where L is the angular momentum and ω the frequency of rotation. Therefore the velocity v will be constant independently of the distance if the following term, which has the dimension of an energy, is the same constant for all the stars of a given galaxy :

$$E_0 = L\omega \tag{18}$$

This last expression is well known in sub-atomic physics, this is the Plank-Eintein's relation, where the Plank's constant h has been replaced by L, both having the same dimension. Accepting such an analogy leads us to consider that the energy E_0 is a fundamental energy level of the galaxy, like energy levels also exist in the atoms. Consequently the stars are populating this energy level as the electrons do in an atomic orbital.

This is how the kinematics explain the observed rotation of galaxies. No current nor passed theory of the gravitation has however expected that the gravitation could be driven by the same fundamental laws as those encountered in the atomic physics. Actually this is even one of the most difficult problem of the physics : the unification of the macroscopic and microscopic physics is still to discover. We then take this result of the kinematics as a track for such a unification. Any theory of the gravitation should consider embedding the notion of energy level to be consistent with both the kinematics and the experiment. Such a theory does not exist yet, and the only kinematics can not replace it, so this track must be investigated.

4.2 Einstein's equivalence principle

We demonstrated that the mechanical force is of a different nature than the gravitational force, the first one providing the translation only, the second one the rotation. This former conclusion leads to a problem : the Einstein's equivalence principle, stating the exact contrary, can not be correct. As a consequence the kinematics forecast an other description of Einstein's thought experiment of the observer letting fall a ball inside an elevator. As we mentioned above about the Galileo's equivalence principle, the observer will be able to figure out if he is at the surface of a planet or thrusted mechanically : in the first case the ball will fall on a conic while in the second case it will fall on a straight line.

Einstein's equivalence principle being false, it does not mean that all the General Relativity should be ignored from now on. We all know the very nice agreement of the GR with the experiment, therefore the present results leads to think that the GR could rely on other bases than the equivalence principle. May be there exixts a special relativity of the non inertial frames of reference, that is still to discover, and could explain the GR better than the equivalence principle. Far from detroying the GR the above results of the kinematics could lead to a better statement and understanding of the GR.

4.3 Beyond Newton

We mentioned above that te kinematics agree with the Newton's postulate of gravitation if $L\, v_R = G\, M = k = constant$ (see equation 14). This expression shows that the definition of k is less restrictive for the kinematics than it is for the Newton's postulate that can only apply to the gravitation. For instance the kinematics definition of k can suite to explain the force of Coulomb, which has the same mathematical structure as the Newton's force but with a different k. Nothing is mathematically opposed to this.

Furthermore it is important to note that the acceleration coming from the derivative of the equation 1 is also compliant with the foundation of the electromagnetism. Indeed we can write

$$\mathbf{a} = (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{v} \times (-\boldsymbol{\omega}) = \mathbf{E} + \mathbf{v} \times \mathbf{H} \tag{19}$$

In this expression $\mathbf{E}$ is similar to an electric field and $\mathbf{H}$ is similar to a magnetic field, both divided by the mass of the orbiter. We then obtain the mathematical formulation of the Lorentz's acceleration [19].

All these mathematical facts can not be ignored and have to be investigated. Such a study could demonstrate that the kinematics can unify the gravitation and the electromagnetism.

5 Conclusion

The aim of this article was to state a very well known property of the Keplerian motion, fully described in the literature, not any more as a consequence of the Newton's law of gravitation, but as a standalone kinematics theorem. With this new perspective we show that the three laws of Kepler are satisfied, as well as the Newton's gravitational acceleration, the Galileo's principle of equivalence and the structure of the classical mechanical energy. Furthermore it provides a simple alternative method to solve the orbit determination. We then demonstrated that the pure kinematics, with no help of any postulate, is consistent with all the experimental observations.

However it also shows that some famous human postulates are not fully correct. Although the kinematics forecast the mathematical structure of the Newton's acceleration, it describes the constant GM rather as the kinematics term Lv_R, which is mathematically less restrictive than the Newton's postulate. This enables to explain the rotation of galaxies without postulate, but with an extension of the Plank-Einstein relation at a macroscopic scale. This also opens the door to an explanation of the Coulomb's force which has the same mathematical structure as the Newton's force, but with a different factor than GM. Concerning the Einstein's principle of equivalence, the kinematics are more straight forward : it can not be correct, because the gravitational acceleration causes the rotation while the mechanical acceleration can only provide the translation. They are then of different natures.

We must insist on the fact that we do not pretend that Newton and Einstein are partially wrong, but we report that the kinematics demonstrate it. We used no postulate to get this result, but only the logic of the kinematics. We used a theorem not a theory, and then you can easily refuse it : just demonstrate that the theorem is false. You can do the same with any theorem, the one of Pythagoras for instance : if you disagree with it, just demonstrate that it is false. This is the beauty of theorems with regards to theories, they can be proven as false, while a postulate can not be proven neither true nor false. But if you can not prove that a theorem is false, you must admit it as far as you are a scientist, whatever your human opinion about it. Let us remind that the ultimate aim of the science is the interpretation of the universe without any human postulate, whoever the pure genius stating it.

Of course such a theorem alone can not pretend to be a complete alternate gravitational theory, competing with those of Newton and Einstein. For instance a central question remains unexplained by the kinematics : why is the mass causing the gravitation ? The theorem just shows the geometric characteristics of the Keplerian motion that any theory of the gravitation must at least respect, but not more. We demonstrate that the gravitation is not a matter of attraction, but of rotation, the Newton's acceleration being only the consequence of the rotation velocity induced by the gravitation. However the only kinematics do not tell us why the nature is working so. A lonely theorem is therefore insufficient to explain the whole gravitation, but it is very useful to guide us towards a better understanding, and finally a better overall theory.

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