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Theorem of the Keplerian kinematics

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Abstract As described in the literature the velocity of a Keplerian orbiter on a fixed orbit is always the sum of a uniform rotation velocity and a uniform translation velocity, both coplanar. This property is stated here as a theorem and demonstrated as true. The consequences are investigated among which the Newton's law of gravitation appears as its derivative with respect to time, the classical mechanical energy is deduced, the Galileo's equivalence principle is respected, an alternative to the Lambert's problem emerges.

Keywords Kinematics · Laws of Kepler · Gravitation · Orbital dynamics

1 Introduction

Although the three laws of Kepler are widely known [1], it exists a special property of the Keplerian motion for a fixed conic that is too often forgotten : the velocity is simply the addition of a uniform circular and a uniform translation velocities. This kinematics aspect of the motion, fully referenced in the literature [2–8], is generally quoted by the means of the hodograph plane representation, although some authors as R.H. Battin state it in a different way [9]. In all cases this special property is presented as a consequence of the Newton's law of gravitation. From this point of view it appears somehow difficult and complex to use, although it could be a powerful tool when presented in an other way.

Our aim in this article is to inverse the usual vision of this property by stating it as a theorem, and then showing its consequences, among which the Newton's law of gravitation appears, as a its trivial derivation with respect to

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time, the classical expression of the mechanical energy emerges as well as an alternative to the Lambert's problem.

In no way at all we will pretend that such a vision of the Keplerian motion could be a satisfactory gravitation theory that could compete with the Newton's or Einstein's ones. Our purpose is only to perform a pure kinematics study without any postulate, nor assumption, nor theory.

We will first state and proof the theorem, and second we will investigate some of its consequences.

2 Stating and proving the theorem

2.1 Statement

Accordingly to what is demonstrated in the literature [2–8] about the mathematical structure of the Keplerian motion, we deduce the possibility of writing the following theorem :

Theorem 1 *The velocity of a Keplerian orbiter on a fixed orbit is always the sum of a uniform rotation velocity and a uniform translation velocity, both coplanar.*

We do not know why nature choosed to exhibit this behavior, and this is not the purpose of this paper, but the works of the literature show that it applies to all Keplerian motions. Our work here will consist to prove that it is mathematically true, i.e. that it fully describes the Keplerian motion.

At a kinematics point of view such a velocity will be written as follows :

$$\mathbf{v} = \mathbf{v}_R + \mathbf{v}_T \tag{1}$$

where $\mathbf{v}_R = \boldsymbol{\omega} \times \mathbf{r}$ is the uniform rotation velocity (its norm is constant), with $\boldsymbol{\omega}$ being the frequency of rotation, $\mathbf{r}$ being the vector radius from the focus of the orbit to the orbiter, and $\mathbf{v}_T$ is the uniform translation velocity (its norm and direction are constant).

It is important to remember that the indice R does not stand for radial, but for rotation, while the indice T does not stand for tangential but for translation. The figure 1 exhibits the different velocities at 4 steps of a conic trajectory. Note that the translation velocity is always perpendicular to the main axis of the conic.

Of course the rotation velocity being uniform by statement, and the frequency of rotation being perpendicular to the velocity/radius plane, we must verify :

$$v_R = \|\mathbf{v}_R\| = \|\boldsymbol{\omega} \times \mathbf{r}\| = \omega r = constant \tag{2}$$

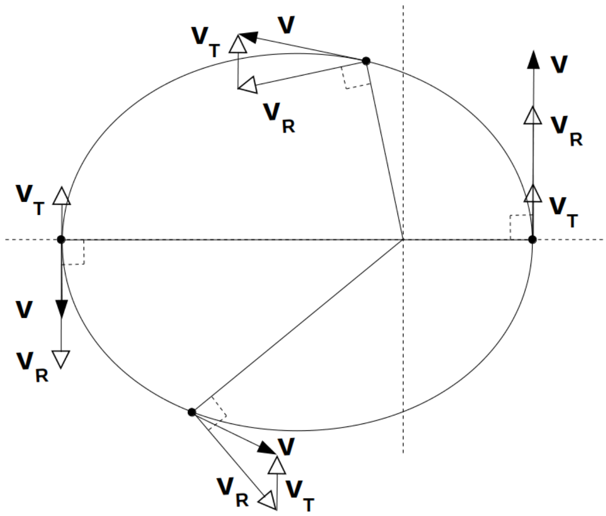

Fig. 1 The velocity of a Keplerian orbiter $\mathbf{v}$ on a fixed orbit is always the sum of a uniform rotation velocity $\mathbf{v}_R$, perpendicular to the vector radius, and a uniform translation velocity $\mathbf{v}_T$, which direction is always perpendicular to the main axis of the conic. Both are coplanar and have a constant norm all along the trajectory.

Derivating this last relationship with report to time we get a trivial but important expression :

$$\dot{\omega}r + \omega\dot{r} = 0 \tag{3}$$

The scalar $\dot{\omega}$ shall correspond to a vector $\dot{\boldsymbol{\omega}}$ which is collinear to the vector $\boldsymbol{\omega}$. Finally because the translation velocity is also uniform by statement we can write

$$v_T = \|\mathbf{v}_T\| = constant \tag{4}$$

This being stated, we are now going to give the proof of this theorem.

2.2 Proof

2.2.1 The angular momentum and its constancy

We define the angular momentum $\mathbf{L}$ as follows :

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} \tag{5}$$

This angular momentum does not refer to the mass as it is only a kinematics vector. R.H. Battin called it *the massless angular momentum* [10]. It is trivial to see that its derivation with respect to time, by including the relation 3, is null, thus the angular momentum is constant as expected for a central field motion.

2.2.2 First law of Kepler

The vector multiplication of the rotation velocity by the momentum leads to

$$\mathbf{v}_R \times \mathbf{L} = v_R^2 \left(1 + \frac{\mathbf{v}_R . \mathbf{v}_T}{v_R^2} \right) \mathbf{r} \tag{6}$$

Therefore the scalar version of this expression is

$$\frac{L}{v_R} = \left(1 + \frac{v_T}{v_R} \cos\theta \right) r \quad \text{or} \quad p = (1 + e\cos\theta)\, r \tag{7}$$

This last equation is the one of a conic where $p = L/v_R$ is the semilatus rectum, $e = v_T/v_R$ is the eccentricity and θ is the angle between the directions of the rotation and the translation velocity, i.e. the true anomaly. We see that both p and e are constant and therefore the equation 7 is nothing else but the first law of Kepler [1].

The theorem 1 also provides an elegant way to describe the eccentricity vector, thus the direction of the periapsis, by the mean of the translation velocity :

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{L}}{k} - \frac{\mathbf{r}}{r} = \frac{\mathbf{v}_T \times \mathbf{L}}{k} \text{ with } k = L v_R \tag{8}$$

2.2.3 Second law of Kepler

The second Kepler's law derives simply from the constancy of the angular momentum, demonstrated above. As explained by L. Landau and E. Lifchitz [1], the momentum can also be written as a function of the position and the derivative of the true anomaly with respect to time :

$$L = r^2 \dot{\theta} \tag{9}$$

In this expression we see that the angular momentum is worth twice the areal velocity. The first being constant, the second will also be, which is the second law of Kepler.

From the equation 7 it is trivial to relate the angular frequency of rotation to the derivative of the true anomaly with respect to time :

$$\dot{\theta} = \omega(1 + e\cos\theta) = \omega \frac{p}{r} \tag{10}$$

2.2.4 Third law of Kepler

The third Kepler's law also derives simply from the constancy of the angular momentum [1]. Indeed the integration with respect to time of the relation 9 , over a complete period T of revolution, gives

$$LT = \int_0^{2\pi} r^2 \, \mathrm{d}\theta \tag{11}$$

For the case where the trajectory is an ellipse, the right side of this equation is worth $2\pi a\, b$, where a is the major semi axis and b the minor one. Knowing that $a = p/(1-e^2)$ and $b = 1/\sqrt{1-e^2}$, and remembering the definition of the semilatus rectum p given by the equations 7, it is easy to finally get the following relation :

$$L\, v_R = 4\pi^2 a^3/T^2 \tag{12}$$

Because L and v_R are constants, this last expression is nothing else but the third law of Kepler stating that the square of the period of revolution is proportional to the cube of the major semi axis [1].

3 Consequences

3.1 Newton's law of gravitation

When we derive the equation 1 with respect to time, we get the acceleration $\mathbf{a}$ of a Keplerian orbiter : $\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}$. Now including the equation 3 we can write $\mathbf{a} = -(\boldsymbol{\omega}/r^2) \times [\mathbf{r} \times (\mathbf{r} \times \mathbf{v})]$, and finally

$$\mathbf{a} = -\frac{L\, v_R}{r^3}\mathbf{r} \tag{13}$$

This is the expression of the Newton's gravitational acceleration if

$$L\, v_R = G\, M \tag{14}$$

where G is the constant of gravitation and M is the attracting mass. We can also notice that the equation 14 is consistent with the expression 12 of the third Kepler's law when compared to the one of the literature [1].

As we see here the kinematics does agree with the mathematical structure of the Newton's acceleration, but this last is not any more a prior to the existence of the velocity. As explained by the theorem 1, it only becomes a trivial consequence, the centripetal acceleration due to the rotation velocity. In a sense we can say that, from a kinematics point of view, the gravitation law is not a law of attraction, but a law of rotation.

The Newton's postulate proposing that $G\, M$ should be the numerator of expression 13 can not be reached, nor discussed, by the kinematics that describe it rather as $L\, v_R$, both factors being however accordingly constant.

3.2 Galileo's equivalence principle

Galileo has shown in the early 17th century that the motion in a gravitational field is mass independent. This is quite consistent with the kinematics structure of the Keplerian motion as the equation 1 is also mass independent.

A body falling, for instance from the top of the tower of Pisa, must follow the equation 1. If the starting velocity of the body is null, we have $\mathbf{v}_T = -\mathbf{v}_R$, which means $e = 1$ in equation 7. If we drop the object, $\mathbf{v}_T$ will slightly decrease, and therefore the mobile will enter a conic with an eccentricity close but lower than 1. The body will fall on a sharp ellipse which focus is at the center of gravity of the two implied bodies, the falling object and the Earth, so nearly at the center of the Earth. Of course at such a distance from the Earth's center, and locally from the top of the tower to the ground, such a conic trajectory could be confused with a straight line at a first approximation.

3.3 Mechanical energy

If we develop the square of the equation 1, and include the result 7, it is trivial to define the massless mechanical energy E_M as follows :

$$E_M = \frac{1}{2}v^2 - \frac{L\,v_R}{r} = \frac{1}{2}v_R^2(e^2 - 1) \tag{15}$$

This expression is interesting because it describes the classical mechanical energy (divided by the mass of the orbiter) made of the addition of the usual kinetic and potential parts. It also shows, with its right member, that this energy is a constant for a fixed conic. Therefore the kinematics does agree with the classical physics of the gravitation [1], as far as, once again, the relation 14 is true.

3.4 Orbit determination and Lambert's problem alternative

The usual way to determine the orbit of a celestial body is to solve the Lambert's problem, provided that you know the position and the velocity of the body at two different times [16]. Concurrently the theorem 1 requires the knowledge of the position and the velocity only at a single time to determine the orbit, provided that we know the value of the attracting mass M.

Indeed if we know $\mathbf{r}$ and $\mathbf{v}$ at a single time t, we can trivially calculate the angular momentum $\mathbf{L}$, and thus get directly the rotation velocity v_R by the means of the equation 14. Now the direction of the rotation velocity must be perpendicular to the vector radius, then we can calculate the vector $\mathbf{v}_R$, so the translation velocity from equation 1 with $\mathbf{v}_T = \mathbf{v} - \mathbf{v}_R$. At this point we are able to calculate the eccentricity, the semilatus rectum, and the true anomaly by the means of the equation 7, i.e. $e = v_T/v_R$, $p = L/v_R$ and $\theta = \arccos[(1/e)(p/r - 1)]$. We then get all the characteristics needed to draw the complete conic of the orbiter, including the direction of the periapsis given by the equation 8.

Of course once we have the former informations it is easy to calculate the velocity of the mobile at any other position on the conic, and this is useful to resolve other problems as space rendezvous, which are usually also handled by the mean of the Lambert's problem [17]. This subject deserves a complete article to be fully discussed from the present kinematics point of view.

The theorem 1 can then be a fine alternative to the Lambert's method, as it requires less calculations, and moreover less input data to perform an orbit determination.

3.5 Accelerating an orbiter with a mechanical thrust

If an orbiter has a null translation velocity, it will only possess the uniform rotation velocity that is well described in the literature [1], i.e. $v = v_R = \sqrt{GM/r}$, which is consistent with the equations 7 and 14, and the eccentricity of the conic is of course null. This velocity is due to the gravitation and we can not get rid of it, as far as the attracting mass exists.

Let us now apply a very tiny thrust during a very short time, so we provide an impulsional momentum to the body, thus a translation velocity, in a specific direction. The consequence is a change of eccentricity (see the definition of the eccentricity in equation 7), and the orbiter can not stay on its initial circular orbit, whatever the direction of $\mathbf{v}_T$, thus of the thrust, is. This result is consistent with what is described in the literature when either a tangential or a radial thrust is applied to an orbiter [11,12].

Generalizing this, we know that the the acceleration equation for any thrusted trajectory of an orbiter is [13–15] :

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r} + \frac{\mathbf{F}}{m} \tag{16}$$

where $\mathbf{F}$ is the thrusting force applied to the orbiter, and m its mass. Straight forward integrating this expression with respect to time, regarding paragraph 3.1, we get :

$$\mathbf{v} = \mathbf{v}_R + \int_{t_0}^{t} \frac{\mathbf{F}}{m} dt \tag{17}$$

While $\mathbf{v}_R$ is the usual rotation velocity due to the gravitation, the integral in the right hand of this equation is of course nothing else but $\mathbf{v}_T$, and we find back the equation 1. This expression is a new way to represent the velocity of a thrusted orbiter.

A very interesting question here is can we produce an acceleration with a mechanical thrust that would have the mathematical structure of a gravitational acceleration, i.e. $a_{thrust} = k'/r^2$, where k' is a constant. By integration of the total acceleration, this would correspond to increasing the rotation velocity (see paragraph 3.1) and keeping null the translation velocity. All would then happen as if the attractive mass would have increased (see relation 14). The orbiter would still be on the same circular orbit as before the thrust, but it would go faster, and the astronauts inside the orbiter would still be in a weightlessness free fall. In these conditions a space rendezvous would not be a problem any more : first go to the desired circular orbit, and then accelerate to catch up the space station where you want to go.

Unfortunately the experiment has shown that this is impossible in the real world, whatever the type of engine used or the direction of the thrust, and furthermore, as far as we know, no one in the literature has ever described such a theoretical possibility. This is why the problem of the space rendezvous is still so complex to solve [17].

A series of micro-thrusts set to simulate a perfect rotation will only be a succession of rectilinear thrusts, changing the translation velocity $\mathbf{v}_T$ of the equation 1, therefore changing the eccentricity, and the direction of the main axis, at each micro-thrust, as short as it could be. If the direction of the thrust changes with the time, $\mathbf{v}_T$ will consequently change, and thus the characteristics of the conic. We then typically get a low-thrust trajectory structure, well known by the space ingineers placing satellites into orbit [13–15]. The experiment shows therefore that a mechanical acceleration is not equivalent to a gravitational acceleration. The gravitation provides the pure rotation $\mathbf{v}_R$ while the mechanical acceleration can only provide the translation $\mathbf{v}_T$, accordingly to the equations 1 and 17.

4 Conclusion

The aim of this article was to state a very well known property of the Keplerian motion, fully described in the literature, not any more as a consequence of the Newton's law of gravitation, but as a standalone kinematics theorem. With this new perspective we show that the three laws of Kepler are satisfied, as well as the Newton's gravitational acceleration, the Galileo's principle of equivalence and the structure of the classical mechanical energy. Furthermore it provides a simpler method than the Lambert's problem to solve the orbit determination.

Of course such a theorem can not pretend to be an alternate gravitational theory, competing with those of Newton and Einstein. At the contrary of these lasts we are stating no postulate, nor assumption, nor theory, but just studying the pure kinematics, so the geometry, and that is all. In these conditions we observe that the simple kinematics can not answer to the most fundamental questions as why such a geometric property, the equation 1, can arise from nature. Obviously what we called here *the attracting mass* must have a key role but the only kinematics is unable to determine which one, and even less why.

Therefore the theorem of the Keplerian kinematics presented here will be mainly useful to simplify some orbital mechanics calculations, as the orbit determination or the space rendezvous, because it gives a simpler alternative to the Lambert's problem. It also provides a new equation to handle the problems of thrusted space motions.

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References

[1] L.Landau, E. Lifchitz, Mechanics, Ed. Mir, Moscow, 1966, 15
[2] Orbit information derived from its hodograph, J. B. Eades, Tech. Rep. TM X-63301, NASA (1968)
[3] W. R. Hamilton, The hodograph, or a new method of expressing in symbolic language the Newtonian law of attraction, Proc. R. Ir. Acad. III , 344353 (1845).
[4] H. Abelson, A. diSessa and L. Rudolph, Velocity space and the geometry of planetary orbits, Am. J. Phys. 43 , 579-589 (1975).
[5] A. Gonzalez-Villanueva, H. N. Nunez-Yepez, and A. L. Salas-Brito, In veolcity space the Kepler orbits are circular, Eur. J. Phys. 17 , 168-171 (1996).
[6] T. A. Apostolatos, Hodograph: A useful geometrical tool for solving some difficult problems in dynamics, Am. J. Phys. 71 , 261-266 (2003).
[7] E. I. Butikov, The velocity hodograph for an arbitrary keplerian motion, Eur. J. Phys. 21 (2000) 1-10
[8] D. Derbes, Reinventing the wheel: Hodographic solutions to the Kepler problems, Am. J. Phys. 69 , 481-489 (2001).
[9] R. H. Battin, An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition, American Institute of Aeronautics and Astronautics, Inc., Reston, 1999, 3.5, 126
[10] R. H. Battin, An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition, American Institute of Aeronautics and Astronautics, Inc., Reston, 1999, 3.3, 115
[11] Karel F. Wakker, Fundamentals of astrodynamics, Institutional Repository Library, Delft University of Technology, 2015, 211-217
[12] Pramod S. Joag, An Introduction to Vectors, Vector Operators and Vector Analysis, Cambridge University Press, 2016, 12.4.3, 473
[13] R. H. Battin, An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition, American Institute of Aeronautics and Astronautics, Inc., Reston, 1999, 11.7, 550
[14] Karel F. Wakker, Fundamentals of astrodynamics, Institutional Repository Library, Delft University of Technology, 2015, 555
[15] B. Bonnar et al, Geometric optimal control of elliptic Keplerian orbits, Discrete and continuous dynamical systems B, 2005, vol 5, No 4, 930
[16] E. R. Lancaster et R. C. Blanchard, A Unified Form of Lamberts Theorem, Goddard Space Flight Center, 1968
[17] Aldrin, Buzz (1963). Line-of-sight guidance techniques for manned orbital rendezvous (Sc.D.). MIT. Archived from the original on May 11, 2015.