myjournal manuscript No.
(will be inserted by the editor)

Theorem of the Keplerian kinematics

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Received: date / Accepted: date

Abstract As described in the literature the speed of a Keplerian orbiter on a fixed orbit is always the sum of a uniform rotation speed and a uniform translation speed, both coplanar. This property is stated here as a theorem and demonstrated as true. The consequences of this theorem are investigated among which the Newton's law of gravitation appears as its derivative with respect to time, the classical mechanical energy is deduced, the Galileo's equivalence principle is respected, the conic determination is simplified, as well as the description of the motion of thrusted orbiters.

Keywords Kinematics · Laws of Kepler · Gravitation · Orbital dynamics

1 Introduction

Although the three laws of Kepler are widely known [1], it exists a special property of the Keplerian motion for a fixed conic that is too often forgotten : the speed is simply the addition of a uniform circular and a uniform translation speeds. This kinematics aspect of the motion, fully referenced in the literature [2–8], is generally quoted by the means of the hodograph plane representation, although some authors as R.H. Battin state it in a different way [9]. In all cases this special property is presented as a consequence of the Newton's law of gravitation. From this point of view it appears somehow difficult and complex to use, although it could be a powerful tool when presented in an other way.

Our aim in this article is to inverse the usual vision of this property by stating it as a theorem, and then showing its consequences, among which the Newton's law of gravitation appears, as a its trivial derivation with respect to

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time, the classical expression of the mechanical energy emerges and the orbit determination is simplified as well as the velocity of a thrusted orbiter.

In no way at all we will pretend that such a vision of the Keplerian motion could be a satisfactory gravitation theory that could compete with the Newton's or Einstein's ones. Our purpose is only to perform a pure kinematics study without any postulate, nor assumption, nor theory.

2 Stating and proving the theorem

2.1 Statement

Let us state the following theorem :

Theorem 1 *The speed of a Keplerian orbiter on a fixed orbit is always the sum of a uniform rotation speed and a uniform translation speed, both coplanar.*

At a kinematics point of view such a speed will be written as follows :

$$\mathbf{v} = \mathbf{v}_R + \mathbf{v}_T \tag{1}$$

where $\mathbf{v}_R = \boldsymbol{\omega} \wedge \mathbf{r}$ is the uniform rotation speed, with $\boldsymbol{\omega}$ being the frequency of rotation and $\mathbf{r}$ being the vector radius from the focus of the orbit to the orbiter, and $\mathbf{v}_T$ is the uniform translation speed. It is important to remember that the indice R does not stand for radial, but for rotation, while the indice T does not stand for tangential but for translation. Of course the rotation speed being uniform, and the frequency of rotation being perpendicular to the speed/radius plane, we must verify :

$$v_R = \|\mathbf{v}_R\| = \|\boldsymbol{\omega} \wedge \mathbf{r}\| = \omega r = cste \tag{2}$$

Derivating this last relationship by the time we get a trivial but important expression :

$$\dot{\omega} r + \omega \dot{r} = 0 \tag{3}$$

The scalar $\dot{\omega}$ shall correspond to a vector $\dot{\boldsymbol{\omega}}$ which is collinear to the vector $\boldsymbol{\omega}$. Finally because the translation speed is also uniform we can write

$$v_T = \|\mathbf{v}_T\| = cste \tag{4}$$

This being stated, we are now going to give the proof of this theorem.

2.2 Proof

2.2.1 The angular momentum and its constancy

We define the angular momentum $\mathbf{L}$ as follows :

$$\mathbf{L} = \mathbf{r} \wedge \mathbf{v} \tag{5}$$

This angular momentum does not refer to the mass as it is only a kinematics vector. R.H. Battin called it *the massless angular momentum* [10]. It is trivial to see that its derivation with respect to time, by including the relation 3, is null, thus the angular momentum is constant as expected for a central field motion.

2.2.2 First law of Kepler

The vector multiplication of the rotation speed by the momentum leads to

$$\mathbf{v}_R \wedge \mathbf{L} = v_R^2 \left(1 + \frac{\mathbf{v}_R . \mathbf{v}_T}{v_R^2} \right) \mathbf{r} \tag{6}$$

Therefore the scalar version of this expression is

$$\frac{L}{v_R} = \left(1 + \frac{v_T}{v_R} \cos\theta \right) r \quad \text{or} \quad p = (1 + e\cos\theta)\, r \tag{7}$$

This last equation is the one of a conic where $p = L/v_R$ is the semilatus rectum, $e = v_T/v_R$ is the eccentricity and θ is the angle between the directions of the rotation and the translation speed, i.e. the true anomaly. We see that both p and e are constant and therefore the equation 7 is nothing else but the first law of Kepler [1].

2.2.3 Second law of Kepler

The second Kepler's law derives simply from the constancy of the angular momentum, demonstrated above. As explained by L. Landau and E. Lifchitz [11], the momentum can also be written as a function of the position and the derivative of the true anomaly with respect to time :

$$L = r^2 \dot{\theta} \tag{8}$$

From the equation 7 it is trivial to relate the angular frequency of rotation to the derivative of the true anomaly with respect to time :

$$\dot{\theta} = \omega(1 + e\cos\theta) \tag{9}$$

2.2.4 Third law of Kepler

The third Kepler's law also derives simply from the constancy of the angular momentum [1]. Indeed the integration with respect to time of the relation 8 , over a complete period T of revolution, gives

$$LT = \int_0^{2\pi} r^2 \, \mathrm{d}\theta \qquad (10)$$

For the case where the trajectory is an ellipse, the right side of this equation is worth $2\pi a\, b$, where a is the major semi axis and b the minor one. Knowing that $a = p/(1-e^2)$ and $b = 1/\sqrt{1-e^2}$, and remembering the definition of the semilatus rectum p given by the equations 7, it is easy to finally get the following relation :

$$L\, v_R = 4\pi^2 a^3/T^2 \qquad (11)$$

Because L and v_R are constants, this last expression is nothing else but the third law of Kepler stating that the square of the period of revolution is proportional to the cube of the major semi axis [1].

3 Consequences

3.1 Newton's law of gravitation

An interesting point is to derive the equation 1 with respect to time in order to get the acceleration $\boldsymbol{\gamma}$ of a Keplerian orbiter. We get $\boldsymbol{\gamma} = \dot{\boldsymbol{\omega}} \wedge \mathbf{r} + \boldsymbol{\omega} \wedge \mathbf{v}$, and including the equation 3 we can write $\boldsymbol{\gamma} = -(\boldsymbol{\omega}/r^2) \wedge [\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{v})]$, and finally

$$\boldsymbol{\gamma} = -\frac{L\, v_R}{r^3}\mathbf{r} \qquad (12)$$

This is the expression of the Newton's gravitational acceleration if

$$L\, v_R = G\, M \qquad (13)$$

where G is the constant of gravitation and M is the attracting mass. We can also notice that the equation 13 is consistent with the expression 11 of the third Kepler's law when compared to the one of the literature [1].

As we see here the kinematics does agree with the mathematical structure of the Newton's acceleration, but this last is not any more a prior to the existence of the velocity, as described by the theorem 1. It only becomes a trivial consequence, the centripetal acceleration due to the rotation speed. The Newton's postulate proposing that $G\, M$ should be the numerator of expression 12 can not be reached, nor discussed, by the kinematics that describe it rather as $L\, v_R$, both factors being however constant.

3.2 Galileo's equivalence principle

Galileo has shown in the early 17th century that the motion in a gravitational field is mass independent. This is quite consistent with the kinematics structure of the Keplerian motion as the equation 1 is also mass independent. A body falling, for instance from the top of the tower of Pisa, must follow the equation 1. If the starting speed of the body is null, we have $\mathbf{v}_T = -\mathbf{v}_R$, which means $e = 1$ in equation (7), and therefore the body will fall on a parabola which focus is at the center of gravity of the two implied bodies, the falling object and the Earth, so nearly at the center of the Earth. Of course at such a distance from the Earth's center, and locally from the top of the tower to the ground, the parabolic trajectory could be confused with a straight line at a first approximation.

3.3 Mechanical energy

If we develop the square of the equation 1, and include the result 7, it is trivial to define the massless mechanical energy E_M as follows :

$$E_M = \frac{1}{2}v^2 - \frac{L\,v_R}{r} = \frac{1}{2}v_R^2(e^2 - 1) \tag{14}$$

This expression is interesting because it describes the classical mechanical energy (divided by the mass of the orbiter) made of the addition of the usual kinetic and potential parts. It also shows, with its right member, that this energy is a constant for a fixed conic. Therefore the kinematics does agree with the classical physics of the gravitation [1], as far as, once again, the relation 13 is true.

3.4 Conic determination

A very interesting property of the theorem 1 is to give straight forward all the characteristics of a conic when knowing only the position and the speed of an orbiter at a single and unique time, provided that we know the value of the attracting mass M. Indeed if we know $\mathbf{r}$ and $\mathbf{v}$ at a single time t, we can trivially calculate the angular momentum $\mathbf{L}$, and thus get directly the rotation velocity v_R by the means of the equation 13. Now the direction of the rotation speed must be perpendicular to the vector radius, then we can calculate the vector $\mathbf{v}_R$. At this point it is also trivial to calculate the translation speed from equation 1 with $\mathbf{v}_T = \mathbf{v} - \mathbf{v}_R$. We can then calculate the eccentricity, the semilatus rectum, and the true anomaly by the means of the equation 7, i.e. $e = v_T/v_R$, $p = L/v_R$ and $\theta = \arccos[(1/e)(p/r - 1)]$. We then get all the characteristics needed to draw the complete conic of the orbiter. Note by the way that the translation speed will always be collinear to the minor axis of the conic, this is obvious when looking at the equation 7.

3.5 Accelerating an orbiter with a mechanical thrust

If an orbiter has a null translation speed, it will only possess the uniform rotation velocity that is well described in the literature [1], i.e. $v = v_R = \sqrt{GM/r}$, which is consistent with the equations 7 and 13, and the eccentricity of the conic is of course null. This velocity is due to the gravitation and we can not get rid of it, as far as the attracting mass exists. Let us now apply a very tiny thrust during a very short time, in a specific direction. Obviously we add a new non null speed to the rotation one, and that respects the theorem 1. Therefore the eccentricity of the conic can not be null any more (see the definition of the eccentricity in equation 7) and consequently the orbiter can not stay on its initial circular orbit, whatever the direction of $\mathbf{v}_T$, thus of the thrust, is. This result is consistent with what is described in the literature when either a tangential or a radial thrust is applied to an orbiter [12,13]. Actually the orbiter will enter a conic, an ellipse if the thrust is small, which minor axis is collinear to $\mathbf{v}_T$.

Generalizing this, we know that the the acceleration equation for any thrusted trajectory of an orbiter is [14–16] :

$$\boldsymbol{\gamma} = -\frac{GM}{r^3}\mathbf{r} + \frac{\mathbf{F}}{m} \tag{15}$$

where $\mathbf{F}$ is the thrusting force applied to the orbiter, and m its mass. Straight forward, with the help of equation 12, and integrating this expression with respect to time, we get :

$$\mathbf{v} = \mathbf{v}_R + \int_{t_0}^{t} \frac{\mathbf{F}}{m} dt \tag{16}$$

While $\mathbf{v}_R$ is the usual rotation speed due to the gravitation, the integral in the right hand of this equation is of course nothing else but $\mathbf{v}_T$, and we find back the equation 1.

A very interesting question here is can we produce an acceleration with a mechanical thrust that would have the mathematical structure of a gravitational acceleration, i.e. $\gamma_T = k/r^2$, where k is a constant. This means that $\mathbf{v}_T$ should also be a rotation speed instead of a translation one. In this case we would be able to accelerate the orbiter by keeping it on the same circular orbit with a total acceleration $\gamma = (GM + k)/r^2$ (see equation 15). Everything would happen as if the attracting mass would have increased, and of course some astronauts inside the orbiter would not feel any acceleration, as they would be still in free fall, in the same way as if the thrust would not exist, but they would be falling faster, i.e. orbiting faster. However as far as we know, no one has ever described this theoretical possibility in the literature, neither it has been experimentally performed, whatever the type of engine used. Until this has been achieved we should then conclude that a mechanical acceleration is not equivalent to a gravitational acceleration. The gravitation provides the rotation while the mechanical acceleration provide the translation. If the direction of the thrust changes with the time, $\mathbf{v}_T$ will consequently change, and thus the characteristics of the conic. We then typically get a patched conic structure for the trajectory.

4 Conclusion

The aim of this article was to state a very well known property of the Keplerian motion, fully described in the literature, not any more as a consequence of the Newton's law of gravitation, but as a standalone kinematics theorem. With this new perspective we show that the three laws of Kepler are satisfied, as well as the Newton's gravitational acceleration, the Galileo's principle of equivalence and the structure of the classical mechanical energy. Furthermore the simplicity of the theorem enables to determine easily, in a new fashion, the characteristics of any Keplerian conic and to simplify a bit more the problem of thrusting a space orbiter.

Of course such a theorem can not pretend to be an alternate gravitational theory, competing with those of Newton and Einstein. At the contrary of these lasts we are stating no postulate, nor assumption, nor theory, but just studying the pure kinematics, so the geometry, and that is all. In these conditions we observe that the simple kinematics can not answer to the most fundamental questions as why such a geometric property, the equation 1, can arise from natures. Obviously what we called here *the attracting mass* must have a key role but the only kinematics is unable to determine which one, and even less why.

Therefore the theorem of the Keplerian kinematics presented here is mainly intended to simplify the calculations concerning the orbital mechanics, even if it could be used as a side help to the theorists who are working on the foundations of the gravitation.

Acknowledgements I wish to thank professor David B. Spencer, form the Department of Aerospace Engineering, at Penn State University, who's attention and encouragements were key factors for writing this article.

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