Theorem of the keplerian kinematics

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Abstract : Any mobile having a velocity which is the addition of a rotation velocity and a translation velocity, both with a constant modulus, will follow a trajectory that respects the three laws of Kepler. This article demonstrate this theorem and discuss it. An important result is to forecast the mathematical structure of the Newton's acceleration of attraction, not any more as a prior, but as a consequence, the subsequent centripetal acceleration due to the rotation velocity.

1 Introduction

Since the work of Kepler we know that, at a first approximation, the trajectories of all celestial satellites are following three peculiar laws[1]. These laws are kinematic ones, they do not refer to any physical consideration, as the mass for instance. We can then expect to forecast them only from the kinematics.

This is what we are going to achieve here by the mean of a kinematic theorem that applies to all keplerian mobiles. We will not postulate any physical reason to explain the existence of this theorem in the real world, but just demonstrate its validity from a mathematical point of view.

We will see however that this theorem forecasts the Newton's attraction law as a consequence, but not any more as a mandatory foundation of the keplerian motion.

2 Theorem of the keplerian kinematics

2.1 Statement

Let us state the following theorem :

Theorem 1 : *Any mobile having a velocity which is the addition of a rotation velocity and a translation velocity, both with a constant modulus, will follow a trajectory that respect the three laws of Kepler.*

At a mathematical point of view the velocity described by this theorem is written as follows :

$$\mathbf{v}=\mathbf{v_R}+\mathbf{v_T}$$
$$\text{with}\quad \mathbf{v_R}=\boldsymbol{\omega}\wedge\mathbf{r}\quad,\quad \|\mathbf{v_R}\|=v_R=\omega r=\text{cste}\quad\text{and}\quad \|\mathbf{v_T}\|=v_T=\text{cste} \tag{1}$$

where $\mathbf{v_R}$ is the rotation velocity, $\mathbf{v_T}$ is the translation velocity, $\boldsymbol{\omega}$ is the frequency of rotation and $\mathbf{r}$ is the vector radius.

2.2 Proof

To prove the validity of this theorem we have to demonstrate that the relation (1) forecasts the three laws. This is what we are going to achieve but we need first to define the momentum and the acceleration for this type of motion.

2.2.1 Momentum and acceleration

We define the vector $\mathbf{L}$ as follows :

$$\mathbf{L}=\mathbf{r}\wedge\mathbf{v} \tag{2}$$

We call it "kinematic momentum" as a reference to the well known physical "kinetic momentum"[2] $\mathbf{M}=m\,\mathbf{r}\wedge\mathbf{v}$, where m is the mass of the mobile. Note that the kinematic momentum $\mathbf{L}$ is collinear to the frequency of rotation, but it is mass independent.

Concerning the acceleration $\boldsymbol{\gamma}$, as far as the translation velocity is a constant, there is no translation acceleration, and the derivative of the relation (1) with respect to time is $\boldsymbol{\gamma}=\dot{\boldsymbol{\omega}}\wedge\mathbf{r}+\boldsymbol{\omega}\wedge\mathbf{v}$. Because $\omega r=\text{cste}$ this expression becomes $\boldsymbol{\gamma}=-\frac{\boldsymbol{\omega}}{r^2}\wedge[\mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{v})]$ and finally :

$$\boldsymbol{\gamma}=-\frac{L\,v_R}{r^3}\mathbf{r} \tag{3}$$

This expression shows that the acceleration and the vector radius are collinear. This fact forces the kinematic momentum to be constant because its derivative with respect to time, $\dot{\mathbf{L}}=\mathbf{r}\wedge\boldsymbol{\gamma}$, is then null. We can note that this expression of the acceleration is also consistent with the mathematical structure of the acceleration of the Newton's attraction[1]. We will discuss this property later on.

2.2.2 First law of Kepler

The vector multiplication of the rotation velocity and the kinematic momentum leads to :

$$\mathbf{v_R}\wedge\mathbf{L}=v_R^2\left(1+\frac{\mathbf{v_R}\cdot\mathbf{v_T}}{v_R^2}\right)\mathbf{r} \tag{4}$$

Thus the modulus of this last expression is

$$\frac{L}{v_R} = \left(1 + \frac{v_T}{v_R}\cos\theta\right) r \quad \text{or} \quad p = (1 + e\cos\theta)\, r \tag{5}$$

This last equation is the one of a conic if $p = L/v_R$ is the parameter of the orbit, $e = v_T/v_R$ is its eccentricity and θ is the angle between the directions of $\mathbf{v_T}$ and $\mathbf{v_R}$, i.e. the true anomaly. Because L, v_T and v_R are constant p and e are also constant. The relation (5) therefore agrees with the first law of Kepler stating that the trajectory must be a conic[1].

2.2.3 Second law of Kepler

The second law, or area law, derives from the constancy of the kinematic momentum. As explained by L. Landau and E. Lifchitz[3], the momentum can also be written as a function of the position and the derivative of the true anomaly with respect to time :

$$L = r^2 \dot{\theta} \tag{6}$$

The right side of this last equation being the double of the areal velocity, and the momentum being a constant, the areal velocity must also be a constant. This is the second law of Kepler[1].

2.2.4 Third law of Kepler

The integration with respect to time of the relation (6) , over a complete revolution, gives

$$L\,T = \int_0^{2\pi} r^2\, d\theta \tag{7}$$

For the case where the trajectory is an ellipse, the right side of this equation is worth $2\pi a b$, where a is the major semi axis and b the minor one. Knowing then that $a = p/(1-e^2)$ and $b = p/\sqrt{1-e^2}$, it is easy to finally get the following relation :

$$L\, v_R = 4\pi^2 a^3 / T^2 \tag{8}$$

Because L and v_R are constant this last expression agrees with the third law of Kepler stating that the square of the period of revolution is proportional to the cube of the major semi axis[1].

3 Discussion

We demonstrated that the theorem 1 forecasts the three laws of Kepler. So far the only way to explain them was the Newton's postulate of attraction, and of course the Einstein's general relativity, which resumes to the Newton's law for slow velocities, regarding the speed of light. When we first state the Newton's postulate as the reason of the keplerian motion, it is possible to demonstrate the existence of the relation (1). Consequently the literature has already noticed that the relation (1) exists[4-10] but the authors present it always as a consequence of the Newton's law, not as a prior.

The kinematic point of view presented here inverses this assumption, making the Newton's acceleration a consequence, the derivative with rapport to time, of the relation (1). This acceleration is only the subsequent centripetal acceleration due to the rotation velocity. Therefore from the kinematic point of view, regarding the equation (3), the Newton's postulate of attraction is reduced to set up the only following assumption :

$$L\, v_R = G\, M \tag{9}$$

where G is the constant of gravitation and M the attracting mass. Usually the most remarkable part of the Newton's postulate is considered to be the “inverse square law”, i.e. the dependency of the acceleration toward the inverse square of the distance to the attracting mass. However for the kinematics this inverse square characteristic is only a geometrical consequence of the theorem 1, so it is not a postulate any more but a kinematic trivial result. Nonetheless the other part of the Newton's assumption consists to state that the coefficient of proportionality should be GM , instead of the strictly expected $L\, v_R$, and this represents indeed a postulate with regards to the kinematics.

At this point we may wonder if the Newton's assumption is true for for all masses, at all scales, while the theorem 1, and thus the equation (3), are always true for all masses at all scales. About the mass, note that remarkably the kinematic approach is consistent with the Galileo's principle of equivalence stating that the motion in a gravitational field is mass independent. Indeed the theorem 1 is also mass independent.

Of course the theorem 1 alone does not explain all the subtleties of the gravitation (precession, nutation, many body problem, ...). It is only a fundamental brick describing the simple pure keplerian motion. It is exactly at the same position as the Newton's law of attraction before the invention of the Lagrangian mechanics. The theorem 1 should be a new starting point to have a new look at the gravitation. For instance if the translation velocity v_T is replaced by a rotation velocity $v_T = \omega_T \wedge r_T$, the precession appears. So this theorem has to be investigated moreover in order to get all that the kinematics can give us.

Even if the theorem 1 proposes a new vision of the gravitation, it is not a physics theory. Sure it is not a postulate, but the fundamental reason why nature chooses to set it up, in the real world, is still a remaining question, that will certainly need a physics postulate to be answered. Any way, whatever this postulate could be, the physicist can not ignore the kinematics, therefore he has to take the theorem 1, and its consequences, into account.

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