On the algebraic property of Brauer character

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ABSTRACT: In this paper we solve a previously formulated conjecture. That is, for the cocommutative irreducible coalgebra $C$, we also introduce the azumaya coalgebras over a cocommutative coalgebra $R$. The computation of the brauer group $BM$ of modified supergroup algebras over a field is performed, yielding the computation of the brauer group of all finite-dimensional triangular hopf algebra. Then we address the question if the cartan matrix of a block of a finite group cannot be arranged as a direct sum of smallest matrices. The aim of our work is to show that the topological hochschild homology of a discrete ring and the maclane homology of $R$ are isomorphic.

INTRODUCTION

We construct unramified function fields over an algebraically closed field of characteristic 0 such that the unramified cohomology group is not trivial. This implies that the field $K$ is not stably rational. For this purpose, we give a sufficient condition for an element to be unramified in the cohomology group and this condition relies on computations in the exterior algebra of a vector space of finite dimension over the finite field $F_{p}$. After a short excursion into the realm of group cohomology, we shall construct and examine the so-called kummer sequence. This sequence of algebraic groups produce an exact sequence of groups when applied to the algebraic closure of a fixed field $k$. The mordell-weil theorem is a fundamental result in the arithmetic of elliptic curves defined over a number field $K$, describing the structure of their $K$-valued points.

We define the cohomology groups of a topological space $X$ with coefficient in a sheaf of abelian group in terms of the derived functors of the global section functor. Using strong equivalence for coalgebras we define the strong brauer group of a cocommutative coalgebra $C$, which is a subgroup of the brauer group $C$.

1. The Brauer group in coalgebra

1.1 Morita-Takeuchi equivalence and the Azumaya coalgebra

Theorem 1.1.1: (see lemma 2.3 in [1])

Theorem 1.1.2: The following sequence is an injective resolution of $TV$ as $TV$-comodule.

$$0 \rightarrow TV \xrightarrow{\lambda} TV \otimes TV \xrightarrow{\delta} TV \otimes V \otimes TV \rightarrow 0$$

Theorem 1.1.3: If $C$ is an involutive $k$-coalgebra and $M$ is a comodule then,

$$Hoch(TV, C, M) = Cotor_{c}(M, C)$$

Firstly, we try to introduce the results in [1] to handle the azumaya coalgebra. That is, the following program:

$$
\begin{align*}
Q \otimes P \otimes M &\xrightarrow{\phi} P \otimes M \otimes Q \\
Q \otimes P \otimes C^{e} \otimes M &\rightarrow P \otimes C \otimes M \otimes C \otimes Q
\end{align*}
$$
Where, \( \phi \) is an isomorphism. And this fact lead us to introduce the concept of descent data.

so that:

\[
P_{24} = M \rightarrow C \otimes M, \quad P_{34} = Q \rightarrow C \otimes Q, \quad P_{21} = M \rightarrow P \otimes M, \quad P_{31} = Q \rightarrow P \otimes Q
\]  \hspace{1cm} (1)

On the other hand, if we realized that:

\[
\Delta^{-1} = \phi = \theta_1 \theta_2 = \theta_1 \circ P_{21}, \quad \theta_2 \circ P_{24} = P_{34} \Rightarrow \Delta^{-1} = \phi = \theta_1 \theta_2 = P_{31} P_{21} P_{24}^{-1} \Rightarrow \Delta = P_{23} = M \rightarrow Q \otimes M
\]  \hspace{1cm} (2)

We shall take the following program into account:

By the property of average mapping, we obtain:

\[
\phi \Delta = \phi = \Lambda^{n+1} \Rightarrow \phi = w \phi w \Rightarrow w = \phi^{-1} \Lambda^{-2} \Rightarrow c_i \otimes \ldots \otimes c_x \otimes m = (-1)^{\frac{n(n+1)}{2}} c_a \otimes \ldots \otimes c_i \otimes m
\]  \hspace{1cm} (3)

Then we finished section 1.1.

1.2 cocycle and hopf algebra

Theorem 1.2: every finite-dimensional triangular hopf algebra over an algebraic closed field of characteristic zero is the drinfeld twist of a modified supergroup algebra.

Firstly, we discuss the cocycle in [2], by the property of \( R \) matrix, we have:

\[
R_{\lambda} = \frac{1}{2} \sum (-1)^{\frac{|t|}{2}} \sum [v \ldots [v] \text{sign} (\eta) a_r (v \otimes v + uv \otimes v + (-1)^{|v|} v \otimes v) - (-1)^{|v|} v \otimes uv - \]  \hspace{1cm} (4)

Recall the results in section 1.1, we can write out the relation immediately,

\[
R_{\lambda} = -\frac{1}{2} \sum [v \ldots [v] \text{sign} (\eta) a_r ((1 + u)v \otimes v + (1 - u)v \otimes v) - \]  \hspace{1cm} (5)

Then we can introduce the exact sequence:

\[
C \otimes P \xrightarrow{10.1} C \otimes M \xrightarrow{10.2} C \otimes Q \rightarrow 0
\]  \hspace{1cm} (6)
By the exactness above, we have:

\[- \sum |v| |r| a_r (uv \otimes v_f) = - \sum |v| |r| a_r (x_r \otimes v_f)\]  

(7)

Also related formula (7) with section 1.1, to get:

\[\Delta = P_{23} = P_{24} P_{40} \Rightarrow 1 \otimes g (\sum c_i \otimes m_j) = \Delta = - \sum |v| |r| a_r (x_r \otimes v_f)\]  

(8)

In which,

\[x_r \in C , \ a_r v_f \in Q\]  

(9)

Lastly, basing on the property of hopf algebra, we can further handle the problem above (see [3]):

\[- \sum |v| |r| a_r (x_r \otimes v_f) = \sum |v| |r| a_r (x_r \otimes v_f)\]  

(10)

Consequently, we achieve to define the following mapping:

\[\Gamma = (C \rightarrow Q) \rightarrow Hom (U \rightarrow A_f)\]  

(11)

1.3 Irreducible coalgebra and Cocommutative coalgebra

Theorem 1.3.1: (see theorem 3.3 and lemma 3.5 in [4])

Theorem 1.3.2: (see theorem 7.1 in [5])

Theorem 1.3.3: if \(D\) is an R-coseparable coalgebra, then \((-\), \(-D\)) is an adjoint pair of functor.

Theorem 1.3.4: if \(D\) is an R-co-central coseparable coalgebra, then \(D\) is simple if and only if \(R\) is simple.

Firstly, we study the irreducible coalgebra (see [4]), that is:

\[FA = \{ J \leq_A A : IA \subseteq J, I \in F \}, \ F \cap R = \{ J \leq_R R : I \cap R \subseteq J, I \in F \}\]  

(12)

Where, \(I\) can be viewed as a two side ideal.

Then we introduce the annihilate ideal and the graded ring (see [5]):

\[0 \rightarrow (0, x_j)(-s_j) \rightarrow B(-s_j) \xrightarrow{x} B \rightarrow B / x_jB \rightarrow 0\]  

(13)

Now we can see it clearly:

\[IB \subseteq Ann _g (x_k) = x_k B \Rightarrow y \cdot x = \sum b_j y_j p_j = 0\]  

(14)

Next, taking the following program into account:

\[\begin{array}{cccc}
M & \xrightarrow{\rho_E} & M \otimes E & \xrightarrow{1 \otimes \epsilon_E} M \otimes C \\
\rho_D & & & \parallel \\
\end{array}\]

\[\begin{array}{cccc}
D \otimes M & \xrightarrow{\epsilon_D \otimes 1} & C \otimes M & \xrightarrow{\tau} M \otimes C \\
\end{array}\]
That is,

\[ \left\langle \sum a_i, \varepsilon^* (\psi_j), a \right\rangle = \sum \sum \left\langle w_i, w_j \right\rangle \varepsilon^* (\psi_j) \right\rangle = \left\langle \varphi, a \right\rangle \]  

(15)

Also note the morphism \( \varphi : Y \rightarrow Y \), we obtain now:

\[ \varphi = h^* (\psi) \Rightarrow \left\langle h^* (w_i \otimes \varepsilon_c), a_{i(i)} \right\rangle = \left\langle w_i \otimes \varepsilon_c, h(a_{i(i)}) \right\rangle \]

(16)

Since \( Y = \lim_{\rightarrow} Y^*_k \) is finite submodule ,we can define the following mapping :

\[ \text{Hom} \ (U \rightarrow \text{Ann}_y (x)) \]

(17)

Where, we use the relations below :

\[ x(\lambda \cdot m) = \sum \lambda_i x(\psi_i, m) + \sum \beta_j (\psi_{i(j)} \cdot m) = 0 \Rightarrow b \cdot a_{m} = \psi_i \]

(18)

Our next step is to introduce \( [6] \) to handle the cocommutative coalgebra.

Firstly, by \( [2] \) and \( [6] \), the cocycle formula holds:

\[ \sum (a \# b)(c \otimes d) = \sum a(R_{i(2)} \cdot c)(R_{i(1)} \cdot b) = \sum (a \# a)(R_{i(2)} \cdot c)(R_{i(1)} \cdot b) \]

\[ = \sum a(c \otimes R_{i(1)}) \varepsilon (R_{i(2)}) = \text{End} (A) = \varphi_X \]

(19)

From :

\[ \sum (1 \otimes \varepsilon)(\tau \otimes \pi) = \sum \varepsilon a \otimes \pi \]

(20)

We obtain :

\[ \varphi_X = \sum \varepsilon a \otimes R_{i(1)} \varepsilon (R_{i(2)}) = \sum \frac{1}{n} \left\langle f(m), c \otimes R_{i(1)} \varepsilon (R_{i(2)}) \right\rangle \]

(21)

Which lead that :

\[ \varphi_X = \sum (1 \otimes d_i)(c \otimes R_{i(1)}) \varepsilon (R_{i(2)}) \]

(22)

On the other hand, we also know :

\[ \sum d_i \otimes \varepsilon (\varepsilon (d_i \otimes r_i = \sum d_i \otimes r_{i(1)} \otimes r_{i(2)} \]

(23)

Which implies ,

\[ d_i \otimes \varepsilon (d_i \otimes r_i = d_i \otimes r_{i(1)} \otimes r_{i(2)} \Rightarrow \varepsilon (d_{i(2)}) = \frac{d_i \otimes r_{i(1)} \otimes r_{i(2)}}{d_i \otimes r_i} \]

\[ \Rightarrow \varphi_X = \sum (1 \otimes d_i)(c \otimes R_{i(1)}) \varepsilon (R_{i(2)}) = \sum (1 \otimes d_i)(c \otimes R_{i(1)} \cdot d_i \otimes r_{i(1)} \otimes r_{i(2)}) \]

\[ \Rightarrow \varphi_X = \sum (R_{i(2)} \otimes d_i)(c \otimes R_{i(1)}) \Rightarrow \text{Hom} (A \rightarrow U) = \text{Ann}_y (x) \]

(24)

Now, we successfully completed our preparation for the main result.
1.4 The strong equivalence in coalgebra

Theorem 1.4: (see proposition 3.6 in [7])

Here, we introduce [7] to handle the cocommutative property. At first, we consider the bimodule. And the following program holds:

\[
\begin{array}{ccc}
P & \xrightarrow{\rho} & P \otimes C \\
\theta_{P,P} & & \tau \\
& \theta_{C \otimes P} & \epsilon \otimes 1
\end{array}
\]

Therefore, we have to prove that:

\[
\tau \rho = (\epsilon \otimes 1) \circ \theta_{P,P} \Rightarrow \theta = P \rightarrow e_{-c}(P) = \epsilon^{-1} m \Delta \Rightarrow \Delta = m^{-1} \Rightarrow U \cong A_f
\] (25)

By imposing the results in 1.1—1.3 and simply handling the structure above, we have:

\[
\Delta = \Delta \circ \text{Id} = v_r \otimes v_r \otimes v_r = m
\] (26)

On the other hand, we note that:

\[
\left(\theta^*_{P,C} \eta_{Q,P} (\Phi^{-1} \otimes 1) \right) = g \Rightarrow \theta^*_{P,C} = g \eta^{-1}_{Q,P} = g \tau \circ h_{-c}(P,C)^* = qm \circ h_{-c}(P,C)^*
\]

\[
\Rightarrow h_{-c}(P,C)^* = \Delta
\] (27)

By the isomorphism: \( h_{-c}(P,C)^* \cong \text{Com}_{-c}(C,P) \), we obtain:

\[
(\epsilon \otimes 1) \theta_r = \Delta \Rightarrow \epsilon(e_{-c}(P)) = h_{-c}(P,C)
\] (28)

The adjoint property leads,

\[
e_{-c}(X)^* \cong \text{Com}_{-c}(X,X)
\] (29)

Then, we achieve to construct the following isomorphism:

\[
\epsilon = e_{-c}(P) \xrightarrow{\sim} C
\] (30)

That is the semisimple property of coalgebra and the existence of the centralizers.

2. THH sequence and the rational problem

2.1 MacLane homology and Hochschild homology

Theorem 2.1: \( THH_q(HZ) \cong \otimes HH_p(R) \otimes qTHH(HZ) \oplus \text{Tor}^2_p(\text{HH}_p(R), THH_q(HZ)) \)

Firstly, we introduce [8] to handle the homology. That is, the following natural embedding:
\[ T(\tilde{R}(Y)) \otimes Z(X) = \oplus_{x \in \mathbb{N}} T(\tilde{R}(Y)) \rightarrow T(\oplus_{x \in \mathbb{N}} \tilde{R}(Y)) = T(\tilde{R}(X \wedge Y)) \tag{1} \]

Here, the commutative below is used:

By the figure above, we can give the following definition:

\[ \alpha \beta = A \rightarrow A / \text{co ker} \quad \rho = A \rightarrow A / \pi_\ast \text{THH} \tag{2} \]

Where, we define the representation as \( \rho = E \rightarrow A \). So that:

\[ \text{Co ker}(E \rightarrow A) = \text{Co ker}(\pi_\ast L_x G_x) = \tilde{\pi}_x T(\tilde{R}(S)) \Rightarrow \]

\[ E / I - G = A = \pi_x T \tag{3} \]

By the property of cokernel, we can find the relation below:

\[ \pi_\ast \beta = 0 \wedge \alpha \beta = 0 \Rightarrow \sigma \pi_\ast = \alpha \tag{4} \]

In which,

\[ \beta = fg = \oplus_{x \in \mathbb{N}} T(\tilde{R}(Y)) \rightarrow T(\oplus_{x \in \mathbb{N}} \tilde{R}(Y)) \tag{5} \]

Equivalently,

\[ E + I = G \wedge gf = fg \Rightarrow \sigma \pi_\ast = g \Leftrightarrow \pi_\ast = f^{-1} = \tilde{R}(Y) \tag{6} \]

Where,

\[ gf = T(\oplus_{x \in \mathbb{N}} \tilde{R}(Y)) \rightarrow T(\oplus_{x \in \mathbb{N}} \tilde{R}(Y)) \tag{7} \]

Next, let us introduce \([9]\) to further study the THH sequence.

\[ \text{THH}(R; M) = R_x \wedge \ldots \wedge M \tag{8} \]

Then, we can use the dual base to make an assumption, such that:

\[ R_1 = e^1 \wedge e^2 + e^3 \wedge e^4, \quad R_2 = e^1 \wedge e^3 + e^2 \wedge e^4 \tag{9} \]

Please note that the definition above is only for the two dimensional case. To the higher dimensional space, we can easily get a similar result by simple generalization.

Lastly, directly computation shows:

\[ \text{THH}(R; M) = \sum \sum e_1 \wedge \ldots \wedge e_{12} \wedge \ldots \wedge e_x \wedge m \tag{10} \]
2.2 Leray-Serre spectral sequence

Theorem 2.2.1: (see lemma 1.3 in [10])
Theorem 2.2.2: (see proposition 3.5 in [13])

Here, we introduce spectral sequence to handle the homology problem in section 2.1. By theorem 2.2.1 in [10], the kummer sequence leads:

\[ \sum_{i=1}^{k+1} (-1)^i \sum_{j=i+1}^{i-2} (-1)^j c(g_1, \ldots, g_i g_{j+1}, \ldots, g_i g_{j+1}, \ldots, g_i g_{i+2}) + \]
\[ \sum_{i=1}^{k+1} (-1)^{i+1} c(g_1, \ldots, g_i g_{i+1}, \ldots, g_i g_{i+1}, \ldots, g_2) - \]
\[ \sum_{i=1}^{k+1} (-1)^i \sum_{j=i+2}^{i+1} (-1)^j c(g_1, \ldots, g_i g_{j+1}, \ldots, g_i g_{j+1}, \ldots, g_2) - \]
\[ \sum_{i=1}^{k+1} (-1)^{i+1} c(g_1, \ldots, g_i g_{i+1}, \ldots, g_i g_{i+1}, \ldots, g_2) + \]
\[ \Rightarrow c(g_1, \ldots, g_i, \ldots, g_i, \ldots, g_i, \ldots, g_2) = c(g_1, \ldots, g_i, \ldots, g_i, \ldots, g_i, \ldots, g_2) \]
\[ c(g_1, \ldots, g_i, g_i, g_i, \ldots, g_i, g_i, g_i, \ldots, g_i, g_i) = c(g_1, \ldots, g_i, g_i, g_i, \ldots, g_i, g_i, g_i, \ldots, g_i, g_i) \]
\[ \text{(11)} \]

Relate formula (11) with section 2.1, we can make such assertion:

\[ THH(R; M) = E_{\oplus, \ast} \wedge m \]
\[ \text{(12)} \]

Next the definition in [11] gives:

\[ G = \text{Gal}(K_j / K), \quad I = \text{Gal}(K_j / K_m), \quad I_p = \text{Gal}(K_j / K_j) \]
\[ \text{(13)} \]

Which lead us to denote the following sequence:

\[ 0 \to K^+ / E_m \to I / U_m \to C_m \to 0 \]
\[ \text{(14)} \]

Therefore, by the deduction above, we can find the relations now:

\[ K^+ / E_m + C_m = I / U_m \Rightarrow K^+ / E \cap U_m + 1 / K^+ U_m = I / U_m \Rightarrow \]
\[ K^+ (E U_m) \cap E \cdot U_m = U_m (I - K^+) \Rightarrow K^+ (E U_m) = U_m (I - K^+ - E) \Rightarrow \]
\[ K^+ E = I - K^+ - E \Rightarrow K^+ (G - I) = I - K^+ - G + I = I - K^+ - G \Rightarrow \]
\[ K^+ G = I - G \wedge gf = fg \Rightarrow \sigma x_n = g \Leftrightarrow \pi_n = f^{-1} = R(Y) \]
\[ \text{(15)} \]

Where,

\[ gf = fg \Leftrightarrow T(\oplus x_i) \sim R(Y)) = \oplus x_i \sim T(R(Y)) \Leftrightarrow T \text{ is injective} \]
\[ \text{(16)} \]

Our next step is to use the results in [12] [13] to handle the spectral sequence. We consider the morphism at first:

\[ \pi_n(X, A) \to H_n(X, A) \]
\[ \text{(17)} \]
Then the associated homology sequence is:

$$\to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to$$ (18)

Now let us recall the five-lemma that:

Equivalent, $$\varphi_5 \tau_2 = \beta_2 \varphi_2 = 0, \varphi_2 \tau_1 = \beta_1 \varphi_1 = \beta_1 = \varphi_2$$ (19)

And, $$\varphi_5 \tau_4 = \beta_4 \varphi_4 = 0, \varphi_4 \tau_3 = \beta_3 \varphi_3 = \beta_3 = \varphi_4$$ (20)

After simply handling, we can see that:

$$H_n(A) \in \text{Ker}H_n(X,A), H_n(X,A) \in \text{Ker}H_{n-1}X$$ (21)

On the other hand, we know: $$A = f^{-1}(U)$$. Then by the isomorphism: $$\oplus (H_j \otimes H_j) \to H_{j+j}$$.

We can make the conclusion that:

$$\sim T(R(Y)) = \frac{H_n(A)}{H_n(X,A)}$$ (22)

2.3 Rational problem

Lastly, we study the rational problem in [14].

By the Hochschild-Serre sequence, we have:

$$H^p(G, H ^qI) \Rightarrow H ^{p+q}K \Rightarrow G(G + I) = I - G$$ (23)

By lemma 1 in [14], that is $$x \cup (-x) = 0$$, we obtain:

$$H_*(X,A) = 0 \wedge K_* = K_{-n} \Rightarrow \pi_{-n} = f^{-1} = A$$ (24)

By:

$$\text{THH}(R;M) = e_{\omega_n} \wedge m$$ (25)

And,

$$\alpha \beta = A \to A / \text{co ker } \rho = A \to A / \pi_n \text{THH}$$ (26)

We can find the existence of the last condition $$\pi_{-n} = f^{-1} = A$$. Then we obtain the following
conclusions:

\[ H_{\alpha} (X, A) = 0 \Rightarrow K_\alpha = K_{\alpha^\prime} \]  \hspace{1cm} (27)

Which means that there exist the galois extension with \( K_\alpha = K_{\alpha^\prime} \). So we can get the completion condition of the maximal unramified extension.

3. **elliptic curves and quadratic form**

3.1 The mordell-weil theorem in elliptic curves

Theorem 3.1: (see proposition 8 and lemma 9 in [15])

Here, we introduce [15] to handle the mordell-weil theorem.

\[
\log \max \left| a \right| \left| b \right| = \log \prod p^{r_p} + \log \max \left( 1, \left| \frac{a}{b} \right| \right) = \log \prod p^{r_p} + \sum \max \left( 0, \log \left| \frac{b}{a} \right| \right) \hspace{1cm} (1)
\]

Where,

\[
f(T) = \frac{G}{H} = \frac{a_n T^n + a_{n-1} T^{n-1} + \ldots + a_0}{b_n T^n + b_{n-1} T^{n-1} + \ldots + b_0}, \quad \forall \rho > 0 \hspace{1cm} (2)
\]

Next, we introduce the concept of annihilate linear operator to get:

\[ A(-\mu_{n}^{-1} A^{n-1} - \mu_{n}^{-1} A^{n-2} - \ldots - \mu_{n}^{-1} A^{n-1}) = e \Rightarrow f(T) = \frac{a_n}{b_0} \hspace{1cm} (3)\]

By easily computation, we can find:

\[
\log \prod p^{r_p} + \sum \max \left( 0, \log \left| \frac{b}{a} \right| \right) \geq d \log \max \left| \frac{b}{a} \right| - C \Rightarrow \hspace{1cm} (4)
\]

In which,

\[ \left| a \right| = \frac{p}{q}; \quad p, q \text{ are primes; and } d \leq n^2 \hspace{1cm} (5)\]

Now we can introduce the enveloping algebra and the sheaf topology to handle the kernel-cokernel sequence (see [16] [17]).

That is,

\[ 0 \rightarrow E_{1} (K_{p}^{\omega}) \xrightarrow{d_{-1}} E (K_{p}^{\omega}) \rightarrow E_{\omega} (K) \rightarrow 0 \hspace{1cm} (6)\]

We denote the map at first:

\[ d = C^{a} \rightarrow C^{1} \hspace{1cm} (7)\]

Then we can introduce the example in [17]:

\[ \]
\[ d \left( f \, dx, g \, dy \right) = \left( f \left( x \right) + \frac{1}{x^2} \right) g \left( -x \right) dx \]  
(8)

From [16], we can inserting the Lie structure to the associated algebra \( A \), such that:

\[ f\left( x \right) = f\left( p, q \right) = f\left( p \right)f\left( q \right) - f\left( q \right)f\left( p \right) \]  
(9)

\[ \frac{1}{x^2} \, g \left( \frac{1}{x} \right) = \frac{1}{\left( pq - qp \right)^2} \left( \left[ \left( g \left( p \right)g \left( q \right) \right)^{-1} - \left( g \left( q \right)g \left( p \right) \right)^{-1} \right] \right) \]  
(10)

\[ d = f\left( p \right)f\left( q \right) - f\left( q \right)f\left( p \right) + \frac{1}{\left( pq - qp \right)^2} \left[ \left( g \left( p \right)g \left( q \right) \right)^{-1} - \left( g \left( q \right)g \left( p \right) \right)^{-1} \right] \]  
(11)

Basing on \( C - \sum \log \left| a \right| \geq d \sum \max \left( 0, \log \left( \left| a \right| \right) \right) \), we can select \( C \geq n \) to ensure:

\[ h_k \left( f \left( a \right) \right) \leq nh_k \left( a \right) + C_r \Rightarrow \frac{h_k \left( f \left( a \right) \right) - C_r}{n} \leq h_k \left( a \right) \Rightarrow \]

\[ C - \sum \log \left| a \right| \geq d \frac{h_k \left( f \left( a \right) \right) - C_r}{n} \Rightarrow C - \sum \log \left| a \right| = n \cdot \left( h_k \left( f \left( a \right) \right) - C_r \right) \Rightarrow \]

\[ d = f\left( p \right)f\left( q \right) - f\left( q \right)f\left( p \right) + \frac{1}{\left( p^2 - q^2 \right)^2} \left[ \left( g \left( p \right)g \left( q \right) \right)^{-1} - \left( g \left( q \right)g \left( p \right) \right)^{-1} \right] \]  
(12)

### 3.2 Labglands group

Here, we introduce [18] to handle the quadratic form.

The calculation holds,

\[ 1 + \left( q - 1 \right) X = \frac{1 \left( X \right) + qX}{1 - q^4 X^2} \Rightarrow \left( 1 + \left( q - 1 \right) X \right) \left( 1 - q^4 X^2 \right) = \left( 1 - X \right) \left( 1 + qX \right) \Rightarrow \]

\[ 1 - q^4 X^2 + \left( q - 1 \right) X \left( 1 - q^4 X^2 \right) = 1 + qX - X - qX^2 \Rightarrow \]

\[ 1 - q^4 X^2 + qX - X + q^4 X^3 - q^5 X^3 = 1 + qX - X - qX^2 \Rightarrow \]

\[ qX^2 + q^4 X^2 + q^4 X^3 - q^4 X^3 = 0 \Rightarrow X = pq - qp = \frac{1 + q^4}{q^4 - q^3} \Rightarrow \]

\[ p^2 - q^2 = \frac{1 + q^4}{q^4 - q^3} \Rightarrow \left( p^2 - q^2 \right) \left( q^4 - q^3 \right) = 1 + q^4 \Rightarrow \]
\[ p^2 \cdot (q^4 - q^3) = 1 + q^3 + q^2 (q^4 - q^3) \quad \Rightarrow \quad p^2 = \frac{1 + q^3 + q^2 (q^4 - q^3)}{q^4 - q^3} \]

\[ q = -1 \iff p^2 = q^2 \tag{13} \]

### 3.3 Residue norm and quadratic form

Theorem 3.3: the following two conditions are equivalent:

1) at least one of the algebras \((a, u, d)\), where \(I\) runs over all the subset of \([1, \ldots, n - 1]\), is zero.

2) \(\beta_I \in \text{res}_{M/F \subset} BrF\)

Lastly, we introduce \([19][20]\) to further study the quadratic form.

From \([19]\) and section 3.2, we can assume: \(p = uq\). Then we try to substitute this relation into reference \([20]\).

Immediately calculation shows:

\[ p = \sqrt{t}, \quad q = \frac{d}{\sqrt{t}} \tag{14} \]

Therefore we can deduce that:

\[ t = \sqrt{d} \cdot i \Rightarrow \frac{d}{\sqrt{d} \cdot i} = i \Rightarrow \frac{d}{\sqrt{d} i} = -1 \Rightarrow \sqrt{d} = -i \Rightarrow d = -1 \Rightarrow t = 1 \tag{15} \]

By the fact that:

\[ \frac{d}{t} = \frac{1}{2} (x - \sqrt{x^2 - 4d}) \tag{16} \]

We can construct the function in section 3.2 as:

\[ d = \frac{1}{(p^2 - q^2)^2} \left[ \left( g(p)^2 - q^2 \right) \right] \Rightarrow g(p) \geq q \Rightarrow f(p) \leq f(q) \tag{17} \]

That is, the following relation:

\[ f \left( \frac{p}{q} \right) \leq 1 \Rightarrow f(p) \leq f(q) \iff \frac{a_d p^d + a_{d-1} p^{d-1} q + \ldots + a_0 q^d}{b_d p^d + b_{d-1} p^{d-1} q + \ldots + b_0 q^d} \leq 1 \Rightarrow \]

\[ \frac{a_d p^d + a_{d-1} p^{d-1} + \ldots + a_0}{b_d q^d + b_{d-1} q^{d-1} + \ldots + b_0} \leq 1 \iff p^2 = q^2 = 1 \tag{18} \]

### 4. Height 0 conjecture and the P-block theory

#### 4.1 Brauer character and the p-block

Theorem 4.1.1: (see main theorem in [21])
Theorem 4.1.2: the sequence is exact,

\[ 0 \to H^1(k_u / k, A) \to H^1(k, A) \to [H^1(k_u, A)]^{G_1} \to 0 \]

Firstly, we introduce [21] to handle the brauer character:

\[ NN_{\phi}(D^{\sigma^{-1}}) \cap I = NN_{\phi}(D^{\sigma^{-1}}) \]  

Recall the following isomorphism:

\[ \frac{N(G \cap I)}{N(G \cap (D^{\sigma^{-1}}))} \cong \frac{G \cap I}{D} \]  

In which,

\[ D = (N \cap I)(G \cap D^{\sigma^{-1}}) = N_{\phi}(D^{\sigma^{-1}}) \]  

On the other hand, from [22], we can define:

\[ \sigma \to (\phi_{\sigma})^{\sigma^{-1}} \]  

Where, \( \phi_{\sigma} \) is the irreducible character of \( NN_{\phi} \cap I \).

By comparing (2) and (4) above, we have:

\[ (\phi_{\sigma})^{\sigma^{-1}} = \xi_{\sigma^{-1}m} \Rightarrow (\phi_{\sigma})^{\sigma^{-1}} = (\phi_{\sigma})^{\sigma^{-1}} = \phi_{\sigma} \phi \]  

4.2 P-solvable and Height 0 conjecture

Theorem 4.2.1: (see lemma 1, 2 and theorem 1 in [23])

Theorem 4.2.2: (see proposition 3 in [24])

Firstly, we use [23] to handle the number of irreducible characters.

That is theorem 4.2.1,

\[ k(B) \leq \frac{p^d - 1}{l(B)} + l(B) \]  

Then we introduce the Frobenius theorem to obtain:

\[ \frac{\partial c'_i}{\partial u^a} \neq \frac{\partial c'_i}{\partial u^a} \neq \frac{\partial c'_i}{\partial u^a} \neq \frac{\partial c'_i}{\partial u^a} \]

That is, when \( k + 1 \leq \lambda \neq \mu \leq m \), we have:

\[ \frac{\partial c'_i}{\partial u^a} \neq \frac{\partial c'_i}{\partial u^a} \neq \frac{\partial c'_i}{\partial u^a} \neq \frac{\partial c'_i}{\partial u^a} \]

On the other hand, we also know:
1 = \det C_i = \sum_{V \in \{1, \ldots, k\}} \det Q'_i Q_V \tag{9}

Therefore, we can get the following conclusion:

\[ c'_x \neq c'_y, \quad c'_x = c'_y \] \tag{10}

Consequently, we achieve to normalize the Brauer block and then we can introduce the map in section 4.1 to deduce:

\[ C = S^T C_2 S = (\phi_0)^{\sigma^{-1}} = (\phi^\sigma)^{\sigma^{-1}} \tag{11} \]

Where, \( \det C = p^d \)

Lastly, we discuss the p-solvable property (see [24]):

\[ \phi_0^\sigma = \phi^{1/s} \Rightarrow \phi_0^\sigma = \phi_0 \Rightarrow p^d = \phi_0 \Rightarrow k(B) \leq l(B) \tag{12} \]

Then we will ask, under what condition, we can get the counterexample that the number of ordinary irreducible character equals the number of Brauer irreducible character? On the other hand, if we let \( eA = eAe \), then \( I^{1/s\phi(1-r)} = \phi^{1/s} \Rightarrow u = lj^{-1}x \rightarrow \infty \). So whether we can get the condition of Height 0 by the relation above?

5. other problems

5.1 Noether problem

Theorem 5.1: (see definition 4.3, 4.5 in [25])

Here, by introducing the Lie structure in theorem 5.1, we try to discuss a phenomenon in Noether problem.

Firstly, we consider the bilinear property and the anti-symmetric property:

\[ \{ f_1, f_2 \} = 2f_2, \quad [ f_1, f_3 ] = -2f_4, \quad [ f_2, f_3 ] = f_1 \] \tag{1}

\[ [ f_1, f_3 ] = [ f_2, f_4 ] \Rightarrow f_1 = -2f_3 \] \tag{2}

\[ f_3 = -\frac{1}{2} f_1 = \{ f_2, f_4 \} = [ f_3, f_4 ] \Rightarrow -\frac{1}{2} f_1 = f_1 f_4 - f_4 f_1 \] \tag{3}

\[ [ f_1, f_2 ] = -2f_1, \quad [ f_3, f_2 ] = 2f_3, \quad [ f_3, f_4 ] = -f_2 \] \tag{4}

\[ [ f_1, f_3 ] = [ f_2, f_4 ] \Rightarrow f_2 = -2f_3 \] \tag{5}

\[ f_3 = -\frac{1}{2} f_2 = \{ f_2, f_4 \} = [ f_3, f_4 ] \Rightarrow -\frac{1}{2} f_2 = f_2 f_4 - f_4 f_2 \] \tag{6}

Consequently,

\[ f_1 = f_2 \wedge f_3 = f_4 \] \tag{7}
Then we proved that when $B_0(G) = 0$, $\Phi_3$ and $\Phi_4$ satisfy the bilinear property and the anti-symmetric property.

Please note that, if $f_1 \neq f_2$, we will encounter $f_1 = -2[f_1, f_2] \neq f_2 = -2[f_2, f_0]$, which is a contradiction. So by $f_1 = f_2$, we have: $[f_1, f_2] = [f_3, f_4] = 0 \Rightarrow f_3 = f_4$.

### 5.2 Jacobson-witt algebra

**Theorem 5.2**: (see lemma 1.3 in [26])

Here, we study the Jacobson-witt algebra in [26].

Firstly, we introduce the Leibnitz formula to get:

$$D = D_1 + D_2 + (\prod x_i^{p_i-1})D_3 = [(D - a \cdot Ad)x, y] + [x, (D - b \cdot Ad)y] + (a + b) \cdot Ad \left( \{x, y\} \right)$$

(8)

By theorem 5.2, we obtain:

$$Ad (D_1 + D_2)^{p_i-1} (\prod x_i^{p_i-1}) D_3 = [(2D - a - b)x, y]^{p_i-1}(a + b) = (a + b) \quad (\text{mod } J^k W_a)$$

(9)

Where,

$$K = (k - 1)(p - 1) + 1$$

(10)

And,

$$a + b = \prod x_i^{p_i-1} \Leftrightarrow D_3 = Ad$$

(11)

Immediately calculation shows:

$$J^k W_a \left( [(2D - a - b)x, y]^{p_i-1} - 1 \right)(a + b) \Rightarrow J^k W_a \left( [(2D - a - b)x, y]^{p_i-1} - 1 \right) \Rightarrow
\left[ x, (D - b)y \right] \left( [(2D - a - b)x, y]^{p_i-1} - 1 \right) \Rightarrow D_2 \left( D_1 + D_2 \right)^{p_i-1} - 1 \Rightarrow D_2 \left( D_1 \right)^{p_i-1} - 1$$

(12)

Then we achieve to reduce the congruent from $D_1^{p_i-1}$ to $D_1^{p_i-1} - 1$.

Please note that, since $(a + b)^{[r]} = a^{[r]} + b^{[r]} + \sum_{i=1}^{p_i-1} x_i(a, b)$, we can use the Leibnitz formula to omit $D_1$.

### 5.3 Brauer semisimple group

Lastly, we discuss the brauer semisimple group (see [27]).

Here, we introduce the Hilbert zero point theorem by:

$$\chi = (E^* \langle x \rangle)^{d_2} \xrightarrow{\sim} H_1^{\alpha}(E, \mu_{\alpha_1}) \xrightarrow{d_2} H_2^{\alpha}(X, G_{\alpha}).$$

(13)
Since $d^{2,2}$ is invertible, we can construct the following morphism:

$$H^{2}_{\eta}(X, G_{\omega}) \rightarrow H^{1}_{\eta}(E, \mu \mu_{\omega})$$

(14)

Recall that $g$ is zero on the zero points of $f$. So that:

$$\ker g \subseteq \ker f = 0$$

(15)

Next, we consider the possibility of shifting:

$$\begin{array}{c}
h \downarrow \\
\bigtriangleup
\end{array}$$

So we will ask when $X$ is a smooth affine $K$-scheme. That is, when $g$ is surjective and $\ker g \subseteq \ker f$. Whether we can deduce $H^{2}_{\eta}(X, G_{\omega}) \rightarrow H^{1}_{\eta}(E, \mu \mu_{\omega})$ is injective? If this assumption holds, we will obtain: $\ker g = \ker f$.

Similarly, if $f$ is invertible and then $Br(X) \xrightarrow{h} H^{2}_{\eta}(X, G_{\omega})$ is injective. That is, $h$ is injective and $\text{Im} f \subseteq \text{Im} h$. Then recall the fact we get in the last paragraph, that is:

$$\ker g = \ker f$$

(16)

Next by this conclusion we can deduce that $g$ is surjective. And this is the condition we need in the problem we ask at the beginning, which means that we can prove this assumption and we can also conclude the equivalent between $X$ is a smooth affine $K$-scheme and the injective of:

$$H^{2}_{\eta}(X, G_{\omega}) \rightarrow H^{1}_{\eta}(E, \mu \mu_{\omega})$$

(17)

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