Some Thoughts on the Notion of Kinetic Energy.

Jeremy Dunning-Davies,
Institute for Basic Research,
Palm Harbor, Florida, U.S.A.
and
Institute for Theoretical Physics and Advanced Mathematics (IFM) Einstein-Galilei,
Via Santa Gonda,
14 - 59100 Prato,
Italy
j.dunning-davies@hull.ac.uk

Abstract.

The entire notion of kinetic energy seems one well and completely understood. Here, however, the topic will be examined again with special emphasis on considering the physical situation when the mass is not constant. The situation in special relativistic mechanics will also be examined in the light of these discussions.

Introduction.

It seems that the whole idea of kinetic energy is invariably introduced into classical mechanics after making the basic assumption that the mass remains constant. Here the derivation of the notion utilising a constant acceleration, as well as a constant mass, will be considered before moving on to the derivation involving a non-constant acceleration but a still a constant mass. However, the traditional form of Newton's second law, which is used here, strictly speaking only has constant mass as a special case. Hence, after examining the question in the usual manner, it will be examined afresh taking a possible variation in the mass into account and, once an appropriate expression has been derived, the additional problem of offering a physical explanation for all the terms appearing will be addressed. This proves to be a non-trivial but interesting exercise and the path followed will be that pioneered by Mandelker[1] which makes use of the example of the collision of a moving body with a stationary one to clarify the situation.

It is then intended to proceed to a new consideration of the situation obtaining in relativistic mechanics since here is a situation involving a very definitely varying mass. The usual expressions will be re-examined but whether derived by traditional methods or by more recent techniques which do not employ the basic ideas of special relativity will be seen irrelevant. Once again, as in the non-relativistic case, it will be seen that the physical explanation of what is occurring is all-important. However, in this case, although a similar explanation to that obtaining in the non-relativistic case will be proffered, for reasons which will become apparent, it is not such an obvious explanation as in the non-relativistic case.

Introducing Kinetic Energy into Mechanics.

In many elementary approaches to classical mechanics, the starting point is to assume a constant acceleration so that

$$\frac{d^2s}{dt^2} = a,$$

where *s* is distance, *t* time, and *a* the constant acceleration.

Integrating once and assuming the velocity, *v*, initially has the value *u*, we get

$$v = \frac{ds}{dt} = u + at.$$

Integrating a second time and assuming s is zero initially, we get

$$s = ut + \frac{1}{2}ft^2.$$

These are very well-known straightforward equations but are included here, together with their derivation, to highlight the fact that they do not depend in any way on the mass of an object. In fact, all assumptions made are clearly stated.

Finally, if s is eliminated between the latter two equations

$$v^2 = u^2 + 2as$$

results and it follows from this equation that, if an object is brought to rest ($v = 0$) by a retardation a then this equation gives

$$0 = u^2 - 2as \Rightarrow \frac{1}{2}u^2 = as$$

If this equation is multiplied throughout by m, one obtains

$$\frac{1}{2}mu^2 = mas.$$

Here the interpretation of the right-hand side follows from considering Newton's Second Law with the mass assumed constant; that is

$$\text{Force} = P = \frac{d}{dt}(mu) = ma.$$

Then, since work done is force multiplied by distance, mas is the work done and is regarded as being equal here to the kinetic energy; that is, $\frac{1}{2}mu^2$ represents the kinetic energy of the object.

This derivation of the usual expression for the kinetic energy is one common in textbooks but may be seen to rely on constant values for both the mass and, in this case, the acceleration. Hence, it is more restrictive than the following approach but is useful to note.

Consider Newton's Second Law with the mass assumed constant,

$$\text{Force} = P = \frac{d}{dt}(mv).$$

Then, since work done is force multiplied by distance, the total work done by a system going from state 1 to state 2 is

$$\int_1^2 Pds = \int_1^2 \frac{d}{dt}(mv)ds = \int_1^2 \left\{\frac{d}{dt}(mv)\frac{ds}{dt}dt\right\} = \int_1^2 v\frac{d}{dt}(mv)dt = \int_1^2 \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt,$$

where ds represents an element of distance, then the above equation may be interpreted as

$$\text{amount of work done} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

or, bearing in mind that the definition of the kinetic energy of a body is the energy the body possesses by virtue of its motion and is measured by the amount of work which it does in coming to rest, the amount of work done here might be interpreted as the difference in the kinetic energies of the two states.

If, however, the mass is assumed variable, then Newton's Second Law takes the form

$$P = \frac{d}{dt}(mv) = ma + v\frac{dm}{dt}.$$

The introduction of the second term on the right-hand side of the above equation to take account of the fact that the mass is assumed to vary with time obviously can have no bearing on the actual definition but, if you consider the work done in a displacement, s, in this case, the work done is seen to be given by

$$\begin{aligned}\text{work done} = \int_1^2 Pds &= \int_1^2 \left(m\frac{dv}{dt} + \frac{dm}{dt}v\right)ds = \int_1^2 \left(m\frac{dv}{dt}\frac{ds}{dt} + \frac{dm}{dt}\frac{ds}{dt}\frac{ds}{dt}\right)dt\\ &= \int_1^2 \left(mv\frac{dv}{dt}dt + v^2dm\right)\\ &= \int_1^2 mv\frac{dv}{dt}dt + [mv^2]_1^2 - \int_1^2 m2v\frac{dv}{dt}dt\\ &= [mv^2]_1^2 - \int_1^2 mv\frac{dv}{dt}dt,\end{aligned}$$

where again it is assumed that the work being done is between states 1 and 2. Note also that this expression reduces to the familiar one above for kinetic energy when the mass, m, is supposed constant. In accordance with what has preceded it in this section, this final result is derived using scalar quantities at all points; the slight generalisation using vector quantities is a trivial extension.

Again, it should be remembered that, here, all that has been assumed about the variability of the mass is that it varies with time. No mention is made of how! The said variation could be through being a direct function of time; through being a function of varying position coordinates or varying velocity components or varying acceleration components, etc. The exact form of this dependence doesn't matter at this juncture.

However, this final expression may be put into a more convenient and more easily interpretable form if a partial integration is performed:

$$\int_1^2 Pds = [mv^2]_1^2 - \left[\frac{1}{2}mv^2\right]_1^2 + \int_1^2 \frac{1}{2}v^2dm = \left[\frac{1}{2}mv^2\right]_1^2 + \int_1^2 \frac{1}{2}v^2dm.$$

In incremental form, this would be written

$$dW = dE + \frac{1}{2}v^2dm.$$

where E retains the usual meaning of kinetic energy.

It now remains to give a physical interpretation to all the terms in this equation. Immediately, bearing in mind the above accepted definition of kinetic energy as being $\frac{1}{2}mass\ x\ (velocity)^2$, it is tempting to regard the final equation but one as offering a generalisation of the expression for the difference in the kinetic energies of the two states to the case where the mass is not constant. This expression was derived in both the appendix to chapter 4 and in chapter 6 of the book *Neo-Newtonian Mechanics with Extension to Relativistic Velocities* by Dennis P. Allen Jr. and Jeremy Dunning-Davies[2], although an explicit physical explanation was not included in either instance. However, is it the case that the expression in question does represent a generalisation of the expression for the difference in the kinetic energies of two states to the case of non-constant mass?

Possibly the most important point to note in this case where the mass is not held constant, though, is that the work done during the transition from state 1 to state 2 has to perform two roles:

(i) part is used to alter the motion of the system,

(ii) part is used to alter the mass of the system.

An initial examination of the right-hand side of either of the final two equations above might seem to indicate that the first term on the right-hand side takes account of the change in motion by registering a change in the accepted expression for the non-relativistic kinetic energy during the transition, while the second term, which does include a term representing an incremental change in the mass, might be felt to take account of any change in the mass of the system during the transition from state 1 to state 2. However, is this a correct physical interpretation?

Here, useful insight is provided by Mandelker[1] in his book *Matter Energy Mechanics* (Philosophical Library, New York, 1954). Since the mass of the system at the beginning and end of the transition changes, Mandelker considers the case of a body of mass m moving with a velocity u and colliding with another mass dm which is at rest. If the two coalesce and move off with a common velocity v, conservation of momentum gives

$$mu = (m + dm)v$$

or

$$v = \frac{m}{m + dm}u.$$

Now the total work performed during the collision must equal the initial kinetic energy of the system; that is

$$E_{in} = W = \frac{1}{2}mu^2,$$

where E_{in} is the initial value of the kinetic energy.

In terms of the final velocity, this becomes

$$W = \frac{1}{2}m\left[\frac{m+dm}{m}v\right]^2 = \left(\frac{m+dm}{m}\right)\frac{1}{2}(m + dm)v^2.$$

Since the final value of the kinetic energy is given by

$$E_f = \frac{1}{2}(m + dm)v^2,$$

it follows that

$$W = \left(\frac{m + dm}{m}\right) E_f.$$

Here it is seen quite clearly that the work done is greater in magnitude than the final kinetic energy so some of the initial quantity of work is used up in a way other than simply changing the kinetic energy of the system. In the case of the coalescence of two bodies as depicted here, the remaining part of the total work represents energy dissipated during the collision. Thus, as Mandelker[1] points out, the kinetic energy of the initially moving body, which equates with the total work, subdivides after coalescence into two parts – a kinetic part and a dissipative part – where the dissipative part is given by

$$E_d = W - E_f = W \frac{dm}{m+dm}.$$

Although the physical situation considered here is somewhat different from the original one, it does indicate that, when a system possessing a variable mass is considered, the right-hand side of the equation expressing the work done during the transition from a state 1 to a state 2 might be interpreted as consisting of a term representing the change in kinetic energy, together with a final term expressing a dissipation of energy due to the change in mass of the system. This could be seen as analogous to the situation obtaining in the operation of a Carnot engine – or any other heat engine for that matter – in thermodynamics, where the efficiency of such an engine is never unity since not all the heat is transferable into work. Here, in the example from classical mechanics under specific consideration, the terminology is slightly different but the principle is the same and it seems that, when a system of non-constant mass is considered, some of the initial energy is dissipated in allowing the mass itself to alter during the transition. However, it might be noted also that, in the collision process considered, the dissipation of energy at the collision is due to a variety of reasons – some might be dissipated as heat and some as sound, for example. In the case specifically under consideration here, neither of these reasons would come into play; rather the dissipated energy would be utilised simply in increasing the mass of the system. Nevertheless, it seems the same basic principles apply and the expression under discussion here should be interpreted as having a term indicating a change in the kinetic energy plus a separate term related to a dissipation of energy. It follows that the final value of the kinetic energy does not equal the total work done and, as shown by Mandelker[1], this has serious consequences for some expressions in special relativity.

Kinetic Energy in Relativistic Mechanics.

Considering the above considerations, as well as the speculations by Mandelker[1] alluded to, it seems sensible now to consider any resulting consequences for the concept of kinetic energy in special relativistic mechanics. In special relativity, an object is taken to possess an energy while at rest, denoted by mc^2 where m is the so-called rest mass of the object. When moving with a speed v, it's energy is given by $m\gamma c^2$ where $\gamma = (1 - v^2/c^2)^{-1/2}$. It is then said that the difference between the rest energy and the energy of the moving object is the kinetic energy which is obviously given by $mc^2(\gamma - 1)$. Interestingly, this very formula has been derived by Wesley[3] using techniques of classical Newtonian mechanics and with no reference to the ideas of special relativity. However, his starting point is to note the experimentally verified link between energy and mass via

$$E = mc^2.$$

He then says that, when a body is moving, its total mass must be the sum of its mass while at rest and an additional contribution given by its kinetic energy divided by c^2. This derivation is perfectly satisfactory except that it too assumes the total energy of a moving object is made up of two parts: - its energy while at rest and its kinetic energy when moving. Both approaches totally ignore the possibility, brought to the fore by Mandelker[1], of there being a third component – some energy dissipated during the motion. In the classical case of the collision of two billiard balls, one of which is at rest initially, it is easy to see that some of the initial energy in the system may be dissipated as heat energy and sound energy. In the case of a body whose mass naturally increases with speed, it is not so easy to differentiate between various possible uses of the initial energy in the system. However, it is undoubtedly the case that, during the motion, both the mass and speed of the system change. It must be wondered though if the two are irrevocably linked? Any such change may be broken down into infinitesimally small pieces, during each of which the mass and speed both change but each of these small pieces might be viewed along the lines of the analysis shown above of a moving mass colliding and coalescing with a small stationary mass. If that is the case, the final kinetic energy will not equal the initial kinetic energy; it will have a slightly lower value in accordance with the formula derived above. In the above, all the initial energy was in the form of kinetic energy and this equalled the maximum amount of work available. The final value of the kinetic energy was found to be given by

$$E_f = \frac{m}{m+dm}W = \frac{m}{m+dm}E_{in},$$

where all the symbols retain the same meanings as above.

If this same analysis applies in the relativistic case, the mass increment dm has to be considered at rest before coalescing with the moving mass m. Such an interpretation leads to

$$m + dm = m/\sqrt{1 - v^2/c^2}.$$

Hence, substituting into the above equation leads to

$$E_f = W\sqrt{1 - v^2/c^2} = E_{in}\sqrt{1 - v^2/c^2}.$$

Then, the final expression for the kinetic energy in relativistic mechanics is seen to be

$$E_f = mc^2\left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]\sqrt{1 - v^2/c^2} = mc^2\left[1 - \sqrt{1 - v^2/c^2}\right].$$

As Mandelker points out[1], this expression represents a new kinetic energy formula which has been derived within classical mechanics but with what might be termed the 'energy concept of matter', $E = mc^2$, as its basis.

It is interesting to note that

$$\frac{dE_f}{dv} = \frac{mv}{\sqrt{1 - v^2/c^2}} = m\gamma v,$$

that is, the derivative of the kinetic energy with respect to velocity equals the linear momentum as in classical Newtonian mechanics, again as shown by Mandelker[1]. Also, this new expression, like the older one, reduces to the kinetic energy being given by $\frac{1}{2}mv^2$ in the non-relativistic limit as it should. Unfortunately, the argument advanced for the validity of the 'new' kinetic energy expression for the relativistic case is not as complete and clear-cut as might be wished. As Mandelker has mentioned already[1], the explanation obtaining in the classical case does not apply so obviously in this case; it is not clear just how the total energy is split between genuine kinetic energy and dissipated energy in this relativistic case. However, it seems more than likely that this explanation is the true one and the 'new' kinetic energy expression as given above is the one which should be used.

Concluding Thoughts.

The above are merely some thoughts on the definition of, and expressions used for, the kinetic energy in mechanics. They show that the presently accepted position is not as secure as some might wish to believe and, while it is freely admitted that more thought is required to make the position completely clear and logically secure, this discussion and extension of the thoughts of Mandelker[1] and Wesley[3] certainly seems a step in the correct direction. Also, it does appear vitally important to recognise the role of what Mandelker[1] calls the 'energy concept of matter'. By this, he is really referring to the importance of the mass – energy equivalence expressed by

$$E = mc^2,$$

and where it must always be remembered that this relation is one which has been known and used well before the advent of special relativity; it is, in fact, an experimentally verifiable one. In fact, it is amazing to realise that the idea of mass-energy equivalence stretches back to Newton who, in the thirtieth of his cosmological queries asked "Are not gross bodies and light convertible into one another?"[4] However, maybe all in science would do well to remember and reflect on this thought from Thomas Merton[5]:

> 'All true learning should therefore be alive with the sense of its own limitations and with the instinct for a vital experience of reality which speculation alone cannot provide.'

When applied specifically to research in physical science, this thought might be interpreted as indicating the fact that limitations must exist to any theory but that these theories, to be truly worthwhile, cannot rely on human mental ingenuity alone; experimentation and observation have vitally important roles to play also and, without these, much highly erudite theory is largely worthless. In truth, what has been written above is merely a very small extension to the well-known Newtonian mechanics which has truly stood the test of time and, indeed, only makes use of relations and ideas originally put forward by Newton himself. It is almost certainly not the end of the story and all should note the thought of Thomas Merton and remain completely open-minded to the possibility of even further extensions within the sphere of classical Newtonian mechanics.

References.

1. J. Mandelker, *Matter Energy Mechanics*, Philosophical Library, New York, 1954.

2. Dennis P. Allen & Jeremy Dunning-Davies, *Neo-Newtonian Mechanics With Extension To Relativistic Velocities: Part 1: Non-Radiative Effects:* chap.6 (2013)**;** and at http://viXra.org/abs/1309.0210

3. J. P. Wesley; *Selected Topics in Scientific Physics*, Benjamin Wesley, Blumberg, 2002.

4. Kevin Brown, *Reflections on Relativity*, Lulu.com 2015.

5. T. Merton; *The Ascent to Truth*, Hollis & Carter, London, 1951.