Stability of the Moons orbits in Solar system (especially of Earth's Moon) in the restricted three-body problem (R3BP, celestial mechanics)

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Abstract: We consider the equations of motion of three-body problem in a *Lagrange form* (which means a consideration of relative motions of 3-bodies in regard to each other). Analyzing such a system of equations, we consider in details the case of moon's motion of negligible mass m_3 around the 2-nd of two giant-bodies m_1, m_2 (*which are rotating around their common centre of masses on Kepler's trajectories*), the mass of which is assumed to be less than the mass of central body.

Under assumtions of R3BP, we obtain the equations of motion which describe the relative mutual motion of the centre of mass of 2-nd giant-body m_2 (Planet) and the centre of mass of 3-rd body (Moon) with additional effective mass $\xi \cdot m_2$ placed in that centre of mass ($\xi \cdot m_2 + m_3$), where ξ is the dimensionless dynamical parameter. They should be rotating around their common centre of masses on Kepler's elliptic orbits.

For negligible effective mass ($\xi \cdot m_2 + m_3$) it gives the equations of motion which should describe a *quasi-circle* orbit of 3-rd body (Moon) around the 2-nd body m_2 (Planet) for most of the moons of the Planets in Solar system. But the orbit of Earth's Moon should be considered as non-constant *elliptic* motion for the effective mass $0.0178 \cdot m_2$ placed in the centre of mass for the 3-rd body (Moon). The position of their common centre of masses should obviously differ for real mass $m_3 = 0.0123 \cdot m_2$ and for the effective mass $(0.0055+0.0123) \cdot m_2$ placed in the centre of mass of the Moon.

Key Words: restricted three-body problem, orbit of the Moon, relative motion

Introduction.

The stability of the motion of the Moon is the ancient problem which leading scientists have been trying to solve during last 400 years. A new derivation to estimate such a problem from a point of view of relative motions in restricted three-body problem (R3BP) is proposed here.

Systematic approach to the problem above was suggested earlier in KAM-(*Kolmogorov-Arnold-Moser*)-theory [1] in which the central KAM-theorem is known to be applied for researches of stability of Solar system in terms of *restricted* three-body problem [2-5], especially if we consider *photogravitational* restricted three-body problem [6-8] with additional influence of *Yarkovsky* effect of non-gravitational nature [9].

KAM is the theory of stability of dynamical systems [1] which should solve a very specific question in regard to the stability of orbits of so-called “small bodies” in Solar system, in terms of *restricted* three-body problem [3]: indeed, dynamics of all the planets is assumed to satisfy to restrictions of *restricted* three-body problem (*such as infinitesimal masses, negligible deviations of the main orbital elements, etc.*).

Nevertheless, KAM also is known to assume the appropriate Hamilton formalism in proof of the central KAM-theorem [1]: the dynamical system is assumed to be *Hamilton* system as well as all the mathematical operations over such a dynamical system are assumed to be associated with a proper Hamilton system.

According to the Bruns theorem [5], there is no other invariants except well-known 10 integrals for three-body problem (*including integral of energy, momentum, etc.*), this is a classical example of Hamilton system. But in case of *restricted* three-body problem, there is no other invariants except only one, Jacobian-type integral of motion [3].

Such a contradiction is the main paradox of KAM-theory: it adopts all the restrictions of *restricted* three-body problem, but nevertheless it proves to use the Hamilton formalism, which assumes the conservation of all other invariants (*the integral of energy, momentum, etc.*).

To avoid ambiguity, let us consider a relative motion in three-body problem [2].

1. Equations of motion.

Let us consider the system of ODE for restricted three-body problem in barycentric Cartesian co-ordinate system, at given initial conditions [2-3]:

$$m_1 \boldsymbol{q}_1'' \;=\; -\gamma\left\{\frac{m_1 m_2(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{m_1 m_3(\boldsymbol{q}_1 - \boldsymbol{q}_3)}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3}\right\},$$

$$m_2 \boldsymbol{q}_2'' \;=\; -\gamma\left\{\frac{m_2 m_1(\boldsymbol{q}_2 - \boldsymbol{q}_1)}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3} + \frac{m_2 m_3(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3}\right\},$$

$$m_3 \boldsymbol{q}_3'' \;=\; -\gamma\left\{\frac{m_3 m_1(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{m_3 m_2(\boldsymbol{q}_3 - \boldsymbol{q}_2)}{|\boldsymbol{q}_3 - \boldsymbol{q}_2|^3}\right\} \;.$$

- here $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ - mean the radius-vectors of bodies m_1, m_2, m_3, accordingly; γ - is the gravitational constant.

System above could be represented for relative motion of three-bodies as shown below (by the proper linear transformations):

$$(\boldsymbol{q}_1 - \boldsymbol{q}_2)'' + \gamma(m_1 + m_2)\frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} \;=\; \gamma m_3\left\{\frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3}\right\},$$

$$(\boldsymbol{q}_2 - \boldsymbol{q}_3)'' + \gamma(m_2 + m_3)\frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \;=\; \gamma m_1\left\{\frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3}\right\},$$

$$(\boldsymbol{q}_3 - \boldsymbol{q}_1)'' + \gamma(m_1 + m_3)\frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} \;=\; \gamma m_2\left\{\frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3}\right\} \;.$$

Let us designate as below:

$$\boldsymbol{R}_{1,2} = (\boldsymbol{q}_1 - \boldsymbol{q}_2), \quad \boldsymbol{R}_{2,3} = (\boldsymbol{q}_2 - \boldsymbol{q}_3), \quad \boldsymbol{R}_{3,1} = (\boldsymbol{q}_3 - \boldsymbol{q}_1) \tag{*}$$

Using of (*) above, let us transform the previous system to another form:

$$\begin{aligned}
&\boldsymbol{R}_{1,2}'' + \gamma(m_1 + m_2)\frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} = \gamma m_3 \left\{\frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} + \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3}\right\}, \\
&\boldsymbol{R}_{2,3}'' + \gamma(m_2 + m_3)\frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} = \gamma m_1 \left\{\frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} + \frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3}\right\}, \\
&\boldsymbol{R}_{3,1}'' + \gamma(m_1 + m_3)\frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} = \gamma m_2 \left\{\frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} + \frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3}\right\} .
\end{aligned} \tag{1.1}$$

Analysing the system (1.1) we should note that if we sum all the above equations one to each other it would lead us to the result below:

$$\boldsymbol{R}_{1,2}'' + \boldsymbol{R}_{2,3}'' + \boldsymbol{R}_{3,1}'' = 0 \ .$$

If we also sum all the equalities (*) one to each other, we should obtain

$$\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + \boldsymbol{R}_{3,1} = 0 \tag{**}$$

Under assumption of restricted three-body problem, we assume that the mass of small 3-rd body $m_3 \ll m_1, m_2$, accordingly; besides, for the case of moving of small 3-rd body m_3 as a moon around the 2-nd body m_2, let us additionaly assume $|\boldsymbol{R}_{2,3}| \ll |\boldsymbol{R}_{1,2}|$.

So, taking into consideration (**), we obtain from the system (1.1) as below:

$$\boldsymbol{R}_{1,2}{}'' + \gamma(m_1 + m_2)\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} = 0,$$

$$\boldsymbol{R}_{2,3}{}'' + \gamma(m_2 + m_3)\frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} = \gamma m_1 \left\{\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} - \frac{(\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3})}{|\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3}|^3}\right\}, \qquad (1.2)$$

$$\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + \boldsymbol{R}_{3,1} = 0 \ ,$$

- where the 1-st equation of (1.2) describes the relative motion of 2 massive bodies (*which are rotating around their common centre of masses on Kepler's trajectories*); the 2-nd describes the orbit of small 3-rd body m_3 (Moon) relative to the 2-nd body m_2 (Planet), for which we could obtain according to the trigonometric "Law of Cosines" [10]:

$$\boldsymbol{R}_{2,3}{}'' + \gamma(m_2 + m_3)\frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} + \frac{\gamma m_1}{|\boldsymbol{R}_{1,2}|^3}\left(1 + 3\cos\alpha\frac{|\boldsymbol{R}_{2,3}|}{|\boldsymbol{R}_{1,2}|}\right)\boldsymbol{R}_{2,3} \cong -3\cos\alpha\left(\frac{\gamma m_1}{|\boldsymbol{R}_{1,2}|^3}\boldsymbol{R}_{1,2}\right)\frac{|\boldsymbol{R}_{2,3}|}{|\boldsymbol{R}_{1,2}|}, \qquad (1.3)$$

- here α – is the angle between the radius-vectors $\boldsymbol{R}_{2,3}$ and $\boldsymbol{R}_{1,2}$.

Equation (1.3) could be simplified under additional assumption $|\boldsymbol{R}_{2,3}| \ll |\boldsymbol{R}_{1,2}|$ for *restricted* mutual motions of bodies m_1, m_2 in R3BP [3] as below:

$$\boldsymbol{R}_{2,3}{}'' + \left(\frac{\gamma\,(m_2+m_3)}{\left|\boldsymbol{R}_{2,3}\right|^3} + \frac{\gamma\,m_1}{\left|\boldsymbol{R}_{1,2}\right|^3} \right) \cdot \boldsymbol{R}_{2,3} = 0 \qquad (1.4)$$

Moreover, if we present Eq. (1.4) in a form below

$$\boldsymbol{R}_{2,3}{}'' + \gamma\,(1+\xi+\eta) \cdot m_2 \cdot \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} = 0\,, \qquad (1.5)$$

$$\xi = \left(\frac{m_1}{m_2} \cdot \frac{\left|\boldsymbol{R}_{2,3}\right|^3}{\left|\boldsymbol{R}_{1,2}\right|^3} \right), \quad \eta = \left(\frac{m_3}{m_2} \right)$$

- then Eq. (1.5) describes the relative motion of the centre of mass of 2-nd giant-body m_2 (Planet) and the centre of mass of 3-rd body (Moon) with the effective mass ($\xi \cdot m_2 + m_3$), which are rotating around their common centre of masses on the stable Kepler's elliptic trajectories.

Besides, if the dimensionless parameters ξ, $\eta \rightarrow 0$ then equation (1.5) should describe a quasi-circle motion of 3-rd body (Moon) around the 2-nd body m_2 (Planet).

2. The comparison of the moons in Solar system.

As we can see from Eq. (1.5), ξ is the key parameter which determines the character of moving of the small 3-rd body m_3 (the Moon) relative to the 2-nd body m_2 (Planet). Let us compare such a parameter for all considerable known cases of orbital moving of the moons in Solar system [12] (Tab.1):

Table 1. Comparison of the parameters of the moons in Solar system.

Masses of the Planets (*Solar system*), kg	Ratio m_1 (Sun) to mass m_2 (Planet)	Distance $\lvert \boldsymbol{R}_{1,2} \rvert$ (*between Sun-Planet*), AU	Parameter η, ratio m_3 (Moon) to mass m_2 (Planet)	Distance $\lvert \boldsymbol{R}_{2,3} \rvert$ (*between Moon-Planet*) in 10^3 km	Parameter $\xi = \left(\frac{m_1}{m_2} \cdot \frac{\lvert \boldsymbol{R}_{2,3} \rvert^3}{\lvert \boldsymbol{R}_{1,2} \rvert^3} \right)$
Mercury, $3.3 \cdot 10^{23}$	$\left(\frac{332{,}946}{0.055} \right)$	0.387 AU			
Venus, $4.87 \cdot 10^{24}$	$\left(\frac{332{,}946}{0.815} \right)$	0.723 AU			
Earth, $5.97 \cdot 10^{24}$	1 Earth = 332,946 kg	1 AU = 149,500,000 km	$\mathbf{12{,}300} \cdot 10^{-6}$	383.4	Moon $\mathbf{5{,}532} \cdot 10^{-6}$
Mars, $6.42 \cdot 10^{23}$	$\left(\frac{332{,}946}{0.107} \right)$	1.524 AU	1) Phobos $0.02 \cdot 10^{-6}$ 2) Deimos $0.003 \cdot 10^{-6}$	1) Phobos 9.38 2) Deimos 23.46	1) Phobos $0.22 \cdot 10^{-6}$ 2) Deimos $3.4 \cdot 10^{-6}$
Jupiter, $1.9 \cdot 10^{27}$	$\left(\frac{332{,}946}{317.8} \right)$	5.2 AU	1) Ganymede $\mathbf{79} \cdot 10^{-6}$ 2) Callisto $\mathbf{58} \cdot 10^{-6}$ 3) Io $\mathbf{47} \cdot 10^{-6}$ 4) Europa $25 \cdot 10^{-6}$	1) Ganymede **1,070** 2) Callisto **1,883** 3) Io 422 4) Europa 671	1) Ganymede $2.73 \cdot 10^{-6}$ 2) Callisto $14.89 \cdot 10^{-6}$ 3) Io $0.17 \cdot 10^{-6}$ 4) Europa $0.67 \cdot 10^{-6}$

Saturn, $5.69 \cdot 10^{26}$	$\left(\frac{332{,}946}{95.16}\right)$	9.54 AU	1) Titan **240**$\cdot 10^{-6}$ 2) Rhea $4.1 \cdot 10^{-6}$ 3) Iapetus $3.4 \cdot 10^{-6}$ 4) Dione $1.9 \cdot 10^{-6}$ 5) Tethys $1.09 \cdot 10^{-6}$ 6) Enceladus $0.19 \cdot 10^{-6}$ 7) Mimas $0.07 \cdot 10^{-6}$	1) Titan **1,222** 2) Rhea 527 3) Iapetus **3,561** 4) Dione 377 5) Tethys 294.6 6) Enceladus 238 7) Mimas 185.4	1) Titan $2.2 \cdot 10^{-6}$ 2) Rhea $0.18 \cdot 10^{-6}$ 3) Iapetus **54.46**$\cdot 10^{-6}$ 4) Dione $0.07 \cdot 10^{-6}$ 5) Tethys $0.03 \cdot 10^{-6}$ 6) Enceladus $0.016 \cdot 10^{-6}$ 7) Mimas $0.008 \cdot 10^{-6}$
Uranus, $8.69 \cdot 10^{25}$	$\left(\frac{332{,}946}{14.37}\right)$	19.19 AU	1) Titania $40 \cdot 10^{-6}$ 2) Oberon $35 \cdot 10^{-6}$ 3) Ariel: $16 \cdot 10^{-6}$ 4) Umbriel: $13.49 \cdot 10^{-6}$ 5) Miranda: $0.75 \cdot 10^{-6}$	1) Titania 436 2) Oberon 584 3) Ariel: 191 4) Umbriel: 266.3 5) Miranda: 129.4	1) Titania $0.08 \cdot 10^{-6}$ 2) Oberon $0.2 \cdot 10^{-6}$ 3) Ariel: $0.01 \cdot 10^{-6}$ 4) Umbriel: $0.019 \cdot 10^{-6}$ 5) Miranda: $0.002 \cdot 10^{-6}$

Neptune, 1.02·10^{26}	$\left(\frac{332{,}946}{17.15}\right)$	30.07 AU	1) Triton **210**·10^{-6} 2) Proteus 0.48·10^{-6} 3) Nereid 0.29·10^{-6}	1) Triton 355 2) Proteus 118 3) Nereid **5,513**	1) Triton 0.01·10^{-6} 2) Proteus 0.0004·10^{-6} 3) Nereid 35.81·10^{-6}
Pluto, 1.3·10^{22}	$\left(\frac{332{,}946}{0.002}\right)$	39.48 AU	Charon **124,620**·10^{-6}	Charon 20	Charon 0.006·10^{-6}

3. Discussion.

As we can see from the Tab.1 above, the dimensionless key parameter ξ, which determines the character of moving of the small 3-rd body m_3 (Moon) relative to the 2-nd body m_2 (Planet), is varying for all variety of the moons of the Planets (in Solar system) from the meaning 0.0004·10^{-6} (for Proteus of Neptune) to the meaning 54.46·10^{-6} (for Iapetus of Saturn); but it still remains to be negligible enough for adopting the stable moving of the effective mass (ξ·m_2 + m_3) on *quasi-elliptic* Kepler's orbit around their common centre of masses with the 2-nd body m_2.

If the total sum of dimensionless parameters (ξ + η) → 0 is negligible then equation (1.5) should describe a stable quasi-circle orbit of 3-rd body (Moon) around the 2-nd body m_2 (Planet). Let us consider the proper examples which deviate (differ) to some extent from the negligibility case (ξ + η) → 0 above (Tab.1) [12]:

1. Charon-Pluto: $(\xi + \eta) = (0.0062+124,620)\cdot 10^{-6}$, eccentricity 0.00
2. Nereid-Neptune: $(\xi + \eta) = (\mathbf{35.81}+0,29)\cdot 10^{-6}$, eccentricity **0.7507**
3. Triton-Neptune: $(\xi + \eta) = (0.01+210)\cdot 10^{-6}$, eccentricity 0.000 016
4. Iapetus-Saturn: $(\xi + \eta) = (\mathbf{54.46}+3.4)\cdot 10^{-6}$, eccentricity 0.0286
5. Titan-Saturn: $(\xi + \eta) = (2.2+240)\cdot 10^{-6}$, eccentricity 0.0288
6. Io-Jupiter: $(\xi + \eta) = (0.168 + 47)\cdot 10^{-6}$, eccentricity 0.0041
7. Callisto-Jupiter: $(\xi + \eta) = (14.89+58)\cdot 10^{-6}$, eccentricity 0.0074
8. Ganymede-Jupiter: $(\xi + \eta) = (2.73+79)\cdot 10^{-6}$, eccentricity 0.0013
9. Phobos-Mars: $(\xi + \eta) = (\mathbf{0,217}+\mathbf{0,02})\cdot 10^{-6}$, eccentricity **0.0151**
10. Moon-Earth: $(\xi + \eta) = (\mathbf{5,532}+\mathbf{12,300})\cdot 10^{-6}$, eccentricity 0.0549

The obvious extreme exception is the Nereid (moon of Neptune) from this scheme: Nereid orbits Neptune in the prograde direction at an average distance of 5,513,400 km, but its high eccentricity of 0.7507 takes it as close as 1,372,000 km and as far as 9,655,000 km [12].

The unusual orbit suggests that it may be either a captured asteroid or Kuiper belt object, or that it was an inner moon in the past and was perturbed during the capture of Neptune's largest moon Triton [12]. One could suppose that the orbit of Nereid should be derived preferably from the assumptions of R4BP (the case of Restricted Four-Body Problem) or more complicated cases.

As we can see from consideration above, in case of the Earth's Moon such a dimensionless key parameters increase simulteneously to the crucial meanings $\xi = 0.0055$ and $\eta = 0.0123$ respectively, $(\xi + \eta) = 0.0178$ (other case is the pair 'Charon-Pluto', but only parameter $\eta \cong 0.125$ is increased for them). It means that the orbit for relative motion of the Moon in regard to the Earth could not be considered as *quasi-elliptic* orbit and should be considered as non-constant *elliptic* orbit with the effective mass $(\xi\cdot m_2 + m_3)$ placed in the centre of mass for the Moon.

As we know [3-4], the elements of that elliptic orbit depend on the position of the common centre of masses for 3-rd small body (Moon) and the planet (Earth). But such a position of their common centre of mass should obviously differ for the real mass m_3 and the effective mass ($\xi \cdot m_2 + m_3$) placed in the centre of mass of the 3-rd body (Moon). So, the elliptic orbit of motion of the Moon derived from the assumtions of R3BP should differ from the elliptic orbit which could be obtained from the assumtions of R2BP (the case of Restricted Two-Body Problem: it means mutual moving of 2 gravitating masses without the influence of other central forces).

4. Remarks about the eccentricities of the orbits.

According to the definition [12], the orbital eccentricity of an astronomical object is a parameter that determines the amount by which its orbit around another body deviates from a perfect orbit:

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} \ ,$$

- where ε - is the specific orbital energy; h - is the specific angular momentum; μ - is the sum of the standard gravitational parameters of the bodies, $\mu = \gamma \cdot m_2 \cdot (1 + \xi + \eta)$, see (1.5).

The specific orbital energy equals to the constant sum of kinetic and potential energy in a 2-body ballistic trajectory [12]:

$$\varepsilon = -\frac{\mu}{2a} = const \ ,$$

- here a – is the semi-major axis. For an elliptic orbit the specific orbital energy is the negative of the additional energy required to accelerate a mass of one kilogram to escape velocity (parabolic orbit).

Thus, assuming $\xi = \xi(t)$, we should obtain from the equality above:

$$-\frac{\gamma \cdot m_2 (1+\xi(t)+\eta)}{2a(t)} = const, \quad \Rightarrow \quad a(t) = a_0 \cdot (1+\xi(t)+\eta), \tag{4.1}$$

- where $\xi(t)$ – is the periodic function depending on time-parameter t, which is slowly varying during all the time-period from the minimal meaning $\xi_{min} > 0$ to the maximal meaning ξ_{max}, preferably $(\xi_{max} - \xi_{min}) \ll \xi_{min}$.

Besides, we should note that in an elliptical orbit, the specific angular momentum h is twice the area per unit time swept out by a chord from the primary to the secondary: this area is referred to by Kepler’s second law of planetary motion.

Since the area of the entire orbital ellipse is swept out in one orbital period, the specific angular momentun h is equal to twice the area of the ellipse divided by the orbital period, as represented by the equation:

$$h = b\sqrt{\frac{\gamma(1+\xi+\eta)\cdot m_2}{a}} \;,$$

- where b – is the semi-minor axis. So, from Eq. (4.1) we should obtain that for the constant specific angular momentum h, the semi-minor axis b should be constant also.

Thus, we could express the components of elliptic orbit as below:

$$x(t) = a_0 \cdot (1+\xi(t)+\eta)\cdot \cos t \;,$$

$$y(t) = b\cdot \sin t \,,$$

- which could be schematically imagined as it is shown at Fig.1-3.

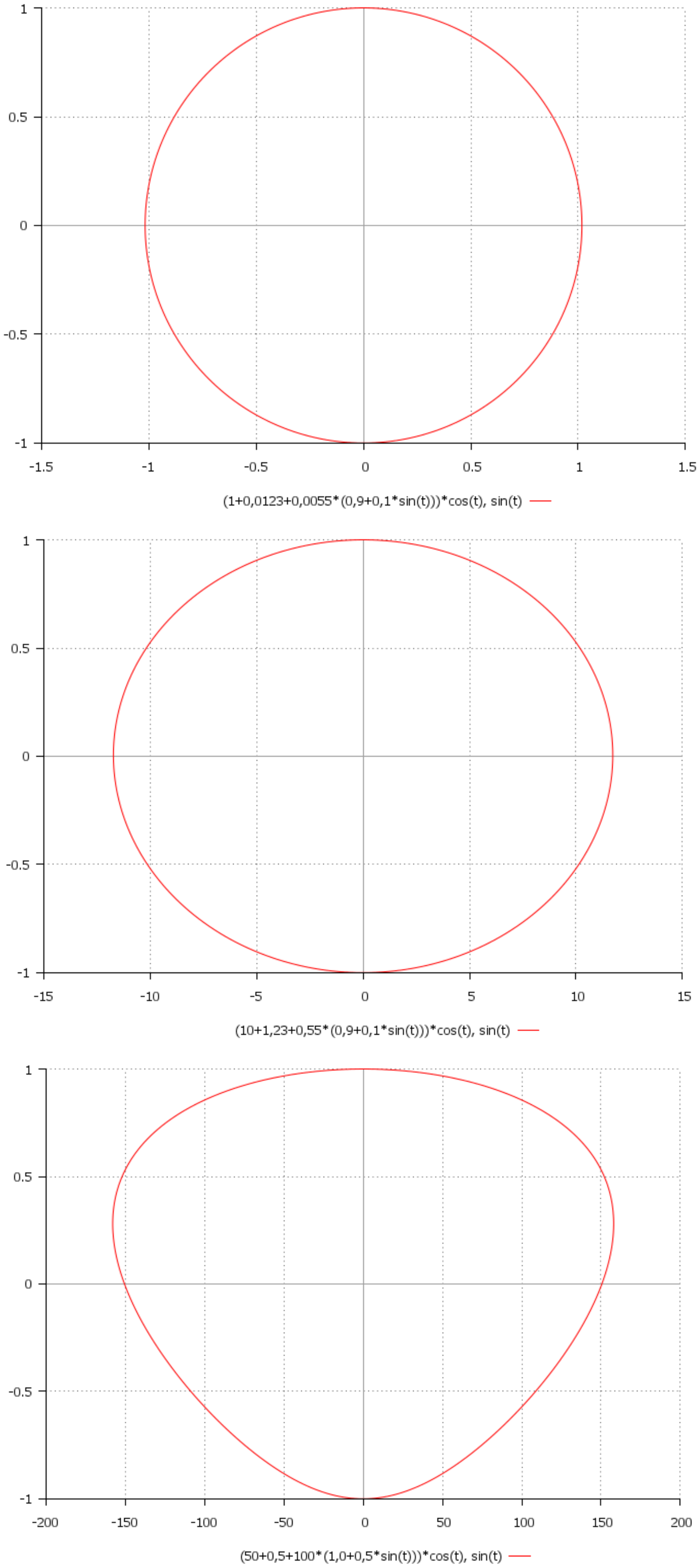

Fig.1-3. Orbits of the moon, schematically imagined.

Conclusion.

We have considered the equations of motion of three-body problem in a *Lagrange form* (which means a consideration of relative motions of 3-bodies in regard to each other). Analyzing such a system of equations, we consider in details the case of moon's motion of negligible mass m_3 around the 2-nd of two giant-bodies m_1, m_2 (*which are rotating around their common centre of masses on Kepler's trajectories*), the mass of which is assumed to be less than the mass of central body.

Under assumtions of R3BP, we obtain the equations of motion which describe the relative mutual motion of the centre of mass of 2-nd giant-body m_2 (Planet) and the centre of mass of 3-rd body (Moon) with additional effective mass $\xi \cdot m_2$ placed in that centre of mass ($\xi \cdot m_2 + m_3$), where ξ is the dimensionless dynamical parameter. They should be rotating around their common centre of masses on Kepler's elliptic orbits.

For negligible effective mass ($\xi \cdot m_2 + m_3$) it gives equations of motion which should describe a *quasi-circle* orbit of 3-rd body (Moon) around the 2-nd body m_2 (Planet) for most of the moons of the Planets in Solar system. But the orbit of Earth's Moon should be considered as non-constant *elliptic* motion for the effective mass $0.0178 \cdot m_2$ placed in the centre of mass for the 3-rd body (Moon). The position of their common centre of masses should obviously differ for the real mass $m_3 = 0.0123 \cdot m_2$ and for the effective mass $(0.0055+0.0123) \cdot m_2$ placed in the centre of mass of the Moon.

Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this article.

Acknowledgements

I am thankful to CNews Russia project (*Science & Technology Forum*, Prof. L.Vladimirov-Paraligon) - for valuable discussions in preparing of this manuscript.

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