A QUANTUM LOGICAL UNDERSTANDING OF BOUND STATES

KAN KUO-HSIAO

ABSTRACT. This short note presents the structures of lattices and continuous geometries in the energy spectrum of a quantum bound state. Quantum logic, in von Neumann's original sense, is used to construct these structures. Finally, a quantum logical understanding of the emergence of discreteness is suggested.

1. INTRODUCTION

Bound states appear almost everywhere in quantum physics. The standard treatment is solving the Schrödinger equation to get the discrete and nondegenerate spectrum. See, for example, the classic discussions in §3.1.6 and §3.2.10 of [1]. A heuristic explaination of the discreteness of the spectrum, which is also frequently adopted in explaining the Casimir effect, is the trunction of possible modes enforced by the boundary conditions. However, we hope to find a deeper understanding of this emergence of discreteness with respect to the underlying mathematical structrues (e.g. the Hilbert space, lattices, etc.).

Here we present an alternative understanding of this discreteness from the perspective of quantum logic in the original sense of Birkhoff and von Neumann[2]. The main idea is very simple: in the seminal work of Birkhoff and von Neumann[2], the lattice structure of the propositional calculus in a Hilbert space is shown to be a projective geometry, in which the failure of the modularity for quantum theory leads von Neumann's preference to the continuous projective geometry[3], and finally, to the type II_1 factor von Neumann algebra[4]. These concepts will be expanded in the next section. And an excellent account of this history can be found in [5]. For our current concern of bound states, one needs first to identify the lattice structure in the energy spectrum of a bound state and then examines other properties on this lattice. In fact, as will be shown below in the simple example of 1-dimensional finite potential well, the subspaces of eigen-energies of a bound state admits a Hilbert lattice, on which one can define a continuous geometry. Then the relationship with all kinds of mathematical structures of quantum mechanics becomes clearer. We suggest that the discreteness is just a manifestation of the noncommutativity of the observable algebra via the quantum logical constructions.

2. PRELIMINARIES

Let us start with some necessary definitions in [2].

Definition 2.1. A lattice L is a partially ordered set, any two elements of which have a unique infimum and a unique supremum. To be more explicit, denote the partial order as $<$, then for any two element $x, y \in L$ there correspond

Date: March 30, 2015.

2010 *Mathematics Subject Classification.* 06C15, 81P10.

(1) an infimum(meet) $x \wedge y$ such that $x \wedge y < x$, $x \wedge y < y$ and $(z < x) \wedge (z < y) \to z < x \wedge y$;
(2) a supremum(join) $x \vee y$ such that $x < x \vee y$, $y < x \vee y$ and $(x < w) \wedge (y < w) \to x \vee y < w$.

Remark 2.2. The partial order can be any logical implication or simply the subset inclusion $\subset$ in which case the set intersection $\cap$ and the union $\cup$ play the roles of meet and join respectively. In terms of these set-theoretical notations, one calls a lattice *complete* if all its subsets are sublattices. And a lattice is *bounded* if it has a maximum and a minimum subset.

To obtain a Boolean algebra, one further needs the complementation,

Definition 2.3. The *complementation* of an element a of a lattice L is an operation $\neg$ such that

(1) $\neg(\neg a) = 0$;
(2) $a \wedge \neg a = \emptyset$, $a \vee \neg a = L$, or simply $(a < \neg a) \to (a = \emptyset)$;
(3) $(a < b) \to (\neg b < \neg a)$.

A bounded lattice is called an orthocomplemented lattice if it has the complementation and the orthogonality, i.e. $a \perp b$ iff $(a < \neg b) \vee (a = \neg b)$ or simply $a \leqslant \neg b$.

A Boolean algebra has the distributive property,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \text{ and } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{1}$$

This is correct in classical mechanics but not in quantum mechanics. Even if one abandons the Boolean algebra, there is still one identity that might be missing in quantum mechanics, that is, the modularity,

$$a < c \to a \vee (b \wedge c) = (a \vee b) \wedge c. \tag{2}$$

The modularity can be weakened to the orthomodularity[6],

$$a < c \to a \vee (\neg a \wedge c) = c, \tag{3}$$

which is the starting point of the abstract quantum logic.

In connection with quantum mechanics, an immediate example is the lattice structure in a Hilbert space $\mathcal{H}$. Indeed, one has the Hilbert lattice $\mathcal{P}(\mathcal{H})$, the set of closed linear subspaces of $\mathcal{H}$. The partial oerder of $\mathcal{P}(\mathcal{H})$ is the subset inclusion $\subseteq$ or the subspace projections, and hence the other operations can be defined set-theoretically. For the properties of $\mathcal{P}(\mathcal{H})$ one can refer [6]. Concerning the modularity in $\mathcal{P}(\mathcal{H})$, one has especially

Proposition 2.4. *If the Hilbert space $\mathcal{H}$ is finite dimensional, then $\mathcal{P}(\mathcal{H})$ is modular. If the dimension is infinite, $\mathcal{P}(\mathcal{H})$ is not mudular but orthomodular.*

Since in quantum mechanics the pertinent Hilbert space might be finitie-dimensional, e.g. the 2-dimensional Hilbert space for qubits, one sees that this is not an essential difference between the classical and the quantum. The reason that Birkhoff and von Neumann insist on the existence of modularity in quantum logic is the existence of an "*a priori thermo-dynamic weight of state*"[3]. For the exact meaning of this vague statement, see below or [5].

The modularity can be recovered if one considers the continuous geometry of von Neumann[3]. The following definition is the version of Birkhoff[7].

Definition 2.5. An irreducible complemented modular lattice L is a *continuous geometry* if it has a metric completion by the distance function

$$|x - y| = \delta(x \vee y) - \delta(x \wedge y),\ \forall\ x, y \in L,$$

where

$$\delta(x) = \frac{d[x] - d[0]}{d[1] - d[0]}$$

is the normed dimension function defined in terms of the commom geometrical dimension $d[x]$.

Remark 2.6. The metric topology determined by this distance function can be constructed without essential difficulties. While the irreducibility means that if $\forall a \in L$ its complementation is unique, then $a = \emptyset$ or $a = L$. Besides, Kaplansky[8] showed that a continuous geometry only needs to be an orthocomplemented modular lattice.

An incisive observation of von Neumann on continuous geometries is the following proposition, the ingenious proof of which is outlined in [3] and detailed in [9]. Consider the lattice $L = L_n$ of all linear subspaces of any $(n-1)$-dimensional projective geometry P_{n-1} and the set-theoretical sum L_∞ of all L_n's. As can be readily checked, they are both continuous geometries (in the axiomatic sense of [3], L_n is in fact discrete though). Then one has

Proposition 2.7. *The ranges of the dimension function $\delta(L_n)$ and $\delta(L_\infty)$ are $\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\} \subset \mathbb{R}$ and $[0, 1] \subset \mathbb{R}$ respectively.*

A further step is the projective lattice structure in a von Neumann algebra $\mathcal{M}$, namely, the von Neumann lattice $\mathcal{P}(\mathcal{M})$. This is defined in analogue to the Hilbert lattice $\mathcal{P}(\mathcal{H})$ with the Hilbert space $\mathcal{H}$ replaced by the von Neumann algebra $\mathcal{M}$. But the properties of $\mathcal{P}(\mathcal{M})$ are different from $\mathcal{P}(\mathcal{H})$ in many aspects, for which one can refer [6].

An important proprty of $\mathcal{P}(\mathcal{M})$ is that the order type of the equivalence classes $\mathcal{P}(\mathcal{M})_\sim$ can be characterized by the range of the dimension function δ. Here the equivalence $\sim$ is in the sense that $\forall$ projections $A, B \in \mathcal{P}(\mathcal{M})$, $A \sim B$ if $\exists$ a partial isometry α such that $\alpha(A^\perp) = \emptyset$ and $\alpha(A) = B$. This property leads to the famous classification of factor von Neumann algebra[4]. For usual concerns in quantum mechanics, the Hilbert spaces belongs to the type I factor von Neumann algebra with the range of δ being $\{0, 1, 2, ...(\infty)\}$. While the type II_1 factor, von Neumann's favorate one, corresponds to the case in which the range of δ is $[0, 1]$. One has the following result, for the proof of which one can refer [10].

Proposition 2.8. *There exists a subspace lattice of a type II_1 factor $\mathcal{M}$ isomorphic to the continuous geometry L_∞.*

Another inportant property of $\mathcal{P}(\mathcal{M})$ is that the dimension function δ on $\mathcal{P}(\mathcal{M})$ defines a unique finite trace τ on $\mathcal{M}$, with which the "*a priori thermo-dynamic weight of state*" is realized as this trace on an infinite-dimensional Hilbert space in the sense of the 'statistical ansatz'[11, 5] that the 'relative' probability arise from the *a prioiri* ensamble as relative frequency. In this respect, for quantum mechanics with infinite-dimensional Hilbert space, the lattice of the observable algebra is just the von Neuman lattice with the type II_1 factor algebra.

Remark 2.9. This kind of frequency interpretation of probability is flawed and abandoned by von Neumann himself later[12]. It is indeed the probability theory on noncommutative spaces, or simply quantum probability, takes over. The last proposal of von Nuemann on this subject is

> "a formal mechanism, in which logics and probability theory arise simultaneously and are derived simultaneously." (quoted in [12])

Recent works of Holik et.al.[13] shows that it is indeed possible to find such a logical theory of quantum probability.

3. Lattices in the Energy Spectrum of a Bound State

Now let us come back to our main objective: to find lattice structures in the energy spectrum of a bound state. For a bound state its eigen-energies are of course bounded no matter whether the state is quantum or classical. Hence one can expect a bounded lattice.

As a first step, let us see an example of bound states in classical mechanics.

Example 3.1 (Holik et.al.[13]**).** Consider a classical harmonic oscillator or a ball confined in an elliptical potential well, with energy E_c. Then "$E_c = E_0$" corresponds to an ellipse in the phase space Γ, and "$E_c < E_0$" correspond to the ellipse and its interior which is of course continuous and infinte-dimensional. One can readily see that the propositional calculus in this case can be represented by the subsets of the phase space Γ. Consequently, one can define a lattice $\mathcal{P}(\Gamma)$ in the phase space Γ by set-theoretical operations. $\mathcal{P}(\Gamma)$ is a complete bounded lattice.

Similarly, in the quantum case the phase space is a Hilbert space $\mathcal{H}$. Thus one can obtain a Hilbert lattice $\mathcal{P}(\mathcal{H})$. Let us check the operations:

- partial ordering: $\forall$ energy eigen-subspaces $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$, $\mathcal{H}_1 \leqslant \mathcal{H}_2$ if $E_1 \leqslant E_2$;
- meet: $\mathcal{H}_1 \wedge \mathcal{H}_2 \equiv \mathcal{H}_i$ with energy$=\min\{E_1, E_2\}$, or set-theoretical intersection $\cap$ of energy eigen-subspaces since by definition all intersections$=\emptyset$;
- join: $\mathcal{H}_1 \vee \mathcal{H}_2 \equiv \mathcal{H}_i$ with energy$=\max\{E_1, E_2\}$ but note that set-theoretical union $\cup$ of energy eigen-subspaces is not closed;
- complementation: set-theoretical orthocomplementation of subspaces $\perp$.

Thus, one indeed obtains a Hilbert lattice for quantum bound states. Now let us see some examples first.

Example 3.2. Consider the 1-dimensional finite square potential well in quantum mechanics with $U_2 < E < U_1 < U_3 < \infty$ (U_2 at $[a, b]$). Following §3.1.6 of [1], one can write the connection condition at the boundary as

$$n\pi - \xi KL = \sin^{-1}\xi + \sin^{-1}(\xi\cos\gamma),$$

where

$$K = \sqrt{U_1 - U_2},\ L = b - a,\ \cos\gamma = \sqrt{\frac{U_1 - U_2}{U_3 - U_2}}\ (0 < \gamma < \frac{\pi}{2}),\ \xi = \sqrt{\frac{E - U_2}{U_1 - U_2}}.$$

Solutions exist iff

$$KL \geqslant (n-1)\pi + \gamma,$$

hence one obtains a discrete and finite energy spectrum, the maximum of which corresponds to the maximal integer in $1 + (KL - \gamma)/\pi$. From this discreteness one gets a finite Hilbert lattice $\mathcal{P}(\mathcal{E})$. And hence $\mathcal{P}(\mathcal{E})$ is modular.

Moreover, for an eigen-state with energy E_n, the number of zeros of the wave function is $n-1$ that equals the number of excited energy levels. If one identifies the energy eigen-subspaces with the number of zeros, or simply nodes, of the eigen wave funtions, the result is actully a continuous geometry. Let us elaborate on this point. Firstly since $\mathcal{P}(\mathcal{E})$ is a finite modular lattice and the complementations of the eigen-subspaces are obvious orthocomplementations, by Kaplansky's theorem[8] one arrives at a continuous geometry. In this finite case, the spectrum has to be discrete to ensure the finiteness of $\mathcal{H}$. Secondly, to be more explicit, let us check the projective properties: when identifying the quantum number n with the node number $(n-1)$, one in fact constructs a projective geometry in the following sense. The eigen subspaces $[\mathcal{H}_1 : ... : \mathcal{H}_n]$ form a right-ratio, with which one can define the projective equivalence, i.e. $[\mathcal{H}_1 : ... : \mathcal{H}_n] \sim [\mathcal{G}_1 : ... : \mathcal{G}_n]$ if $\exists \mathcal{H}_j \in \mathcal{F}$ such that $\mathcal{H}_i = \mathcal{H}_j \vee \mathcal{G}_i$. One can take $\mathcal{H}_j = \mathcal{H}_1 = \mathcal{G}_1 =$ ground state, which gives a $(n-1)$-dimensional projective geometry $P_{n-1}(\mathcal{E})$. Intuitively, a state with energy E in the bounded region is an eigenstate iff the logarithmic derivatives of the two aymptotic boundary solutions are equal $f_- = f_+$[1], which gives a visual picture of the projective identification. The linear subspaces of $P_{n-1}(\mathcal{E})$ are defined by the constraints of linear independence

$$\alpha_{k1}\mathcal{H}_1 + ... + \alpha_{kn}\mathcal{H}_n = 0,\ k = 1, ...m.$$

Since the "points"($m = 1$ or of dimension $1/n$) in these linear subspaces are in a 1-1 correspondence with the right-ratios, one can inductively see that the number $(n-1)$ of nodes characterizing the energy levels shows the same structure as the geometric dimension from 0 to $n-1$ (or $1/n$ to 1). Hence one obtains a discrete geometry similar to L_n, in other words, a successful "coördinatization"[14]. In this respect, the discreteness of the spectrum corresponds to the discrete range of the dimension function. Since the energy spectrum of a bound state can be countably infinite-dimensional, the second argument is still valid for the continuous geometry L_∞ while the Kaplansky theorem does not apply here.

From this example one sees that beyond the usual Hilbert space structure, there are also lattices and continuous geometry entering in this simple 1-diemnsional quantum mechanical system in quite noval ways. One can even expect a type II_1 factor von Neumann algebra, since the continuous geometry L_∞ is isomorphic to some subspace lattice of a type II_1 factor algebra. This seems magical right now, so let us explain this magic a little bit:

In the operational approach to physics(see e.g. [15]), the observable algebra is a C^*-algebra. Based on this observation one can develop the whole mathematical structures underlying quantum mechanics:

$$\begin{array}{ccccc}
C^*\text{-algebra} & \xrightarrow{\text{GNS}} & \text{Hilbert space} & \xrightarrow{\text{B\&vN}} & \text{Hilbert lattice} \\
\Big\downarrow{\scriptstyle \text{measurable}} & & \Big\downarrow{\scriptstyle \text{lattice gas, vN?}} & & \Big\downarrow{\scriptstyle (*)} \\
\text{von Neumann algebra} & \xrightarrow{\text{M\&vN}} & \text{type II}_1 \text{ factor} & \xrightarrow{\text{vN},L_\infty} & \text{continuous geometry}
\end{array}$$

The $(*)$ in the above diagram denotes our construction in the previous text. One can see that in some special cases, especially when the quantum system has infinite but discrete spectrum, the Hilbert space can actually admits a type II_1 factor von Neumann algebra. This might corroborate the unpublished work (cited in [12]) of von Neumann that the type II_1 factor von Neumann algebra can be constructed from the infinite-dimensional Hilbert space.

In the light of this understanding, we conclude that the discreteness in the spectrum of a quantum bound state can be understood as originating from the noncommutativity of the observable algebra for either the reason of solving the Schrödinger equations or the reason that the generic quantum mechanics admits a continuous geometry or a type II_1 factor von Neumann algebra. The proof of the latter is still lacking, and the example of bound states in this note is too special to be illuminating. However, again from von Neumann's last proposal, the logical quantum probability, as is recently illustrated in [13], suggests a new paradigm to understand Nature, to wit, from discrete logics to continuous mechanics, and from probability to stochastic behaviors and then to certain observations.

Remark 3.3. Most researchers today seem to ignore von Neumann's insights. In fact, many modern models share similar features with von Neumann's veteran theory. Cf. the causal set approach to quantum gravity[16] and the celluar automaton interpretation of quantum mechancs[17].

Aknowledgement. This problem was brought to my attention by Zhiqiang Huang. I thank him for helpful discussions.

References

1. A. Messiah, *Mécanique Quantique*, Dunod, Paris, 1973.
2. G. Birkhoff, J. von Neumann, The Logic of Quantum Mechanics, Ann. Math. **37**, 823 (1936).
3. J. von Neumann, Continuous Geometry, Proc. Nat. Acad. Sci. USA **22**, 92 (1936).
4. F. J. Murray, J. von Neumann, On Rings of Algebras, Ann. Math. **37**, 116 (1936).
5. M. Rédei, Why John von Neumann did not Like the Hilbert Space Formalism of Quantum Mechanics, Stud. Hist. Phil. Mod. Phys. **27**, 493 (1996).
6. M. Rédei, *Quantum Logic in Algebraic Approach*, Kluwer, Dordrecht, 1998.
7. G. Birkhoff, von Neumann and Lattice Theory, Bull. Amer. Math, Soc. **64**, 50 (1958).
8. I. Kaplansky, Any Orthocomplemented Complete Modular Lattice is a Continuous Geometry, Ann. Math. **61**, 524 (1955).
9. J. von Neumann, *Continuous Geometry*, Princeton University Press, Princeton, 1960.
10. J. von Neumann, *Continuous Geometry with A Transition Probability*, Memoirs of AMS, No.252, 1981.
11. J. von Neumann, *Collected Works*, Vol.I, pp.151, Pergamon, London, 1961.
12. M. Rédei, "Unsolved Problems in Mathematics", Math. Intelligencer **21**(4), 7 (1999).
13. F. Holik, M. Sáenz, A. Plastino, A Discussion on the Origin of Quantum Probabilities, Ann. Phys. **340**, 293 (2014).
14. J. von Neumann, Algebraic Theory of Continuous Geometry, Proc. Nat. Acad. Sci. USA **23**, 16 (1937).
15. F. Strocchi, *An Introduction to the Mathematical Structures of Quantum Mechanics*, 2nd Ed., World Scientific, Singapore, 2008.
16. B. F. Dribus, On the Axioms of Causal Set Theory, arXiv:1311.2148v3.
17. G. 't Hooft, The Cellular Automaton Interpretation of Quantum Mechancis, arXiv:1405.1548v2.

E-mail address: `kankuohsiao@whu.edu.cn`