Simplified Calculation of Component Number in the Curvature Tensor

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Abstract

The number of independent components in the Riemann-Christoffel curvature tensor, being composed of the metric tensor and its first and second derivatives, varies considerably with the dimension of space. Since few texts provide an explicit derivation of component number, we present here a simplified method using only the curvature tensor's antisymmetry property and the cyclicity condition. For generality and comparison, the method for computing component number in both Riemannian and non-Riemannian space is presented.

Introduction

For any vector or tensor quantity one can take the difference of its double covariant derivatives to obtain a derivation of the Riemann-Christoffel curvature tensor $R^{\lambda}{}_{\mu\alpha\beta}$. For example, given the arbitrary covariant vector ξ_μ it is easy to show that

$$\xi_{\mu||\alpha||\beta} - \xi_{\mu||\beta||\alpha} = -\xi_\lambda R^{\lambda}{}_{\mu\alpha\beta} = -\xi^\lambda R_{\lambda\mu\alpha\beta} \tag{1}$$

(the double-bar convention of Adler-Bazin-Schiffer is assumed), where

$$R^{\lambda}{}_{\mu\alpha\beta} = \Gamma^{\lambda}_{\mu\alpha|\beta} - \Gamma^{\lambda}_{\mu\beta|\alpha} + \Gamma^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\nu}_{\mu\beta}$$

where the $\Gamma^{\lambda}_{\mu\alpha}$ quantities are *connection coefficients* and the single subscripted bar represents ordinary partial differentiation. Lowering the upper index with the metric tensor $g_{\mu\lambda}$ gives us $R_{\mu\nu\alpha\beta}$. The indices μ, ν, which are unconstrained for the time being, contribute n^2 terms, while the antisymmetry of the last two indices contributes $n(n-1)/2$ terms. Initially, this gives a total of $n^3(n-1)/2$ independent components to the curvature tensor, or 96 in 4-dimensional space.

Riemannian space is characterized by a connection coefficient that is identified with the familiar Christoffel symbol of differential geometry, or

$$\Gamma^{\alpha}_{\mu\nu} = \begin{Bmatrix} \alpha \\ \mu\,\nu \end{Bmatrix} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta} \right)$$

When this is the case, the covariant derivative of the metric tensor can be shown to vanish identically:

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda} \begin{Bmatrix} \lambda \\ \alpha\,\nu \end{Bmatrix} - g_{\lambda\nu} \begin{Bmatrix} \lambda \\ \alpha\mu \end{Bmatrix} = 0$$

The vanishing of $g_{\mu\nu||\lambda}$ (called the *non-metricity tensor*) is usually simply referred to as *metricity*. When $g_{\mu\nu||\lambda} \neq 0$, then the space is called non-Riemannian.

However, there is one other symmetry property of the curvature tensor that applies to both Riemannian and non-Riemannian spaces. It is called the *cyclicity condition*, and is given by

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0, \tag{2}$$

which can easily be verified by inspection upon cyclic permutation of the indices ν, α, β. This symmetry property signficantly reduces the number of independent components in the curvature tensor, especially in higher-dimensional spaces. We shall see that the cyclicity condition, along with the antisymmetry of the last two indices in $R_{\mu\nu\alpha\beta}$, completely determines the number of independent components in the curvature tensor for both the Riemannian and non-Riemannian cases.

Riemannian Case

As in (1), the difference of the double covariant derivatives of the metric tensor can be written as

$$g_{\mu\nu||\alpha||\beta} - g_{\mu\nu||\beta||\alpha} = -g_{\mu\lambda} R^{\lambda}{}_{\nu\alpha\beta} - g_{\lambda\nu} R^{\lambda}{}_{\mu\alpha\beta}$$

or

$$g_{\mu\nu||\alpha||\beta} - g_{\mu\nu||\beta||\alpha} = -(R_{\mu\nu\alpha\beta} + R_{\nu\mu\alpha\beta}) \tag{3}$$

For $g_{\mu\nu||\alpha} = 0$ we then have the additional antisymmetry property $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$, and so the initial number of components in the curvature tensor becomes $n^2(n-1)^2/4$, or 36 for a 4-dimensional space. As for the cyclicity condition, let us first consider the case where the second and third indices are the same, as in $R_{\mu\nu\nu\beta}$ (no summation on ν). For this case cyclicity gives us

$$R_{\mu\nu\nu\beta} + R_{\mu\beta\nu\nu} + R_{\mu\nu\beta\nu} = 0$$

In view of the antisymmetry property noted earlier for the last two indices, this is identically zero for any arbitrary leading index μ. Consequently, for a fixed initial index with any other two indices being equal, cyclicity is trivial and provides no reduction in the number of components. This leaves the case $\nu \neq \alpha \neq \beta$. The cyclicity condition is then equivalent to the permutation of 3 quantities taken 3 at a time which, for a total of n objects, is given mathematically by

$$\frac{n!}{3!(n-3)!} = \frac{1}{6} n(n-1)(n-2)$$

But this is the number of terms for every leading index μ (which has n terms), so cyclicity reduces the number of components by a total of $n^2(n-1)(n-2)/6$. Therefore, the number of independent terms in the curvature tensor becomes $n^2(n-1)^2/4 - n^2(n-1)(n-2)/6 = n^2(n^2-1)/12$. In a 4-dimensional space, the Riemann-Christoffel tensor exhibits a total of 20 independent components.

There is yet another symmetry property we can derive for the curvature tensor in Riemannian space, although at this juncture it is irrelevant. Using the two equivalent cyclicity expressions

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0$$

and

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

we can easily show, by repeated exchange of the antisymmetric index pairs in each, that

$$R_{\mu\alpha\beta\nu} + R_{\alpha\mu\beta\nu} = \left(R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta}\right) + \left(R_{\mu\beta\alpha\nu} - R_{\alpha\nu\mu\beta}\right) = 0$$

For this expression to vanish we must therefore have the *index-pair exchange symmetry* $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$. However, this symmetry provides no further reduction of components in the curvature tensor, as it represents just another expression of the cyclicity condition itself.

Non-Riemannian Case

When the space is non-Riemannian we have $g_{\mu\nu||\alpha} \neq 0$ and we lose, in accordance with (3), the antisymmetry property $R_{\mu\nu\alpha\beta} + R_{\nu\mu\alpha\beta} = 0$. Consequently, the number of independent components in the curvature tensor is simply $n^3(n-1)/2 - n^2(n-1)(n-2)/6$, which simplifies to $n^2(n^2-1)/3$. In four dimensions, this comes out to a rather whopping 80 terms. Precisely what this large number of components means geometrically or what additional degrees of freedom it conveys to the curvature tensor in a non-Riemannian space is not known.

References

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