The Eigen-Cover Ratio of a Graph: Asymptotes, Domination and Areas

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Abstract

The separate study of the two concepts of energy and vertex coverings of graphs has opened many avenues of research. In this paper we combine these two concepts in a ratio, called the *eigen-cover ratio*, to investigate the domination effect of the subgraph induced by a vertex covering of a graph G (called the *cover graph* of G), on the original energy of G, where large number of vertices are involved. This is referred to as the *eigen-cover domination* and has relevance, in terms of conservation of energy, when a molecule's atoms and bonds are mapped onto a graph with vertices and edges, respectively. If this energy-cover ratio is a function of n, the order of graphs belonging to a class of graph, then we discuss its horizontal *asymptotic* behavior and attach the graphs average degree to the Riemann integral of this ratio, thus associating *eigen-cover area* with classes of graphs. We found that the eigen-cover domination had a strongest effect on the complete graph, while this chromatic-cover domination had zero effect on star graphs. We show that the eigen-cover asymptote of discussed classes of graphs belong to the interval [0,1], and conjecture that the class of complete graphs has the largest eigen-cover area of all classes of graphs.

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1. Introduction

All graphs in this paper are simple and loopless and on n vertices and m edges. We shall use the graph-theoretical notation of Harris, Hirst and Mossinghoff [8].

Energy, vertex covers and ratios of graphs

Much research has been done involving the energy of a graph (see Adiga, Bayad, Gutman and Srinivas [1] and Coulson, O'Leary and Mallion [4]) and (minimum) vertex coverings of graph (see Adiga, Bayad, Gutman and Srinivas [1]). Ratios have been an important aspect of graph theoretical definitions. Examples of ratios are: expanders (see Alon and Spencer [2]), the central ratio of a graph (see Buckley [3]), eigen-pair ratio of classes of graphs (see Winter and Jessop [13]), Independence and Hall ratios (see Gábor [4]), tree-cover ratio of graphs (see Winter and Adewusi [12]), the eigen-energy formation ratio of graphs (see Winter and Sarvate [16]), the t-complete eigen ratio of graphs (see Winter, Jessop and Adewusi [14]), the chromatic-cover ratio of graphs (see Winter [9]), the chromatic-complete difference ratio of graphs (see Winter [10], the tree-3-cover ratio of graphs (see Winter [11]) and the eigen-complete difference ratio (see Winter and Ojako [15]), .

In this paper we combine the two concepts of energy and vertex covering of graphs to form a ratio, the eigen-cover ratio, associated with a connected graph G, involving the energy of the sugraph $H(S)$ of G induced by a vertex cover S of G, called the *cover graph of* G, and the energy of G. This eigen-cover ratio allows for the investigation of the domination effect of the energy of the cover graph on the original energy of G, where a large number of vertices are involved – referred to as the *eigen-cover domination*. This eigen-cover ratio has relevance when molecules with atoms and bonds are mapped onto a graph with vertices and edges, respectively. In terms of conservation of energy, one requires the smallest set S of atoms whose excitation would affect all atoms which can be reached from S by a path of length at most 1, and where a large amount of atoms are involved the eigen-domination effect becomes relevant. If the eigen-cover ratio is a function of n, the order of graphs belonging to a particular class of graphs, then we investigated its asymptotic behavior (see Winter and Adewusi [12] ,Winter and

Jessop [13], Winter, Jessop and Adewusi [14], Winter and Sarvate [16],Winter and Ojako [15], Winter[9],[10], and[11],). The eigen-cover domination was determined for known classes of graph. We found that, for the complete graph, the eigen-cover domination was the strongest for complete graphs, and for star graphs with rays of length one, no effect at all. By introducing the average degree of a graph together with the Riemann integral of the eigen-cover ratio we associated eigen–cover area with classes of graphs (see Winter and Adewusi[12] , Winter and Jessop[13], Winter and Sarvate[16], Winter, Jessop and Adewusi [14],Winter and Ojako[15], Winter[9],[10], and[11]).

2. Eigen-cover ratio, asymptotes, domination and area

We combine the idea of energy (defined below) and vertex cover (the minimum set of vertices S of a graph G such that every edge of G has at least one end in S) of graphs in the following definitions, to allow for the measure of the domination of the energy of a cover graph over the energy of original graph for large values of n. If one considers a molecule in a graph-theoretical way, where the atoms are the vertices and the edges the bonds between the atoms, then the idea of *energizing* the whole molecule is relevant. Conserving energy will involve the *smallest* set S of atoms which can be energized so that all atoms outside S, and connected to S by a path of length at most one, also be energized. This is equivalent, graphically, to finding a minimum vertex covering of a graph, i.e. every path of length 1 has at least one end in S (see Adiga, Bayad, Gutman and Srinivas [1]). When a large number of atoms are involved the asymptotic behavior of this eigen-cover ratio becomes significant.

Definition 2.1

The Huckel Molecular Orbital theory provided the motivation for the idea of the *energy* of a graph - the sum of the absolute values of the eigenvalues associated with the graph (Adiga, Bayad, Gutman and Srinivas [1] and Coulson, O'Leary and Mallion [4]):

$E(G)=\sum_{1}^{n}|\lambda_i|,$ where $\lambda_i,\ 1\le i\le n$ are eigenvalues of adjacency matrix of graph G.

The definitions below allows for the investigation of the energy-domination of, and the eigen-cover areas, of classes of graphs (see Winter and Adewusi [12] , Winter and Jessop [13], Winter and Sarvate [16] , Winter, Jessop and Adewusi [14], Winter and Ojako[15], Winter[9],[10], and[11] for similar definitions).

Definition 2.2

Let G be a connected graph with minimum covering S of vertices. Let $H(S)$, the subgraph of G induced by S, be the *cover graph* of G .

The *eigen-cover ratio* of a graph G of order n, with respect to S , is defined as:

$$\operatorname{cov}\{E^S(G)\} = \frac{|S|E(H(S))}{nE(G)}$$

where $E(G)$ is the energy of G defined above. If H(S) consists of isolated vertices only then we define E(H(S)) as 1.

Definition 2.3

If $\operatorname{cov}\{E^S(G)\} = f(n)$ for every $G \in \Im$, where $\Im$ is a class of graphs, then the asymptotic behavior of $f(n)$ is called the *eigen-cover asymptote of* $\Im$ and denoted by:

$$as \operatorname{cov}\{E^S(\Im)\}.$$

Eigen–cover domination

This asymptote give a measure of the *domination effect* of the energy of the cover graph on the energy of the original graph, for large values of n, referred to as the *eigen-cover domination.*

Definition 2.4

If $\operatorname{cov}\{E^S(G)\} = f(n)$ for every $G \in \Im$, where $\Im$ is a class of graphs ,then the *eigen-cover area* is defined as :

$$A_{\Im(n)}^{E^S} = \frac{2m}{n}\int f(n)dn$$

with $A_{\Im(k)}^{E^S} = 0$ where k is the smallest number of vertices for which $\operatorname{cov}\{E^S(G)\} = f(n)$ is defined, and $\frac{2m}{n}$ is the average degree of $G \in \Im$, referred to as the *length* of G while the integral part is its *height* which we always make positive.

Examples:

2.1 The complete graph K_n

For the complete graph K_n, we have its cover graph $H(S) = K_{n-1}$. The energy of the complete graph is

$$E(K_n) = \sum_1^n |\lambda_i| = [(n-1) + (n-1)(1)] = 2n - 2$$

and therefore the energy of its cover graph is

$$E(H(S)) = E(K_{n-1}) = 2(n-1) - 2 = 2n - 4.$$

Hence,

$$\operatorname{cov}\{E^S(K_n)\} = \frac{|S|E(H(S))}{nE(G)} = \frac{(n-1)(2n-4)}{n(2n-2)} = \frac{2(n-1)(n-2)}{2n(n-1)} = \frac{n-2}{n} \quad \text{and}$$

$$as\operatorname{cov}\{E^S(K_n)\} = 1.$$

Then $A_{K_n}^{E^S} = \frac{2m}{n} \int f(n)dn = (n-1)\int \frac{n-2}{n} dn = (n-1)(n - 2\ln n + c).$

Now, $A_{K_2}^{E^S} = 0 \Rightarrow c = 2\ln 2 - 2,$ so

$$A_{K_n}^{E^S} = (n-1)(n - 2\ln n + 2\ln 2 - 2).$$

2.2 The complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$

For the complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$, we have S consisting of one of the partite sets on $\frac{n}{2}$ vertices, and its cover graph the set of $\frac{n}{2}$ isolated vertices with zero energy. Therefore,

$$\operatorname{cov}\{E^S(K_{\frac{n}{2},\frac{n}{2}})\} = \frac{|S|E(H(S))}{nE(G)} = \frac{\frac{n}{2}.1}{nE(G)} = \frac{n}{n.n} = \frac{1}{n} \text{ and}$$

$$as \operatorname{cov}\{E^S(K_{\frac{n}{2},\frac{n}{2}})\} = 0.$$

The energy of G is n so that $A_{K_{\frac{n}{2},\frac{n}{2}}}^{E^S} = \frac{2m}{n}[\int \frac{1}{n} dn] = \frac{n}{2}(\ln n + c)$ and

$$A_{K_{1,1}}^{\chi^S} = 0 \Rightarrow c = -\ln 2$$

Hence the eigen-cover area of the complete-split bipartite graph is $\frac{n^2}{4} - 1.$ or $\frac{n}{2}\left(\frac{n}{2} - 1\right)$

2.3 The cycle graph C_n

For the cycle graph C_n on an even number of vertices ,we have with S having size $\frac{n}{2}$ by considering every second vertex of the cycle so that the cover graph consists of $\frac{n}{2}$ isolated vertices. The energy of the cycle is: $2\sum_{0}^{n-1}\left|\cos(\frac{2\pi j}{n})\right| \leq 2n$

$$\operatorname{cov}\{E^S(C_n)\} = \frac{|S|E(H(S))}{nE(G)} = \frac{\frac{n}{2}.1}{nE(G)} = \frac{n}{2n2\sum_{0}^{n-1}\left|\cos(\frac{2\pi j}{n})\right|} \geq \frac{n}{4n.n} = \frac{1}{4n}.$$

$$A_{C_n}^{E^S} = 2\int \frac{n}{2n2\sum_{0}^{n-1}\left|\cos(\frac{2\pi j}{n})\right|} dn \geq 2(\ln 4n + c).$$

2.4 The cycle on an odd number of vertices

We take S as a pair of adjacent vertices and then alternating vertices. The triangle will have just the two adjacent vertices. The cycle on five vertices will have two adjacent vertices and an isolated vertex. Generally S will consist of $2 + \frac{n-3}{2} = \frac{n+1}{2}$ vertices and will have energy 2.

$$\operatorname{cov}\{E^S(C_n)\} = \frac{|S|E(H(S))}{nE(C_n)} = \frac{(n+1)2}{2nE(C_n)} = \frac{(n+1)}{nE(C_n)} = \frac{(n+1)}{2n\sum_{j=0}^{n-1}\left|\cos\left(\frac{2\pi j}{n}\right)\right|} \geq \frac{(n+1)}{2n^2}$$

$$A_{C_n}^{E^S} = \frac{2m}{n}\int f(n)dn \geq \int \frac{1}{n} + n^{-2}dn = \ln n - n^{-1} + c.$$

2.5 The path graph P_n

For the path graph P_n on an even number of vertices, we have S having size $\frac{n}{2}$ by considering the first vertex of the path and then skipping a vertex so that the cover graph consists of $\frac{n}{2}$ isolated vertices. The energy of the path graph is:

$$2\sum_{1}^{n}\left|\frac{\cos \pi j}{n+1}\right| \leq 2n$$

$$\operatorname{cov}\{E^S(P_n)\} = \frac{|S|E(H(S))}{nE(P_n)} = \frac{\frac{n}{2}}{nE(P_n)} = \frac{1}{4\sum_1^n \left|\frac{\cos \pi j}{n+1}\right|} \geq \frac{1}{4n}.$$

$$A_{P_n}^{\chi^S} = \frac{2m}{n}\int \frac{1}{4n} = \frac{2n-2}{n}(\ln 4n + c).$$

2.6 Path graphs on an odd number of vertices

Let P_n be a path on an odd number of vertices. S will consist of the second vertex and then skipping every vertex so that S will be on $\frac{n-1}{2}$ vertices.

$$\operatorname{cov}\{E^S(P_n)\} = \frac{|S|E(H(S))}{nE(P_n)} = \frac{\frac{n-1}{2}}{nE(P_n)} = \frac{n-1}{4n\sum_1^n \left|\frac{\cos \pi j}{n+1}\right|} \geq \frac{n-1}{4n^2}.$$

$$A_{P_n}^{E^S} = \frac{2m}{n}\int \frac{n-1}{4n\sum_1^n \left|\frac{\cos \pi j}{n+1}\right|} dv \geq \frac{2n-2}{n}\int \frac{n-1}{4n^2} dn = \frac{n-1}{n}\left(\frac{1}{2}\ln n + \frac{1}{2}n^{-1} + c\right).$$

2.7 The wheel W_n, with $n-1$ spokes, and with n odd

For the wheel graph W_n, with $n-1$ spokes and where n is odd, we have the central vertex and every second vertex of the cycle as the vertices in S.

The energy of the wheel graph is:

$$E(W_n) = 1 + \sqrt{n} - 1 + \sqrt{n} + \sum_{k=1}^{n-2} 2\left|\cos \frac{2\pi k}{n-1}\right| = 2\sqrt{n} + \sum_{k=1}^{n-2}\left|2\cos \frac{2\pi k}{n-1}\right| \leq 2\sqrt{n} + 2(n-2).$$

And, the cover graph is the star graph with $\frac{n-1}{2}$ rays of length 1.

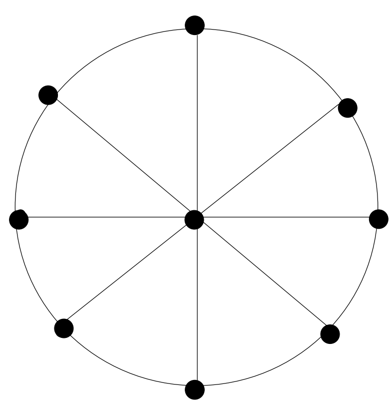

Figure 2.7.1: Wheel graph W_9

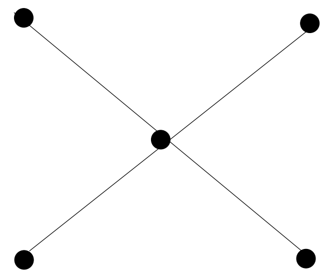

Figure 2.7.2: Cover graph of W_9, $H(S)$

The energy of $H(S)$ is

$$E(H(S)) = \sum_{i=1}^{\frac{n+1}{2}} |\lambda_i| = \left(\frac{n+1}{2} - 2\right)(0) + \left|\sqrt{\frac{n+1}{2} - 1}\right| + \left|-\sqrt{\frac{n+1}{2} - 1}\right| = 2\sqrt{\frac{n-1}{2}} = \sqrt{2(n-1)} \ .$$

$$\operatorname{cov}\{E^S(W_n)\} = \frac{|S|E(H(S))}{nE(W_n)} = \frac{\left(\frac{n+1}{2}\right)\sqrt{2(n-1)}}{nE(W_n)} = \frac{(n+1)\sqrt{n-1}}{\sqrt{2}n(2\sqrt{n} + \sum_{k=1}^{n-2}\left|2\cos\frac{2\pi k}{n-1}\right|)}$$

$$\geq \frac{(n+1)\sqrt{n-1}}{2\sqrt{2}n(\sqrt{n} + (n-2))}$$

$$as \operatorname{cov}\{E^S(W_n)\} = 0$$

$$A_{W_n}^{E^S} = \frac{2m}{n}\int f(n)dn = \frac{4n-4}{n}\int \frac{(n+1)\sqrt{n-1}}{\sqrt{2}n(2\sqrt{n} + \sum_{k=1}^{n-2}\left|2\cos\frac{2\pi k}{n-1}\right|)}dn \geq \frac{4n-4}{n}\int \frac{(n+1)\sqrt{n-1}}{n2\sqrt{2}(\sqrt{n} + (n-2))}dn.$$

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2.8 Star graphs $S_{r,1}$ on n vertices with r rays of length 1

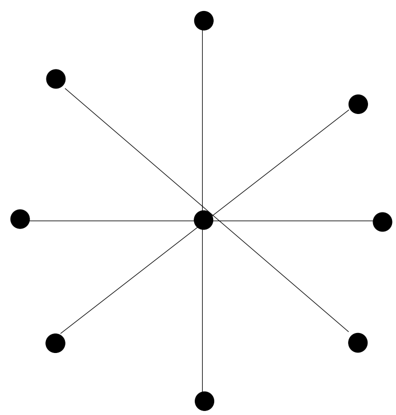

Figure 2.8.1: Star graph $S_{8,1}$

The vertex cover S comprises of the central vertex only. Then

$$\operatorname{cov}\{E^S(S_{r,1})\} = \frac{|S|E(H(S))}{nE(S_{r,1})} = \frac{1}{2n\sqrt{n-1}}.$$

$$as\operatorname{cov}\{E^S(S_{r,1})\} = 0$$

$$A_{S_{r,1}}^{E^S} = \frac{2m}{n}\int f(n)dn = \frac{(n-1)}{n}\int \frac{1}{n\sqrt{n-1}}dn \Rightarrow n = \sec^2 x \Rightarrow dn = 2\sec^2 x \tan x dx$$

$$\Rightarrow \frac{(n-1)}{n}\int 2dx = \frac{(n-1)}{n}(2arc\sec\sqrt{n} + c).$$

$$A_{S_{1,1}}^{E^S} = 0 \Rightarrow c = -2arc\sec\sqrt{2}.$$

2.9 Star graphs $S_{r,2}$ with r rays of length 2

We have $n = 2r+1$, and S consists of the middle vertex of each ray, so that the cover graph consists of $r = \frac{n-1}{2}$ isolated vertices.

The eigenvalues of $S_{r,2}$ are : 1 and -1, each of multiplicity $r-1 = \frac{n-3}{2}$, one eigenvalue 0, and two eigenvalues $\lambda = \pm\sqrt{r+1} = \pm\sqrt{\frac{n+1}{2}}$.

The energy of this graph is therefore:

$n-3+\sqrt{2}\sqrt{n+1}$

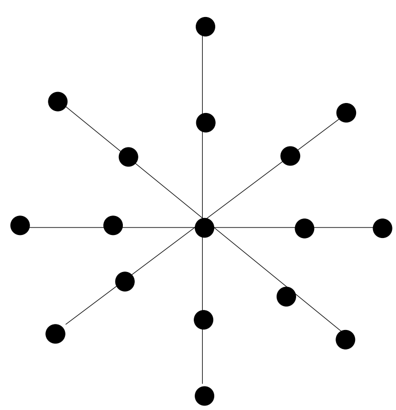

Figure 2.9.1: Star graph $S_{r,2}$

Then

$$\text{cov}\{E^S(S_{r,2})\} = \frac{|S|E(H(S))}{nE(S_{r,2})} = \frac{\frac{n-1}{2}}{n(n-3+\sqrt{2}\sqrt{n+1})} = \frac{n-1}{2n(n-3+\sqrt{2}\sqrt{n+1})}.$$

$as\ \mathrm{cov}\{\chi^S(S_{r,2})\} = 0.$

$$A^{E^S}_{S_{r,1}} = \frac{2m}{n}\int f(n)dn = \frac{2(n-1)}{n}\int \frac{n-1}{2n(n-3+\sqrt{2}\sqrt{n+1})}dn\,.$$

2.10 Lollipop graph

Lemma

Let G be a graph with an end vertex x_1 adjacent to vertex x_2, and let G' be the subgraph of G induced by removing the vertex x_1. and let G'' be the subgraph of G induced by removing the vertex x_2. Then (see Haemers [7]):

$$P_{A(G)}(\lambda) = \lambda P_{A(G')}(\lambda) - P_{A(G'')}(\lambda)$$

Where $P_{A(G)}(\lambda)$ is the characteristic polynomial $\det(A(G) - \lambda I)$

Example with complete graph joined to end vertex

So if $G = LP_n$ is the complete graph F on n-1 vertices joined to a single end vertex x_2 by an edge x_1x_2, we have:

$$P_{A(G)}(\lambda) = \lambda P_{A(G')}(\lambda) - P_{A(G'')}(\lambda) = \lambda(\lambda+1)^{n-2}(\lambda-(n-2)) - \lambda(\lambda+1)^{n-3}(\lambda-(n-3))$$

$$\lambda(\lambda+1)^{n-3}[(\lambda+1)(\lambda-(n-2)-(\lambda-(n-3))]$$

$$\lambda(\lambda+1)^{n-3}[(\lambda^2-\lambda(n-2)+\lambda-(n-2)-\lambda+(n-3)]$$

$$\lambda(\lambda+1)^{n-3}[(\lambda^2-\lambda(n-2)-1]$$

Roots of quadratic are:

$\lambda = \frac{(n-2) \pm \sqrt{n^2 - 4n + 4 + 4}}{2}$; we have roots:

$$\lambda = 0; \lambda = -1(multipliti\ cy\ \ n-3); \lambda = \frac{(n-2) + \sqrt{n^2 - 4n + 8}}{2}; \lambda = \frac{(n-2) - \sqrt{n^2 - 4n + 8}}{2}$$

Energy of this graph is therefore:

$$0 + 1(n-3) + \frac{(n-2) + \sqrt{n^2 - 4n + 8}}{2} + \frac{\sqrt{n^2 - 4n + 8} - (n-2)}{2} \quad \text{since } n \geq 4$$

$$= (n-3) + \sqrt{n^2 - 4n + 8}.$$

Theorem

The energy of the lollipop graph is: $E(LP_n) = (n-3) + \sqrt{n^2 - 4n + 8}$.

A cover graph is the subgraph induced by n-2 vertices of the complete graph F including x_1 so that its energy is 2n-6.

Hence:

$$\text{cov}\{E^S(LP_n)\} = \frac{|S|E(H(S))}{nE(S_{r,1})} = \frac{(n-2)(2n-6)}{n[(n-3) + \sqrt{n^2 - 4n + 8}]}.$$

For large n this behaves like: $\frac{2n^2}{n(n+n)} = \frac{2n^2}{2n^2} = 1$

$as\ \text{cov}\{\chi^S(LP_n)\} = 1.$

$$A_{LP_n}^{E^S} = \frac{2m}{n}\int f(n)dn = \frac{[(n-1)(n-2)+2]}{n}\int \frac{(n-2)(2n-6)}{n[(n-3) + \sqrt{n^2 - 4n + 8}]}dn.$$

2.11 Dual star graph

A dual star DuS_n is defined as two star graphs with m rays of length 1 (each on $\frac{n}{2}$ vertices) joined by an edge (its center edge) connecting their centers. This graph has 4 non-zero eigenvalues found as solutions of the following equation (see Haemers [6]):

$$x^4-(2m+1)x^2+m^2=0 \Rightarrow x^4-(n-1)x^2+\frac{(n-2)^2}{4}$$

$$\Rightarrow x^2=\frac{(n-1)\pm\sqrt{(n-1)^2-(n-2)^2}}{2}=\frac{(n-1)\pm\sqrt{2n-3}}{2}$$

$$\Rightarrow x=\pm\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}} \; or \; \pm\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}$$

Thus $E(DuS_n)=2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}$.

A cover set S consists of the vertices of the center edge of the graph.

Hence:

$$\operatorname{cov}\{E^S(DuS_n)\}=\frac{|S|E(H(S))}{nE(G)}=\frac{2.2}{nE(G)}=\frac{4}{n[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}]}$$

And

$as\operatorname{cov}\{E^S(DuS_n)\}=0.$

$$A_{DuS_n}^{E^S} = \frac{2(n-1)}{n}\int f(n)dn = \frac{2(n-1)}{n}\int \frac{4}{n[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}]}dn.$$

For large n this ratio behaves like:

$$\frac{4}{n[4\frac{\sqrt{n}}{\sqrt{2}}} = \frac{\sqrt{2}}{n^{\frac{3}{2}}} = \sqrt{2}n^{-\frac{3}{2}}..$$

Thus for large n the area associated with this ratio is:

$$A_{DuS_n}^{E^S} = \frac{2m}{n}[\int \sqrt{2}n^{-\frac{3}{2}}.dn] = \frac{\sqrt{2}(2n-2)}{n}(-2n^{-\frac{1}{2}}+c);\text{. n large.}$$

Making the height aspect positive we have:

$$A_{DuS_n}^{E^S} = \frac{2\sqrt{2}(n-2)}{n}(2n^{-\frac{1}{2}}+c);$$

$$A_{DuS_4}^{\chi^S} = 0 \Rightarrow c = -1$$

Theorem

The eigen-cover ratio, asymptote and area respectively for the following classes $\mathfrak{I}$ of graphs are:

1. $K_n : \frac{n-2}{n}$;1;$(n-1)(n-2\ln n+2\ln 2-2)$. and

2. $K_{\frac{n}{2},\frac{n}{2}} : \frac{1}{n}$; 0; $\frac{n}{2}(\ln n+-\ln 2)$ and

3. $C_n : \frac{n}{2n\sum_{j=0}^{n-1}\left|\cos\left(\frac{2\pi j}{n}\right)\right|} \geq \frac{1}{4n}$;-;$\geq 2(\ln 4n+c)$. n even.

4. $C_n : \frac{(n+1)}{2n\sum_{j=0}^{n-1}\left|\cos\left(\frac{2\pi j}{n}\right)\right|} \geq \frac{(n+1)}{2n^2}$;-;$\geq \ln n - n^{-1}+c$. n odd.

5. $P_n : \frac{1}{4\sum_{1}^{n}\left|\frac{\cos \pi j}{n+1}\right|} \geq \frac{1}{4n}$;-;$\geq \frac{2n-2}{n}(\ln 4n+c)$. n even.

6. $P_n : \frac{n-1}{4n\sum_{1}^{n}\left|\frac{\cos \pi j}{n+1}\right|} \geq \frac{n-1}{4n^2}$;-;$\geq \frac{n-1}{n}(\frac{1}{2}\ln n+\frac{1}{2}n^{-1}+c)$

7. $W_n : \frac{(n+1)\sqrt{n-1}}{\sqrt{2}n(2\sqrt{n}+\sum_{k=1}^{n-2}\left|2\cos\frac{2\pi k}{n-1}\right|)} \geq \frac{(n+1)\sqrt{n-1}}{2\sqrt{2}n(\sqrt{n}+(n-2))}$;0;

$\geq \frac{4n-4}{n}\int\frac{(n+1)\sqrt{n-1}}{2\sqrt{2}n(\sqrt{n}+n-2)}dn$

8. $S_{r,1} : \frac{1}{2n\sqrt{n-1}}$; 0; $\frac{(n-1)}{n}\int 2dx = \frac{(n-1)}{n}(2arc\sec\sqrt{n}+-2arcs\sec\sqrt{2})$.

9. $S_{r,2} : \frac{n-1}{2n(n-3+\sqrt{2}\sqrt{n+1})}$; 0; $\frac{2(n-1)}{n}\int\frac{n-1}{2n(n-3+\sqrt{2}\sqrt{n+1})}dn$.

$$10.\ LP_n : \frac{(n-2)(2n-6)}{n[(n-3)+\sqrt{n^2-4n+8}]};1;\frac{[(n-1)(n-2)+2]}{n}\int\frac{(n-2)(2n-6)}{n[(n-3)+\sqrt{n^2-4n+8}]}dn$$

.

$$11.\ DuS_n : \frac{4}{n[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}]};0;$$

$$\frac{2(n-1)}{n}\int\frac{4}{n[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}]}dn.$$

$$A_{DuS_n}^{E^S} = \frac{2\sqrt{2}(n-2)}{n}(2n^{-\frac{1}{2}}-1);\ \text{n large.}$$

Corollary

If $\operatorname{cov}\{E^S(G)\} = \frac{|S|E(H(S))}{nE(G)} = f(n)$ for each $G \in \Im$, then $as \operatorname{cov}\{\chi^S(\Im)\} \in [0,1]$ for all classes of graphs examined in the theorem above.

Conjecture

The complete graph possesses the largest eigen-cover area of all classes of graphs.

3. Conclusion

In this paper we combined the concepts of energy and vertex covering S of G(the covering graph is the subgraph H(S) of G induced by S) to determine the domination effect of the energy of the covering graph on the main graph G. This involved the eigen-cover ratio:

$$\operatorname{cov}\{E^S(G)\} = \frac{|S|E(H(S))}{nE(G)}.$$

Regarding a molecule in a graph-theoretical way, where the atoms are the vertices and the edges the bonds between the atoms, then the idea of *energizing* the whole molecule is relevant. Conserving energy will involve the *smallest* set S of atoms which can be energized so that all atoms outside S, and connected to S by a path of length at most one, also be energized. This is equivalent, graphically, to finding a minimum vertex covering of a graph. When a large number of atoms are involved the asymptotic behavior of this eigen-cover ratio becomes significant as illustrated by complete graphs having the strongest eigen-cover domination, implying that activation of its vertex cover will result in *immediate* activation of all atoms when a large number of atoms are involved.

If this eigen-cover ratio is a function of n, the order of G belonging to a class of graphs, then we determined the horizontal asymptote of the ratio, and by attaching the average degree of G to the Riemann integral we found the eigen-cover area of classes of graphs. We found that the eigen-cover asymptote value (the domination effect) for classes of graphs investigated belongs to the interval [0,1] and claim that the eigen-cover area is greatest of all eigen-cover areas of classes of graphs.

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