Tree-3-Cover Ratio of Graphs: Asymptotes and Areas

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Abstract

The graph theoretical ratio, the tree-cover ratio, involving spanning trees of a graph G, and a 2-vertex covering (a minimum set S of vertices such that every edge (or path on 2 vertices) of G has at least vertex end in S) of G has been researched. In this paper we introduce a ratio, called the *tree-3-covering ratio with respect to S*, involving spanning trees and a 3-vertex covering (a minimum set S of vertices of G such that every path on 3 vertices has at least one vertex in S) of graphs. We discuss the *asymptotic convergence* of this tree-3-cover ratio for classes of graphs, which may have application in ideal communication situations involving spanning trees and 3-vertex coverings of extreme networks. We show that this asymptote lies on the interval $[0,\infty)$ with the dumbbell graph (a complete graph on n-1 vertices appended to an end vertex) has tree-3-cover asymptotic convergence of 1/e, identical to the convergence in the secretary problem, and the tree-cover asymptotic convergence of complete graphs. We also introduce the idea of a *tree-3-cover area* by integrating this tree-3-cover ratio.

AMS classification: 05C99

Key words: spanning trees of graphs, vertex cover, 3-vertex cover, ratios, social interaction, network communication, convergence, asymptotes.

1. Introduction

We shall use the graph theoretical notation of [8] where our graphs are simple and connected. The order of graphs will be n and size m.

Spanning trees

The graph-theoretical concept of *spanning trees* can be found in many real world applications, especially in social networking scenarios. For example, research in [2] involves work on sexual networks in an American high school which suggest that sexual networking involving individuals at the school are characterized by long chains or “spanning trees”, implying that a large part of the school had sexual contact with each another.

Vertex cover

The importance of minimum 2-vertex coverings of a graph G, i.e. a minimum set S of vertices such that every path of G on 2 vertices has at least one vertex in S, occurs often in real life applications involving (extreme) networks with a large number of nodes (see the parameterized Vertex Cover problem in [5] and [9]). The idea of a 3-vertex covering of a graph G was introduced in [10]- this involved the smallest set S of vertices such that every path of G on 3 vertices has at least one vertex in S. This allowed for the investigation of the effect of the “activation” of S on all other vertices on paths of length at most 2 connected to S.

Ratios

Ratios, such as expanders, Raleigh quotient (see [1]), the central ratio of a graph (see [4]) and eigen-pair ratio of classes of graphs (see [14]), Independence and Hall ratios (see [7]), tree-cover ratio (see [13]), h-eigen formation ratio (see [17]), t-complete sequence ratio (see [15]), chromatic-cover ratio (see [11]), chromatic-complete difference ratio (see [12]) and the eigen-complete difference ratio (see [16]), have been investigated.

Spanning trees and 3-vertex cover

In this paper we combine the two ideas of spanning trees and (minimum) 3-vertex cover to introduce the idea of a *tree-3-cover ratio* of a graph. The importance of large numbers of vertices, which occurs in (extreme) networks, allowed for the investigation *asymptotic convergent* of this tree-3-cover ratio for different classes of graphs. We found that this asymptote lies on the interval $[0,\infty)$ with the dumbbell graph (the graph consisting of a complete graph on n-1 vertices appended to an end vertex) having tree-3-cover asymptotic convergence of 1/e identical to the secretary problem and the tree-2-cover asymptotic convergence of the complete graph (see [6] and [13]). The idea of *area* is also introduced which involves the Riemann integral of this tree-3-cover ratio.

This ratio $\frac{|S|t(H(S))}{t(K_n)}$ involving spanning trees and 3-vertex cover S with its asymptotic property and area of classes of graphs is presented below:

1.1.1 Definition

A *(minimim) 3-vertex cover* is of G is a smallest set of vertices of G such that every path on 3 vertices has at least one vertex in G. If u is a vertex in S and v a vertex not in S connected to u, we say that v is connected to S by a path of length at most 2.

1.1 .2 Definition

Let $t(G)$ be the number of spanning trees of a connected graph G of order n. Let S be a set of vertices of a *minimum 3-vertex cover* of G, and $H(S)$ the subgraph of G induces by S . We consider only the 2 cases (i) *Either* H(S) is connected or (ii) H(S) is disconnected and consists of trees as components . In case (ii) $t(H(S))$ is *defined as* $t(H(S))=1$.

Then the ratio:

$tc(G)_3{}^s = \frac{|S|t(H(S))}{t(G)}$ is the *tree-3-cover ratio* of G with respect to S .

Note: If $H(S)$ is disconnected, and not trees as components, then one can consider spanning forests involving the components of $H(S)$, but such cases are not considered in this paper.

1.1.2 Definition

The importance of graphs with a large number of vertices is well known. If ¶ is a class of graphs and $tc(G)_3{}^s = \frac{|S|t(H(S))}{t(G)} = f(n)$ for each $G \in$ ¶ , where n is the order of G, then the horizontal asymptote of f(n) is debited by:

$$tcasymp(\P)^{s}{}_3 = \lim_{n\to\infty} f(n)$$

This asymptote is called the *tree-3-cover asymptote* of ¶ which is an indication of the behavior of the tree cover ratio when the graph has a large number of vertices, such as in extreme networks.

An ideal communication problem and tree-cover asymptote

In [9] the communication problem is to select a minimal set S of placed sensor devices in a service area so that the all the nodes of service area is accessible by the minimal set of sensors. This can be adapted to a situation where there is a need for a minimal set S of placed sensor devices to communicate with all nodes that can be reached by paths of length at most 2 from S. Finding the minimal set of sensors can be modelled as a 3-vertex cover problem, where the 3-vertex cover set S facilitates the communications between the sensors and the nodes (on paths of length at most 2 from S) of the service area in networks with a large number of nodes (vertices), i.e. in extreme networks. If H(S), in the 3-tree cover definition, is connected, and M represents the vertices of G not in S, then each vertex of M is connected *by an path of length at most 2*(an *out-3-vertex path*) to vertex of H(S) which is part of a spanning tree. Thus the *ease* of communication between vertices of H(S) and M through the out-3-vertex paths, involving spanning trees, may be represented by this tree-3-cover ratio – the "ideal" case, involving large number of nodes, -which we believe is in the case of complete graphs. The more difficult communication case may be in the situation involving paths, where this tree-cover asymptote is infinite.

2. EXAMPLES OF TREE-3-COVER RATIOS AND ASYMPTOTES

2.1 Complete graph

Let G be the compete graph K_n on n vertices.

Then a minimum 3-covering set of K_n is any subset of n-2 vertices of K_n , and since $t(K_n) = n^{n-2};\ t'(K_{n-2}) = (n-2)^{n-4}$ we have:

$tc(K_n)_3^s = \dfrac{|S|t(K_{n-1})}{t(K_n)} = f(n) = \dfrac{(n-2)(n-2)^{n-4}}{n^{n-2}} = \dfrac{1}{(n-2)}\left(\dfrac{n-2}{n}\right)^{n-2}$ which behaves like $\dfrac{1}{n}$ for n large, so that:

$$\Rightarrow tcasymp(K_n)^S{}_3 = \lim_{n\to\infty} f(n) = 0.$$

2.2 Cycles

The cycle C_n on $n = 3k$ vertices has $t(C_n) = n$, and a minimum 3-vertex cover S will be the $\dfrac{n}{3}$ vertices of the disconnected graph induced by every third vertex of the cycle, so that $t(H(S)) = 1$ and $|S| = \dfrac{n}{3}$. Thus:

$tc(C_n)^s{}_3 = \dfrac{|S|t(H(S))}{t(C_n)} = f(n) = \dfrac{1}{3}$ so that

$$tcasympC_n)^s{}_3 = \frac{1}{3}.$$

2.3 Complete split-bipartite graph

Let $K_{\frac{n}{2},\frac{n}{2}}$ be the complete split-bipartite graph on n vertices.

Then $t(K_{\frac{n}{2},\frac{n}{2}}) = \left(\frac{n}{2}\right)^{n-2}$ and either partite set can be taken as a minimum 3-vertex cover S which yields $t(H(S)) = 1$ so that

$$tc(K_{\frac{n}{2},\frac{n}{2}})^{s}{}_{3} = \frac{|S|t(H(S))}{t(K_{\frac{n}{2},\frac{n}{2}})} = \frac{n}{2\left(\frac{n}{2}\right)^{n-2}} = \left(\frac{2}{n}\right)^{n-3} = f(n) \text{ so}$$

$$tc(K_{\frac{n}{2},\frac{n}{2}})^{S}{}_{3} = \left(\frac{2}{n}\right)^{n-3} \text{ and}$$

$$tcasymp(K_{\frac{n}{2},\frac{n}{2}})^{s}{}_{3} = 0\,.$$

2.4 Paths

Let P_n be a path on $n = 3k$ of vertices. A minimum vertex cover S consists of every third vertiex of P_n . Since $|S| = \frac{n}{3}$, $t(H(S)) = 1$ and $t(P_n) = 1$ we have:

$$tc(P_n)^{s}{}_{3} = \frac{|S|t(H(S))}{t(P_n)} = f(n) = \frac{n}{3} \text{ so that}$$

$$tcasymp(P_n)^{s}{}_{3} = \infty$$

2.5 Wheel graph

The wheel graph W_n on $n = 3k+1$ vertices has a cycle of length $3k$ with each vertex joined to a center. The number of spanning trees of this wheel is:

$t(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^{n} + \left(\frac{3-\sqrt{5}}{2}\right)^{n} - 2$ and the minimum vertex cover S will involve every third vertex of the cycle and the center vertex. Thus:

$t'(H(S)) = 1$ and:

$$tc(W_n)^s{}_3 = \frac{|S|t(H(S))}{t(W_n)} = f(n) = \frac{\frac{n-1}{3}+1}{\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2} \approx \frac{n}{6\left(\frac{3}{2}\right)^n} =; \; n \text{ large, so}$$

that:

$$tcasymp(W_n)^s{}_3 = 0$$

2.5 Ladder graph

The ladder graph $L_{\frac{n}{2},\frac{n}{2}}$ on an even number n of vertices has:

$t\left(L_{\frac{n}{2},\frac{n}{2}}\right) = \frac{(2+\sqrt{3})^{\frac{n}{2}} - (2-\sqrt{3})^{\frac{n}{2}}}{\sqrt{3}}$ and $t(H(S)) = 1$, where S is taken as follows:

Let P and P’ be the two paths, each having $\frac{n}{2}$ vertices, of the ladder, with edges between matched vertices of the two paths. Take S as the set of alternating vertices on P and P’, where the first vertex of P is selected and the second vertex of P’ is selected, so that S will have $\frac{n}{2}$ vertices. Then we have:

$$tc(L_{\frac{n}{2},\frac{n}{2}})^s{}_3 = \frac{|S|t(H(S))}{t(L_{\frac{n}{2},\frac{n}{2}})} = f(n) = \frac{n\sqrt{3}}{2\left(2+\sqrt{3}\right)^n - 2\left(2-\sqrt{3}\right)^n}.$$

Since $\left(2+\sqrt{3}\right)^n$ dominates $\left(2-\sqrt{3}\right)^n$ for large n we have:

$f(n) = \frac{n\sqrt{3}}{2\left(2+\sqrt{3}\right)^n - 2\left(2-\sqrt{3}\right)^n} \approx \frac{n\sqrt{3}}{2\left(2+\sqrt{3}\right)^n}$ for large n so that:

$tcasymp(L_{\frac{n}{2},\frac{n}{2}})^{s}{}_{3} = 0$

2.6 Star graph with rays of length 1

Let $S_{n,1}$ be the star graph on n vertices with n-1 rays of length 1. Then its centre is its minimum 3-covering set so that:

$tc(S_{n,1})^{s}{}_{3} = \frac{|S|t(H(S))}{t(S_{n,1})} = f(n) = 1$. Hence:

$tcasymp(S_{n,1})^{s}{}_{3} = 1$

2.7 Star graph with k rays of length 2.

Let $S_{n,k(2)}$ be the star graph in n vertices with k rays of length 2 from its center so that n=2k+1 (odd). The center is the minimum 3-vertex cover so that $|S| = 1$and $t(H(S)) = 1$ so that:

$tc(S_{n,k(2)})^{s}{}_{3} = \frac{|S|t(H(S))}{t(S_{n,k(2)})} = 1$ and

$tcasymp(S_{n,k(2)})^{s}{}_{3} = 1.$

2.8 Sun graph

Take a cycle on $\frac{n}{2}$ vertices, $n = 4k$, and attach an end vertex to each vertex of the cycle to form the sun graph SN_n on n vertices. Since $t(SN_n) = n$ and S consists of every alternate vertex of the cycle so that $t(H(S)) = 1; |S| = \frac{n}{4}$. Hence:

$tc(SN_n)^s{}_3 = \frac{|S|t(H(S))}{t(SN_n)} = \frac{n1}{4n} = \frac{1}{4}$ so that:

$tc(SN_n)^s{}_3 = \frac{1}{4}$ and $tcasymp(SN_n)^s{}_3 = \frac{1}{4}$.

2.8 Dumbbell graph

Let D_n^2 be the dumbbell graph consisting of two disjoint copies, A and B, of $K_{\frac{n}{2}}$ joined by and edge uv.

For each spanning tree of A we get $(\frac{n}{2})^{\frac{n}{2}-2}$ spanning trees of D_n^2 through the edge uv. Thus:

$$t(D_n^2) = (\frac{n}{2})^{\frac{n}{2}-2}(\frac{n}{2})^{\frac{n}{2}-2} = (\frac{n}{2})^{n-4}$$

A 3-vertex cover of A will consist of any set P of $\frac{n}{2}-2$ vertices of A containing u.

A 3-vertex cover of B will consist of any set Q of $\frac{n}{2}-2$ vertices of B containing v.

Since each spanning tree of a 3-covering of D_n^2 must contain uv, the subgraph $H(P \cup Q)$ induced by $S = P \cup Q$ will contain the following number of spanning trees:

$$t(H(S)) = (\frac{n}{2}-2)^{\frac{n}{2}-4}(\frac{n}{2}-2)^{\frac{n}{2}-4} = (\frac{n}{2}-2)^{n-8}$$

.

Thus:

$$tc(D_n^2)^s{}_3 = \frac{|S|t(H(S))}{t(SN_n)} = \frac{(n-4)(\frac{n}{2}-2)^{n-8}}{(\frac{n}{2})^{n-4}} = \frac{(n-4)^{n-7}}{2^{n-8}2^{-n+4}n^{n-4}} = \frac{2^4}{(n-4)^3}(\frac{n-4}{n})^{n-4}.$$

Thus: $tcasymp(SN_n)^s{}_3 = 0$.

2.9 Lollipop graph

Let $LP_{n-1,1}$ be the lollipop graph consisting of a complete graph F on n-1 vertices with vertex u joined to a single end vertex.

The number of spanning of $LP_{n-1,1}$ will be $(n-1)^{n-3}$.

A 3-vertex cover of $LP_{n-1,1}$ will consist of a set S of n-2 vertices of F *not* including u. Thus:

$t(H(S)) = (n-2)^{n-4}$ so that:

$$tc(LP_{n-1,1})^s{}_3 = \frac{|S|t(H(S))}{t(SN_n)} = \frac{(n-2)(n-2)^{n-4}}{(n-1)^{n-3}} = (\frac{n-2}{n-1})^{n-3} = (1-\frac{1}{n-1})^{n-3}.$$

$$let\ \ y = (1-\frac{1}{n-1})^{n-3} \Rightarrow \ln y = (n-3)\ln(1-\frac{1}{n-1}) = \frac{\ln(1-\frac{1}{n-1})}{\frac{1}{n-3}}.$$

Letting n go to infinity we get:

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(1-\frac{1}{n-1})}{\frac{1}{n-3}} = \lim_{n\to\infty} \frac{\left(\frac{1}{1-\frac{1}{n-1}}\right)(n-1)^{-2}}{-(n-3)^{-2}} = \lim_{n\to\infty} -\left(\frac{1}{1-\frac{1}{n-1}}\right)\left(\frac{n-3}{n-1}\right)^2 = -1$$

Thus: $\lim_{n\to\infty} y = e^{-1} = tcasymp(LP_{n-1,1})^s{}_3$, identical to the secretary problem.

Theorem

The tree-cover ratios and tree-cover asymptotes of the following graphs are:

$tc(K_n)^{s}{}_3 = \frac{1}{n-2}\left(\frac{n-2}{n}\right)^{n-2}$ and $tcasymp(K_n)^{S}{}_3 = 0$.

$tc(C_n)^{s}{}_3 = \frac{1}{3}$ and $tcasymp(C_n)^{s}{}_3 = \frac{1}{3}; n = 3k$.

$tc(K_{\frac{n}{2},\frac{n}{2}})^{s}{}_3 = \left(\frac{2}{n}\right)^{n-3}$ and $tcasymp(K_{\frac{n}{2},\frac{n}{2}})^{s}{}_3 = 0$

$tc(P_n)^{s}{}_3 = \frac{n}{3}$ and $tcasymp(P_n)^{s}{}_3 = \infty; n = 3k$

$tc(W_n)^{s}{}_3 = \frac{\frac{n-1}{3}+1}{\left(\frac{3+\sqrt{5}}{2}\right)^{n}+\left(\frac{3-\sqrt{5}}{2}\right)^{n}-2}$ and $tcasymp(W_n)^{s}{}_3 = 0; n = 3k+1$

$tc(L_{\frac{n}{2},\frac{n}{2}})^{s}{}_3 = \frac{n\sqrt{3}}{\left(2+\sqrt{3}\right)^{n}-\left(2-\sqrt{3}\right)^{n}}$ and $tcasymp(L_{\frac{n}{2},\frac{n}{2}})^{s}{}_3 = 1$

$tc(S_{n,1})^{s}{}_3 = 1$ and $tcasymp(S_{n,1})^{s}{}_3 = 1$

$tc(S_{n,k(2)})^{s}{}_3 = 1$ and $tcasymp(S_{n,k(2)})^{s}{}_3 = 1$

$tc(SN_n)^{s}{}_4 = \frac{1}{4}$ and $tcasymp(SN_n)^{s}{}_3 = \frac{1}{4}; n = 4k$.

$tc(D_n^2)^{s}{}_3 = \frac{2^4}{(n-4)^3}(\frac{n-4}{n})^{n-4}$ and $tcasymp(D_n^2)^{s}{}_3 = 0$.

$tc(LP_{n-1,1})^{s}{}_3 = (1-\frac{1}{n-1})^{n-3}$ and $tcasymp(LP_{n-1,1})^{s}{}_3 = \frac{1}{e}$.

Corollary

The tree-3-cover asymptote for all classes of graphs lies on the interval $[0,\infty]$.

3. TREE-COVER AREA OF CLASSES OF GRAPHS

We introduce another dimension by integrating this tree-cover ratio.

3.1 Definition

If ¶ is a class of graphs and $tc(G)^{s}{}_{3} = \dfrac{|S|t(H(S))}{t(G)} = f(n)$ for each $G \in ¶$, where n is the size of G and G has m edges, then the *tree cover area* of ¶ is defined as:

$$tcA^{3}{}_{¶(n)} = \frac{2m}{n}\int f(n)dn;\ tcA^{3}{}_{¶(p)} = 0\ \ for\ \ \min p\ \ defined$$

Average degree

The value $\frac{2m}{n}$ represents the *average degree* of a graph G.

Tree-cover height

For complete graphs, the length of the longest path is (n-1) so that we refer to the integral part of the definition as the *tree-3-cover height* of the graph.

3.1 Example- cycle

If C_n is a cycle on $n = 3k$ vertices, then:

$tc(C_n)^{s}{}_{3} = \dfrac{|S|t(H(S))}{t(C_n)} = f(n) = \dfrac{1}{3}$ so that the tree-cover height of cycles is:

$\int \frac{1}{3}dn$ which gives the tree-cover area of cycles as:

$$tcA_{C_n}{}^3 = \frac{2n}{n}\int\frac{1}{3}dn = 2(\frac{n}{3}+c);\ tcA_{C_3}{}^3 = 0 \Rightarrow c = -1$$

Theorem

$$tcA_{C_n}{}^3 = \frac{2}{3}n - 2;\ n = 3k\ .$$

3.2 Example- the path

If P_n is a path on $n = 3k$ number of vertices then:

$$tc(P_n)^s = \frac{|S|t(H(S))}{t(P_n)} = f(n) = \frac{n}{3}$$ so that:

$$tcA_{P_n}{}^3 = \frac{(2n-2)}{n}\int\frac{n}{3}dn = \frac{2(n-1)}{n}(\frac{n^2}{3}+c);\ tcA_{P_3}{}^3 = 0 \Rightarrow c = -3$$

Theorem

$$tcA_{P_n}{}^3 = \frac{2(n-1)}{n}(\frac{n^2}{3}-3); n = 3k\ .$$

3.3 Example- star graph with rays of length 1

$$tcA_{S_{n,1}}{}^3 = \frac{(2n-2)}{n}\int dn = \frac{(2n-2)}{n}(n+c); tcA_{S_{1,1}}{}^3 = 0 \Rightarrow c = -2$$

Theorem

$$tcA_{S_{n,1}}{}^3 = \frac{(2n-2)}{n}(n-2)$$

3.4 Example= star graph with rays of length 2

$$tcA_{Sk(2)}{}^{3} = \frac{2n-2}{n}\int dn = \frac{(2n-2)}{n}(n+c); tcA_{S1(2)}{}^{3} = 0 \Rightarrow c = -3$$

Theorem

$$tcA_{Sk(2)}{}^{3} = \frac{(2n-2)}{n}(n-3)$$

3.5 Sun graph

$$tcA_{SN_n}{}^{3} = 2\int \frac{1}{4} dn = 2(\frac{n}{4} + c); n = 4k; tcA_{SN_8}{}^{3} = 0 \Rightarrow c = -2$$

Theorem

$$tcA_{SN_n}{}^{3} = \frac{n}{2} - 4\,.$$

4. CONCLUSION: KNOWN AND NEW RESULTS

4.1 Combining spanning trees and 3-vertex coverings

In this paper we combined the concepts of spanning trees t(G) and a minimum 3-vertex cover, S, of a graph G, to introduce a *new* concept of a tree-3-cover ratio of G (where H(S) is the induced subgraph of G induced by a minimum 3-vertex covering S of G):

$$\frac{|S|t(H(S))}{t(G)}$$

This ratio was motivated by the possible importance of 3-vertex coverings in sensor activation, the tree-cover ratio of [13], and that the general tree-3-cover ratio for lollipop graphs, as a function of the order n of such graphs, is

$$\left(1-\frac{1}{n-1}\right)^{n-3}.$$

This ratio has the asymptotic convergence of 1/e, which is identical to the probability of the best applicant being selected in the secretary problem. These considerations resulted in the investigation of the asymptotic convergence of the tree-3-cover ratio of classes of graphs. We introduced integration of the tree-3-cover ratio which allowed for the idea of tree-cover area of classes of graphs.

We propose that the tree-cover asymptote of the sun graph on n=4k vertices is the smallest amongst all such possible positive tree-3-cover asymptotes of classes of graphs. Future research may involve considering the tree-3-cover ratio of the complement of classes of graphs discussed here. We could have considered the reciprocal of the tree-cover ratio, i.e. the ratio:

$$\left(tc(G)_3\right)^{-1} = \frac{t(G)}{|S|t(H(S))}.$$

For example, the reciprocal of the tree-3-cover ratio of lollipop graphs would have the asymptotic convergence of e, while paths on 3k number of vertices would have a reciprocal tree-cover asymptote of 0 (which is the same as the tree-cover asymptote of complete-split bipartite graphs) and (reciprocal) tree-cover area of

$$\frac{(2n-2)}{n}\int \frac{3}{n}dn = \frac{2(n-1)}{n}(3\ln n + c).$$

4.2 known and new results: ratios, asymptotes and areas

For the complete graph on n vertices the following are **known results**:

The *vertex expansion ratio*: $\min_{|S|\leq \frac{n}{2}} \frac{|\partial(S)|}{|S|} = \frac{n/2}{n/2} = 1$ which has *asymptote* 1 (see [1])

The *Hall ratio:* $°(G) = \max\left(\frac{|V(H)|}{\sim(H)}\right) = \frac{n}{1}$ which *converges* to infinity (see [7]).

The *integral eigen-ratio*, i.e the ratio of a+b to ab, where a and b and two, distinct non-zero eigenvalues whose sum and product is integral, is:

$\frac{n-2}{1-n}$ which *converges* to -1 and:

The *eigen- area*: $(n-1)(n-\ln(n-1))$ (see [14]).

The *central radius ratio* is $\frac{rad(G)}{n} = \frac{n}{n} = 1$ which has *asymptote* 1 (see [4]).

The *tree-cover ratio* (or tree-2-cover ratio) is $tc(G)^s = \frac{|S|t(H(S))}{t(G)} = \left(1-\frac{1}{n}\right)^{n-2}$ with *asymptote* 1/e (see [13]).

The *H-eigen formation ratio* of the graph G, on m edges, with H-decomposition. Is:

$ratio_H E(G) = [E(G) - E^H(G)]/m$ so that for the complete graph we get:

$ratio_{K_2}(K_n) = \frac{2(-n^2+3n-2)}{n(n-1)}$ with asymptote -2 (see[17]).

The *chromatic-cover ratio* is $\mathrm{cov}\{\text{¡}^S(K_n)\} = \frac{|S|\text{¡}(H(S))}{n\text{¡}(K_n)} = \frac{(n-1)^2}{n^2}$ with *asymptote* 1 (see [11]).

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