Tree-3-Cover Ratio of Graphs: Asymptotes and Areas
Paul August Winter*
*winter@ukzn.ac.za

Abstract
The graph theoretical ratio, the tree-cover ratio, involving spanning trees of a graph G, and a 2-vertex covering (a minimum set S of vertices such that every edge (or path on 2 vertices) of G has at least vertex end in S) of G has been researched. In this paper we introduce a ratio, called the tree-3-covering ratio with respect to S, involving spanning trees and a 3-vertex covering (a minimum set S of vertices of G such that every path on 3 vertices has at least one vertex in S) of graphs. We discuss the asymptotic convergence of this tree-3-cover ratio for classes of graphs, which may have application in ideal communication situations involving spanning trees and 3-vertex coverings of extreme networks. We show that this asymptote lies on the interval \([0, \infty)\) with the dumbbell graph (a complete graph on n-1 vertices appended to an end vertex) has tree-3-cover asymptotic convergence of 1/e, identical to the convergence in the secretary problem, and the tree-cover asymptotic convergence of complete graphs. We also introduce the idea of a tree-3-cover area by integrating this tree-3-cover ratio.

AMS classification: 05C99
Key words: spanning trees of graphs, vertex cover, 3-vertex cover, ratios, social interaction, network communication, convergence, asymptotes.
1. Introduction

We shall use the graph theoretical notation of [8] where our graphs are simple and connected. The order of graphs will be $n$ and size $m$.

**Spanning trees**

The graph-theoretical concept of *spanning trees* can be found in many real world applications, especially in social networking scenarios. For example, research in [2] involves work on sexual networks in an American high school which suggest that sexual networking involving individuals at the school are characterized by long chains or “spanning trees”, implying that a large part of the school had sexual contact with each another.

**Vertex cover**

The importance of minimum 2-vertex coverings of a graph $G$, i.e. a minimum set $S$ of vertices such that every path of $G$ on 2 vertices has at least one vertex in $S$, occurs often in real life applications involving (extreme) networks with a large number of nodes (see the parameterized Vertex Cover problem in [5] and [9]). The idea of a 3-vertex covering of a graph $G$ was introduced in [10]- this involved the smallest set $S$ of vertices such that every path of $G$ on 3 vertices has at least one vertex in $S$. This allowed for the investigation of the effect of the “activation” of $S$ on all other vertices on paths of length at most 2 connected to $S$.

**Ratios**

Ratios, such as expanders, Raleigh quotient (see [1]), the central ratio of a graph (see [4]) and eigen-pair ratio of classes of graphs (see [14]), Independence and Hall ratios (see [7]), tree-cover ratio (see [13]), h-eigen formation ratio (see [17]), t-complete sequence ratio (see [15]), chromatic-cover ratio (see [11]), chromatic-complete difference ratio (see [12]) and the eigen-complete difference ratio (see [16]), have been investigated.
Spanning trees and 3-vertex cover

In this paper we combine the two ideas of spanning trees and (minimum) 3-vertex cover to introduce the idea of a tree-3-cover ratio of a graph. The importance of large numbers of vertices, which occurs in (extreme) networks, allowed for the investigation asymptotic convergent of this tree-3-cover ratio for different classes of graphs. We found that this asymptote lies on the interval \([0, \infty)\) with the dumbbell graph (the graph consisting of a complete graph on \(n-1\) vertices appended to an end vertex) having tree-3-cover asymptotic convergence of \(1/e\) identical to the secretary problem and the tree-2-cover asymptotic convergence of the complete graph (see [6] and [13]). The idea of area is also introduced which involves the Riemann integral of this tree-3-cover ratio.

This ratio \(\frac{|S|t(H(S))}{t(K_n)}\) involving spanning trees and 3-vertex cover \(S\) with its asymptotic property and area of classes of graphs is presented below:

1.1.1 Definition

A (minimum) 3-vertex cover is of \(G\) is a smallest set of vertices of \(G\) such that every path on 3 vertices has at least one vertex in \(G\). If \(u\) is a vertex in \(S\) and \(v\) a vertex not in \(S\) connected to \(u\), we say that \(v\) is connected to \(S\) by a path of length at most 2.

1.1.2 Definition

Let \(t(G)\) be the number of spanning trees of a connected graph \(G\) of order \(n\). Let \(S\) be a set of vertices of a minimum 3-vertex cover of \(G\), and \(H(S)\) the subgraph of \(G\) induces by \(S\). We consider only the 2 cases (i) Either \(H(S)\) is connected or (ii) \(H(S)\) is disconnected and consists of trees as components. In case (ii) \(t(\ H(S))\) is defined as \(t(\ H(S)) = 1\).

Then the ratio:
\[ tc(G) = \frac{|S|t(H(S))}{t(G)} \] is the tree-3-cover ratio of \( G \) with respect to \( S \).

Note: If \( H(S) \) is disconnected, and not trees as components, then one can consider spanning forests involving the components of \( H(S) \), but such cases are not considered in this paper.

1.1.2 Definition

The importance of graphs with a large number of vertices is well known. If \( \xi \) is a class of graphs and \( tc(G) = \frac{|S|t(H(S))}{t(G)} = f(n) \) for each \( G \in \xi \), where \( n \) is the order of \( G \), then the horizontal asymptote of \( f(n) \) is debited by:

\[ tc_{\text{asymp}}(\xi) = \lim_{n \to \infty} f(n) \]

This asymptote is called the tree-3-cover asymptote of \( \xi \) which is an indication of the behavior of the tree cover ratio when the graph has a large number of vertices, such as in extreme networks.

An ideal communication problem and tree-cover asymptote

In [9] the communication problem is to select a minimal set \( S \) of placed sensor devices in a service area so that the all the nodes of service area is accessible by the minimal set of sensors. This can be adapted to a situation where there is a need for a minimal set \( S \) of placed sensor devices to communicate with all nodes that can be reached by paths of length at most 2 from \( S \). Finding the minimal set of sensors can be modelled as a 3-vertex cover problem, where the 3-vertex cover set \( S \) facilitates the communications between the sensors and the nodes (on paths of length at most 2 from \( S \)) of the service area in networks with a large number of nodes (vertices), i.e. in extreme networks. If \( H(S) \), in the 3-tree cover definition, is connected, and \( M \) represents the vertices of \( G \) not in \( S \), then each vertex of \( M \) is connected by an path of length at most 2 (an out-3-vertex path) to vertex of \( H(S) \) which is part of a spanning tree. Thus the ease of communication between vertices of \( H(S) \) and \( M \) through the out-3-vertex paths, involving spanning trees, may be represented by this tree-3-cover ratio – the “ideal” case, involving large number of nodes, -which we believe is in the case of complete graphs. The more
difficult communication case may be in the situation involving paths, where this 
tree-cover asymptote is infinite.

2. EXAMPLES OF TREE-3-COVER RATIOS AND ASYMPTOTES

2.1 Complete graph
Let $G$ be the complete graph $K_n$ on $n$ vertices.

Then a minimum 3-covering set of $K_n$ is any subset of $n-2$ vertices of $K_n$, and
since $t(K_n) = n^{n-2}$; $t'(K_{n-2}) = (n-2)^{n-4}$ we have:

$$tc(K_n)^s_3 = \frac{|S|t(K_{n-1})}{t(K_n)} = f(n) = \frac{(n-2)(n-2)^{n-4}}{n^{n-2}} = \frac{1}{(n-2)} \left( \frac{n-2}{n} \right)^{n-2}$$

which behaves like $\frac{1}{n}$ for $n$ large, so that:

$$\Rightarrow tcasymp(K_n)^s_3 = \lim_{n \to \infty} f(n) = 0.$$

2.2 Cycles
The cycle $C_n$ on $n = 3k$ vertices has $t(C_n) = n$, and a minimum 3-vertex cover $S$
will be the $\frac{n}{3}$ vertices of the disconnected graph induced by every third vertex of
the cycle, so that $t(H(S)) = 1$ and $|S| = \frac{n}{3}$. Thus:

$$tc(C_n)^s_3 = \frac{|S|t(H(S))}{t(C_n)} = f(n) = \frac{1}{3}$$

so that

$$tcasympC_n)^s_3 = \frac{1}{3}.$$

2.3 Complete split-bipartite graph

Let $K_{n-n/2}$ be the complete split-bipartite graph on $n$ vertices.
Then \( t(K_{n/2,n/2}) = \left(\frac{n}{2}\right)^{n-2} \) and either partite set can be taken as a minimum 3-vertex cover \( S \) which yields \( t(H(S)) = 1 \) so that

\[
tc(K_{n/2,n/2})^S_3 = \left| S \right| t(H(S)) = n \left(\frac{2}{n}\right)^{n-3} = \left(\frac{2}{n}\right)^{n-3} = f(n) \text{ so}
\]

\[
_tc(K_{n/2,n/2})^S_3 = \left(\frac{2}{n}\right)^{n-3} \text{ and}
\]

\[
tcasmp(K_{n/2,n/2})^S_3 = 0.
\]

### 2.4 Paths

Let \( P_n \) be a path on \( n = 3k \) of vertices. A minimum vertex cover \( S \) consists of every third vertex of \( P_n \). Since \( |S| = \frac{n}{3} \), \( t(H(S)) = 1 \) and \( t(P_n) = 1 \) we have:

\[
tc(P_n)^S_3 = \left| S \right| t(H(S)) \frac{t(P_n)}{t(P_n)} = f(n) = \frac{n}{3} \text{ so that}
\]

\[
tcasmp(P_n)^S_3 = \infty
\]

### 2.5 Wheel graph

The wheel graph \( W_n \) on \( n = 3k + 1 \) vertices has a cycle of length \( 3k \) with each vertex joined to a center. The number of spanning trees of this wheel is:

\[
t(W_n) = \left(\frac{3 + \sqrt{5}}{2}\right)^n + \left(\frac{3 - \sqrt{5}}{2}\right)^n - 2 \text{ and the minimum vertex cover } S \text{ will involve every third vertex of the cycle and the center vertex. Thus:}
\]

\[
t'(H(S)) = 1 \text{ and:}
\]
$tc(W_n)^3 = \frac{|S|f(H(S))}{t(W_n)} = f(n) = \frac{n-1}{3} + 1 - 2 \approx \frac{n}{6} \text{ for large } n.$

that:

$t\text{asym}(W_n)^3 = 0$

2.5 Ladder graph

The ladder graph $L_{\frac{n}{2}}$ on an even number $n$ of vertices has:

$t L_{\frac{n}{2}} = \left(\frac{2 + \sqrt{3}}{2}\right)^n - \left(\frac{2 - \sqrt{3}}{2}\right)^n$ and $t(H(S)) = 1$, where $S$ is taken as follows:

Let $P$ and $P'$ be the two paths, each having $\frac{n}{2}$ vertices, of the ladder, with edges between matched vertices of the two paths. Take $S$ as the set of alternating vertices on $P$ and $P'$, where the first vertex of $P$ is selected and the second vertex of $P'$ is selected, so that $S$ will have $\frac{n}{2}$ vertices. Then we have:

$t(L_{\frac{n}{2}})^3 = \frac{|S|t(H(S))}{t(L_{\frac{n}{2}})} = f(n) = \frac{n\sqrt{3}}{2(2 + \sqrt{3})^n - 2(2 - \sqrt{3})^n}$.

Since $(2 + \sqrt{3})^n$ dominates $(2 - \sqrt{3})^n$ for large $n$ we have:

$f(n) = \frac{n\sqrt{3}}{2(2 + \sqrt{3})^n - 2(2 - \sqrt{3})^n} \approx \frac{n\sqrt{3}}{2(2 + \sqrt{3})^n}$ for large $n$ so that:
\( tcasymp(L_{n,n})^\ast_3 = 0 \)

### 2.6 Star graph with rays of length 1

Let \( S_{n,1} \) be the star graph on \( n \) vertices with \( n-1 \) rays of length 1. Then its centre is its minimum 3-covering set so that:

\[
tc(S_{n,1})^\ast_3 = \frac{|S|t(H(S))}{t(S_{n,1})} = f(n) = 1. \text{ Hence:} \]

\( tcasymp(S_{n,1})^\ast_3 = 1 \)

### 2.7 Star graph with k rays of length 2.

Let \( S_{n,k(2)} \) be the star graph in \( n \) vertices with \( k \) rays of length 2 from its center so that \( n=2k+1 \) (odd). The center is the minimum 3-vertex cover so that \( |S|=1 \) and \( t(H(S))=1 \) so that:

\[
tc(S_{n,k(2)})^\ast_3 = \frac{|S|t(H(S))}{t(S_{n,k(2)})} = 1 \text{ and} \]

\( tcasymp(S_{n,k(2)})^\ast_3 = 1. \)

### 2.8 Sun graph

Take a cycle on \( \frac{n}{2} \) vertices, \( n = 4k \), and attach an end vertex to each vertex of the cycle to form the sun graph \( SN_n \) on \( n \) vertices. Since \( t(SN_n) = n \) and \( S \) consists of every alternate vertex of the cycle so that \( t(H(S)) = 1; |S| = \frac{n}{4} \). Hence:
\[ tc(SN_n)^s_3 = \frac{|S|t(H(S))}{t(SN_n)} = \frac{n1}{4n} = \frac{1}{4} \] so that:

\[ tc(SN_n)^s_3 = \frac{1}{4} \quad \text{and} \quad tcasymp(SN_n)^s_3 = \frac{1}{4}. \]

2.8 Dumbbell graph

Let \( D_n^2 \) be the dumbbell graph consisting of two disjoint copies, A and B, of \( K_{n/2} \) joined by an edge uv.

For each spanning tree of A we get \( \left(\frac{n}{2}\right)^{n-2} \) spanning trees of \( D_n^2 \) through the edge uv. Thus:

\[ t(D_n^2) = \left(\frac{n}{2}\right)^{n-2} \left(\frac{n}{2}\right)^{n-2} = \left(\frac{n}{2}\right)^{n-4} \]

A 3-vertex cover of A will consist of any set P of \( \frac{n}{2} - 2 \) vertices of A containing u.

A 3-vertex cover of B will consist of any set Q of \( \frac{n}{2} - 2 \) vertices of B containing v.

Since each spanning tree of a 3-covering of \( D_n^2 \) must contain uv, the subgraph \( H(P \cup Q) \) induced by \( S = P \cup Q \) will contain the following number of spanning trees:

\[ t(H(S)) = \left(\frac{n}{2} - 2\right)^{n-4} \left(\frac{n}{2} - 2\right)^{n-4} = \left(\frac{n}{2} - 2\right)^{n-8} \]

Thus:

\[ tc(D_n^2)^s_3 = \frac{|S|t(H(S))}{t(SN_n)} = \frac{(n-4)\left(\frac{n}{2} - 2\right)^{n-8}}{\left(\frac{n}{2}\right)^{n-4}} = \frac{(n-4)^{n-7}}{2^{n-8}2^{n-4}n^{n-4}} = \frac{2^4(n-4)^{n-4}}{(n-4)^3\left(\frac{n}{n}\right)^{n-4}}. \]
Thus: \( t_{\text{casym}}(SN_n)^3 = 0 \).

2.9 Lollipop graph

Let \( LP_{n-1,1} \) be the lollipop graph consisting of a complete graph \( F \) on \( n-1 \) vertices with vertex \( u \) joined to a single end vertex.

The number of spanning \( \text{ of } LP_{n-1,1} \) will be \((n-1)^{n-3}\).

A 3-vertex cover of \( LP_{n-1,1} \) will consist of a set \( S \) of \( n-2 \) vertices of \( F \) not including \( u \). Thus:

\[
t(H(S)) = (n-2)^{n-4}\ 
\]

so that:

\[
tc(LP_{n-1,1})^3 = \left\lfloor \frac{\left\lfloor t(H(S)) \right\rfloor}{t(SN_n)} \right\rfloor = \frac{(n-2)(n-2)^{n-4}}{(n-1)^{n-3}} = \frac{n-2}{n-1} = (1 - \frac{1}{n-1})^{n-3}.
\]

Let \( y = (1 - \frac{1}{n-1})^{n-3} \Rightarrow \ln y = (n-3)\ln(1 - \frac{1}{n-1}) = \frac{\ln(1 - \frac{1}{n-1})}{1} \cdot \frac{n-1}{n-3} \).

Letting \( n \) go to infinity we get:

\[
\lim_{n \to \infty} \ln y = \lim_{n \to \infty} \ln(1 - \frac{1}{n-1}) = \lim_{n \to \infty} \left( \frac{1}{n-1} \right)^2 = \lim_{n \to \infty} \left( \frac{n-3}{n-1} \right)^2 = -1
\]

Thus: \( \lim_{n \to \infty} y = e^{-1} = t_{\text{casym}}(LP_{n-1,1})^3 \), identical to the secretary problem.

Theorem
The tree-cover ratios and tree-cover asymptotes of the following graphs are:

\[ \text{tc}(K_n)^s_3 = \frac{1}{n-2} \left( \frac{n-2}{n} \right)^{n-2} \quad \text{and} \quad \text{tcasym}(K_n)^s_3 = 0. \]

\[ \text{tc}(C_n)^s_3 = \frac{1}{3} \quad \text{and} \quad \text{tcasym}(C_n)^s_3 = \frac{1}{3}; n = 3k. \]

\[ \text{tc}(K_{n,n})^{s_3} = \left( \frac{2}{n} \right)^{n-3} \quad \text{and} \quad \text{tcasym}(K_{n,n})^{s_3} = 0 \]

\[ \text{tc}(P_n)^s_3 = \frac{n}{3} \quad \text{and} \quad \text{tcasym}(P_n)^s_3 = \infty; n = 3k \]

\[ \text{tc}(W_n)^s_3 = \frac{n-1}{3} + 1 \quad \text{and} \quad \text{tcasym}(W_n)^s_3 = 0; n = 3k + 1 \]

\[ \text{tc}(L_{n,n})^{s_3} = \frac{n\sqrt{3}}{\left( 2 + \sqrt{3} \right)^n - \left( 2 - \sqrt{3} \right)^n} \quad \text{and} \quad \text{tcasym}(L_{n,n})^{s_3} = 1 \]

\[ \text{tc}(S_{n,1})^{s_3} = 1 \quad \text{and} \quad \text{tcasym}(S_{n,1})^{s_3} = 1 \]

\[ \text{tc}(S_{n,k(2)})^{s_3} = 1 \quad \text{and} \quad \text{tcasym}(S_{n,k(2)})^{s_3} = 1 \]

\[ \text{tc}(SN_n)^s_4 = \frac{1}{4} \quad \text{and} \quad \text{tcasym}(SN_n)^s_3 = \frac{1}{4}; n = 4k. \]

\[ \text{tc}(D_{n}^2)^s_3 = \frac{2^4}{(n-4)^3} \left( \frac{n-4}{n} \right)^{n-4} \quad \text{and} \quad \text{tcasym}(D_{n}^2)^s_3 = 0. \]

\[ \text{tc}(LP_{n-1,1})^{s_3} = \left( 1 - \frac{1}{n-1} \right)^{n-3} \quad \text{and} \quad \text{tcasym}(LP_{n-1,1})^{s_3} = \frac{1}{e}. \]
Corollary
The tree-3-cover asymptote for all classes of graphs lies on the interval \([0, \infty]\).

3. TREE-COVER AREA OF CLASSES OF GRAPHS

We introduce another dimension by integrating this tree-cover ratio.

3.1 Definition

If \( \xi \) is a class of graphs and \( tc(G) = \frac{|S| t(H(S))}{t(G)} = f(n) \) for each \( G \in \xi \), where \( n \) is the size of \( G \) and \( G \) has \( m \) edges, then the tree cover area of \( \xi \) is defined as:

\[
tcA^3_{\xi(n)} = \frac{2m}{n} \int f(n) \, dn; \quad tcA^3_{\xi(p)} = 0 \quad \text{for min } p \text{ defined}
\]

Average degree

The value \( \frac{2m}{n} \) represents the average degree of a graph \( G \).

Tree-cover height

For complete graphs, the length of the longest path is \((n-1)\) so that we refer to the integral part of the definition as the tree-3-cover height of the graph.

3.1 Example- cycle

If \( C_n \) is a cycle on \( n = 3k \) vertices, then:

\[
tc(C_n) = \frac{|S| t(H(S))}{t(C_n)} = f(n) = \frac{1}{3}
\]

so that the tree-cover height of cycles is:

\[
\frac{1}{3} \int dn
\]

which gives the tree-cover area of cycles as:
\[tcA_{c_n}^3 = \frac{2n}{n} \int \frac{1}{3} dn = 2 \left( \frac{n}{3} + c \right); \quad tcA_{c_3}^3 = 0 \Rightarrow c = -1\]

Theorem

\[tcA_{c_n}^3 = \frac{2}{3} n - 2; \quad n = 3k.\]

3.2 Example- the path

If \(P_n\) is a path on \(n = 3k\) number of vertices then:

\[tc(P_n)^3 = \frac{\left| S \right| t(H(S))}{t(P_n)} = f(n) = \frac{n}{3} \text{ so that:}\]

\[tcA_{p_n}^3 = \frac{(2n-2)}{n} \int \frac{n}{3} dn = \frac{2(n-1)}{n} \left( \frac{n^2}{3} + c \right); \quad tcA_{p_3}^3 = 0 \Rightarrow c = -3\]

Theorem

\[tcA_{p_n}^3 = \frac{2(n-1)}{n} \left( \frac{n^2}{3} - 3 \right); n = 3k.\]

3.3 Example- star graph with rays of length 1

\[tcA_{s_{n,1}}^3 = \frac{(2n-2)}{n} \int dn = \frac{(2n-2)}{n} (n + c); \quad tcA_{s_{1,1}}^3 = 0 \Rightarrow c = -2\]

Theorem

\[tcA_{s_{n,1}}^3 = \frac{(2n-2)}{n} (n - 2)\]
3.4 Example: star graph with rays of length 2

\[ tcA_{S_{k(2)}}^3 = \frac{2n-2}{n} \int dn = \frac{(2n-2)}{n}(n+c); tcA_{S_{1(2)}}^3 = 0 \Rightarrow c = -3 \]

Theorem

\[ tcA_{S_{k(2)}}^3 = \frac{(2n-2)}{n}(n-3) \]

3.5 Sun graph

\[ tcA_{S_{n,n}}^3 = 2\left[\frac{1}{4}dn = 2\left(\frac{n}{4}+c\right); n = 4k; tcA_{S_{n,n}}^3 = 0 \Rightarrow c = -2 \]

Theorem

\[ tcA_{S_{n,n}}^3 = \frac{n}{2} - 4. \]

4. CONCLUSION: KNOWN AND NEW RESULTS

4.1 Combining spanning trees and 3-vertex coverings

In this paper we combined the concepts of spanning trees t(G) and a minimum 3-vertex cover, S, of a graph G, to introduce a new concept of a tree-3-cover ratio of G (where H(S) is the induced subgraph of G induced by a minimum 3-vertex covering S of G):

\[ \frac{|S|t(H(S))}{t(G)} \]

This ratio was motivated by the possible importance of 3-vertex coverings in sensor activation, the tree-cover ratio of [13], and that the general tree-3-cover ratio for lollipop graphs, as a function of the order n of such graphs, is

\[ \left(1 - \frac{1}{n-1}\right)^{n-3}. \]
This ratio has the asymptotic convergence of $1/e$, which is identical to the probability of the best applicant being selected in the secretary problem. These considerations resulted in the investigation of the asymptotic convergence of the tree-3-cover ratio of classes of graphs. We introduced integration of the tree-3-cover ratio which allowed for the idea of tree-cover area of classes of graphs.

We propose that the tree-cover asymptote of the sun graph on $n=4k$ vertices is the smallest amongst all such possible positive tree-3-cover asymptotes of classes of graphs. Future research may involve considering the tree-3-cover ratio of the complement of classes of graphs discussed here. We could have considered the reciprocal of the tree-cover ratio, i.e. the ratio:

$$\left(\frac{tc(G)}{S}\right)_t^{-1} = \frac{t(G)}{|S|I(H(S))}.$$  

For example, the reciprocal of the tree-3-cover ratio of lollipop graphs would have the asymptotic convergence of $e$, while paths on $3k$ number of vertices would have a reciprocal tree-cover asymptote of $0$ (which is the same as the tree-cover asymptote of complete-split bipartite graphs) and (reciprocal) tree-cover area of

$$\frac{(2n-2)}{n} \int n \frac{3}{n} \frac{2(n-1)}{n} = 2(3\ln n + c).$$

### 4.2 known and new results: ratios, asymptotes and areas

For the complete graph on $n$ vertices the following are **known results**:

The **vertex expansion ratio**:

$$\min_{|S| \leq \frac{n}{2}} \frac{|\partial(S)|}{|S|} = \frac{n/2}{n/2} = 1$$ which has **asymptote** $1$ (see [1]).

The **Hall ratio**:

$$\rho(G) = \max \left( \frac{|V(H)|}{\alpha(H)} \right) = \frac{n}{1}$$ which **converges** to infinity (see [7]).

The **integral eigen-ratio**, i.e the ratio of $a+b$ to $ab$, where $a$ and $b$ and two, distinct non-zero eigenvalues whose sum and product is integral, is:
\[ \frac{n-2}{1-n} \text{ which converges to -1 and:} \]

The **eigen-area**: \((n-1)(n - \ln(n-1))\) (see [14]).

The **central radius ratio** is \( \frac{\text{rad}(G)}{n} = \frac{n}{n} = 1 \) which has **asymptote** 1 (see [4]).

The **tree-cover ratio** (or tree-2-cover ratio) is \( tc(G)^{n-2} = \frac{|S_f(H(S))|}{i(G)} = \left( 1 - \frac{1}{n} \right)^{n-2} \)

with **asymptote** \(1/e\) (see [13]).

The **H-eigen formation ratio** of the graph \( G \), on \( m \) edges, with \( H \)-decomposition. Is:

\[ \text{ratio}_H E(G) = \frac{|E(G) - E^H(G)|}{m} \text{ so that for the complete graph we get:} \]

\[ \text{ratio}_K^2 (K_n) = \frac{2(-n^2 + 3n - 2)}{n(n-1)} \text{ with asymptote -2 (see[17]).} \]

The **chromatic-cover ratio** is \( \text{cov}\{\chi^2(K_n)\} = \frac{|S_f(H(S))|}{\chi(K_n)} = \frac{(n-1)^2}{n^2} \text{ with asymptote 1} \) (see [11]).

5. REFERENCES


