CONSECUTIVE NATURAL NUMBERS CONSTANT

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ABSTRACT. The goal of this paper is to demonstrate that there exists a constant, a supposedly irrational and transcendental number, that relates all consecutive natural numbers *n* (taken from 1) when mutually divided as $\frac{n+1}{n}$ and $\frac{n}{n+1}$.

1. INTRODUCTION

Theorem 1.1 Let *n* be a natural number equal or greater than 1. There is a constant that relates all consecutive natural numbers through the result of the summation $\sum$ of $\frac{n+1}{n}-\frac{n}{n+1}$ where $n \to \infty$.

Proof. To demonstrate the existence of a consecutive natural numbers constant, it is necessary to create the function ***ϙ(x)*** (***ϙ-function*** or ***Qoppa function***).

(1.2)
$$\text{ϙ}(x)=\sum_{n=1}^{x}\left(\frac{n+1}{n}-\frac{n}{n+1}\right)$$

This function allows us to obtain a result from the exponential sum of $\frac{n+1}{n}-\frac{n}{n+1}$ whenever *x* is a natural number equal to or greater than 1.

Using this function with the *x* (the sum upper limit) set to 10^{X}-1 (ensuring in this way that the greatest number in the function is a power of 10) we obtain these results (starting from 10^1-1 and limited to 10^7-1 for explanatory purposes).

(1.3)
$$\text{ϙ}(10^1-1)=\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{10}{9}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{9}{10}\right)=4{,}7579365079365079365079365079365\ldots$$
$$\text{ϙ}(10^2-1)=\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{100}{99}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{99}{100}\right)=9{,}3647550352792405216102353513165\ldots$$
$$\text{ϙ}(10^3-1)=\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{1000}{999}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{999}{1000}\right)=13{,}969941721100689825313036408668 5\ldots$$
$$\text{ϙ}(10^4-1)=\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{10000}{9999}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{9999}{10000}\right)=18{,}575112072088764528356955809707 9\ldots$$
$$\text{ϙ}(10^5-1)=\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{100000}{99999}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{99999}{100000}\right)=23{,}180282259726855894726438727023 2\ldots$$
$$\text{ϙ}(10^6-1)=\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{1000000}{999999}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{999999}{1000000}\right)=27{,}785452445731447262762254986422 9\ldots$$
$$\text{ϙ}(10^7-1)=\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{10000000}{9999999}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{9999999}{10000000}\right)=32{,}390622631719703630798237879227 7\ldots$$

Between the results of the above operations appears a regular gap for every greater value of *x* when *x* is a power of 10 minus 1. Knowing this, we are able to build the function ***ͳ(x)*** (***ͳ-function*** or ***Sampi function***) that calculates the size of the gap.

Date: March 5, 2015.
2000 *Mathematics Subject Classification.* Primary 11Y60, 11B25; Secondary 05A10, 05B10, 11A05, 11B34, 11N99.
Key words and phrases. consecutive, constant, irrational, multiplication, natural numbers, scaling factor, transcendental.

(1.4)

$$ϡ(x)=\left[\sum_{n=1}^{10^x-1}\left(\frac{n+1}{n}-\frac{n}{n+1}\right)\right]-\left[\sum_{n=1}^{\frac{10^x}{10}-1}\left(\frac{n+1}{n}-\frac{n}{n+1}\right)\right]$$

The result from this function, in first analysis, seems to be a constant forming up at every greater value of the exponent *x*. Let's set *x* from 2 to 7 (the result of x=1 is just the x=2 subtrahend minus 0).

(1.5)

ϡ (2) = 9,3647550352792405216102353513165…−4,7579365079365079365079365079365…=$\underline{4}$,6068185273427325851022988433800…

ϡ (3) = 13,9699417211006898253130364086685…−9,3647550352792405216102353513165…=$\underline{4,60}$51866858214493037028010573519…

ϡ (4) = 18,5751120720887645283569558097079…−13,9699417211006898253130364086685…=$\underline{4,6051}$703509880747030439194010394…

ϡ (5) = 23,1802822597268558947264387270232…−18,5751120720887645283569558097079…=$\underline{4,605170}$1876380913663694829173152…

ϡ (6) = 27,7854524457314472627622549864229…−23,1802822597268558947264387270232…=$\underline{4,60517018}$60045913680358162593997…

ϡ (7) = 32,3906226317197036307982378792277…−27,7854524457314472627622549864229…=$\underline{4,6051701859}$882563680359828928047…

To confirm the constant nature of the results, let's underline the convergence of their fixed decimal part while we mutually subtract the above results.

(1.6)

4,6068185273427325851022988433800…−4,6051866858214493037028010573519…=0,$\underline{00}$16318415212832813994977860281…

4,6051866858214493037028010573519…−4,6051703509880747030439194010394…=0,$\underline{0000}$163348333746006588816563125…

4,6051703509880747030439194010394…−4,6051701876380913663694829173152…=0,$\underline{000000}$1633499833366744364837242…

4,6051701876380913663694829173152…−4,6051701860045913680358162593997…=0,$\underline{00000000}$16334999983336666579155…

4,6051701860045913680358162593997…−4,6051701859882563680359828928047…=0,$\underline{0000000000}$163349999998333665950…

Analyzing the results of the ϡ-*function* and the operations that led us to them, we are now able to build a formula that calculates the ***constant* Ϡ** (*Sampi constant*) by extending the *x* of the ϡ-*function* to the infinity.

(1.7)

$$Ϡ=\lim_{x\to\infty}ϡ(x)=\lim_{x\to\infty}\left\{\left[\sum_{n=1}^{10^x-1}\left(\frac{n+1}{n}-\frac{n}{n+1}\right)\right]-\left[\sum_{n=1}^{\frac{10^x}{10}-1}\left(\frac{n+1}{n}-\frac{n}{n+1}\right)\right]\right\}=4{,}6051701859\ldots$$

The analysis of the results must take into account that all the same digits in the same positions, through any greater value of *x*, are the constant's fixed part.

A possible demonstration of the presumed irrational and transcendental properties of the *constant* Ϡ, and its equivalence to *ln(100)*, must consider that the convergence of the constant's fixed decimal part, calculated by the ϡ-*function*, is equal to 2 times the exponent *x* of the function, minus 4 (*x*+*x*-4 or simply (*x*·2)-4) as can be seen in (1.5) and (1.6).

A larger decimal scale constant calculation, through the use of the ϡ-*function*, requires to use a powerful computer since an approach by hand may require a considerable amount of time, increasing the risk of possible errors.

The proof is complete. □

Below we can find the **source code** of a “POSIX bc” [1] program that translates the ϡ-*function* (1.4) in a **computer executable code**.

(1.8)

```
/* Begin of the sampi function executable code */
scale=17 /* This value sets the number of digits after the decimal point */
a=0 /* Starting value of the function buffer, don't change it*/
b=0 /* Starting value of the result buffer, don't change it */
c=10^8 /* Change this value to set the upper limit */
define func (x) {
while (x<c) {
        while (x<(c/10)) {
                a=((x+1)/x)-(x/(x+1))+a
                x=x+1
                b=a
                }
        a=((x+1)/x)-(x/(x+1))+a
        x=x+1}
        b=a-b
        return b
}
func (1) /* Function starting command, always set to 1, don't change it */
/* End of the sampi function executable code */
```

In this program the value of the variable *c*, editable in the source code, represents the ϡ-*function*'s *x* value. If *c* is set to 10^8, then the *scale* value has to be set to at least 17 (12, the fixed decimal scale of $x=10^8$, plus 5, five exceeding digit for the round-off error).

Since we are dealing with natural numbers, it is good to remember that the fixed decimal part for *x* equal to 10^1 is 0 (i.e. nonexistent) instead of -2.

2. MULTIPLICATIVE SCALING FACTORS

The ϱ-*function* plays an important role in the *constant* Ϡ formula. Let's analyze the ϱ-*function* to better understand its structure.

Lemma 2.1 *The ϱ-function represent the sum of the multiplicative scaling factors of all consecutive natural numbers n when x and n are natural numbers equal or greater than 1.*

Proof. Examining the structure of the ϱ-*function*, we can safely assume that all the multiplicative **scaling factors** [6] of the (*n*+1) times *n* multiplications are added together to form the function's result.

$$\varrho(x)=\sum_{n=1}^{x}\left(\frac{n+1}{n}-\frac{n}{n+1}\right)$$

Where $\frac{n+1}{n}-\frac{n}{n+1}$ is the scaling factor of (*n*+1) times *n*.

The scaling factor is a fundamental element in the demonstration of the *multiplication is not a repeated addition* (briefly *MI¬RA*) theory [2], [3], [4], [5], [6]. The *MI¬RA* theory can be expressed through the use of this algebraic expression, where the sum of two consecutive numbers is divided by their common scaling factor.

Let *x* be equal to *y*+1 (*x* = *y*+1).

(2.1)

$$\frac{x+y}{\frac{x}{y}-\frac{y}{x}}$$

The expression has been extracted from this identity.
Let $x = y+1$.

(2.2)
$$\frac{x+y}{\frac{x}{y}-\frac{y}{x}}=x\cdot y$$

Here is an example of the above identity to demonstrate its left-hand side equivalence to a classic multiplicative operation.

Let $x = 3$ and $y = 2$.

(2.3)
$$\frac{3+2}{\frac{3}{2}-\frac{2}{3}}=3\cdot 2$$

Continuing with the proof.

(2.4)
$$\frac{5}{1{,}5-0{,}\overline{6}}=6$$
$$\frac{5}{0{,}8\overline{3}}=6$$
$$6=6$$

What we have just examined is a simplified variation of a more generalized identity where the true scaling factor nature of multiplication is exposed, disproving in this way, with tangible evidence, the wrong theory stating that the *multiplication is a repeated addition* (briefly *MIRA*) [6].
Let x be not equal to y ($x \neq y$).

(2.5)
$$\frac{\left(\frac{\left(\frac{x+y}{x-y}\right)}{\left(\frac{x}{y}-\frac{y}{x}\right)}\right)}{\left(\frac{\left(\frac{\left(\frac{\left(\frac{x+y}{x-y}\right)}{\left(\frac{x}{y}-\frac{y}{x}\right)}\right)}{x+y}\right)}{x}\right)+\left(\frac{\left(\frac{\left(\frac{\left(\frac{x+y}{x-y}\right)}{\left(\frac{x}{y}-\frac{y}{x}\right)}\right)}{x+y}\right)}{y}\right)}=x\cdot y$$

In this identity the scaling factors are again playing the main role.
Given the generalized nature of this identity, the scaling factors, now that $x - y$ may not be equal to 1, need to be defined in every main numerator and denominator, so that we are able to use every type of number allowed in the classic multiplicative operation.
The left-hand side of this identity must not be mistaken as a multiplication replacement, but rather as a deep explanation of its inner concept.

Let's see an example of the identity assigning two different numbers to its x and y.

Let x = 50 and y = 60.

(2.6)
$$\frac{\left(\frac{\left(\frac{50+60}{50-60}\right)}{\left(\frac{50}{60}-\frac{60}{50}\right)}\right)}{\left(\frac{\left(\frac{\left(\frac{50+60}{50-60}\right)}{\left(\frac{50}{60}-\frac{60}{50}\right)}\right)}{\frac{50+60}{50}}\right)+\left(\frac{\left(\frac{\left(\frac{50+60}{50-60}\right)}{\left(\frac{50}{60}-\frac{60}{50}\right)}\right)}{\frac{50+60}{60}}\right)}=50\cdot 60$$

Let's solve the identity.

(2.7)
$$\frac{\left(\frac{\left(\frac{110}{-10}\right)}{\left(0{,}8\overline{3}-1{,}2\right)}\right)}{\left(\frac{\left(\frac{\left(\frac{110}{-10}\right)}{\left(0{,}8\overline{3}-1{,}2\right)}\right)}{\frac{110}{50}}\right)+\left(\frac{\left(\frac{\left(\frac{110}{-10}\right)}{\left(0{,}8\overline{3}-1{,}2\right)}\right)}{\frac{110}{60}}\right)}=3000$$

$$\frac{\left(\frac{-11}{-0{,}3\overline{6}}\right)}{\left(\frac{\left(\frac{\left(\frac{-11}{-0{,}3\overline{6}}\right)}{110}\right)}{50}\right)+\left(\frac{\left(\frac{\left(\frac{-11}{-0{,}3\overline{6}}\right)}{110}\right)}{60}\right)}=3000$$

$$\frac{30}{\left(\frac{\left(\frac{30}{110}\right)}{50}\right)+\left(\frac{\left(\frac{30}{110}\right)}{60}\right)}=3000$$

$$\frac{30}{\left(\frac{0{,}2\overline{7}}{50}\right)+\left(\frac{0{,}2\overline{7}}{60}\right)}=3000$$

$$\frac{30}{0{,}00\overline{54}+0{,}00\overline{45}}=3000$$

$$\frac{30}{0{,}01}=3000$$

$$3000=3000$$

As we can see, the most important part of these identities is $\frac{x}{y}-\frac{y}{x}$ that is the scaling factor.

In the specific case of the *ϙ-function*, where the scaling factor can be written as $\frac{n+1}{n}-\frac{n}{n+1}$, we have the following function.
Let *n* be a natural number equal or greater than 1 ($n \geq 1$).

(2.8)
$$f(n)=\frac{n+1+n}{\frac{n+1}{n}-\frac{n}{n+1}}$$

This function multiplies (n+1) times *n*.

When *x* is equal to *y* (as when we want to calculate a square), the (2.5) identity can't be proved because, in this case, the denominator of $\frac{x+y}{x-y}$ is 0.

This division by 0 helps us to better understand that the scaling factor nature of the multiplicative operation has nothing in common with the exponential concept, to which the square belongs. In fact, any power has a scaling factor equal to 0, which means that the base does not *scale* by the exponent, as we would have erroneously expected from such a seemingly multiplicative operation. All this endorse the demonstration that, just like the *multiplication is not a repeated addition* (*MI¬RA*), the *exponentiation is not a repeated multiplication* (*EI¬RM*) [3], [4], [5].

The proof is complete. □

Below we can find the source code of a "POSIX bc" [1] program that translates the *ϙ-function* (1.2) in a computer executable code.

(2.9)

```
/* Begin of the qoppa function executable code */
scale=32 /* This value sets the number of digits after the decimal point */
a=0 /* Starting value of the function buffer, don't change it*/
b=0 /* Starting value of the result buffer, don't change it */
c=10 /* Change this value to set the upper limit*/
define func (x) {
while (x<c) {
        a=((x+1)/x)-(x/(x+1))+a
        x=x+1}
        return a
}
func (1) /* Function starting command, always set to 1, don't change it */
/* End of the qoppa function executable code */
```

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