Modules and Universal Constructions

Takahiro Kato

December 5, 2019
ABSTRACT. Modules (also known as profunctors or distributors) and morphisms among them subsume categories and functors and provide more general and abstract framework to explore the theory of structures, in particular, the notion of universal property. In this book we generalize and redevelop the basic notions and results of various universal constructions in category theory using this framework of modules.

About modules and profunctors

The terms “module” and “profunctor” are synonymous in the sense that both are defined as a functor of the form $F : X^{op} \times A \to \text{Set}$. The notion given by this definition is seen by many as a generalization of the notion of functor. The term “profunctor” is quite appropriate in this regard. However, the notion has another, probably more important, aspect: it is also seen as an extension of categories. We use the term “module” when this aspect of the notion is emphasized; roughly, modules are to categories what, in the sense of abstract algebra, modules are to rings (or vector spaces are to fields). Modules constitute the objects of the category $\text{MOD}$, while profunctors constitute the morphisms of the category $\text{Prof}$. This book studies modules, not profunctors; in other words, this book studies an extension of categories.

Topics

Chapter 1 introduces modules and cells among them. A module $M : X \to A$ from a category $X$ to $A$ is a functor of the form $M : X^{op} \times A \to \text{Set}$, assigning a set of “arrows” to each pair of objects $x \in X$ and $a \in A$. A cell from a module $M$ to $N$ sends each arrow of $M$ to an arrow of $N$. The hom-functor of a category $C$ forms an endomodule $\hom C : C \to C$ called the hom of $C$, and the arrow function of a functor $F : C \to D$ forms a cell $(F) : \hom C \to \hom D$ from the hom of $C$ to the hom of $D$. Modules and cells thus subsume categories and functors in this way and set up a more general and conceptual framework to explore the structure of mathematics. Presheaves and copresheaves are called right and left modules respectively in this book and studied as special instances of modules.

Chapter 2 discusses the action of a module on its domain and codomain, the operation that yields the Yoneda embedding functor in the case of a hom endomodule. The chapter introduces an important class of modules called representable, which each functor produces by composition with the hom of its codomain.

Chapter 3 presents two variants of modules, namely collages and commas¹, which are special sorts of cospans and spans between two categories. We will establish an isomorphism between the category of modules and the category of collages, and, later in Chapter 9, construct an adjoint equivalence between the category of commas and the category of collages. Two forgetful functors from $\text{MOD}$ (the category of modules and cells) to $\text{CAT}$ (the category of categories and functors) are defined through constructions of collages and commas, and it is shown that they form left and right adjoints of the embedding $\text{CAT} \to \text{MOD}$ given by the hom assignment $C \mapsto \hom C$ (and thus that $\text{CAT}$ is a reflective and coreflective subcategory of $\text{MOD}$).

Chapter 4 introduces the notion of frames of a module. A cylindrical frame of an endomodule abstracts a natural transformation between two functors, and a conical frame of a right (resp. left) module abstracts a cone between an object and a functor. Ordinary and extraordinary cylinders, which are, so to speak, ordinary and extraordinary natural transformations spanning a module², are

¹The term “two-sided discrete fibration” is used in the literature to refer to what this book calls a comma. The term “comma” is adopted because every two-sided discrete fibration is given by a comma category and its projection functors (see [LR18] Theorem 2.3.3).

²The notion is called “het natural transformation” in [Ell07].
defined as instances of cylindrical frames. Likewise, cones are defined along a module as instances of conical frames.

Chapter 5 succeeds Chapter 2 and discusses the actions of the domain and codomain of a module on its arrows and frames. It is shown, as a generalization of the Yoneda embedding, that these actions embed a module \( X \to A \) in (the hom of) the category of right modules (i.e. presheaves) over \( X \) and the category of left module (i.e. copresheaves) over \( A \). The Yoneda lemma is presented in a general form to state that the morphisms from the representable module of a functor \( F : X \to A \) to an arbitrary module \( \mathcal{M} : X \to A \) correspond one-to-one with the cylinders defined between \( F \) and \( \mathcal{M} \). Using the lemma, we establish a variety of bijective correspondences between frames and cells.

The remaining chapters explore universal constructions (such as limits, ends, lifts, extensions, adjoint functors, etc) in the framework of modules using the machinery developed so far. Universality is formulated in terms of universal arrows in a module; the embedding of a module in the category of presheaves makes the definition of a universal arrow simple enough: an arrow in a module is universal if and only if its image under the embedding is an isomorphism. Universal constructions are defined as universal arrows of some module, and we study if they are preserved by a cell. Here are some highlights.

- Limits and ends are defined along a module as universal cones and as universal extraordinary cylinders, respectively. Preservation of limits and ends are treated as an instance of the general notion of a cell preserving universal arrows.

- Extensions are defined as universal cells, and lifts are defined as universal (ordinary) cylinders. Kan extensions and Kan lifts are presented as special instances of the general notions of extensions and lifts, respectively.

- The notion of a pointwise lift is introduced. We show that the notion subsumes that of pointwise extensions and vice versa. Parameterized limits, ends, and adjunctions are all treated as instances of pointwise lifts.

- The notion of a symmetric cell is introduced to define adjunctions between two modules as well as between two categories. In fact, adjunctions are proved to constitute universal arrows of the module of symmetric cells, allowing the treatment of them in the general framework of modules and universal arrows.

- The concept of an adjoint is extended to a cell. We show that a cell preserves universal arrows if it has adjoints, and deduce RAPL (right adjoints preserve limits) as a corollary of this general mechanism.

- The notion of an equivalence of categories is extended to that of an equivalence of modules. It is shown that an equivalence cell preserves and reflects universal arrows.

- Epicity and monicity are defined not only for arrows in a category but for arrows in a module. A proof of the special adjoint functor theorem is given using epi-mono-factorizations for modules.

- The concept of density is also generalized for modules (in fact, density is most naturally defined with modules). The fact “every presheaf is a colimit of representables” is proved using the concept of density to show how the concept works in the framework of modules.
**Advice on reading**

The exposition is carried out within the framework of set-enriched 1-dimensional categories. Chapter 0 presents the notations and some elementary facts of category theory used in the sequel.

Size issues are not treated rigorously. The specification of a universe is almost always implicit; a universe $\mathfrak{U}$ is chosen so that given categories and modules become locally $\mathfrak{U}$-small. We just say “small” instead of “$\mathfrak{U}$-small” for $\mathfrak{U}$ chosen implicitly. The choice of a universe is fixed in each context unless otherwise stated.

This book is made up of sequences of “Definition” - “Proposition” - “Theorem” - “Corollary”, with each optionally preceded by “Note” and followed by “Remark”. Each “Proposition” is tightly associated with the preceding “Definition” and states some straightforward consequences of it. Many “Theorem”s in fact state facts too obvious to deserve the name (“there are no theorems in category theory”). Two statements dual to each other are indicated by the symbols $\triangleright$ and $\triangleright$; proofs are given for the assertions indicated by $\triangleright$.

Parentheses and brackets serve mainly as punctuation to enhance readability; parentheses are used to delimit objects and arrows, whereas square (resp. angle) brackets are used to delimit categories and functors (resp. modules and cells).
## Contents

0. Preliminaries 7

1. Modules and Cells 12
   1.1. Modules ............................................. 12
   1.2. Cells ............................................. 23
   1.3. Cell morphisms .................................... 32

2. Slicing and Action 35
   2.1. Slicing of modules ................................. 35
   2.2. Action of modules ................................ 39
   2.3. Yoneda functors and representations ............... 41

3. Collages and Commas 46
   3.1. Collages ........................................... 46
   3.2. Commas ............................................ 51

4. Frames 64
   4.1. Cylindrical frames ................................. 64
   4.2. Conical frames .................................... 67
   4.3. Cylinders ......................................... 72
   4.4. Extraordinary cylinders ........................... 85
   4.5. Weighted cylinders ................................ 92
   4.6. Cones .............................................. 98
   4.7. Bicylinders ....................................... 113
   4.8. Wedges ............................................ 117
   4.9. Cones and wedges in a category .................... 121
   4.10. Cones and wedges in Set .......................... 127

5. Yoneda Lemma 131
   5.1. Yoneda modules ................................... 131
   5.2. Yoneda morphisms .................................. 135
   5.3. Yoneda morphisms for cylinders .................... 145
   5.4. Yoneda morphisms for cones ....................... 161
   5.5. Correspondences between frames and cells ........... 167

6. Universals 179
   6.1. Units of one-sided modules ....................... 179
   6.2. Universal arrows .................................. 181
   6.3. Conjugation ....................................... 188
   6.4. Units of two-sided modules ....................... 191
   6.5. Lifts .............................................. 194
   6.6. Kan lifts ......................................... 200

7. Limits 203
   7.1. Limits ............................................ 203
   7.2. Limits with parameters ............................ 206
   7.3. Limits in a category ............................... 212
7.4. Limits of modules .................................................. 215
7.5. Limits and Yoneda morphisms ................................. 219
7.6. Limits in comma categories ................................. 223

8. Adjunctions ................................................................. 226
8.1. Symmetric cells .................................................. 226
8.2. Adjunctions for modules ....................................... 231
8.3. Adjunctions for categories .................................... 234
8.4. Adjunctions as universal arrows ............................ 240
8.5. Conjugation for adjunctions .................................. 243
8.6. Adjunctions with parameters .................................. 248
8.7. Composition of adjunctions .................................. 250
8.8. Exponentials in a category ..................................... 253
8.9. Adjoints of a cell .................................................. 257
8.10. Equivalence of categories ..................................... 261
8.11. Equivalence of modules ......................................... 265

9. Adjoint Functor Theorem ........................................... 271
9.1. Cofinality .............................................................. 271
9.2. General adjoint functor theorem ............................. 277
9.3. Epimorphisms and Monomorphisms .......................... 279
9.4. Subobjects ............................................................. 281
9.5. Epi-mono factorizations ......................................... 284
9.6. Generators ............................................................. 286
9.7. Special adjoint functor theorem ............................... 287

10. Collages and Commas (continued) .............................. 289
10.1. Cylinder modules .................................................. 289
10.2. Equivalence $\text{COM} \simeq \text{CLG}$ ............................. 295
10.3. Equivalence $[X \downarrow A] \simeq [X \uparrow A]$ .......................... 299
10.4. Equivalences $[X \downarrow ] \simeq [X :]$ and $[\downarrow A] \simeq [: A]$ .................................................. 303

11. Extensions ................................................................. 310
11.1. coYoneda lemma .................................................. 310
11.2. Weighted limits .................................................... 319
11.3. Extensions ............................................................. 326
11.4. Kan extensions ..................................................... 335
11.5. Density ................................................................. 338

12. Ends ................................................................. 345
12.1. Ends ................................................................. 345
12.2. Ends with parameters .......................................... 347
12.3. Ends of modules .................................................. 353

A. List of Symbols ............................................................ 361
0. Preliminaries

1. Given a universe $\mathcal{U}$, $\text{Set}_\mathcal{U}$, $\text{Cat}_\mathcal{U}$, and $\text{CAT}_\mathcal{U}$ denote the categories of $\mathcal{U}$-small sets, $\mathcal{U}$-small categories, and locally $\mathcal{U}$-small categories respectively. By $\text{Set}$ we mean $\text{Set}_\mathcal{U}$ for $\mathcal{U}$ chosen implicitly, and similarly for $\text{Cat}$ and $\text{CAT}$.

2. A category is defined in terms of hom-sets. Pairwise disjointness of hom-sets is not required. Given a category $C$, an element $f$ of a hom-set $\text{hom}_C(a, b)$ or a triple $(a, f, b)$ such that $f \in \text{hom}_C(a, b)$ is called an arrow of $C$ (or a $C$-arrow) and written $f : a \to b$, $a \xrightarrow{f} b$, or just $f$ if its domain and codomain are understood or unimportant. The composite of arrows $f : a \to b$ and $g : b \to c$ is written as $f \circ g = g \circ f$.

   An invertible arrow is called iso (or an isomorphism).

3. The set of objects of a category $C$ is denoted by $\|C\|$ and the object function of a functor $F$ is denoted by $\|F\|$.

4. Italic small letters $a, b, c, \ldots$ vary over objects and arrows. When we write $c \in C$ for a category $C$, $c$ stands for an object or arrow of $C$.

5. The value of a functor $F : C \to B$ at $c \in C$ is written as $c \circ F = F(c) = Fc = F \cdot c$

   , and the component of a natural transformation $\tau : S \to T : C \to B$ at an object $c \in \|C\|$ is written as $c \circ \tau = \tau_c = \tau \cdot c$

6. The opposite of a category $C$ is denoted by $C^\sim$; for a $C$-arrow $f : a \to b$, the corresponding $C^\sim$-arrow (the opposite of $f$) is written as $f^\sim : b \to a$. The opposite of a functor $F : C \to B$ is denoted by $F^\sim : C^\sim \to B^\sim$, or often by $F : C^\sim \to B^\sim$ using the same name as its original counterpart.

7. Given categories $A$ and $B$, the product, coproduct, and functor categories are denoted by $A \times B$, $A + B$, and $[A, B]$ respectively. The terminal category and its sole object are denoted by $*$; the identity arrow $* \to *$ is also denoted by $*$. The interval category is denoted by $2$ and its objects are denoted by $0$ and $1$.

8. Italic capital letters $F, G, \ldots$ vary over both functors and natural transformations. When we write $F \in [A, B], F : A \to B$, or $A \xrightarrow{F} B$ for categories $A$ and $B$, $F$ stands for an object or an arrow of the functor category $[A, B]$, i.e. a functor or a natural transformation.

9. Let $A$, $B$, and $C$ be categories. The composite of $F : A \to B$ and $G : B \to C$ is written as $F \circ G = G \circ F$

   , and its value $a \circ [F \circ G] = (a \circ F) \cdot G$ at $a \in A$ is often written as $a \circ F : G = G \circ F \cdot a$
10. The bifunctorial operation 
\( (F,G) \mapsto F \circ G \) defines a bifunctor \([A,B] \times [B,C] \to [A,C]\), and the right and left exponential transpositions yield functors
\[
[A,-] : [B,C] \to [[A,B],[A,C]] ; \quad G \mapsto (F \mapsto F \circ G)
\]
and
\[
[-,C] : [A,B] \to [[B,C],[A,C]] ; \quad F \mapsto (G \mapsto F \circ G)
\]
. The functor \([A,-]\) sends each functor \(G : B \to C\) to the postcomposition functor
\[
[A,G] : [A,B] \to [A,C] ; \quad F \mapsto F \circ G
\]
and sends each natural transformation \(\tau : S \to T : B \to C\) to the postcomposition natural transformation
\[
\]
defined by
\[
[A,\tau]_F = F \circ \tau : F \circ S \to F \circ T : A \to C
\]
for \(F\) a functor \(A \to B\). Dually, the functor \([-,C]\) sends each functor \(F : A \to B\) to the precomposition functor
\[
[F,C] : [B,C] \to [A,C] ; \quad G \mapsto F \circ G
\]
and sends each natural transformation \(\sigma : S \to T : A \to B\) to the precomposition natural transformation
\[
[\sigma,C] : [S,C] \to [T,C] : [B,C] \to [A,C]
\]
defined by
\[
[\sigma,C]_G = \sigma \circ G : S \circ G \to T \circ G : A \to C
\]
for \(G\) a functor \(B \to C\).

11. The bifunctorial operation \([-,-]\) on \(\text{CAT}^{-} \times \text{CAT}\) is defined in the following way:

a) for a pair of categories \(X\) and \(A\), \([X,A]\) is the category of functors \(X \to A\); 

b) for a functor \(P : X \to Y\) and a category \(A\), \([P,A] : [Y,A] \to [X,A]\) is the precomposition functor; 

c) for a functor \(Q : A \to B\) and a category \(X\), \([X,Q] : [X,A] \to [X,B]\) is the postcomposition functor. 

The commutativity of
\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
[Y,B] & \xrightarrow{[P,B]} & [X,B]
\end{array}
\]
expresses the associativity of functor composition.

11. The right and left exponential transposes of a bifunctor \(K : E \times D \to C\) are denoted by
\[
K^+ : D \to [E,C] \quad K^- : E \to [D,C]
\]
, and the (simple) transpose of a functor \(K : E \to [D,C]\) is denoted by \(K^ : D \to [E,C]\). These transpositions yield the iso functors shown in the commutative diagram
\[
\begin{array}{ccc}
[D,[E,C]] & \xrightarrow{T} & [E,[D,C]] \\
\downarrow & & \downarrow \\
[E \times D,C] & \xrightarrow{T} & [E,[D,C]]
\end{array}
\]
, natural in \(E, D,\) and \(C\).
12. The identity functor $1_E : E \to E$ is also denoted by $E$ using the name of the category. More generally, if $D$ is a subcategory $E$, the inclusion functor $D \to E$ is also denoted by $D$ using the name of its domain; the restriction of $F : E \to C$ to $D$ is written as

$$D \circ F = F \circ D$$

Precomposition with the inclusion

$$D \longrightarrow E$$

yields a functor

$$[E, C] \longrightarrow [D, C]$$

"restriction to $D"$, for any category $C$.

13. By the obvious isomorphism $[\ast, E] \cong E$, a functor (resp. a natural transformation) $\ast \to E$ is identified with an object (resp. an arrow) of $E$. Given categories $E$ and $D$ and given an object $e \in \|E\|$, the product functor $e \times D$ as in

$$
\begin{array}{ccc}
* & \longleftarrow & e \times D \\
\downarrow e & & \downarrow e \times D \\
E & \longleftarrow & E \times D
\end{array}
$$

gives the section

$$e \times D : D \to E \times D$$

of $E \times D$ at $e$, under the identification $\ast \times D \cong D$. The section

$$E \times d : E \to E \times D$$

of $E \times D$ at $d \in \|D\|$ is defined symmetrically.

14. Given categories $E$ and $C$, the evaluation

$$(e, F) \mapsto e \circ F : E \times [E, C] \to C$$

is identified with the composition

$$(e, F) \mapsto e \circ F : [\ast, E] \times [E, C] \to [\ast, C]$$

Given an object $e \in \|E\|$, the precomposition functor

$$[e, C] : [E, C] \to C$$

"evaluation at $e"$, takes each $F \in [E, C]$ and yields the composite

$$* \longrightarrow^e E \longrightarrow^F C$$

i.e. the value of $F$ at $e$.

15. The left slice of a bifunctor $K : E \times D \to C$ at $e \in \|E\|$ is the functor $D \to C$ given by the value of the left exponential transpose $K^e : E \to [D, C]$ at $e$. The naturality square

$$
\begin{array}{ccc}
[E \times D, C] & \longrightarrow & [E, [D, C]] \\
\downarrow_{[\ast \times D, C]} & & \downarrow_{[\ast, [D, C]]} \\
[\ast \times D, C] & \longrightarrow & [\ast, [D, C]]
\end{array}
$$
shrinks to the commutative triangle

\[
\begin{array}{c}
\text{[E \times D, C]} \\
\text{[e \times D, C]}
\end{array} \xrightarrow{\tau} \begin{array}{c}
\text{[E, [D, C]]} \\
\text{[e, [D, C]]}
\end{array}
\]

by the identifications \[\ast \times D, C] \cong [D, C] \cong [* , [D, C]]\]; the precomposition

\[D \xrightarrow{e \times D} E \times D \xrightarrow{K} C\]

of \(K\) with the section \(e \times D\) thus yields the same functor as the one given by the evaluation

\[\ast \xrightarrow{e} E \xrightarrow{K^\tau} [D, C]\]

, i.e. the left slice of \(K\) at \(e\).

16. Similarly, the slice of an (exponentially transposed) bifunctor \(K : D \to [E, C]\) at \(e \in [E]\) is given by the value of the transpose \(K^\tau : E \to [D, C]\) at \(e\) (i.e. by the left slice of the bifunctor \(K^\tau : E \times D \to C\) at \(e\)). The naturality square

\[
\begin{array}{c}
\text{[D, [E, C]]} \\
\text{[D, [e, C]]}
\end{array} \xrightarrow{\tau} \begin{array}{c}
\text{[E, [D, C]]} \\
\text{[e, [D, C]]}
\end{array}
\]

shrinks to the commutative triangle

\[
\begin{array}{c}
\text{[D, [E, C]]} \\
\text{[D, [e, C]]}
\end{array} \xrightarrow{\tau} \begin{array}{c}
\text{[E, [D, C]]} \\
\text{[e, [D, C]]}
\end{array}
\]

by the identifications \([D, [* , C]] \cong [D, C] \cong [* , [D, C]]\); the postcomposition

\[D \xrightarrow{K} [E, C] \xrightarrow{[e, C]} C\]

of \(K\) with the evaluation \([e, C]\) thus yields the same functor as the one given by the evaluation

\[\ast \xrightarrow{e} E \xrightarrow{K^\tau} [D, C]\]

, i.e. the slice of \(K\) at \(e\).

17. Given a category \(E\), the unique functor \(E \to \ast\) is denoted by \(!E\) or just by \(!\). Given categories \(E\) and \(D\), the product functor \(!E \times D : E \times D \to \ast \times D\) as in

\[
\begin{array}{ccc}
E & \xleftarrow{!E} & E \times D \\
* & \xleftarrow{!E \times D} & \ast \times D \\
* & \xrightarrow{D} & D
\end{array}
\]

gives the projection

\(!E \times D : E \times D \to D\)

of \(E \times D\) on \(D\) under the identification \(\ast \times D \cong D\). Similarly, the product functor \(E \times !D : E \times D \to E \times \ast\) gives the projection

\(E \times !D : E \times D \to E\)

of \(E \times D\) on \(E\) under the identification \(E \times \ast \cong E\).
18. Given categories $D$, $E$, and $C$, the $D$-ary diagonal functor of the functor category $[E, C]$ is the precomposition functor

$$[E \times D, C] : [E, C] \to [E \times D, C]$$

sending each functor $F : E \to C$ to the composite bifunctor

$$E \times D \xrightarrow{E \times !} E \xrightarrow{F} C$$

. The diagonal functor $[E \times D, C]$ thus duplicates each $F : E \to C$ across $D$ by introducing to it a dummy variable varying over $D$. As a special case, the $D$-ary diagonal functor

$$[1_D, C] : C \to [D, C]$$

of a category $C$ sends each object $c \in \|C\|$ to the constant functor $\Delta_D c : D \to C$ given by the composition

$$D \xrightarrow{!} * \xrightarrow{c} C$$

. The commutativity of

$$[E \times 1_D, C] \xrightarrow{\sim} [E, C] \xrightarrow{1_D} [D, [E, C]]$$

follows from the commutativity of the naturality squares

$$[E \times *, C] \xrightarrow{\sim} [*, [E, C]] \xrightarrow{1_D} [D, [E, C]]$$

by the identifications $[E \times *, C] \equiv [E, C] \equiv [*, [E, C]]$ and $[E \times *, C] \equiv [E, C] \equiv [E, [* , C]]$.

19. A subcategory $D \subseteq E$ is called essentially wide if for every object $e \in \|E\|$ there exists an object $d \in \|D\|$ isomorphic to $e$. A functor is called essentially surjective if its image is an essentially wide subcategory of the codomain. Note that a subcategory is essentially wide if and only if the inclusion functor is essentially surjective. We use the following lemma in the sequel.

**Lemma.** Let $\tau : S \to T : E \to C$ be a natural transformation and $D$ be an essentially wide subcategory of $E$. Then $\tau$ is a natural isomorphism if and only if so is its restriction to $D$; that is, if and only if the component $\tau_d$ is an isomorphism for each $d \in \|D\|$.

**Proof.** Since a natural transformation is iso iff each component is an isomorphism, the forward implication is obvious. Assume now that $\tau$ is iso on $D$. We need to show that the component $\tau_e$ is an isomorphism for each $e \in \|E\|$. Since $D$ is essentially wide in $E$, there is an object $d \in \|D\|$ and an iso $E$-arrow $h : d \to e$, giving a naturality square

$$d : S \xrightarrow{\tau_d} T \cdot d$$

$$\xrightarrow{h} \downarrow \quad \downarrow T \cdot h$$

$$\xrightarrow{\tau_e} e : S \xrightarrow{\tau_e} T \cdot e$$

with $h : S$ and $T \cdot h$ iso (because any functor preserves isomorphisms). Now, since $\tau_d$ an isomorphism by assumption, so is $\tau_e = (h : S)^{-1} \circ \tau_d \circ (T : h)$ as required. ■
1. Modules and Cells

1.1. Modules

Definition 1.1.1.

- A right module $M$ over a category $X$, written $M : X \to \ast$, is a functor $M : X^\to \to \text{Set}$. Given a pair of right modules $M, N : X \to \ast$, a morphism $\Phi$ from $M$ to $N$, written $\Phi : M \to N : X \to \ast$, is a natural transformation $\Phi : M \to N : X^\to \to \text{Set}$.
- A left module $M$ over a category $A$, written $M : \ast \to A$, is a functor $M : A \to \text{Set}$. Given a pair of left modules $M, N : \ast \to A$, a morphism $\Phi$ from $M$ to $N$, written $\Phi : M \to N : \ast \to A$, is a natural transformation $\Phi : M \to N : A \to \text{Set}$.

Remark 1.1.2.

1. For a right module $M : X \to \ast$, the image of an object/arrow $x \in X$ under the functor $M : X^\to \to \text{Set}$ is written $(x)(M)$ or just $x(M)$; and for a left module $M : \ast \to A$, the image of an object/arrow $a \in A$ under the functor $M : A \to \text{Set}$ is written $(M)(a)$ or just $(M)a$.
2. For a right module morphism $\Phi : M \to N : X \to \ast$, the component of the natural transformation $\Phi : M \to N : X^\to \to \text{Set}$ at $x \in \|X\|$ is written $x(\Phi)$; and for a left module morphism $\Phi : M \to N : \ast \to A$, the component of the natural transformation $\Phi : M \to N : A \to \text{Set}$ at $a \in \|A\|$ is written $\langle \Phi \rangle a$.
3. For a universe $\mathcal{U}$, a right module $M : X \to \ast$ is called $\mathcal{U}$-small (resp. locally $\mathcal{U}$-small) if $X$ is $\mathcal{U}$-small (resp. locally $\mathcal{U}$-small) and $M$ is a functor $X^\to \to \text{Set}_\mathcal{U}$ (i.e. the sets $x(M)$ are $\mathcal{U}$-small for all objects $x \in \|X\|$), and similarly a left module $M : \ast \to A$ is called $\mathcal{U}$-small (resp. locally $\mathcal{U}$-small) if $A$ is $\mathcal{U}$-small (resp. locally $\mathcal{U}$-small) and $M$ is a functor $A \to \text{Set}_\mathcal{U}$. We just say “small” instead of “$\mathcal{U}$-small” for $\mathcal{U}$ chosen implicitly.
4. Right and left modules are referred to as one-sided modules to distinguish them from two-sided modules to be introduced below in Definition 1.1.11.

Notation 1.1.3. The notations $[X^\to, \text{Set}]$ and $[A, \text{Set}]$ (see Preliminary 10) are abbreviated to $[X:]$ and $[:A]$ respectively:

1. For a category $X$, $[X:]$ (an abbreviation of $[X^\to, \text{Set}]$) denotes the category of right modules over $X$, and for a category $A$, $[:A]$ (an abbreviation of $[A, \text{Set}]$) denotes the category of left modules over $A$.
2. For a functor $K : E \to X$, $[K:] : [X:] \to [E:]$ (an abbreviation of $[K^\to, \text{Set}] : [X^\to, \text{Set}] \to [E^\to, \text{Set}]$) denotes the precomposition functor given by the assignment $M \mapsto K \circ M$, and for a functor $K : E \to A$, $[K] : [:A] \to [:E]$ (an abbreviation of $[K, \text{Set}] : [A, \text{Set}] \to [E, \text{Set}]$) denotes the precomposition functor given by the assignment $M \mapsto K \circ M$.

Remark 1.1.4.

1. If $X$ (resp. $A$) is a small category, then the category $[X:]$ (resp. $[:A]$) is locally small.
2. The assignment $K \mapsto [K:]$ (resp. $K \mapsto [:K]$) is contravariant functorial.
3. By definition, $[X:] = [:X^\to]$ and $[:A] = [A^\to:]$:

   : a right module over $X$ is the same thing as a left module over the opposite category $X^\to$, and a left module over $A$ is the same thing as a right module over the opposite category $A^\to$.

Definition 1.1.5.

- A right module morphism $\Phi : M \to N : X \to \ast$ is called iso (or an isomorphism) if it is invertible in the category $[X:]$. 

12
- A left module morphism $\Phi : M \to N : * \to A$ is called iso (or an isomorphism) if it is invertible in the category $[\mathcal{A}]$.

**Proposition 1.1.6.**
- A right module morphism $\Phi : M \to N : X \to *$ is iso if and only if each component $x(\Phi) : x(M) \to x(N)$ is a bijection.
- A left module morphism $\Phi : M \to N : * \to A$ is iso if and only if each component $(\Phi) a : (M) a \to (N) a$ is a bijection.

**Proof.** Since a right module morphism $\Phi : M \to N : X \to *$ is the same thing as a natural transformation $\Phi : M \to N : X \to \text{Set}$, this is an instance of the general fact that a natural transformation is iso iff each component is an isomorphism. \qed

**Definition 1.1.7.**
- If $M : X \to *$ is a right module, for any object $x \in |X|$, an element $m$ of the set $x(M)$, or a pair $(x, m)$ such that $m \in x(M)$, is called an arrow of $M$ (or an $M$-arrow) at $x$, and written $m : x \to * \text{ or } x \cdot m \to *$ (or just $m$ if its domain is understood or unimportant).
- If $M : * \to A$ is a left module, for any object $a \in |\mathcal{A}|$, an element $m$ of the set $(M) a$, or a pair $(m, a)$ such that $m \in (M) a$, is called an arrow of $M$ (or an $M$-arrow) at $a$, and written $m : * \to a$ or $* \cdot m \to a$ (or just $m$ if its codomain is understood or unimportant).

**Remark 1.1.8.** For a right module morphism $\Phi : M \to N : X \to *$, the image of an $M$-arrow $m : x \to *$ under the function $x(\Phi) : x(M) \to x(N)$ is written variously as

\[
m : x(\Phi) = m : \Phi = \Phi(m) = \Phi : m = x(\Phi) \cdot m
\]

Similarly, for a left module morphism $\Phi : M \to N : * \to A$, the image of an $M$-arrow $m : * \to a$ under the function $(\Phi) a : (M) a \to (N) a$ is written variously as

\[
m : (\Phi) a = m : \Phi = \Phi(m) = \Phi : m = (\Phi) a \cdot m
\]

**Definition 1.1.9.**
- For a right module $M : X \to *$, the composite

$$\xymatrix{ g \circ m = m \circ g }$$

of an $X$-arrow $g : y \to x$ and an $M$-arrow $m : x \to *$, as shown in

$$\xymatrix{ x \ar[r]^m \ar[rd]_g & * \ar@{.>}[ld]^g \ar@{.>}[ld]_m \\
& y }
$$

is the $M$-arrow $y \to *$ defined by

$$g \circ m = g(M) \cdot m$$

the image of $m$ under the function $g(M) : x(M) \to y(M)$.
- For a left module $M : * \to A$, the composite

$$\xymatrix{ m \circ f = f \circ m }$$

of an $M$-arrow $m : * \to a$ and an $A$-arrow $f : a \to b$, as shown in

$$\xymatrix{ * \ar[r]^m \ar[rd]_m & a \ar@{.>}[ld]^f \\
& b }
$$

is the $M$-arrow $* \to b$ defined by

$$m \circ f = m : (M) f$$

the image of $m$ under the function $(M) f : (M) a \to (M) b$. 

1.1. Modules 13
Remark 1.1.10. The naturality of a right module morphism \( \Phi : \mathcal{M} \to \mathcal{N} : X \to * \) is expressed by the identity

\[
\Phi (g \circ m) = g \circ \Phi (m)
\]

for every composable pair of an \( X \)-arrow \( g \) and an \( \mathcal{M} \)-arrow \( m \); and the naturality of a left module morphism \( \Phi : \mathcal{M} \to \mathcal{N} : * \to A \) is expressed by the identity

\[
\Phi (m \circ f) = \Phi (m) \circ f
\]

for every composable pair of an \( \mathcal{M} \)-arrow \( m \) and an \( A \)-arrow \( f \).

Definition 1.1.11. A [two-sided] module \( \mathcal{M} \) from a category \( X \) to a category \( A \), written \( \mathcal{M} : X \to A \), is a bifunctor \( \mathcal{M} : X^\times A \to \text{Set} \). Given a pair of modules \( \mathcal{M}, \mathcal{N} : X \to A \), a morphism \( \Phi \) from \( \mathcal{M} \) to \( \mathcal{N} \), written \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \), is a natural transformation \( \Phi : \mathcal{M} \to \mathcal{N} : X^\times A \to \text{Set} \).

Remark 1.1.12.
(1) If \( \mathcal{M} : X \to A \) is a module, then the image of an object/arrow \((x, a) \in X^\times A\) under the functor \( \mathcal{M} : X^\times A \to \text{Set} \) is written \((x) (\mathcal{M}) (a)\) or just \( x (\mathcal{M}) a \).

(2) If \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \) is a module morphism, then the component of the natural transformation \( \Phi : \mathcal{M} \to \mathcal{N} : X^\times A \to \text{Set} \) at \((x, a) \in X^\times A\) is written \( x (\Phi) a \).

(3) For a universe \( \mathcal{U} \), a module \( \mathcal{M} : X \to A \) is called \( \mathcal{U} \)-small (resp. locally \( \mathcal{U} \)-small) if \( X \) and \( A \) are \( \mathcal{U} \)-small (resp. locally \( \mathcal{U} \)-small) and \( \mathcal{M} \) is a functor \( X^\times A \to \text{Set}_\mathcal{U} \) (i.e. the sets \( x (\mathcal{M}) a \) are \( \mathcal{U} \)-small for all pairs of objects \( x \in \| X \| \) and \( a \in \| A \| \)). We just say “small” instead of “\( \mathcal{U} \)-small” for \( \mathcal{U} \) chosen implicitly.

(4) A two-sided module \( \mathcal{M} : X \to A \) is called an endomodule when \( X = A \).

Notation 1.1.13. The notation \( \langle X^\times A, \text{Set} \rangle \) (see Preliminary 10) is abbreviated to \( \langle X : A \rangle \);

(1) for a pair of categories \( X \) and \( A \), \( \langle X : A \rangle \) (an abbreviation of \( \langle X^\times A, \text{Set} \rangle \)) denotes the category of modules \( X \to A \);

(2) for a pair of functors \( S : E \to X \) and \( T : D \to A \), \( \langle S : T \rangle : \langle X : A \rangle \to \langle E : D \rangle \) (an abbreviation of \( \langle S^\times T, \text{Set} \rangle : \langle X^\times A, \text{Set} \rangle \to \langle E^\times D, \text{Set} \rangle \)) denotes the precomposition functor given by the assignment \( M \mapsto \langle S^\times T \rangle (M) \).

Remark 1.1.14.
(1) If \( X \) and \( A \) are small categories, then the category \( \langle X : A \rangle \) is locally small.

(2) The assignment \( (S, T) \mapsto \langle S : T \rangle \) is contravariant bifunctorial.

(3) By definition,

\[
\langle X \times A \rangle = \langle X : A \rangle = \langle [\mathcal{M} : X^\times A] \rangle
\]

a two-sided module \( \mathcal{M} : X \to A \) is the same thing as a right module \( \mathcal{M} : X \times A^\to \to \to * \) (resp. a left module \( \mathcal{M} : * \to X \times A \)).

(4) The canonical isomorphisms \( X^\times \cong X^\times \times \star \) and \( A \cong \star \times A \) yield canonical isomorphisms

\[
\langle X : \star \rangle \cong \langle X : \star \rangle \quad \text{and} \quad \langle A : \star \rangle \cong \langle [\mathcal{M} : \star : A] \rangle
\]

By these isomorphisms, a right module over \( X \) is identified with a two-sided module from \( X \) to the terminal category, and a left module over \( A \) is identified with a two-sided module from the terminal category to \( A \).

(5) There are obvious isomorphisms

\[
\langle \text{Set} \rangle \cong \mathcal{M} = \mathcal{N} = \langle [\star : \star] \rangle
\]

by which a set is identified with a module \( \star \to \star \) over the terminal category.

Definition 1.1.15. A module morphism \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \) is called iso (or an isomorphism) if it is invertible in the category \( \langle X : A \rangle \).
**Proposition 1.1.16.** A module morphism $\Phi : M \to N : X \to A$ is iso if and only if each component $x(\Phi) a : x(M) a \to x(N) a$ is a bijection.

**Proof.** See the proof of Proposition 1.1.6. □

**Definition 1.1.17.** If $M : X \to A$ is a module, for any pair of objects $x \in \mathcal{X}$ and $a \in \mathcal{A}$, an element $m$ of the set $x(M) a$, or a triple $(x, m, a)$ such that $m \in x(M) a$, is called an arrow of $M$ (or an $M$-arrow), and written $m : x \to a$ or $x \stackrel{m}{\to} a$ (or just $m$ if its domain and codomain are understood or unimportant).

**Remark 1.1.18.** For a module morphism $\Phi : M \to N : X \to A$, the image of an $M$-arrow $m : x \to a$ under the function $x(\Phi) a : x(M) a \to x(N) a$ is written variously as

$$m : x(\Phi) a = m : \Phi = \Phi (m) = \Phi : m = x(\Phi) a : m$$

**Definition 1.1.19.** For a module $M : X \to A$,

1. the composite

$$g \circ m = m \circ g$$

of an $X$-arrow $g : y \to x$ and an $M$-arrow $m : x \to a$, as shown in

![Diagram](#)

, is the $M$-arrow $y \to a$ defined by

$$g \circ m = g(M) a : m$$

, the image of $m$ under the function $g(M) a : x(M) a \to y(M) a$.

2. the composite

$$m \circ f = f \circ m$$

of an $M$-arrow $m : x \to a$ and an $A$-arrow $f : a \to b$, as shown in

![Diagram](#)

, is the $M$-arrow $x \to b$ defined by

$$m \circ f = m : x(M) f$$

, the image of $m$ under the function $x(M) f : x(M) a \to x(M) b$.

**Remark 1.1.20.**

1. A module $M : X \to A$ thus induces a composition law among the arrows of $X$, $A$, and $M$.

Conversely, a module $M : X \to A$ may be defined by giving a set of $M$-arrows and a composition law of them with $X$-arrows and $A$-arrows satisfying the associativity and identity axioms, i.e. by giving a collage (see Section 3.1) from $X$ to $A$. 
(2) The composite
\[(g \circ m) \circ f = g \circ m \circ f = g \circ (m \circ f)\]
of an \(X\)-arrow \(g : y \to x\), an \(M\)-arrow \(m : x \to a\), and an \(A\)-arrow \(f : a \to b\), as shown in
\[
\begin{array}{ccc}
  x & \xrightarrow{m} & a \\
g \downarrow & & \downarrow f \\
y & \xrightarrow{g \circ m \circ f} & b
\end{array}
\]
, is the \(M\)-arrow \(y \to b\) defined by
\[g \circ m \circ f = m \circ g(M) f\]
, the image of \(m\) under the function \(g(M) f : x(M) a \to y(M) b\).

(3) The naturality of a module morphism \(\Phi : \mathcal{M} \to \mathcal{N} : \mathcal{X} \to \mathcal{A}\) is expressed by the identity
\[\Phi(g \circ m \circ f) = g \circ \Phi(m) \circ f\]
for every composable triple of an \(X\)-arrow \(g\), an \(M\)-arrow \(m\), and an \(A\)-arrow \(f\).

(4) When a two-sided module \(\mathcal{M} : \mathcal{X} \to \mathcal{A}\) is regarded as a right module \(\mathcal{M} : \mathcal{X} \times \mathcal{A}^\lor \to \ast\) (resp. a left module \(\mathcal{M} : \ast \to \mathcal{X} \times \mathcal{A}\)), an \(M\)-arrow \(m : x \to a\) is written as \(m : (a,x) \to \ast\) (resp. \(m : \ast \to (x,a)\)), and the compositions
\[
\begin{array}{ccc}
  x & \xrightarrow{m} & a \\
g \downarrow & & \downarrow f \\
y & \xrightarrow{g \circ m \circ f} & b
\end{array}
\]
are written as
\[
\begin{array}{ccc}
  (x,a) & \xrightarrow{m} & \ast \\
(g,a) \downarrow & & (g,f) \downarrow \\
(y,a) & \xrightarrow{(g,a) \circ m} & (y,b)
\end{array}
\]
(resp.
\[
\begin{array}{ccc}
  \ast & \xrightarrow{m} & (x,a) \\
(g,a) \downarrow & & (g,f) \downarrow \\
(y,a) & \xrightarrow{(g,a) \circ m} & (y,b)
\end{array}
\]
).

**Definition 1.1.21.** The hom of a category \(\mathcal{C}\) is the endomodule \(\langle \mathcal{C} \rangle : \mathcal{C} \to \mathcal{C}\) given by the assignment \((x,a) \mapsto \hom_{\mathcal{C}}(x,a)\) for \(x,a \in \mathcal{C}\).

**Remark 1.1.22.**
(1) The endomodule given by the hom of a category is called a hom endomodule.
(2) For a pair of objects \(a,b \in \mathcal{C}\), the hom-set \(\hom_{\mathcal{C}}(a,b)\) of a category \(\mathcal{C}\) is the same thing as the hom-set \(\mathcal{A}(C) b\) of the endomodule \(\langle \mathcal{C} \rangle\).
(3) Hereafter a hom-set is written as \(\mathcal{A}(C) b\) rather than \(\hom_{\mathcal{C}}(a,b)\); likewise, for an object \(c \in \|\mathcal{C}\|\) and a \(\mathcal{C}\)-arrow \(f : a \to b\), the functions
\[
\begin{align*}
\hom_{\mathcal{C}}(c,f) : \hom_{\mathcal{C}}(c,a) & \to \hom_{\mathcal{C}}(c,b) \\
\hom_{\mathcal{C}}(f,c) : \hom_{\mathcal{C}}(b,c) & \to \hom_{\mathcal{C}}(a,c)
\end{align*}
\]
are written as
\[
\begin{align*}
c(C) f : c(C) a & \to c(C) b \\
f(C) c : b(C) c & \to a(C) c
\end{align*}
\]
(4) For categories $\mathbf{A}$ and $\mathbf{B}$, the hom of the functor category $[\mathbf{A}, \mathbf{B}]$ is denoted by $(\mathbf{A}, \mathbf{B})$; given a pair of functors $F, G : \mathbf{A} \to \mathbf{B}$, $(F) (\mathbf{A}, \mathbf{B}) (G)$ denotes the set of natural transformations from $F$ to $G$.

(5) For categories $\mathbf{X}$ and $\mathbf{A}$, the hom of the category $[\mathbf{X} : \mathbf{A}]$ is denoted by $(\mathbf{X} : \mathbf{A})$; given a pair of modules $M, N : \mathbf{X} \to \mathbf{A}$, $M (\mathbf{X} : \mathbf{A}) N$ denotes the set of module morphism from $M$ to $N$. Likewise, the hom of the category $[\mathbf{X} : \mathbf{A}]$ (resp. $[\mathbf{X} : \mathbf{A}]$) is denoted by $(\mathbf{X} : \mathbf{A})$ (resp. $\mathbf{X} : \mathbf{A}$).

(6) For specific categories (such as $\mathbf{Set}$ or $\mathbf{CAT}$), a hom-set $a(\mathbf{C})b$ is often written as $\mathbf{C}[a,b]$ or, when $\mathbf{C}$ is understood, just $[a,b]$. For example, the set of functions from a small set $\mathbf{S}$ to a small set $\mathbf{T}$ is written as $[\mathbf{S}, \mathbf{T}]$ rather than $\mathbf{S}(\mathbf{Set}) \mathbf{T}$.

**Notation 1.1.23.**

(1) In a diagram, a module $\mathbf{M} : \mathbf{X} \to \mathbf{A}$ and a module morphism $\Phi : \mathbf{M} \to \mathbf{N} : \mathbf{X} \to \mathbf{A}$ are depicted as

$$\xymatrix{ \mathbf{X} \ar[r]^\mathbf{M} & \mathbf{A} \ar@{-->}[d]_{\Phi} \ar@{-->}[u]_{\mathbf{N}} }$$

(2) Italic capital letters $\mathbf{M}$, $\mathbf{N}$, ... vary over both modules and module morphisms. When we write $M \in [\mathbf{X} : \mathbf{A}]$, $M : \mathbf{X} \to \mathbf{A}$, or $\mathbf{X} \mathbf{M} \to \mathbf{A}$, $M$ stands for an object or an arrow of the module category $[\mathbf{X} : \mathbf{A}]$, i.e. a module or a module morphism (cf. Preliminary 8).

**Definition 1.1.24.**

- Given a right module (or module morphism) $M$ and a functor (or natural transformation) $S$ as in

$$\xymatrix{ \mathbf{E} \ar[r]^S & \mathbf{X} \ar[r]^M & * \ar@{-->}[u] }$$

, their composite, written $[\mathbf{S}] (\mathbf{M})$ or just $\mathbf{S} (\mathbf{M})$, is the right module (or module morphism) $\mathbf{E} \to *$ defined by the composition

$$\xymatrix{ \mathbf{E} \ar[r]^S & \mathbf{X} \ar[r]^M & \mathbf{Set} }$$

- Given a left module (or module morphism) $M$ and a functor (or natural transformation) $T$ as in

$$\xymatrix{ * \ar[r]^-M & \mathbf{A} \ar[r]^T & \mathbf{D} \ar@{-->}[u] }$$

, their composite, written $(\mathbf{M}) [\mathbf{T}]$ or just $(\mathbf{M}) T$, is the left module (or module morphism) $* \to \mathbf{D}$ defined by the composition

$$\xymatrix{ \mathbf{D} \ar[r]^T & \mathbf{A} \ar[r]^M & \mathbf{Set} }$$

**Remark 1.1.25.**

- The composition

$$(\mathbf{S}, \mathbf{M}) \mapsto \mathbf{S} (\mathbf{M}) : [\mathbf{E}, \mathbf{X}] \times [\mathbf{X} : \mathbf{]} \to [\mathbf{E} : \mathbf{]}$$

is functorial in each variable, contravariant in $\mathbf{S}$ and covariant in $\mathbf{M}$. If $\mathbf{S} : \mathbf{E} \to \mathbf{X}$ is a functor, then

$$\mathbf{S} (\mathbf{M}) = \mathbf{S} \circ \mathbf{M} = \mathbf{M} \circ [\mathbf{S} :]$$

for any $M \in [\mathbf{X} : \mathbf{]}$.

- The composition

$$(\mathbf{M}, \mathbf{T}) \mapsto (\mathbf{M}) T : [\mathbf{A}] \times [\mathbf{D}, \mathbf{A}] \to [\mathbf{D}]$$

is functorial in each variable, covariant in both $\mathbf{M}$ and $\mathbf{T}$. If $\mathbf{T} : \mathbf{D} \to \mathbf{A}$ is a functor, then

$$(\mathbf{M}) T = \mathbf{T} \circ \mathbf{M} = \mathbf{M} : [\mathbf{T}]$$

for any $M \in [\mathbf{A}]$. 

**Definition 1.1.26.** Given a module (or module morphism) $M$, a functor (or natural transformation) $S$, and a second functor (or natural transformation) $T$, all as in

$$E \xrightarrow{S} X \xrightarrow{M} A \xleftarrow{T} D$$

, their composite, written $[S] \langle M \rangle [T]$ or just $S \langle M \rangle T$, is the module (or module morphism) $E \to D$ defined by the composition

$$E^- \times D \xrightarrow{S \times T} X^- \times A \xrightarrow{M} \text{Set}$$

.

**Remark 1.1.27.**

1. The composition

$$(S, M, T) \mapsto S \langle M \rangle T : [E, X] \times [X : A] \times [D, A] \to [E : D]$$

is functorial in each variable, contravariant in $S$ and covariant in $M$ and $T$. If $S : E \to X$ and $T : D \to A$ are functors, then

$$S \langle M \rangle T = [S \times T] \circ M = M \circ [S : T]$$

for any $M \in [X : A]$.

2. As a special case,

- given

$$E \xrightarrow{S} X \xrightarrow{M} A$$

, their composite $S \langle M \rangle : E \to A$ is defined by the composition

$$E^- \times A \xrightarrow{S \times A} X^- \times A \xrightarrow{M} \text{Set}$$

; if $A$ is the terminal category, the composite $S \langle M \rangle : E \to \ast$ coincides with that defined in Definition 1.1.24 under the identification $[E : ] \cong [E : \ast]$ and $[X : ] \cong [X : \ast]$.

- given

$$X \xrightarrow{M} A \xleftarrow{T} D$$

, their composite $\langle M \rangle T : X \to D$ is defined by the composition

$$X^- \times D \xrightarrow{X \times T} X^- \times A \xrightarrow{M} \text{Set}$$

; if $X$ is the terminal category, the composite $\langle M \rangle T : \ast \to D$ coincides with that defined in Definition 1.1.24 under the identification $[A] \cong [\ast : A]$ and $[D] \cong [\ast : D]$.

**Proposition 1.1.28.**

1. Given

$$E' \xrightarrow{S'} E \xrightarrow{S} X \xrightarrow{M} A \xleftarrow{T} D \xleftarrow{T'} D'$$

, the associative law

$$S' \langle S \langle M \rangle T \rangle T' = [S' \circ S] \langle M \rangle [T \circ T']$$

holds.

2. Given

$$E \xrightarrow{S} X \xrightarrow{M} A \xleftarrow{T} D$$

, the associative law

$$S \langle \langle M \rangle T \rangle = S \langle M \rangle T = \langle S \langle M \rangle \rangle T$$

holds.
Proof.

(1) Indeed,

\[
S'(S(M)T)T' = [S' \times T'] \circ [S \times T] \circ M \\
= [[S' \times T'] \circ [S \times T]] \circ M \\
= [[S' \circ S] \times [T' \circ T]] \circ M \\
= [S' \circ S](M)[T \circ T']
\]

(2) By what we have just seen,

\[
S(M)T = S([1_X](M)T) = [S \circ 1_X](M)T = S(M)T
\]

and

\[
\langle S(M) \rangle T = \langle S(M)[1_{A}] \rangle T = S(M)[1_{A} \circ T] = S(M)T
\]

Example 1.1.29.

(1) Given a module and functors as in

\[
\begin{array}{cccc}
E & \xrightarrow{S} & X & \xrightarrow{\mathcal{M}} & A & \xrightarrow{T} & D \\
& \downarrow{\mathcal{S}} & & \downarrow{\mathcal{M}} & & \downarrow{T} & & \\
\end{array}
\]

, the composition yields the module \( S(\mathcal{M})T : E \rightarrow D \) defined by

\[
e(\mathcal{S}(\mathcal{M})T)d = (e \circ S)(\mathcal{M})(T \circ d)
\]

for \( e \in E \) and \( d \in D \). An \( S(\mathcal{M})T \)-arrow \( m : e \rightarrow d \) is given by an \( \mathcal{M} \)-arrow \( m : e \circ S \rightarrow T \circ d \); for an \( E \)-arrow \( h : e' \rightarrow e \), an \( S(\mathcal{M})T \)-arrow \( m : e \rightarrow d \), and a \( D \)-arrow \( k : d \rightarrow d' \), their composite \( h \circ m \circ k : e' \rightarrow d' \) is given by the \( \mathcal{M} \)-arrow \( (h \circ S) \circ m \circ (T \circ h) : e' \circ S \rightarrow T \circ d' \) as indicated in

\[
\begin{array}{cccc}
e & \xrightarrow{\mathcal{S}} & e' & \xrightarrow{\mathcal{M}} & \xrightarrow{\mathcal{S}} & \xrightarrow{T} & \xrightarrow{\mathcal{D}} \\
\downarrow{h} & & \downarrow{h \circ S} & & \downarrow{T \circ k} & & \downarrow{k} \\
\end{array}
\]

(2) Given a module morphism and functors as in

\[
\begin{array}{cccc}
E & \xrightarrow{S} & X & \xrightarrow{\mathcal{M}} & A & \xrightarrow{T} & D \\
& \xrightarrow{\mathcal{M} \circ \Phi} & \xrightarrow{\mathcal{N}} & \xrightarrow{\mathcal{A}} & \xrightarrow{T} & \xrightarrow{\mathcal{D}} \\
\end{array}
\]

, the composition yields the module morphism \( S(\Phi)T : S(\mathcal{M})T \rightarrow S(\mathcal{N})T : E \rightarrow D \) defined by

\[
e(\mathcal{S}(\Phi)T)d = (e \circ S)(\Phi)(T \circ d)
\]

for \( e \in \mathcal{E} \) and \( d \in \mathcal{D} \).

(3) Given a module \( \mathcal{M} \) and natural transformations \( \sigma : S' \rightarrow S \) and \( \tau : T \rightarrow T' \) as in

\[
\begin{array}{cccc}
E & \xrightarrow{\mathcal{S}} & X & \xrightarrow{\mathcal{M}} & A & \xrightarrow{\mathcal{T}} & D \\
& \xrightarrow{\mathcal{S}' \circ \tau} & \xrightarrow{\mathcal{M}} & \xrightarrow{\mathcal{A}} & \xrightarrow{\mathcal{T}' \circ \tau} & \xrightarrow{\mathcal{D}} \\
\end{array}
\]

, the composition yields the module morphism \( \sigma(\mathcal{M}) \tau : S(\mathcal{M})T \rightarrow S'(\mathcal{M})T' : E \rightarrow D \) which maps each \( S(\mathcal{M})T \)-arrow \( m : e \rightarrow d \) to the \( S'(\mathcal{M})T' \)-arrow \( \sigma_{e} \circ m \circ \tau_{d} : e \rightarrow d \) as indicated in

\[
\begin{array}{cccc}
e & \xrightarrow{\mathcal{S}} & e' & \xrightarrow{\mathcal{M}} & \xrightarrow{\mathcal{S}} & \xrightarrow{T} & \xrightarrow{\mathcal{D}} \\
\downarrow{\sigma_{e}} & & \downarrow{\mathcal{M}} & & \downarrow{\tau_{d}} & & \downarrow{\tau_{d}} \\
\end{array}
\]
(4) The restriction of a module $\mathcal{M} : X \to A$ to subcategories $Y \subseteq X$ and $B \subseteq A$ is given by the composition

$$Y \xrightarrow{Y} X \xrightarrow{\mathcal{M}} A \leftarrow B$$

of $\mathcal{M}$ with the two inclusion functors. Similarly, the restriction of a module morphism $\Phi : \mathcal{M} \to \mathcal{N} : X \to A$ to $Y$ and $B$ is given by the composition

$$Y \xrightarrow{Y} X \xleftarrow{\mathcal{M}} A \leftarrow B \xrightarrow{\Phi} B$$

(5) The evaluation of a module $\mathcal{M} : X \to A$ at $(x,a) \in X \times A$ is identified with the composition

$$\star \xrightarrow{x} X \xrightarrow{\mathcal{M}} A \xleftarrow{a} \star$$

(6) Given a module $\mathcal{M} : X \times Y \to A \times B$ and an object $(x,a) \in \|X \times A\|$, the composition

$$Y \xrightarrow{x \times Y} X \times Y \xrightarrow{\mathcal{M}} A \times B \xleftarrow{a \times B} B$$

of $\mathcal{M}$ with the sections $x \times Y$ and $a \times B$ yields the slice of $\mathcal{M}$ at $(x,a)$, i.e. the module $[x \times Y] \langle \mathcal{M} \rangle [a \times B] : Y \to B$ such that

$$y ([x \times Y] \langle \mathcal{M} \rangle [a \times B]) b = (x,y) \langle \mathcal{M} \rangle (a,b)$$

for $y \in Y$ and $b \in B$.

(7) Given a right module $\mathcal{M} : X \to \star$ and a category $E$, the composition

$$X \xrightarrow{\mathcal{M}} \star \leftarrow 1 \xrightarrow{1} E$$

yields a two-sided module $\langle \mathcal{M} \rangle [1_E] : X \to E$ by duplicating $\mathcal{M}$ across $E$. The $E$-ary diagonal functor of $[X:]$ is the precomposition functor

$$[X : 1_E] : [X :] \to [X : E]$$

sending each right module $\mathcal{M} : X \to \star$ to the composite module $\langle \mathcal{M} \rangle [1_E] : X \to E$. As a special case, the $E$-ary diagonal functor

$$[1 : 1_E] : \text{Set} \to [\_ : E]$$

of $\text{Set}$ sends each small set $S$ to the constant left module $\langle S \rangle [1_E] : \star \to E$ given by the composition

$$\star \xrightarrow{S} \star \xleftarrow{1} E$$

Given a left module $\mathcal{M} : \star \to A$ and a category $E$, the composition

$$E \xrightarrow{1} \star \xrightarrow{\mathcal{M}} A$$

yields a two-sided module $[1_E] \langle \mathcal{M} \rangle : E \to A$ by duplicating $\mathcal{M}$ across $E$. The $E$-ary diagonal functor of $[\_ : A]$ is the precomposition functor


sending each left module $\mathcal{M} : \star \to A$ to the composite module $[1_E] \langle \mathcal{M} \rangle : E \to A$. As a special case, the $E$-ary diagonal functor

$$[1_E : \_] : \text{Set} \to [E : \_]$$
of \textbf{Set} sends each small set \( S \) to the constant right module \([1_{\text{E}}]\{S\} : \text{E} \to *\) given by the composition
\[
\text{E} \xrightarrow{i} * \xrightarrow{S} * \xrightarrow{s} * \xrightarrow{E} \text{E}
\]

(8) \>
\>

- Given a right module \( \mathcal{M} : \text{X} \to * \) and a category \( \text{E} \), the composition
\[
\text{X} \times \text{E}^\rightarrow \xrightarrow{\text{X} \times i} \text{X} \xrightarrow{\mathcal{M}} \text{X} \xrightarrow{\text{E}^\rightarrow} *
\]

yields a right module \([\text{X} \times 1_{\text{E}}]\{\mathcal{M}\} : \text{X} \times \text{E}^\rightarrow \to *\) by introducing to \( \mathcal{M} \) a dummy variable varying over \( \text{E}^\rightarrow \). Note that this right module is the same thing as the two-sided module \([1_{\text{E}}]\{\mathcal{M}\} : \text{E} \to \text{E} \) in the preceding example.

- Given a left module \( \mathcal{M} : * \to \text{A} \) and a category \( \text{E} \), the composition
\[
* \xrightarrow{\mathcal{M}} A \xleftarrow{1 \times A} \text{E}^\rightarrow \times \text{A}
\]

yields a left module \([1_{\text{E}}-\times \text{A}]\{\mathcal{M}\} : * \to \text{E}^\rightarrow \times \text{A} \) by introducing to \( \mathcal{M} \) a dummy variable varying over \( \text{E}^\rightarrow \). Note that this left module is the same thing as the two-sided module \([1_{\text{E}}]\{\mathcal{M}\} : \text{E} \to \text{A} \) in the preceding example.

(9) Given a category \( \text{E} \), the compositions
\[
\text{E} \xrightarrow{i} * \xrightarrow{\langle \ast \rangle} * \xrightarrow{\langle \ast \rangle} * \xrightarrow{1} \text{E}
\]

(where \( \langle \ast \rangle \) is the hom of the terminal category) yield the right and left constant modules
\[
\Delta_{\text{E}*} : \text{E} \to * \quad * \Delta_{\text{E}} : * \to \text{E}
\]

over \( \text{E} \) such that
\[
e(\Delta_{\text{E}*}) = \{*\} = (\ast \Delta_{\text{E}}) e
\]

for every object \( e \in \|\text{E}\| \). They are given respectively by the constant functor \( \Delta_{\text{E}*} : \{\ast\} : \text{E}^\rightarrow \to \text{Set} \) and the constant functor \( \Delta_{\text{E}} \{\ast\} : \text{E} \to \text{Set} \) sending every object to the singleton set \( \{\ast\} \).

(10) As a special case of (1) above, given a pair of functors
\[
\text{E} \xrightarrow{s} \text{C} \xrightarrow{(C)} \text{C} \xrightarrow{T} \text{D}
\]

, the composition yields the module \( \text{S}(\text{C}) \text{T} : \text{E} \to \text{D} \) defined by
\[
e(\text{S}(\text{C}) \text{T}) d = (e \circ \text{S})(\text{C})(\text{T} \circ d)
\]

for \( e \in \text{E} \) and \( d \in \text{D} \). An \( \text{S}(\text{C}) \text{T}\)-arrow \( f : e \to d \) is given by a \( \text{C}\)-arrow \( f : e : \text{S} \to \text{T} \circ d \); for an \( \text{E}\)-arrow \( h : e' \to e \), an \( \text{S}(\text{C}) \text{T}\)-arrow \( f : e \to d \), and a \( \text{D}\)-arrow \( k : d \to d' \), their composite \( h \circ f \circ k : e' \to d' \) is given by the \( \text{C}\)-arrow \( (h : \text{S}) \circ f \circ (\text{T} \circ k) : e' : \text{S} \to \text{T} \circ d' \) as indicated in

\[
e \quad e : \text{S} \xrightarrow{f} \text{T} \circ d \quad d
\]

\[
h \upharpoonright \quad h : \text{S} \upharpoonright \quad \text{T} \circ k \downarrow \quad \downarrow k
\]

\[
e' \quad e' : \text{S} \xrightarrow{h \circ f \circ k} \text{T} \circ d' \quad d'
\]

\[\text{Proposition 1.1.30.} \quad \text{In Example 1.1.29(4), suppose that } \text{Y and B are essentially wide (see Preliminary 19) subcategories of X and A}. \quad \text{Then } \Phi : \mathcal{M} \to \mathcal{N} : \text{X} \to \text{A} \text{ is an isomorphism if and only if its restriction to } \text{Y and B are isomorphisms; that is, if and only if the component } \text{y} \langle \Phi \rangle \text{b} : \text{y} \langle \mathcal{M} \rangle \text{b} \to \text{y} \langle \mathcal{N} \rangle \text{b} \text{ is a bijection for each pair of objects } y \in \|\text{Y}\| \text{ and b} \in \|\text{B}\| (cf. Proposition 1.1.16).} \]
Proof. Since a module morphism $\Phi : M \to N : X \to A$ is the same thing as a natural transformation $\Phi : M \to N : X \times A \to \text{Set}$, this is an instance of the lemma in Preliminary 19.  

**Proposition 1.1.31.** If $\Phi$ in Example 1.1.29(2) is a module isomorphism, so is the composite $S(\Phi) T$. The converse holds if $S$ and $T$ are essentially surjective.

**Proof.** Note that $S(\Phi) T$ is given by the image of $\Phi$ under the precomposition functor $[S : T]$. Since any functor preserves isomorphisms, if $\Phi$ is an isomorphism, so is $S(\Phi) T$. Since the image of an essentially surjective functor is an essentially wide subcategory of its codomain, the second assertion follows from Proposition 1.1.30.  

**Remark 1.1.32.** Proposition 1.1.30 is a special case of Proposition 1.1.31 where $S$ and $T$ are inclusion functors.

**Proposition 1.1.33.** If, in Example 1.1.29(10), $h$, $f$, and $k$ are isomorphisms, so is the $C$-arrow $(h : S) \circ f \circ (T : k)$.

**Proof.** Immediate because any functor preserves isomorphisms.

**Definition 1.1.34.** Given a module $M : X \to A$, the opposite module $M^\sim : A^\sim \to X^\sim$ is defined by the composition

$$
A \times X^\sim, \xrightarrow{\sim} X^\sim \times A, \xrightarrow{M} \text{Set}
$$

where $\sim$ denotes the simple transposition $(a, x) \mapsto (x, a)$; similarly, given a module morphisms $\Phi : M \to N : X \to A$, the opposite module morphism $\Phi^\sim : M^\sim \to N^\sim : A^\sim \to X^\sim$ is defined by the composition

$$
A \times X^\sim, \xrightarrow{\sim} X^\sim \times A, \xrightarrow{\Phi} \text{Set}
$$

**Remark 1.1.35.**

1. The assignments $M \mapsto M^\sim$ and $\Phi \mapsto \Phi^\sim$ yield an isomorphism

$$[X : A] \cong [A^\sim : X^\sim]$$

2. For any module $M$ and any module morphism $\Phi$,

$$\langle M^\sim \rangle^\sim = M \quad \langle \Phi^\sim \rangle^\sim = \Phi$$

3. For any category $C$,

$$\langle C^\sim \rangle = \langle C \rangle^\sim$$

4. That is, the hom of the opposite category is the opposite module of the hom.

(4) For any composite module $S(M) T$,

$$\langle S(M) T \rangle^\sim = T \langle M^\sim \rangle S$$

; that is, the opposite of the composite

$$
E \xrightarrow{S} X \xrightarrow{M} A \xrightarrow{T} D
$$

is given by the composite

$$
D^\sim \xrightarrow{T} A^\sim \xrightarrow{M^\sim} X^\sim \xrightarrow{S} E^\sim
$$
1.2. Cells

Definition 1.2.1. Given a pair of modules \( M : X \to A \) and \( N : Y \to B \), and given a pair of functors \( P : X \to Y \) and \( Q : A \to B \), a module cell (or just a cell) \( \Phi : P/\sim Q : M \to N \), written diagrammatically as

\[
\begin{array}{c}
X \xrightarrow{\Phi} M \xrightarrow{\sim} A \\
\downarrow \downarrow \\
Y \xrightarrow{\sim} N \xrightarrow{\Phi} B
\end{array}
\]

is defined by a module morphism \( \Phi : M \to P/\sim N/\sim Q : X \to A \).

Remark 1.2.2.

(1) The functors \( P \) and \( Q \) are called the left and right components of a cell \( \Phi : P/\sim Q : M \to N \). A cell is sometimes denoted just by \( \Phi : M \to N \), and when this is the case, its left and right components are denoted by \( \Phi_0 : M_0 \to N_0 \) and \( \Phi_1 : M_1 \to N_1 \) respectively.

(2) A cell \( \Phi : P/\sim Q : M \to N \) sends each \( M \)-arrow \( m : x \to a \) to the \( N \)-arrow \( m : x : P \to Q : a \), the image of \( m \) under the function

\[
x \langle M \rangle a \xrightarrow{x(\Phi)a} x \langle P \langle N \rangle Q \rangle a = (x : P) \langle N \rangle (Q : a)
\]

(3) Cells and module morphisms are regarded as special instances of each other. A cell \( \Phi : P/\sim Q : M \to N \) is thought of as a module morphism from \( M \) to the composite module \( P \langle N \rangle Q \). Conversely, a module morphism \( \Phi : M \to N : X \to A \) is expressed by a cell

\[
\begin{array}{c}
X \xrightarrow{\sim} M \xrightarrow{\Phi} A \\
\downarrow \downarrow \\
Y \xrightarrow{\sim} N \xrightarrow{\sim} A
\end{array}
\]

(4) The identity module morphism \( M \to M \) yields the identity cell

\[
\begin{array}{c}
X \xrightarrow{\sim} M \xrightarrow{\sim} A \\
\downarrow \downarrow \\
X \xrightarrow{\sim} M \xrightarrow{\sim} A
\end{array}
\]

(5) Any composition

\[
\begin{array}{c}
X \xrightarrow{P} Y \xrightarrow{\sim} N \xrightarrow{\sim} B \xrightarrow{Q} A
\end{array}
\]

trivially yields a cell

\[
\begin{array}{c}
X \xrightarrow{P} Y \xrightarrow{N} B \xrightarrow{Q} A \\
\downarrow \downarrow \downarrow \\
X \xrightarrow{N} B
\end{array}
\]
Proposition 1.2.3. A cell is compatible with composition of arrows. Specifically, a cell \( \Phi : P \to Q : \mathcal{M} \to \mathcal{N} \) sends a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{m} & A \\
\downarrow{g} & & \downarrow{f} \\
y & \xleftarrow{g \circ m \circ f} & b
\end{array}
\]

in \( \mathcal{M} \) to a commutative diagram

\[
\begin{array}{ccc}
x : P & \xrightarrow{m \cdot \Phi} & Q : a \\
\downarrow{g \cdot p} & & \downarrow{Q \cdot f} \\
y : P & \xrightarrow{(g \circ m \circ f) \cdot \Phi} & Q : b
\end{array}
\]

in \( \mathcal{N} \); that is, the identity

\[(g \circ m \circ f) \cdot \Phi = (g \cdot P) \circ (m \cdot \Phi) \circ (Q \cdot f)\]

holds.

Proof. The commutativity of the second diagram follows from the naturality of \( \Phi \) (see Remark 1.1.20(3)) and noting Example 1.1.29(1). \( \square \)

Definition 1.2.4.

- Let \( \mathcal{M} : X \to * \) and \( \mathcal{N} : Y \to * \) be right modules. Given a functor \( P : X \to Y \), a right module cell \( \Phi : P \to * : \mathcal{M} \to \mathcal{N} \), written diagrammatically as \( X \xrightarrow{M} * \), is defined by a right module morphism \( \Phi : \mathcal{M} \to P \langle \mathcal{N} \rangle : X \to * \).

- Let \( \mathcal{M} : * \to A \) and \( \mathcal{N} : * \to B \) be left modules. Given a functor \( Q : A \to B \), a left module cell \( \Phi : * \to Q : \mathcal{M} \to \mathcal{N} \), written diagrammatically as \( * \xrightarrow{M} A \), is defined by a left module morphism \( \Phi : \mathcal{M} \to \langle \mathcal{N} \rangle Q : * \to A \).

Remark 1.2.5. A right (resp. left) module cell in Definition 1.2.4 is regarded as a special instance of a two-sided module cell in Definition 1.2.1 where \( A \) and \( B \) (resp. \( X \) and \( Y \)) are the terminal category under the identification in Remark 1.1.14(4). Conversely, by Remark 1.1.14(3), a two-sided module cell in Definition 1.2.1 is the same thing as a right module cell \( X \times A \xrightarrow{M} * \) (resp. left module cell \( Y \times B \xleftarrow{N} * \)).

module cell \( * \xrightarrow{M} X \times A \).

\[
\begin{array}{ccc}
* & \xrightarrow{M} & X \times A \\
\downarrow{1} & & \downarrow{P \times Q} \\
* & \xrightarrow{N} & Y \times B
\end{array}
\]
Definition 1.2.6.
- Let \( \mathcal{M} : X \to * \) be a right module and \( \mathcal{N} : Y \to B \) be a two-sided module. Given a functor \( P : X \to Y \) and an object \( b \in \|B\| \), a right conical cell \( \Phi : P \to b : \mathcal{M} \to \mathcal{N} \), written diagrammatically as \( X - \mathcal{M} \to * \), is defined by a right module morphism \( \Phi : \mathcal{M} \to P(\mathcal{N}) b : X \to * \).

\[
\begin{array}{c}
Y - \mathcal{N} \to B \\
| \quad \downarrow \quad \Phi \\
\mathcal{M} \quad \Phi \\
| \quad \downarrow \quad b \\
X - \mathcal{M} \to *
\end{array}
\]

- Let \( \mathcal{M} : * \to A \) be a left module and \( \mathcal{N} : Y \to B \) be a two-sided module. Given an object \( y \in \|Y\| \) and a functor \( Q : A \to B \), a left conical cell \( \Phi : y \sim Q : \mathcal{M} \to \mathcal{N} \), written diagrammatically as \( * - \mathcal{N} \to A \), is defined by a left module morphism \( \Phi : \mathcal{M} \to y(\mathcal{N}) Q : * \to A \).

\[
\begin{array}{c}
\mathcal{N} \quad \Phi \\
\downarrow \quad \downarrow \quad y \\
\mathcal{M} \quad b \\
| \quad \downarrow \quad \downarrow \quad Q \\
Y - \mathcal{N} \to B \\
\mathcal{M}
\end{array}
\]

Remark 1.2.7.
(1) A conical cell in Definition 1.2.6 is regarded as a special instance of a two-sided module cell in Definition 1.2.1 where \( A \) (resp. \( X \)) is the terminal category under the identification in Remark 1.1.14(4).

(2) A right (resp. left) module cell in Definition 1.2.4 is regarded as a special instance of a right (resp. left) conical cell in Definition 1.2.6 where \( B \) (resp. \( Y \)) is the terminal category. Conversely, a right (resp. left) conical cell in Definition 1.2.6 is the same thing as a right module cell \( X - \mathcal{M} \to * \) (resp. left module cell \( * - \mathcal{M} \to A \)).

\[
\begin{array}{c}
Y - \mathcal{N} \to B \\
| \quad \downarrow \quad \Phi \\
\mathcal{M} \quad \Phi \\
| \quad \downarrow \quad b \\
X - \mathcal{M} \to *
\end{array}
\]

Definition 1.2.8. Given a pair of modules \( J : E \to D \) and \( M : X \to A \), the module of cells \( J \to M \),

\[
\langle J, M \rangle : [E, X] \to [D, A]
\]

is defined by

\[
(S) (J, M) (T) = (J) (E : D) (S (M) T)
\]

for \( S \in [E, X] \) and \( T \in [D, A] \), where \( (E : D) \) is the hom of the module category \([E : D]\).

Remark 1.2.9.
(1) For a pair of functors \( S : E \to X \) and \( T : D \to A \), the set \( (S) (J, M) (T) \) consists of all cells \( S \sim T : J \to M \).

(2) Given a cell \( \Theta : S \sim T \) and natural transformations \( \sigma : S' \to S \) and \( \tau : T \to T' \) as in

\[
\begin{array}{c}
E - \mathcal{J} \to D \\
S' \downarrow \quad \Theta \quad \tau \downarrow \tau' \\
X - \mathcal{M} \to A
\end{array}
\]

, their composite in the module \( \langle J, M \rangle \) is the cell

\[
\begin{array}{c}
E - \mathcal{J} \to D \\
S' \downarrow \quad \sigma \circ \Theta \circ \tau \downarrow \tau' \\
X - \mathcal{M} \to A
\end{array}
\]

defined by the module morphism \( \sigma \circ \Theta \circ \tau : J \to S' (M) T' \) given by the composition

\[
\mathcal{J} \quad \Theta \quad S (M) T \quad \sigma (M) \tau \quad S' (M) T'
\]
By Example 1.1.29(3), the cell \(\sigma \circ \Theta \circ \tau\) sends each \(\mathcal{J}\)-arrow \(j: e \leadsto d\) to the \(\mathcal{M}\)-arrow 

\[ j: (\sigma \circ \Theta \circ \tau) = \sigma_e \circ (j \circ \Theta) \circ \tau_d : e \leadsto T' \leadsto d \]

as indicated in

\[ e \overset{j}{\leadsto} d \]

\[ e \circ S \overset{j \circ \Theta}{\leadsto} T \circ d \]

\[ \sigma_e \uparrow \sigma_{T'} \]

\[ e \circ S' \overset{j \circ \Theta}{\leadsto} T' \circ d \]

(3) If \(\mathcal{J}\) is small and \(\mathcal{M}\) is locally small, then the module \(\langle \mathcal{J}, \mathcal{M} \rangle\) is locally small.

**Proposition 1.2.10.** Given a module \(\mathcal{J}\) and a composite module \(P \langle \mathcal{N} \rangle Q\) as in

\[
\begin{array}{ccl}
E & \overset{\mathcal{J}}{\leadsto} & D \\
X & \overset{P \langle \mathcal{N} \rangle Q}{\leadsto} & A \\
\downarrow_{P} & & \downarrow_{Q} \\
Y & \overset{\mathcal{N}}{\leadsto} & B
\end{array}
\]

, the identity

\[
\begin{array}{ccl}
[E, X] & \overset{\langle \mathcal{J}, P \langle \mathcal{N} \rangle Q \rangle}{\leadsto} & [D, A] \\
\downarrow_{[E, P]} & & \downarrow_{[D, Q]} \\
[E, Y] & \overset{\langle \mathcal{J}, \mathcal{N} \rangle}{\leadsto} & [D, B]
\end{array}
\]

, i.e.

\[ \langle \mathcal{J}, P \langle \mathcal{N} \rangle Q \rangle = [E, P] \langle \mathcal{J}, \mathcal{N} \rangle [D, Q] \]

, holds.

**Proof.** For any \(S \in [E, X]\) and \(T \in [D, A]\),

\[
S \langle \mathcal{J}, P \langle \mathcal{N} \rangle Q \rangle T = (\mathcal{J}) \langle E : D \rangle (S \langle P \langle \mathcal{N} \rangle Q \rangle T)
\]

\[= (\mathcal{J}) \langle E : D \rangle ([S \circ P] \langle \mathcal{N} \rangle [Q \circ T])\]

\[=(S \circ P) \langle \mathcal{J}, \mathcal{N} \rangle (Q \circ T)\]

\[=(S \circ [E, P]) \langle \mathcal{J}, \mathcal{N} \rangle ([D, Q] \circ T)\]

\[=(S) \langle [E, P] \langle \mathcal{J}, \mathcal{N} \rangle [D, Q] \rangle (T)\]

\[\square\]

**Remark 1.2.11.** The cell \(\langle \mathcal{J}, P \langle \mathcal{N} \rangle Q \rangle \overset{1}{\leadsto} \langle \mathcal{J}, \mathcal{N} \rangle\) above sends each cell

\[
\begin{array}{ccl}
E & \overset{\mathcal{J}}{\leadsto} & D \\
\downarrow_{S} & \Theta & \downarrow_{T} \\
X & \overset{P \langle \mathcal{N} \rangle Q}{\leadsto} & A
\end{array}
\]

to the cell

\[
\begin{array}{ccl}
E & \overset{\mathcal{J}}{\leadsto} & D \\
\downarrow_{S \circ P} & \Theta & \downarrow_{Q \circ T} \\
Y & \overset{\mathcal{N}}{\leadsto} & B
\end{array}
\]

defined by the same module morphism.
Definition 1.2.12. If $\Theta : J \to M$ is a cell and $\Phi : M \to N$ is a module morphism as in

\[
\begin{array}{ccc}
E & \xrightarrow{J} & D \\
\downarrow^s & & \downarrow^T \\
X & \xrightarrow{M} & A
\end{array}
\]

, then their composite $\Theta \circ \Phi = \Phi \circ \Theta$ is the cell

\[
\begin{array}{ccc}
E & \xrightarrow{J} & D \\
\downarrow^s & & \downarrow^T \\
X & \xrightarrow{N} & A
\end{array}
\]

defined by the module morphism $\Theta \circ \Phi : J \to S\langle N \rangle T$ given by the composition

\[
J \xrightarrow{\Theta} S\langle M \rangle T \xrightarrow{S(\Phi)T} S\langle N \rangle T
\]

.

Remark 1.2.13. See Example 1.1.29(2) for the module morphism $S(\Phi) T$. The cell $\Theta \circ \Phi$ sends each $J$-arrow $j : e \rightsquigarrow d$ to the $N$-arrow

\[
j : (\Theta \circ \Phi) = j : \Theta \circ \Phi : e \rightsquigarrow S \rightsquigarrow T \rightsquigarrow d
\]

, the image of $j$ under the composite function

\[
e(J) d \xrightarrow{e(\Theta)d} e(S\langle M \rangle T) d = (e \circ S) \langle M \rangle (T \circ d) \xrightarrow{(e \circ S)(\Phi)(T \circ d)} (e \circ S) \langle N \rangle (T \circ d)
\]

.

Definition 1.2.14. Given a module $J : E \to D$ and a module morphism $\Phi : M \to N : X \to A$, the module morphism

\[
\langle J, \Phi \rangle : \langle J, M \rangle \to \langle J, N \rangle : [E, X] \to [D, A]
\]

, "postcomposition with $\Phi$", is defined by

\[
(S) \langle J, \Phi \rangle (T) = (J) \langle E : D \rangle (S(\Phi) T)
\]

for each pair of functors $S : E \to X$ and $T : D \to A$.

Remark 1.2.15.

(1) The module morphism $\langle J, \Phi \rangle$ maps each cell $\Theta : S \rightsquigarrow T : J \to M$ to the cell $\Theta \circ \Phi : S \rightsquigarrow T : J \to N$ defined in Definition 1.2.12.

(2) The assignment $\Phi \mapsto \langle J, \Phi \rangle$ is functorial; indeed, the functor

\[
\langle J, - \rangle : [X : A] \to [[E, X] : [D, A]]
\]

is defined by

\[
S\langle J, M \rangle T = (J) \langle E : D \rangle (S \langle M \rangle T)
\]

for $S \in [E, X], T \in [D, A]$, and $M \in [X : A]$.

Note. By Remark 1.2.2(3), the following definition is regarded as a special case of Definition 1.2.12, and vice versa.
Definition 1.2.16. Given a pair of cells as in
\[
\begin{align*}
E &\rightarrow^J D \\
S &\rightarrow^P T \\
X &\rightarrow^M A \\
P &\rightarrow^\Phi Q \\
Y &\rightarrow^N B
\end{align*}
\]
their composite \( \Theta \circ \Phi = \Phi \circ \Theta \) is the cell
\[
\begin{align*}
E &\rightarrow^J D \\
S &\rightarrow^P T \\
Y &\rightarrow^N B
\end{align*}
\]
defined by the module morphism \( \Theta \circ \Phi : J \rightarrow [S \circ P] (N) [Q \circ T] \) given by the composition
\[
J \xrightarrow{\Theta} S(M) T \xrightarrow{S(\Phi)T} S(P(N) Q) T = [S \circ P] (N) [Q \circ T]
\]

Remark 1.2.17. The cell \( \Theta \circ \Phi \) sends each \( J \)-arrow \( j : e \sim d \) to the \( N \)-arrow
\[
j : (\Theta \circ \Phi) = j : \Theta : \Phi : e : S : P \sim Q : T : d
\]
, the image of \( j \) under the composite function
\[
\begin{align*}
e(J) d \\
\downarrow \quad e(\Theta) d \\
e(S(M) T) d = (e : S)(M) (T : d) \\
\downarrow (e : S)(\Phi)(T : d) \\
(e : S)(P(N) Q) (T : d) = (e : S : P)(N) (Q : T : d)
\end{align*}
\]

Proposition 1.2.18. Modules and cells among them form a category with the composition given in Definition 1.2.16 and the identities given in Remark 1.2.2(4).

Proof. The only non-trivial part is the verification of the associativity of the composition. Consider cells as in
\[
\begin{align*}
X &\rightarrow^M A \\
P &\rightarrow^\Phi Q \\
X' &\rightarrow^{M'} A' \\
P' &\rightarrow^{\Phi'} Q' \\
X'' &\rightarrow^{M''} A'' \\
P'' &\rightarrow^{\Phi''} Q'' \\
X'''' &\rightarrow^{M''''} A''''
\end{align*}
\]
The two cell compositions \((\Phi \circ \Phi') \circ \Phi''\) and \(\Phi \circ (\Phi' \circ \Phi'')\) are given by the module morphisms \((\Phi \circ P)(\Phi') Q\) \([P \circ P'](\Phi'')(Q' \circ Q)\) and \(\Phi \circ P(\Phi' \circ P')(\Phi'')(Q'')\) respectively. But by the functoriality (see Remark 1.1.27(1)) and associativity (see Proposition 1.1.28) of the composition, we have
\[
\begin{align*}
(\Phi \circ P)(\Phi') Q \circ [P \circ P'](\Phi'') (Q' \circ Q) &= \Phi \circ P(\Phi') Q \circ P(\Phi'(\Phi'')(Q')) Q \\
&= \Phi \circ P(\Phi' \circ P'(\Phi'')) Q' Q
\end{align*}
\]
Remark 1.2.19.

(1) Given a universe $\mathfrak{U}$, $\text{MOD}_\mathfrak{U}$ denotes the category consisting of all locally $\mathfrak{U}$-small modules and all cells among them. By $\text{MOD}$ we mean $\text{MOD}_\mathfrak{U}$ for $\mathfrak{U}$ chosen implicitly.

(2) Given a pair of categories $\mathbf{X}$ and $\mathbf{A}$, there is a canonical embedding $[\mathbf{X}:\mathbf{A}] \to \text{MOD}$, identical on objects, given by the obvious arrow function (see Remark 1.2.2(3)). The embedding is not, in general, full.

**Proposition 1.2.20.** A cell $\Phi : P \rightsquigarrow Q : \mathcal{M} \to \mathcal{N}$ is an isomorphism in the category $\text{MOD}$ if and only if the functors $P$ and $Q$ are iso and the module morphism $\Phi : \mathcal{M} \to P(\mathcal{N})Q$ is iso.

**Proof.** If $\Phi : P \rightsquigarrow Q : \mathcal{M} \to \mathcal{N}$ has an inverse $\Psi : S \rightsquigarrow T : \mathcal{N} \to \mathcal{M}$, then $P$ (resp. $Q$) and $S$ (resp. $T$) are inverse to each other and $\Phi : \mathcal{M} \to P(\mathcal{N})Q$ and $P(\Psi)Q : P(\mathcal{N})Q \to \mathcal{M}$ are inverse to each other. Conversely, if $P$, $Q$, and $\Phi : \mathcal{M} \to P(\mathcal{N})Q$ are iso, the inverse of $\Phi : P \rightsquigarrow Q : \mathcal{M} \to \mathcal{N}$ is given by the cell $\Psi : S \rightsquigarrow T : \mathcal{N} \to \mathcal{M}$ with $S = P^{-1}$, $T = Q^{-1}$, and the module morphism $\Psi : \mathcal{N} \to S(\mathcal{M})T$ defined by $\Psi = [P^{-1}](\Phi^{-1})[Q^{-1}]$.

**Remark 1.2.22.** In Proposition 1.2.29 we will see the relation between the notion of fully faithfulness for cells and that for functors.

**Note.** The postcomposition in Definition 1.2.14 and the identity in Proposition 1.2.10 allow the following definition.

**Definition 1.2.23.** Given a module $\mathcal{J} : \mathbf{E} \to \mathbf{D}$ and a cell

\[
\begin{array}{ccc}
X \xrightarrow{M} & A \\
\downarrow{P} & \Phi & \downarrow{Q} \\
Y \xrightarrow{N} & B \\
\end{array}
\]

, the cell

\[
\begin{array}{ccc}
[E,X] & \xrightarrow{(\mathcal{J},M)} & [D,A] \\
\downarrow{(E,P)} & \downarrow{(\mathcal{J},\Phi)} & \downarrow{[D,Q]} \\
[E,Y] & \xrightarrow{(\mathcal{J},N)} & [D,B] \\
\end{array}
\]

, “postcomposition with $\Phi$”, is defined by the module morphism

\[
(\mathcal{J},M) \xrightarrow{(\mathcal{J},\Phi)} (\mathcal{J},P(\mathcal{N})Q) = [E,P](\mathcal{J},\mathcal{N})[D,Q]
\]

, postcomposition with $\Phi : \mathcal{M} \to P(\mathcal{N})Q$.

**Remark 1.2.24.** The cell $\langle \mathcal{J}, \Phi \rangle$ sends each cell $\Theta : S \rightsquigarrow T : \mathcal{J} \to \mathcal{M}$ to the cell $\Theta \circ \Phi : S \circ P \rightsquigarrow Q \circ T : \mathcal{J} \to \mathcal{N}$ defined in Definition 1.2.16.

**Proposition 1.2.25.** The assignment $\Phi \mapsto \langle \mathcal{J}, \Phi \rangle$ of the postcomposition cell is functorial.

**Proof.** Clearly, the assignment $\Phi \mapsto \langle \mathcal{J}, \Phi \rangle$ preserves the identities. To verify that it preserves the composition, let $\Phi$ and $\Psi$ be a composable pair of cells and consider the cells $\langle \mathcal{J}, \Phi \rangle$, $\langle \mathcal{J}, \Psi \rangle$, and
\( \langle \mathcal{J}, \Phi \ast \Psi \rangle \) depicted in

\[
\begin{array}{ccc}
X \rightarrow^{\mathcal{M}} A & & [\mathbf{E}, \mathbf{X}] \rightarrow^{\langle \mathcal{J}, \mathcal{M} \rangle} [\mathbf{D}, \mathbf{A}] \\
\mathbf{P} \downarrow \Phi \downarrow \mathbf{Q} & & [\mathbf{E}, \mathbf{P}] \rightarrow^{\langle \mathcal{J}, \Phi \rangle} [\mathbf{D}, \mathbf{Q}]
\end{array}
\]

\[
\begin{array}{ccc}
Y \rightarrow^{\mathcal{N}} B & & [\mathbf{E}, \mathbf{Y}] \rightarrow^{\langle \mathcal{J}, \mathcal{N} \rangle} [\mathbf{D}, \mathbf{B}] \\
\mathbf{P}' \downarrow \Psi \downarrow \mathbf{Q}' & & [\mathbf{E}, \mathbf{P}'] \rightarrow^{\langle \mathcal{J}, \Psi \rangle} [\mathbf{D}, \mathbf{Q}']
\end{array}
\]

\[
\begin{array}{ccc}
X \rightarrow^{\mathcal{M}} A & & [\mathbf{E}, \mathbf{X}] \rightarrow^{\langle \mathcal{J}, \mathcal{M} \rangle} [\mathbf{D}, \mathbf{A}] \\
\mathbf{P} \uparrow \Phi \downarrow \mathbf{Q} & & [\mathbf{E}, \mathbf{P}] \rightarrow^{\langle \mathcal{J}, \Phi \rangle} [\mathbf{D}, \mathbf{Q}]
\end{array}
\]

\[
\begin{array}{ccc}
Z \rightarrow^{\mathcal{L}} C & & [\mathbf{E}, \mathbf{Z}] \rightarrow^{\langle \mathcal{J}, \mathcal{L} \rangle} [\mathbf{D}, \mathbf{C}]
\end{array}
\]

We need to verify that the composition of the cells \( \langle \mathcal{J}, \Phi \rangle \) and \( \langle \mathcal{J}, \Psi \rangle \) yields the cell \( \langle \mathcal{J}, \Phi \ast \Psi \rangle \).

First note that \( [\mathbf{E}, \mathbf{P} \circ \mathbf{P}'] = [\mathbf{E}, \mathbf{P}] \ast [\mathbf{E}, \mathbf{P}'] \) and \( [\mathbf{D}, \mathbf{Q} \circ \mathbf{Q}'] = [\mathbf{D}, \mathbf{Q}] \ast [\mathbf{D}, \mathbf{Q}'] \) by the functoriality of the operations \( [\mathbf{E}, -] \) and \( [\mathbf{D}, -] \). The cell \( \langle \mathcal{J}, \Phi \ast \Psi \rangle \) is defined by the module morphism \( \langle \mathcal{J}, \Phi \ast \Psi \rangle : [\mathbf{E}, \mathbf{P}] \langle \mathcal{J}, \Psi \rangle [\mathbf{D}, \mathbf{Q}] \) and the cell \( \langle \mathcal{J}, \Phi \ast \Psi \rangle \) is defined by the module morphism \( \langle \mathcal{J}, \Phi \ast \Psi \rangle : [\mathbf{E}, \mathbf{P}] \langle \mathcal{J}, - \rangle [\mathbf{D}, \mathbf{Q}] \).

But by the functoriality of the operation \( \langle \mathcal{J}, - \rangle \) (see Remark 1.2.15(2)) and Proposition 1.2.10,

\[
\langle \mathcal{J}, \Phi \ast \Psi \rangle \circ \langle \mathcal{J}, - \rangle = \langle \mathcal{J}, \Phi \rangle \circ \langle \mathcal{J}, - \rangle = \langle \mathcal{J}, \Phi \rangle \circ \langle \mathcal{J}, \Psi \rangle = \langle \mathcal{J}, \Phi \ast \Psi \rangle \circ \langle \mathcal{J}, - \rangle 
\]

\[ \square \]

**Remark 1.2.26.** Given a small module \( \mathcal{J} : \mathbf{E} \rightarrow \mathbf{D} \), the functor

\[
\langle \mathcal{J}, - \rangle : \text{MOD} \rightarrow \text{MOD}
\]

is defined by the object function \( \mathcal{M} \mapsto \langle \mathcal{J}, \mathcal{M} \rangle \) and the arrow function \( \Phi \mapsto \langle \mathcal{J}, \Phi \rangle \), extending the functor \( \langle \mathcal{J}, - \rangle \) in Remark 1.2.15(2) as shown in

\[
\begin{array}{ccc}
[X : \mathbf{A}] & \rightarrow^{\langle \mathcal{J}, - \rangle} & [[\mathbf{E}, \mathbf{X}] : [\mathbf{D}, \mathbf{A}]] \\
\downarrow^{\langle \mathcal{J}, - \rangle} & & \downarrow^{\langle \mathcal{J}, - \rangle}
\end{array}
\]

\[
\begin{array}{ccc}
\text{MOD} & \rightarrow^{\langle \mathcal{J}, - \rangle} & \text{MOD}
\end{array}
\]

where \( \rightarrow \) denotes the canonical embedding in Remark 1.2.19(2).

**Definition 1.2.27.** The hom of a functor \( \mathbf{H} : \mathbf{C} \rightarrow \mathbf{B} \) is the cell

\[
\begin{array}{ccc}
\mathbf{C} \rightarrow^{\langle \mathcal{C} \rangle} \mathbf{C} \\
\mathbf{H} \downarrow^{\langle \mathbf{H} \rangle} \downarrow^{\mathbf{H}}
\end{array}
\]

\[
\begin{array}{ccc}
\mathbf{B} \rightarrow^{\langle \mathbf{B} \rangle} \mathbf{B}
\end{array}
\]

given by the arrow function of \( \mathbf{H} \); that is, for each pair of objects \( \mathbf{x}, \mathbf{a} \in \mathbb{C} \), the function

\[
\mathbf{a} : \mathbf{H} \cdot \mathbf{b} : \mathbf{a} \cdot \mathbf{C} \cdot \mathbf{b} \rightarrow (\mathbf{a} : \mathbf{H}) \cdot (\mathbf{b} : \mathbf{B})
\]

is given by

\[
\mathbf{H}_{\mathbf{a}, \mathbf{b}} : \text{hom}_{\mathbf{C}}(\mathbf{a}, \mathbf{b}) \rightarrow \text{hom}_{\mathbf{B}}(\mathbf{H} \cdot \mathbf{a}, \mathbf{H} \cdot \mathbf{b}) \quad f \mapsto \mathbf{H} \cdot f
\]

.
Remark 1.2.28.
(1) The cell given by the hom of a functor is called a hom cell.
(2) The naturality of the module morphism \( \hom(H) : (C) \to H(B) \) follows from the functoriality of \( H \).
(3) Hereafter, given a functor \( H \), each component of the arrow function of \( H \) is written as \( a(H)b \).

Proposition 1.2.29. A functor \( H : C \to B \) is iso (resp. fully faithful) if and only if the hom cell \( \hom(H) : (C) \to (B) \) is iso (resp. fully faithful) (see Definition 1.2.21).

Proof. Immediate from the definitions.

Theorem 1.2.30. The assignment of the hom defined in Definition 1.1.21 and Definition 1.2.27 embeds \( \text{CAT} \) in \( \text{MOD} \). Specifically, the assignment \( C \mapsto (C) \) forms a faithful functor \( (-) : \text{CAT} \to \text{MOD} \), injective on objects.

Proof. The verification of the functoriality is straightforward. The faithfulness and the injectivity on objects are evident.

Definition 1.2.31. Given a cell and commutative squares of functors as in

\[
\begin{array}{cccc}
X' & \xrightarrow{R} & X & \xrightarrow{M} & A & \xleftarrow{F} & A' \\
\downarrow{P'} & & \downarrow{P} & & \downarrow{\Phi} & & \downarrow{Q} & & \downarrow{Q'} \\
Y' & \xrightarrow{S} & Y & \xrightarrow{N} & B & \xleftarrow{G} & B'
\end{array}
\]

, their pasting composite

\[
\begin{array}{cccc}
X' & \xrightarrow{R(M)F} & A' \\
\downarrow{P'} & & \downarrow{R(\Phi)F} & & \downarrow{Q'} \\
Y' & \xrightarrow{S(N)G} & B'
\end{array}
\]

is defined by module morphism

\[
R(\Phi)F : R(M)F \to R(P(N)Q)F = P'(S(N)G)Q'
\]
given by the composition

\[
\begin{array}{cccc}
X' & \xrightarrow{R} & X & \xrightarrow{M} & A & \xleftarrow{F} & A' \\
\downarrow{1} & & \downarrow{P} & & \downarrow{\Phi} & & \downarrow{Q} & & \downarrow{1} \\
X' & \xrightarrow{S} & Y & \xrightarrow{N} & B & \xleftarrow{G} & A'
\end{array}
\]

Remark 1.2.32. A pasting composition

\[
\begin{array}{cccc}
X' & \xrightarrow{R(M)F} & A' \\
\downarrow{1} & & \downarrow{R(\Phi)F} & & \downarrow{1} \\
X' & \xrightarrow{S(N)G} & A'
\end{array}
\]

with both ends being identities, yields a cell

\[
\begin{array}{cccc}
X' & \xrightarrow{R(M)F} & A' \\
\downarrow{1} & & \downarrow{R(\Phi)F} & & \downarrow{1} \\
X' & \xrightarrow{S(N)G} & A'
\end{array}
\]

, i.e. a module morphism \( R(\Phi)F : R(M)F \to S(N)G : X' \to A' \).
Proposition 1.2.33. In Definition 1.2.31, if $\Phi$ is fully faithful, so is the cell $R(\Phi)F$.

Proof. Immediate from Proposition 1.1.31.

Proposition 1.2.34. Cells and commutative squares of functors as in

\[
\begin{array}{cc}
X' & X - \overset{M}{\longrightarrow} A \\
\downarrow^{P'} & \downarrow^{P} \\
Y' & Y - \overset{N}{\longrightarrow} B \\
\downarrow^{K'} & \downarrow^{K} \\
Z' & Z - \overset{L}{\longrightarrow} C
\end{array}
\]

yield the same cell

\[R(\Phi \circ \Psi)F = R(\Phi)F \circ S(\Psi)G : P' \circ K' \to L' \circ Q' : R(M)F \to T(L)H\]

irrespective of the order of the horizontal and vertical compositions.

Proof. The horizontal composition followed by the vertical composition yields the cell $R(\Phi)F \circ S(\Psi)G$ as shown in

\[
\begin{array}{cc}
X' & X - \overset{R(M)F}{\longrightarrow} A' \\
\downarrow^{P'} & \downarrow^{P \circ K'} \\
Y' & Y - \overset{S(\Psi)G}{\longrightarrow} B' \\
\downarrow^{K} & \downarrow^{T(L)H} \\
Z' & Z - \overset{L'}{\longrightarrow} C'
\end{array}
\]

and the vertical composition followed by the horizontal composition yields the cell $R(\Phi \circ \Psi)F$ as shown in

\[
\begin{array}{cc}
X' & X - \overset{R(M)}{\longrightarrow} A' \\
\downarrow^{P'} & \downarrow^{P \circ K'} \\
Y' & Y - \overset{R(\Phi)F \circ S(\Psi)G}{\longrightarrow} B' \\
\downarrow^{K} & \downarrow^{T(L)H} \\
Z' & Z - \overset{L'}{\longrightarrow} C'
\end{array}
\]

The cell $R(\Phi)F \circ S(\Psi)G$ is defined by the module morphism $R(\Phi)F \circ P'(S(\Psi)G)Q'$ and the cell $R(\Phi \circ \Psi)F$ is defined by the module morphism $R(\Phi \circ P(\Psi)Q)F$.

\[
R(\Phi \circ P(\Psi)Q)F = R(\Phi)F \circ R(P(\Psi)Q)F \\
= R(\Phi)F \circ [R \circ P]Q[Q \circ F] \\
= R(\Phi)F \circ [P' \circ S]Q[G \circ Q'] \\
= R(\Phi)F \circ P'(S(\Psi)G)Q'
\]

\[
\square
\]

1.3. Cell morphisms

Note. In the following, the left and right components of a cell $\Phi : M \to N$ are denoted by $\Phi_0 : M_0 \to N_0$ and $\Phi_1 : M_1 \to N_1$ respectively (see Remark 1.2.2(1)). (This is a common practice when discussing cell morphisms.)

Definition 1.3.1. Given a parallel pair of cells $\Phi; \Psi : M \to N$, a morphism from $\Phi$ to $\Psi$, written $\tau : \Phi \to \Psi : M \to N$, is defined by a pair of natural transformations

\[\tau_0 : \Phi_0 \to \Psi_0 \quad \tau_1 : \Phi_1 \to \Psi_1\]
such that the square
\[
\begin{array}{ccc}
x: \Phi_0 & \xymatrix{\ar[r]^m\ar[d]^\tau_0} & \Phi_1 \cdot a \\
x: \Psi_0 & \xymatrix{\ar[r]^m\ar[d]^\tau_1} & \Psi_1 \cdot a
\end{array}
\]
commutes for every \( \mathcal{M} \)-arrow \( m: x \leadsto a \).

\textbf{Remark 1.3.2.}

(1) The natural transformations \( \tau_0 \) and \( \tau_1 \) are called the left and right components of a cell morphism \( \tau \).

(2) The identity morphism of a cell \( \Phi: \mathcal{M} \to \mathcal{N} \) is given by the pair of identity natural transformations; that is,
\[
1_{\Phi} := (1_{\Phi_0}, 1_{\Phi_1})
\]

(3) The composition of two cell morphisms \( \tau: \Phi \to \Psi: \mathcal{M} \to \mathcal{N} \) and \( \sigma: \Psi \to \Omega: \mathcal{M} \to \mathcal{N} \) is given componentwise; that is
\[
\tau \circ \sigma := (\tau_0 \circ \sigma_0, \tau_1 \circ \sigma_1)
\]

(4) Given a pair of modules \( \mathcal{M} \) and \( \mathcal{N} \), all cells \( \mathcal{M} \to \mathcal{N} \) and morphisms among them define the cell category \([\mathcal{M}: \mathcal{N}]\) with the identities and the composition defined above.

(5) If \( \mathcal{M} \) is small and \( \mathcal{N} \) is locally small, then the category \([\mathcal{M}: \mathcal{N}]\) is locally small.

\textbf{Definition 1.3.3.} A cell morphism \( \tau: \Phi \to \Psi: \mathcal{M} \to \mathcal{N} \) is called iso (or an isomorphism) if it is invertible in the category \([\mathcal{M}: \mathcal{N}]\).

\textbf{Proposition 1.3.4.} A cell morphism \( \tau: \Phi \to \Psi: \mathcal{M} \to \mathcal{N} \) is iso if and only if its components \( \tau_0 \) and \( \tau_1 \) are natural isomorphisms.

\textbf{Proof.} Immediate from the definitions.

\textbf{Definition 1.3.5.} The hom of a natural transformation \( \tau: \Phi \to \Psi: \mathcal{M} \to \mathcal{N} \) is the cell morphism \( \langle \tau \rangle: \langle \Phi \rangle \to \langle \Psi \rangle: \langle \mathcal{M} \rangle \to \langle \mathcal{N} \rangle \) given by the pair \( \langle \tau, \tau \rangle \).

\textbf{Remark 1.3.6.} The cell morphism given by the hom of a natural transformation is called a hom cell morphism.

\textbf{Theorem 1.3.7.} Given a pair of categories \( \mathcal{C} \) and \( \mathcal{B} \), the assignment of the hom defined in Definition 1.2.27 and Definition 1.3.5 embeds the functor category \([\mathcal{C}, \mathcal{B}]\) in the cell category \([\langle \mathcal{C} \rangle: \langle \mathcal{B} \rangle]\). Specifically, the assignment \( H \mapsto \langle H \rangle \) forms a faithful functor \( \langle - \rangle: [\mathcal{C}, \mathcal{B}] \to [\langle \mathcal{C} \rangle: \langle \mathcal{B} \rangle] \), injective on objects. Moreover, the functor reflects isomorphisms.

\textbf{Proof.} The verification of the functoriality is straightforward. The faithfulness and the injectivity on objects are evident. The last assertion is immediate from Definition 1.3.5 and Proposition 1.3.4.

\textbf{Definition 1.3.8.} Given a module \( \mathcal{J} \) and a cell morphisms \( \tau: \Phi \to \Psi: \mathcal{M} \to \mathcal{N} \), the cell morphism \( \langle \mathcal{J}, \tau \rangle: \langle \mathcal{J}, \Phi \rangle \to \langle \mathcal{J}, \Psi \rangle: \langle \mathcal{J}, \mathcal{M} \rangle \to \langle \mathcal{J}, \mathcal{N} \rangle \), “postcomposition with \( \tau \)”, is defined by the pair of postcomposition natural transformations
\[
[\mathcal{J}_0, \tau_0]: [\mathcal{J}_0, \Phi_0] \to [\mathcal{J}_0, \Psi_0] \quad [\mathcal{J}_1, \tau_1]: [\mathcal{J}_1, \Phi_1] \to [\mathcal{J}_1, \Psi_1]
\]
(see Preliminary 9).
Remark 1.3.9.

(1) Given a cell $\Theta : \mathcal{J} \to \mathcal{M}$, the commutativity of
\[
\begin{align*}
\Theta_0 : \mathcal{J}_0 \cong \mathcal{J}_1, \\
\Theta_0 : [\mathcal{J}_0, \Phi_0] \cong [\mathcal{J}_1, \Phi_1],
\end{align*}
\]
\[\xymatrix{\Theta_0 : [\mathcal{J}_0, \Phi_0] \ar[r]_{\Theta : (\mathcal{J}, \Phi)} & [\mathcal{J}_1, \Phi_1] \ar[r]_{\Theta_1} & \mathcal{M}, \}
\]

, i.e.
\[
\begin{align*}
\Theta_0 \circ \Phi_0 & \cong \Phi_1 \circ \Theta_1, \\
\Theta_0 \circ \tau_0 & \cong \tau_1 \circ \Theta_1, \\
\Theta_0 \circ \Psi_0 & \cong \Psi_1 \circ \Theta_1,
\end{align*}
\]
follows from the commutativity of
\[
\begin{align*}
e : \Theta_0 : \Phi_0 \cong \Phi_1 : \Theta_1, \\
e : \Theta_0 : \tau_0 \cong \tau_1 : \Theta_1, \\
e : \Theta_0 : \Psi_0 \cong \Psi_1 : \Theta_1,
\end{align*}
\]
for each $\mathcal{J}$-arrow $m : e \to d$.

(2) The assignment $\tau \mapsto \langle \mathcal{J}, \tau \rangle$ defines the functor
\[
\langle \mathcal{J}, - \rangle : [\mathcal{M} : \mathcal{N}] \to [[\mathcal{J}, \mathcal{M}] : \langle \mathcal{J}, \mathcal{N} \rangle],
\]
; indeed, by the definition of the cell morphism $\langle \mathcal{J}, \tau \rangle$, the functoriality of $\langle \mathcal{J}, - \rangle$ is reduced to that of $[\mathcal{J}_i, -] : [\mathcal{M}_i, \mathcal{N}_i] \to [[\mathcal{J}_i, \mathcal{M}_i] : [\mathcal{J}_i, \mathcal{N}_i]]$ for $i = 0, 1$. 

2. Slicing and Action

2.1. Slicing of modules

*Notation* 2.1.1. Let $X$ and $A$ be categories.

- The right exponential transposition

$$[X \times A, \text{Set}] \xrightarrow{\sim} [A, [X, \text{Set}]]$$

(see Preliminary 11) is denoted by

$$[X : A] \xrightarrow{\sim} [A, [X :]]$$

: the right exponential transpose of a module $\mathcal{M} : X \to A$ is a covariant functor

$$[\mathcal{M} :] : A \to [X :]$$

and the right exponential transpose of a module morphism $\Phi : \mathcal{M} \to \mathcal{N} : X \to A$ is a natural transformation

$$[\Phi :] : [\mathcal{M} :] \to [\mathcal{N} :] : A \to [X :]$$

. The value of $\mathcal{M} : a \in A$ is written $\mathcal{M} a$, $(\mathcal{M}) (a)$, or just $(\mathcal{M}) a$. Similarly, the component of $\Phi : a \in \|A\|$ is written $\Phi a$ or $(\Phi) a$.

- The left exponential transposition

$$[X \times A, \text{Set}] \xrightarrow{\sim} [X, [A, \text{Set}]]$$

(see Preliminary 11) is denoted by

$$[X : A] \xrightarrow{\sim} [X, [:A]]$$

or

$$[X : A] \xrightarrow{\sim} [X, [:A]^{-1}]$$

: the left exponential transpose of a module $\mathcal{M} : X \to A$ is a contravariant functor

$$[\mathcal{M}] : X \to [:A]^{-1}$$

and the left exponential transpose of a module morphism $\Phi : \mathcal{M} \to \mathcal{N} : X \to A$ is a natural transformation

$$[\Phi :] : [\mathcal{N}] \to [\mathcal{M}] : X \to [:A]^{-1}$$

. The value of $\mathcal{M} : x \in X$ is written $x \mathcal{M}$, $(\mathcal{M}) (x)$, or just $x (\mathcal{M})$. Similarly, the component of $\Phi$ at $x \in \|X\|$ is written $x \Phi$ or $x (\Phi)$. 

2.1. Slicing of modules

Remark 2.1.2.

(1) Given a module $\mathcal{M} : X \to A$,

\begin{itemize}
  \item the right exponential transpose $[\mathcal{M},.] : A \to [X :]$ sends each object $a \in \|A\|$ to the right slice of $\mathcal{M}$ at $a$, i.e. the right module
  \[
  (\mathcal{M}) a : X \to *
  \]
  given by
  \[
  x ((\mathcal{M}) a) = x (\mathcal{M}) a
  \]
  for each $x \in X$, and sends each $A$-arrow $f : s \to t$ to the right module morphism
  \[
  (\mathcal{M}) f : (\mathcal{M}) s \to (\mathcal{M}) t : X \to *
  \]
  which maps each $\mathcal{M}$-arrow $m : x \leadsto s$ to the $\mathcal{M}$-arrow $m \circ f : x \leadsto t$ as indicated in
  \[
  \xymatrix{
  x \ar[r]^m & s \\
  m : (\mathcal{M}) & t 
  }
  \]

\end{itemize}

\begin{itemize}
  \item the left exponential transpose $[.,\mathcal{M}] : X \to [::A]$ sends each object $x \in \|X\|$ to the left slice of $\mathcal{M}$ at $x$, i.e. the left module
  \[
  x (\mathcal{M}) : * \to A
  \]
  given by
  \[
  (x (\mathcal{M})) a = x (\mathcal{M}) a
  \]
  for each $a \in A$, and sends each $X$-arrow $f : s \to t$ to the left module morphism
  \[
  f (\mathcal{M}) : t (\mathcal{M}) \to s (\mathcal{M}) : * \to A
  \]
  which maps each $\mathcal{M}$-arrow $m : t \leadsto a$ to the $\mathcal{M}$-arrow $f \circ m : s \leadsto a$ as indicated in
  \[
  \xymatrix{
  t \ar[r]^m & a \\
  s & f (\mathcal{M}) : m
  }
  \]

\end{itemize}

(2) Given a module morphism $\Phi : \mathcal{M} \to \mathcal{N} : X \to A$,

\begin{itemize}
  \item the component of the right exponential transpose $[\Phi,.] : [\mathcal{M},.] \to [\mathcal{N},.] : A \to [X :]$ at $a \in \|A\|$ gives the right slice of $\Phi$ at $a$, i.e. the right module morphism
  \[
  (\Phi) a : (\mathcal{M}) a \to (\mathcal{N}) a : X \to *
  \]
  given by
  \[
  x ((\Phi) a) = x (\Phi) a
  \]
  for each $x \in \|X\|$.

  \item the component of the left exponential transpose $\langle \kappa, \Phi \rangle : \langle \kappa, \mathcal{N} \rangle \to \langle \kappa, \mathcal{M} \rangle : X \to [::A]$ at $x \in \|X\|$ gives the left slice of $\Phi$ at $x$, i.e. the left module morphism
  \[
  x (\Phi) : x (\mathcal{M}) \to x (\mathcal{N}) : * \to A
  \]
  given by
  \[
  (x (\Phi)) a = x (\Phi) a
  \]
  for each $a \in \|A\|$.
\end{itemize}
(3) Given a module \( \mathcal{M} : X \to A \),
- the evaluation of the right exponential transpose \( \mathcal{M}^\land \) at \( a \in A \) is identified with the composition
  \[
  X - \xrightarrow{\mathcal{M}^\land} A - \xrightarrow{a} *
  \]
  (cf. Example 1.1.29(5)).
- the evaluation of the left exponential transpose \( \mathfrak{M} \) at \( x \in X \) is identified with the composition
  \[
  * \xrightarrow{x} X - \xrightarrow{\mathfrak{M}^\land} A
  \]
  (cf. Example 1.1.29(5)).

(4) For any module \( \mathcal{M} \),
\[
[\mathcal{M}^\land]^{-} = [\mathfrak{M}^\land]
\]
that is, the opposite of the right exponential transpose of \( \mathcal{M} \) is the left exponential transpose of the opposite module \( \mathcal{M}^\land \).

(5) The diagonal functors in Example 1.1.29(7) and the exponential transpositions form the following commutative diagrams (cf. Preliminary 18):

- \[
\begin{array}{c}
  [\mathfrak{X}^\land E] \xrightarrow{\mathfrak{X}[\mathfrak{X}]} [\mathfrak{X}^\land E, [\mathfrak{X}]] \\
  [X : E] \xrightarrow{\sim} [E, [X]] \\
  [\mathfrak{X}^\land E] [X : E] = [\mathfrak{X}^\land X, \mathfrak{X}^\land E] \\
  [\mathfrak{X}^\land E, [\mathfrak{X}]] \xrightarrow{\sim} [\mathfrak{X}^\land E, [\mathfrak{X}], [\mathfrak{X}, E]]
\end{array}
\]

- \[
\begin{array}{c}
  [\mathfrak{E}^\land A] \xrightarrow{\mathfrak{E}[\mathfrak{E}]} [\mathfrak{E}^\land A, [\mathfrak{E}]] \\
  [\mathfrak{E} : A] \xrightarrow{\sim} [E, [A]] \\
  [\mathfrak{E}^\land A] [\mathfrak{E} : A] = [\mathfrak{E}^\land A, \mathfrak{E}^\land A] \\
  [\mathfrak{E}^\land A, [\mathfrak{E}]] \xrightarrow{\sim} [\mathfrak{E}^\land A, [\mathfrak{E}], [A, E]]
\end{array}
\]

**Proposition 2.1.3.** For a module morphism \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \), the following conditions are equivalent:

1. \( \Phi \) is a module isomorphism;
2. the right exponential transpose \( [\Phi^\land] : [\mathcal{M}^\land] \to [\mathcal{N}^\land] : A \to [X : X] \) is a natural isomorphism;
3. each right slice \( \langle \Phi \rangle a : \langle \mathcal{M} \rangle a \to \langle \mathcal{N} \rangle a : X \to * \) is a right module isomorphism;
4. the left exponential transpose \( [\mathfrak{X} \Phi] : [\mathfrak{X}^\land] : X \to [A]^- \) is a natural isomorphism;
5. each left slice \( \mathfrak{x}(\Phi) : \mathfrak{x}(\mathcal{M}) \to \mathfrak{x}(\mathcal{N}) : * \to A \) is a left module isomorphism.

**Proof.** Since the right and left exponential transpositions give isomorphisms \( [A, [X : X]] \cong [X : A] \cong [X, [A]]^- \), the equivalence \( 1 \Leftrightarrow 2 \Leftrightarrow 4 \) holds. Since a natural transformation is iso iff each component is an isomorphism, the equivalences \( 2 \Leftrightarrow 3 \) and \( 4 \Leftrightarrow 5 \) hold. \( \square \)

**Remark 2.1.4.** Proposition 2.1.3 also follows from Proposition 1.1.6 and Proposition 1.1.16, noting that \( \mathfrak{x}(\langle \Phi \rangle a) = \mathfrak{x}(\Phi) a = \langle \mathfrak{x}(\Phi) \rangle a \).

**Proposition 2.1.5.** Let \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \) be a module morphism.

- If \( B \) is an essentially wide subcategory of \( A \), then \( \Phi \) is an isomorphism if and only if its right slice \( \langle \Phi \rangle b : \langle \mathcal{M} \rangle b \to \langle \mathcal{N} \rangle b : X \to * \) is a right module isomorphism for each \( b \in \|B\| \).
- If \( Y \) is an essentially wide subcategory of \( X \), then \( \Phi \) is an isomorphism if and only if its left slice \( \mathfrak{y}(\Phi) : \mathfrak{y}(\mathcal{M}) \to \mathfrak{y}(\mathcal{N}) : * \to A \) is a left module isomorphism for each \( y \in \|Y\| \).

**Proof.** By Proposition 2.1.3, \( \Phi \) is an isomorphism iff its right exponential transpose \( \Phi^\land \) is a natural isomorphism, and by the lemma in Preliminary 19, this is the case iff the restriction of \( \Phi^\land \) to \( B \) is a natural isomorphism, and this is the case iff for each \( b \in \|B\| \), the right slice of \( \Phi \) at \( b \) (i.e. the component of \( \Phi^\land \) at \( b \)) is an isomorphism. \( \square \)
Proposition 2.1.6.

- Given a module (or module morphism) $M$ and a functor (or natural transformation) $K$ as in

$$X \xrightarrow{M} A \xleftarrow{K} E$$

, the right exponential transpose of the composite $(M)K$ is given by the composition

$$[X:] \xrightarrow{M\cdot} A \xleftarrow{K} E$$

; that is,

$$[(M)K]\cdot = [M\cdot] \circ K$$

- Given a module (or module morphism) $M$ and a functor (or natural transformation) $K$ as in

$$E \xrightarrow{K} X \xrightarrow{M} A$$

, the left exponential transpose of the composite $K(M)$ is given by the composition

$$E \xrightarrow{K} X \xleftarrow{\cdot M} [: A]^\sim$$

; that is,

$$[\cdot(K(M))] = K \circ [\cdot M]$$

Proof. For any $e \in E$,

$$[(M)K]\cdot e = (M)K e = (M)(K \cdot e) = [M\cdot] \cdot (K \cdot e) = [[M\cdot] \circ K] \cdot e$$

\qed

Proposition 2.1.7.

- Given a functor $K$ and a module $M$ as in

$$E \xrightarrow{K} X \xrightarrow{M} A$$

, the right exponential transpose of the composite module $K(M)$ is given by the composition

$$[E:] \xrightarrow{[K]} [X:] \xleftarrow{M\cdot} A$$

; that is,

$$[(K(M))]\cdot = [K] \circ [M\cdot]$$

- Given a functor $K$ and a module $M$ as in

$$X \xrightarrow{\cdot M} A \xleftarrow{K} E$$

, the left exponential transpose of the composite module $(M)K$ is given by the composition

$$X \xleftarrow{\cdot M} [: A]^\sim \xrightarrow{[K]} [: E]^\sim$$

; that is,

$$[\cdot(M)K] = [\cdot M] \circ [: K]$$
Proof. For any $a \in A$,
\[
[(K \langle M \rangle, X) \cdot a = (K \langle M \rangle) a = K \langle M \rangle a] = [K : \cdot (M \langle M \rangle, X) \cdot a] = [[K : \cdot [M \langle M \rangle]] : a] \quad .
\]

Definition 2.1.8. Given a cell $\xymatrix{X \ar[r]^-M & A \ar[d]_-P \ar[r]_-\phi & Y \ar[r]^-N & B}$,

- the right slice of $\Phi$ at $a \in \|A\|$ is the right conical cell
\[
\xymatrix{X \ar[r]^-{(M)a} & * \ar[d]_-{(\Phi)a} \ar[r]^-M & A \ar[d]_-Q \ar[r]^-\phi & Y \ar[r]^-N \ar[d]_-\phi & B}
\]
defined by the right slice of the module morphism $\Phi : M \to P \langle N \rangle Q : X \to A$ at $a$, i.e. by the right module
\[
\langle \Phi \rangle a : (M)a \to (P \langle N \rangle Q)a : X \to *
\]

- the left slice of $\Phi$ at $x \in \|X\|$ is the left conical cell
\[
\xymatrix{* \ar[r]^-{x(M)} & A \ar[d]_-x \ar[r]^-P & X \ar[d]_-{\Phi} \ar[r]^-Q & Y \ar[r]^-N \ar[d]_-\phi & B}
\]
defined by the left slice of the module morphism $\Phi : M \to P \langle N \rangle Q : X \to A$ at $x$, i.e. by the left module
\[
x(\Phi) : x(M) \to x(P \langle N \rangle Q) : * \to A
\]

Proposition 2.1.9. A cell is fully faithful if and only if each right (resp. left) slice is fully faithful.

Proof. Immediate from Proposition 2.1.3.

\[\square\]

2.2. Action of modules

Definition 2.2.1. Let $E$ be a category and $M : X \to A$ be a module.

- The right action of $M$ on the functor category $[E, A]$ is the covariant functor
\[
[M \cdot E] : [E, A] \to [X : E]
\]
defined by
\[
[M \cdot E] : K = (M) K
\]
for $K \in [E, A]$.

- The left action of $M$ on the functor category $[E, X]$ is the contravariant functor
\[
[E \cdot M] : [E, X] \to [E : A]^{\sim}
\]
defined by
\[
K \cdot [E \cdot M] = K (M)
\]
for $K \in [E, X]$.  

Remark 2.2.2.

1. A module $\mathcal{M} : X \to A$ acts on a functor $K : E \to A$ by the composition

$$X \xrightarrow{\mathcal{M}} A \xrightarrow{K} E$$

and yields the module $(\mathcal{M}) K : X \to E$, and acts on a natural transformation $\tau : S \to T : E \to A$ by the composition

$$X \xrightarrow{\mathcal{M}} A \xrightarrow{\tau} E$$

and yields the module morphism $(\mathcal{M}) \tau : (\mathcal{M}) S \to (\mathcal{M}) T : X \to E$ which maps each $(\mathcal{M}) S$-arrow $m : x \to e$ to the $(\mathcal{M}) T$-arrow $m \circ \tau_e : x \to e$ as indicated in

$$\xymatrix{ x \ar[r]^{m} \ar[rd]_{m \cdot (\mathcal{M}) \tau} & e \\ & T \cdot e }$$

(cf. Example 1.1.29(3)).

2. A module $\mathcal{M} : X \to A$ acts on a functor $K : E \to X$ by the composition

$$E \xrightarrow{K} X \xrightarrow{\mathcal{M}} A$$

and yields the module $K(\mathcal{M}) : E \to A$, and acts on a natural transformation $\tau : S \to T : E \to X$ by the composition

$$E \xrightarrow{T \tau} X \xrightarrow{\mathcal{M}} A$$

and yields the module morphism $\tau(\mathcal{M}) : T(\mathcal{M}) \to S(\mathcal{M}) : E \to A$ which maps each $T(\mathcal{M})$-arrow $m : e \to a$ to the $S(\mathcal{M})$-arrow $\tau_e \circ m : e \to a$ as indicated in

$$\xymatrix{ e : T \ar[r]^{m} \ar[rd]_{\tau_e \cdot m} & a \\ & e : S \ar[u]_{\tau(\mathcal{M}) : m} }$$

(cf. Example 1.1.29(3)).

(3) For a module $\mathcal{M} : X \to A$, by Remark 2.1.2(3),

- the right exponential transpose $[\mathcal{M} \cdot \ast] : A \to [X :]$ is identified with the right action $[\mathcal{M} \cdot \ast] : [\ast, A] \to [X : \ast]$ of $\mathcal{M}$ on the functor category $[\ast, A]$.

- the left exponential transpose $[\ast \cdot \mathcal{M}] : X \to [\ast : A]$ is identified with the left action $[\ast \cdot \mathcal{M}] : [\ast, X] \to [\ast : A]$ of $\mathcal{M}$ on the functor category $[\ast, X]$.

(3) For any category $E$ and any module $\mathcal{M} : X \to A$,

$$[\mathcal{M} \cdot E] \cong [E \cdot \mathcal{M}]$$

; that is, the opposite of the right action of $\mathcal{M} : X \to A$ on $[E, A]$ is (identified with) the left action of the opposite module $\mathcal{M}^\ast : A^\ast \to X^\ast$ on $[E^\ast, A^\ast]$.

Proposition 2.2.3. Given a category $E$ and a module $\mathcal{M} : X \to A$, the diagrams

$$\xymatrix{ \mathcal{M} \cdot [E, A] \ar[r] & [E, \mathcal{M} \cdot] \\ X : E \ar[ur]_{\mathcal{M} \cdot} & [E, [X :]] \ar[l] }$$

$$\xymatrix{ E \cdot [E, A] \ar[r] & [E, [\ast : A]] \\ [E : A] \ar[ur]_{E \cdot} & [E, [\ast : A]] \ar[l] }$$

commute.
2.3. Yoneda functors and representations

Proof. Immediate from Proposition 2.1.6.

Proposition 2.2.4. Given a category $E$ and a module $M : X \to A$, the diagrams

\[ \begin{array}{ccc}
A & \xrightarrow{\lbrack E, A \rbrack} & [E, A] \\
\downarrow M \circ & \downarrow M \circ E & \downarrow E \circ M \\
[X : E] & \xrightarrow{\lbrack X : E \rbrack} & [X : E] \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
[E, X] & \xrightarrow{\lbrack E, X \rbrack} & X \\
\downarrow E \circ M & \downarrow E \circ M & \downarrow E \circ M \\
[E : A] & \xrightarrow{\lbrack E : A \rbrack} & [E : A] \\
\end{array} \]

commute.

Proof. The diagrams

\[ \begin{array}{ccc}
A & \xrightarrow{\lbrack E, A \rbrack} & [E, A] \\
\downarrow M \circ & \downarrow M \circ E & \downarrow E \circ M \\
[X : E] & \xrightarrow{\lbrack X : E \rbrack} & [X : E] \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
[E, X] & \xrightarrow{\lbrack E, X \rbrack} & X \\
\downarrow E \circ M & \downarrow E \circ M & \downarrow E \circ M \\
[E : A] & \xrightarrow{\lbrack E : A \rbrack} & [E : A] \\
\end{array} \]

commute by Proposition 2.2.3, and together with the commutative diagrams in Remark 2.1.2(5), yield the desired commutative diagrams. \qed

2.3. Yoneda functors and representations

Definition 2.3.1.

- The right Yoneda functor for a category $X$ is the functor

\[ [X \Rightarrow] : X \to [X :] \]

given by the right exponential transpose of the hom $\langle X \rangle : X \to X$; in short,

\[ [X \Rightarrow] := \langle (X) \Rightarrow \rangle \]

- The left Yoneda functor for a category $A$ is the functor

\[ [\Leftrightarrow A] : A \to [\Leftrightarrow A] \]

given by the left exponential transpose of the hom $\langle A \rangle : A \to A$; in short,

\[ [\Leftrightarrow A] := \langle \Leftrightarrow \rangle (A) \]

Remark 2.3.2.

- The right Yoneda functor $X \Rightarrow$ sends each object $x \in [X]$ to the right module

\[ \langle X \rangle x : X \to * \]

called the representable right module of $x$, and sends each $X$-arrow $f : s \to t$ to the right module morphism

\[ \langle X \rangle f : \langle X \rangle s \to \langle X \rangle t : X \to * \]
which maps each $\mathbf{X}$-arrow $h : x \to s$ to the $\mathbf{X}$-arrow $h \circ f : x \to t$ as indicated in

\[
\begin{array}{c}
x \\ h \downarrow \\
\downarrow f \\
t
\end{array}
\quad \quad \quad
\begin{array}{c}
h \circ f \\
\downarrow
\end{array}
\]

(cf. Remark 2.1.2(1)).

- The left Yoneda functor $\mathbf{L} \mathbf{A}$ sends each object $a \in \| \mathbf{A} \|$ to the left module

\[
a(\mathbf{A}) : \ast \to \mathbf{A}
\]

, called the representable left module of $a$, and sends each $\mathbf{A}$-arrow $f : s \to t$ to the left module morphism

\[
f(\mathbf{A}) : t(\mathbf{A}) \to s(\mathbf{A}) : \ast \to \mathbf{A}
\]

which maps each $\mathbf{A}$-arrow $h : t \to a$ to the $\mathbf{A}$-arrow $f \circ h : s \to a$ as indicated in

\[
\begin{array}{c}
t \\ h \\
\downarrow f \\
a
\end{array}
\quad \quad \quad
\begin{array}{c}
f(\mathbf{A}) \circ h \\
\downarrow
\end{array}
\]

(cf. Remark 2.1.2(1)).

**Definition 2.3.3.**

- A representation of a right module $\mathcal{M} : \mathbf{X} \to \ast$ is a pair $(r, \Upsilon)$ consisting of an object $r \in \| \mathbf{X} \|$, “representing object”, and a right module isomorphism $\Upsilon : \mathcal{M} \cong \langle X \rangle r$. A right module is called representable if it has a representation.

- A representation of a left module $\mathcal{M} : \ast \to \mathbf{A}$ is a pair $(r, \Upsilon)$ consisting of an object $r \in \| \mathbf{A} \|$, “representing object”, and a left module isomorphism $\Upsilon : \mathcal{M} \cong r \langle \mathbf{A} \rangle$. A left module is called representable if it has a representation.

**Remark 2.3.4.**

1. A right module $\mathcal{M} : \mathbf{X} \to \ast$ is representable if it is isomorphic to “the” representable right module $\langle X \rangle r$ for some object $r \in \| \mathbf{X} \|$, and a left module $\mathcal{M} : \ast \to r \langle \mathbf{A} \rangle$ is representable if it is isomorphic to “the” representable left module $r \langle \mathbf{A} \rangle$ for some object $r \in \| \mathbf{A} \|$.

2. A representation $(r, \Upsilon)$ of a right module $\mathcal{M} : \mathbf{X} \to \ast$ (resp. left module $\mathcal{M} : \ast \to \mathbf{A}$) is expressed by a fully faithful conical cell

\[
\begin{array}{c}
\mathbf{X} \\
\Upsilon \\
r \\
\downarrow \\
\mathbf{A}
\end{array}
\quad \quad \quad
\begin{array}{c}
\mathbf{X} \\
\Upsilon \\
r \\
\downarrow \\
\mathbf{A}
\end{array}
\]

**Definition 2.3.5.** For a functor $F : \mathbf{D} \to \mathbf{E}$,

- the module $(\mathbf{E}) F : \mathbf{E} \to \mathbf{D}$ given by the composition

\[
\begin{array}{c}
\mathbf{E} \\
(\mathbf{E}) F \\
\mathbf{D}
\end{array}
\]

is called the corepresentable module of $F$.

- the module $F(\mathbf{E}) : \mathbf{D} \to \mathbf{E}$ given by the composition

\[
\begin{array}{c}
\mathbf{D} \\
F \\
\mathbf{E} \\
(\mathbf{E}) F
\end{array}
\]

is called the representable module of $F$.

**Remark 2.3.6.** The composition in the above definition is a special case of Example 1.1.29(10):
The corepresentable module $(E)F : E \to D$ is defined by
\[ e \langle (E)F \rangle d = e \langle E \rangle (F \cdot d) \]
for \( e \in E \) and \( d \in D \). An \( (E)F \)-arrow \( f : e \to d \) is given by an \( E \)-arrow \( f : e \to F \cdot d \); for an \( E \)-arrow \( h : e' \to e \), an \( (E)F \)-arrow \( f : e \to d \), and a \( D \)-arrow \( k : d \to d' \), their composite \( h \circ f \circ k : e' \to d' \) is given by the \( E \)-arrow \( h \circ f \circ (F \cdot k) : e' \to F \cdot d' \) as indicated in

\[
\begin{array}{ccc}
e & \xrightarrow{f} & F \cdot d \\
h \uparrow & & \downarrow F \cdot k \\
e' & \xrightarrow{h \circ f \circ k} & F \cdot d'
\end{array}
\]

The representable module \( F \langle E \rangle : D \to E \) is defined by
\[ d \langle F \langle E \rangle \rangle e = (d : F) \langle E \rangle e \]
for \( e \in E \) and \( d \in D \). An \( F \langle E \rangle \)-arrow \( f : d \to e \) is given by an \( E \)-arrow \( f : d : F \to e \); for an \( D \)-arrow \( h : d' \to d \), an \( F \langle E \rangle \)-arrow \( f : d \to e \), and a \( E \)-arrow \( k : e \to e' \), their composite \( h \circ f \circ k : d' \to e' \) is given by the \( E \)-arrow \( (h : F) \circ f \circ k : d' \to F \cdot e' \) as indicated in

\[
\begin{array}{ccc}
d & \xrightarrow{d : F} & e \\
h \uparrow & & \downarrow F \cdot k \\
d' & \xrightarrow{h \circ f \circ k} & e'
\end{array}
\]

**Definition 2.3.7.** Let \( X \) and \( A \) be categories.

- The right general Yoneda functor for the functor category \([A, X]\) is the functor
  \[ [X \triangleright A] : [A, X] \to [X : A] \]
given by the right action of the hom \( \langle X \rangle : X \to X \) on the functor category \([A, X]\); in short,
  \[ [X \triangleright A] := \langle [X] \triangleright A \rangle \]

- The left general Yoneda functor for the functor category \([X, A]\) is the functor
  \[ [X \triangleleft A] : [X, A] \to [X : A]^-\]
given by the left action of the hom \( \langle A \rangle : A \to A \) on the functor category \([X, A]\); in short,
  \[ [X \triangleleft A] := \langle [X] \triangleleft \langle A \rangle \rangle \]

**Remark 2.3.8.**

1. The right general Yoneda functor \( X \triangleright A \) sends each functor \( G : A \to X \) to the corepresentable module \( (X)G : X \to A \) of \( G \), and sends each natural transformation \( \tau : S \to T : A \to X \) to the module morphism \( \langle X \rangle \tau : (X)S \to (X)T : X \to A \) which maps each \( (X)S \)-arrow \( h : x \to a \) to the \( (X)T \)-arrow \( h \circ \tau_a : x \to a \) as indicated in

\[
\begin{array}{ccc}
x & \xrightarrow{h} & S \cdot a \\
\downarrow h \cdot \tau_a & & \downarrow \tau_a \\
T \cdot a
\end{array}
\]
(cf. Remark 2.2.2(1)).

- The left general Yoneda functor $\mathbf{X} \times A$ sends each functor $F : X \to A$ to the representable module $F(A) : X \to A$ of $F$, and sends each natural transformation $\tau : S \to T : X \to A$ to the module morphism $\tau(A) : T(A) \to S(A) : X \to A$ which maps each $T(A)$-arrow $h : x \sim a$ to the $S(A)$-arrow $\tau \circ h : x \sim a$ as indicated in

$$
x : T \xrightarrow{h} a \\
\tau \downarrow \quad \tau(A) \downarrow h
$$

(cf. Remark 2.2.2(1)).

(2) The right Yoneda functor $[X, \cdot] : X \to [X :]$ for a category $X$ is identified with the right general Yoneda functor $[X, \star] : [*] \to [X : \star]$ for the functor category $[*] ; X$; and the left Yoneda functor $[\cdot, A] : A \to [\cdot : A]^\sim$ for a category $A$ is identified with the left general Yoneda functor $[\cdot, A] : [\cdot, A] \to [\cdot : A]^\sim$ for the functor category $[\cdot, A]$.

**Proposition 2.3.9.** Given categories $E$, $X$, and $A$, the diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{[\cdot, X]} & [E, X] \\
\downarrow X & & \downarrow X \times E \\
[X :] & \xrightarrow{[\cdot, E]} & [X : E]
\end{array}
\quad
\begin{array}{ccc}
[E, A] & \xrightarrow{[\cdot, A]} & A \\
\downarrow E \times A & & \downarrow A \\
[E : A]^\sim & \xrightarrow{[\cdot : A]^\sim} & [\cdot : A]^\sim
\end{array}
$$

commute.

**Proof.** This is a special case of Proposition 2.2.4 where $M$ is given by the hom of $X$ (resp. $A$).

**Definition 2.3.10.** Let $M : X \to A$ be a module.

- A corepresentation of $M$ is a pair $(R, \Upsilon)$ consisting of a functor $R : A \to X$, “corepresenting functor”, and a module isomorphism $\Upsilon : M \cong (X)R$. $A$ is called corepresentable if it has a corepresentation.

- A representation of $M$ is a pair $(R, \Upsilon)$ consisting of a functor $R : X \to A$, “representing functor”, and a module isomorphism $\Upsilon : M \cong R(A)$. $A$ is called representable if it has a representation.

**Remark 2.3.11.**

1. A module $M : X \to A$ is corepresentable if it is isomorphic to “the” corepresentable module $(X)R$ for some functor $R : A \to X$; and is representable if it is isomorphic to “the” representable module $R(A)$ for some functor $R : X \to A$.

2. A representation of a right module $M : X \to \star$ is identified with a corepresentation of a two-sided module $M : X \to A$ where $A$ is the terminal category; and a representation of a left module $M : \star \to A$ is identified with a representation of a two-sided module $M : X \to A$ where $X$ is the terminal category.

3. A corepresentation (resp. representation) $(R, \Upsilon)$ of a module $M : X \to A$ is expressed by a fully faithful cell $\begin{array}{ccc}X & \xrightarrow{M} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Upsilon} & A
\end{array}$ (resp. $\begin{array}{ccc}X & \xleftarrow{M} & A \\
\downarrow & & \downarrow \\
X & \xleftarrow{\Upsilon} & A
\end{array}$).

4. Example 2.3.13 shows that not all modules are representable.

**Proposition 2.3.12.** Let $M : X \to A$ be a module.

- A functor $R : A \to X$ and a module morphism $\Upsilon : M \to (X)R : X \to A$ form a corepresentation of $M$ if and only if for every $a \in [A]$ the object $R(a)$ and the right module morphism $(\Upsilon) a : (M)a \to (X)(R(a)) : X \to \star$ form a representation of the right module $(M)a : X \to \star$. 
2.3. Yoneda functors and representations 45

- A functor \( R : X \to A \) and a module morphism \( \Upsilon : M \to R(A) : X \to A \) form a representation of \( M \) if and only if for every \( x \in |X| \) the object \( x \cdot R \) and the left module morphism \( x(\Upsilon) : x(M) \to (x \cdot R)(A) : \ast \to A \) form a representation of the left module \( x(M) : \ast \to A \).

\[
\langle \Upsilon \rangle a : \langle M \rangle a \to \langle (X) R \rangle a = \langle X \rangle (R \cdot a)
\]
is iso for every \( a \in |A| \).

Example 2.3.13.

(1) Let \( X \) and \( A \) be discrete categories. A correspondence \( R \) from \( |X| \) to \( |A| \), i.e. a subset of \( |X| \times |A| \), defines a module \( R : X \to A \) by

\[
x(\mathcal{R})a = \begin{cases} \{\ast\} & \text{if } (x, a) \in R \\ \emptyset & \text{otherwise} \end{cases}
\]

\( \mathcal{R} \) is representable (resp. corepresentable) if and only if \( \mathcal{R} \) is a function \( |X| \to |A| \) (resp. \( |A| \to |X| \)); \( \mathcal{R} \) is both corepresentable and representable if and only if \( \mathcal{R} \) is a one-to-one correspondence.

(2) Let \( X \) and \( A \) be thin categories, i.e. preordered sets. A correspondence \( \mathcal{R} \) from \( |X| \) to \( |A| \) defines a module \( \mathcal{R} : X \to A \) if and only if it satisfies the following conditions:

a) \((x, a) \in \mathcal{R} \) and \( a \leq b \) implies that \((x, b) \in \mathcal{R}\),

b) \((x, a) \in \mathcal{R} \) and \( y \leq x \) implies that \((y, a) \in \mathcal{R}\).

When this is the case,

- \( \mathcal{R} \) is corepresentable if and only if for every \( a \in |A| \) the set \( \{x \in |X| \mid (x, a) \in \mathcal{R}\} \) has a maximum element.

- \( \mathcal{R} \) is representable if and only if for every \( x \in |X| \) the set \( \{a \in |A| \mid (x, a) \in \mathcal{R}\} \) has a minimum element.

If \( \mathcal{R} \) is both corepresentable and representable, then a corepresenting functor \( G : A \to X \) and a representing functor \( F : X \to A \) form a Galois connection between \( X \) and \( A \), with

\[
x \leq G \cdot a \iff (x, a) \in \mathcal{R} \iff x \cdot F \leq a
\]
3. Collages and Commas

3.1. Collages

**Definition 3.1.1.** A [two-sided] collage \( \mathcal{M} \) from a category \( X \) to a category \( A \), written \( \mathcal{M} : X \to A \), is defined by a category \([\mathcal{M}]\), “collage category”, satisfying the following conditions:

1. the coproduct category \( X + A \) is a wide full subcategory of \([\mathcal{M}]\);
2. \( a \in [\mathcal{M}] \) if \( x \in \|X\| \) and \( a \in \|A\| \).

**Remark 3.1.2.**

1. The inclusion of the coproduct category \( X + A \) into the collage category \([\mathcal{M}]\) is denoted by
   \[
   \mathcal{M} : X + A \to [\mathcal{M}]
   \]
   or by
   \[
   X \xrightarrow{M_0} [\mathcal{M}] \xleftarrow{M_1} A
   \]

2. A right (resp. left) collage is defined as a special case of a two-sided collage:
   - a right collage over a category \( X \), written \( \mathcal{M} : X \to * \), is defined as a collage from \( X \) to the terminal category; the inclusion of \( X \) into the collage category \([\mathcal{M}]\) is denoted by
   \[
   \mathcal{M} : X \to [\mathcal{M}]
   \]
   - a left collage over a category \( A \), written \( \mathcal{M} : * \to A \), is defined as a collage from the terminal category to \( A \); the inclusion of \( A \) into the collage category \([\mathcal{M}]\) is denoted by
   \[
   \mathcal{M} : A \to [\mathcal{M}]
   \]

3. A collage \( \mathcal{M} \) is called small (resp. locally small) if the collage category \([\mathcal{M}]\) is small (resp. locally small).

4. Any collage \( \mathcal{M} : X \to A \) defines the unique functor \( !_{\mathcal{M}} \) from the collage category \([\mathcal{M}]\) to the interval category \( 2 \) making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}_0} & [\mathcal{M}] & \xleftarrow{\mathcal{M}_1} A \\
\downarrow{!_X} & & \downarrow{!_{\mathcal{M}}} & & \downarrow{!_A} \\
* & \xrightarrow{0} & 2 & \xleftarrow{1} & *
\end{array}
\]

commute. Conversely, any functor \( H : C \to 2 \) defines a unique collage \( \mathcal{M} : X \to A \) with \([\mathcal{M}] = C\) and \([!_{\mathcal{M}}] = H\).

**Definition 3.1.3.** Given a pair of collages \( \mathcal{M}, \mathcal{N} : X \to A \), a morphism \( \Phi \) from \( \mathcal{M} \) to \( \mathcal{N} \), written \( \Phi : \mathcal{M} \to \mathcal{N} \), is defined by a functor \( [\Phi] : [\mathcal{M}] \to [\mathcal{N}] \), “collage functor”, satisfying the following equivalent conditions:

1. \( [\Phi] \) is identity on \( X + A \);
3.1. Collages

(2) the triangle

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{M} & \mathcal{N} \\
\downarrow M & \Phi & \downarrow N \\
\mathcal{M} & \xleftarrow{\Phi} & \mathcal{N}
\end{array}
\]

commutes;

(3) the two triangles

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\mathcal{M}_0} & \mathcal{M} \\
\downarrow \mathcal{X} & \Phi & \downarrow \mathcal{N} \\
\mathcal{X} & \xleftarrow{\Phi} & \mathcal{N}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathcal{M}_1} & \mathcal{A} \\
\downarrow \mathcal{M} & \Phi & \downarrow \mathcal{N} \\
\mathcal{M} & \xleftarrow{\Phi} & \mathcal{N}
\end{array}
\]

commute;

(4) the triangle

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathcal{M}_0} & \mathcal{N} \\
\downarrow \mathcal{M} & 2 & \downarrow \mathcal{N} \\
\mathcal{M} & \xleftarrow{\mathcal{M}_1} & \mathcal{N}
\end{array}
\]

commutes.

**Remark 3.1.4.** Given a pair of categories \( \mathbf{X} \) and \( \mathbf{A} \), all locally small collages \( \mathbf{X} \rightarrow \mathbf{A} \) and morphisms among them define the category \([\mathbf{X} \uparrow \mathbf{A}]\) with the obvious identities and the composition. Indeed, \([\mathbf{X} \uparrow \mathbf{A}]\) is fully embedded into the coslice category under \( \mathbf{X} + \mathbf{A} \). The category \([\mathbf{X} \uparrow]\) of right collages over \( \mathbf{X} \) and the category \([\uparrow \mathbf{A}]\) of left collages over a category \( \mathbf{A} \) are defined by

\[
[X \uparrow] = [X \uparrow \ast] \quad [\uparrow A] = [\ast \uparrow A]
\]

(recall Remark 3.1.2(2)).

**Definition 3.1.5.** Given a pair of collages \( \mathcal{M} : \mathbf{X} \rightarrow \mathbf{A} \) and \( \mathcal{N} : \mathbf{Y} \rightarrow \mathbf{B} \), and given a pair of functors \( P : \mathbf{X} \rightarrow \mathbf{Y} \) and \( Q : \mathbf{A} \rightarrow \mathbf{B} \), a collage cell \( \Phi : P \sim Q : \mathcal{M} \rightarrow \mathcal{N} \) is defined by a functor \( [\Phi] : [\mathcal{M}] \rightarrow [\mathcal{N}] \), “collage functor”, satisfying the following equivalent conditions:

1. \([\Phi]\) is identical to \( P + Q \) on \( \mathbf{X} + \mathbf{A} \);
2. the diagram

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{M_0} & [\mathcal{M}] \\
\downarrow P & \Phi & \downarrow Q \\
\mathbf{Y} & \xleftarrow{M_1} & [\mathcal{N}]
\end{array}
\]

commutes;

3. the triangle

\[
\begin{array}{ccc}
[\mathcal{M}] & \xrightarrow{\Phi} & [\mathcal{N}] \\
\downarrow [\Phi] & 2 & \downarrow [\mathcal{N}]
\end{array}
\]

commutes.

**Remark 3.1.6.**

1. All locally small collages and cells among them define the category \( \text{CLG} \) with the obvious identities and the composition. Indeed, there is an obvious isomorphism between \( \text{CLG} \) and the slice category \( \text{CAT}/2 \).
2. There is an obvious forgetful functor \([-] : \text{CLG} \rightarrow \text{CAT} \), which sends each collage \( \mathcal{M} \) to the collage category \([\mathcal{M}]\) and sends each collage cell \( \Phi \) to the collage functor \([\Phi]\).
3.1. Collages

(3) Given a pair of categories $\mathbf{X}$ and $\mathbf{A}$, there is a canonical embedding $[\mathbf{X} \uparrow \mathbf{A}] \to \mathbf{CLG}$, identical on objects, defined by the arrow function $\Phi \mapsto (\Phi : 1_{\mathbf{X}} \to 1_{\mathbf{A}})$. The embedding is not, in general, full.

**Definition 3.1.7.** Given a parallel pair of collage cells $\Phi : \mathcal{M} \to \mathcal{N}$ and $\Psi : \mathcal{M} \to \mathcal{N}$, a morphism from $\Phi$ to $\Psi$, written $\tau : \Phi \to \Psi : \mathcal{M} \to \mathcal{N}$, is defined by a natural transformation $[\tau] : [\Phi] \to [\Psi] : [\mathcal{M}] \to [\mathcal{N}]$, “collage natural transformation”.

**Remark 3.1.8.** Given a pair of collages $\mathcal{M}$ and $\mathcal{N}$, all collage cells $\mathcal{M} \to \mathcal{N}$ and morphisms among them define the category $[\mathcal{M} \uparrow \mathcal{N}]$ with the obvious identities and the composition. Indeed, the assignment $\tau \mapsto [\tau]$ fully embeds $[\mathcal{M} \uparrow \mathcal{N}]$ into the functor category $[[\mathcal{M}],[\mathcal{N}]]$.

**Note.** In what follows we show a one-to-one correspondence between modules and collages. In Definition 3.1.9 and Definition 3.1.12, a module and a collage corresponding to each other are given the same name.

**Definition 3.1.9.**

1. Given a module $\mathcal{M} : \mathbf{X} \to \mathbf{A}$, the corresponding collage is defined by the collage category $[\mathcal{M}]$ given in the following way:
   a) the objects of $[\mathcal{M}]$ consist of all objects of the coproduct category $\mathbf{X} + \mathbf{A}$;
   b) the arrows of $[\mathcal{M}]$ consist of all $\mathbf{X}$-arrows, $\mathbf{A}$-arrows and $\mathcal{M}$-arrows:
      - $\mathbf{x}(\mathcal{M})y = \mathbf{x}(\mathbf{X})y$ for $x,y \in \|\mathbf{X}\|$;
      - $\mathbf{a}(\mathcal{M})b = \mathbf{a}(\mathbf{A})b$ for $a,b \in \|\mathbf{A}\|$;
      - $\mathbf{x}(\mathcal{M}) \mathbf{a} = \mathbf{x}(\mathbf{A}) \mathbf{a}$ for $x \in \|\mathbf{X}\|$ and $a \in \|\mathbf{A}\|$;
      - $\mathbf{a}(\mathcal{M})x = \emptyset$ for $a \in \|\mathbf{A}\|$ and $x \in \|\mathbf{X}\|$.
   c) the composition law of $[\mathcal{M}]$ is those of $\mathbf{X}$, $\mathbf{A}$, and $\mathcal{M}$ (see Definition 1.1.19).

2. Given a module morphism $\Phi : \mathcal{M} : \mathbf{X} \to \mathbf{A}$, the corresponding collage morphism is defined by the collage functor $[\Phi] : [\mathcal{M}] \to [\mathcal{N}]$ given in the following way:
   a) $[\Phi]$ is the identity on $\mathbf{X} + \mathbf{A}$;
   b) $\mathbf{x}(\mathcal{M}) \mathbf{a} = \mathbf{x}(\Phi) \mathbf{a}$ for $x \in \|\mathbf{X}\|$ and $a \in \|\mathbf{A}\|$.

3. Given a module cell
   \[
   \begin{array}{ccc}
   \mathbf{X} & \xrightarrow{\Phi} & \mathbf{A} \\
   \mathbf{Y} & \xrightarrow{\Psi} & \mathbf{B}
   \end{array}
   \]
   , the corresponding collage cell
   \[
   \begin{array}{ccc}
   \mathbf{X} & \xrightarrow{\mathcal{M}} & \mathbf{A} \\
   \mathbf{Y} & \xrightarrow{\mathcal{N}} & \mathbf{B}
   \end{array}
   \]
   is defined by the collage functor $[\Phi] : [\mathcal{M}] \to [\mathcal{N}]$ given in the following way:
   a) $[\Phi]$ is identical to $P + Q$ on $\mathbf{X} + \mathbf{Y}$;
   b) $\mathbf{x}(\mathcal{M}) \mathbf{a} = \mathbf{x}(\Phi) \mathbf{a}$ for $x \in \|\mathbf{X}\|$ and $a \in \|\mathbf{A}\|$.

4. Given a module cell morphism $\tau : \mathcal{M} \to \mathcal{N}$, the corresponding collage cell morphism is defined by the collage natural transformation $[\tau] : [\mathcal{M}] \to [\mathcal{N}]$ given by:
   - $[\tau]x = [\tau_0]x$ for $x \in \|\mathbf{X}\|$;
   - $[\tau]a = [\tau_1]a$ for $a \in \|\mathbf{A}\|$.
Remark 3.1.10.
(1) For a module morphism $\Phi : M \to N$, the functoriality of $[\Phi] : [M] \to [N]$ follows from the naturality of $\Phi$ (see Remark 1.1.20(3)), and for a module cell $\Phi : M \to N$, the functoriality of $[\Phi] : [M] \to [N]$ follows from Proposition 1.2.3.
(2) For a right module $M : X \to \ast$, the corresponding right collage (see Remark 3.1.2(2)) is constructed as above by identifying $M$ with the two-sided module from $X$ to the terminal category.
Dually for a left module $M : \ast \to A$.

Proposition 3.1.11. The module-to-collage correspondence given in Definition 3.1.9 is functorial and defines the following functors:
(1) $[X : A] \downarrow [X \uparrow A]$ for categories $X$ and $A$;
(2) $\text{MOD} \downarrow \text{CLG}$;
(3) $[M : N] \downarrow [M \uparrow N]$ for modules $M$ and $N$.

Proof. The functoriality is easily verified. \qed

Definition 3.1.12.
(1) Given a collage $M : X \to A$, the corresponding module is defined by the composition

$$X \xrightarrow{M_0} [M] \xrightarrow{\langle \{M\} \rangle} [M] \xleftarrow{M_1} A$$

, where $\langle \{M\} \rangle$ is the hom of the collage category of $M$.
(2) Given a collage morphism $\Phi : M \to N : X \to A$, the corresponding module morphism is defined by the pasting composition

$$X \xrightarrow{M_0} [M] \xrightarrow{\langle \{M\} \rangle} [M] \xmapsto{\Phi} [N] \xleftarrow{\langle \{N\} \rangle} N_1$$

(see Remark 1.2.32), where $\langle \{\Phi\} \rangle$ is the hom of the collage functor of $\Phi$.
(3) Given a collage cell

$$X \xrightarrow{M_0} [M] \xleftarrow{N_0} [N] \xrightarrow{\Phi} A \xrightarrow{Q} B$$

, the corresponding module cell

$$X \xrightarrow{M_0} [M] \xleftarrow{N_0} [N] \xrightarrow{\Phi} A \xrightarrow{Q} B$$

is defined by the pasting composition

$$X \xrightarrow{M_0} [M] \xrightarrow{\langle \{M\} \rangle} [M] \xleftarrow{M_1} A \xrightarrow{Q} B$$

, where $\langle \{\Phi\} \rangle$ is the hom of the collage functor of $\Phi$. 
The composition of the isomorphism 
\[
\tau : \Phi \to \Psi : \mathcal{M} \to \mathcal{N},
\]
the corresponding module cell morphism is defined by the pair of natural transformations
\[
\tau_0 : \Phi_0 \to \Psi_0 \quad \tau_1 : \Phi_1 \to \Psi_1
\]
the restrictions of the collage natural transformation \([\tau] : [\Phi] \to [\Psi] : [\mathcal{M}] \to [\mathcal{N}]\) to the domain and the codomain of the collage \(\mathcal{M}\).

**Proposition 3.1.13.** The collage-to-module correspondence given in Definition 3.1.12 is functorial and defines the following functors:

1. \([\mathbf{X} : \mathbf{A}] \xrightarrow{\mathbf{h}} [\mathbf{X} \uparrow \mathbf{A}]\) for categories \(\mathbf{X}\) and \(\mathbf{A}\);
2. \(\mathbf{CLG} \xrightarrow{\mathbf{h}} \mathbf{MOD}\);
3. \([\mathcal{M} \uparrow \mathcal{N}] \xrightarrow{\mathbf{h}} [\mathcal{M} : \mathcal{N}]\) for modules \(\mathcal{M}\) and \(\mathcal{N}\).

**Proof.** The functoriality is easily verified. \(\square\)

**Theorem 3.1.14.** The corresponding functors

\[
\begin{array}{ccc}
[\mathbf{X} : \mathbf{A}] & \xrightarrow{\mathbf{h}} & [\mathbf{X} \uparrow \mathbf{A}] \\
\mathbf{MOD} & \xrightarrow{\mathbf{h}} & \mathbf{CLG} \\
[\mathcal{M} : \mathcal{N}] & \xrightarrow{\mathbf{h}} & [\mathcal{M} \uparrow \mathcal{N}]
\end{array}
\]

in Proposition 3.1.11 and Proposition 3.1.13 are isomorphisms inverse to each other.

**Proof.** Easily verified. \(\square\)

**Remark 3.1.15.**

1. As noted earlier, a module and a collage corresponding to each other are given the same name and freely identified with each other.
2. A module \(\mathcal{M} : \mathbf{X} \to \mathbf{A}\) is recovered from its collage by the composition
\[
\mathbf{X} \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{[\mathcal{M}]} [\mathcal{M}] \xleftarrow{[\mathcal{M}]} \mathbf{A}
\]
that is,
\[
\mathcal{M} = \mathcal{M}_0([\mathcal{M}]) \mathcal{M}_1
\]
This identity yields a cell

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\mathcal{M}} & \mathbf{A} \\
\mathcal{M}_0 & \xleftarrow{\mathcal{M}_1} & [\mathcal{M}]
\end{array}
\]
called the unit cell of \(\mathcal{M}\).

3. The composition of the isomorphism \(\mathbf{MOD} \xrightarrow{\mathbf{h}} \mathbf{CLG}\) and the forgetful functor \([-] : \mathbf{CLG} \to \mathbf{CAT}\) in Remark 3.1.6(2) yields the forgetful functor \([-] : \mathbf{MOD} \to \mathbf{CAT}\), which sends each module \(\mathcal{M}\) to the collage category \([\mathcal{M}]\) and sends each cell \(\Phi\) to the collage functor \([\Phi]\).

4. The composition of the isomorphism \([\mathcal{M} : \mathcal{N}] \xrightarrow{\mathbf{h}} [\mathcal{M} \uparrow \mathcal{N}]\) and the full embedding of \([\mathcal{M} \uparrow \mathcal{N}]\) into the functor category \([\mathcal{M}, \mathcal{N}]\) described in Remark 3.1.8 yields the full embedding \([-] : [\mathcal{M} : \mathcal{N}] \to [\mathcal{M}, \mathcal{N}]\), which sends each cell \(\Phi : \mathcal{M} \to \mathcal{N}\) to the collage functor \([\Phi] : [\mathcal{M}] \to [\mathcal{N}]\) and sends each cell morphism \(\tau : \Phi \to \Psi : \mathcal{M} \to \mathcal{N}\) to the collage natural transformation \([\tau] : [\Phi] \to [\Psi] : [\mathcal{M}] \to [\mathcal{N}]\).

**Theorem 3.1.16.** There is a canonical adjunction between the forgetful functor \([-] : \mathbf{MOD} \to \mathbf{CAT}\) (see Remark 3.1.15(3)) and the embedding \((-) : \mathbf{CAT} \to \mathbf{MOD}\) (see Theorem 1.2.30), with the unit given by the family of unit cells \(1_{\mathcal{M}} : \mathcal{M} \to ([\mathcal{M}])\). Specifically, for each module \(\mathcal{M} : \mathbf{X} \to \mathbf{A}\) and each category \(\mathbf{E}\),
(1) the adjunct of a cell
\[ \begin{array}{c}
X \xrightarrow{M} A \\
\phi \xrightarrow{\Phi} T \\
E \xrightarrow{(E)} E
\end{array} \]

is given by the functor \( F : [M] \rightarrow E \) defined in the following way:

a) \( F \) is identical to \( S + T \) on \( X + A \);

b) \( x(\Phi) a = x(\Phi) a \) for \( x \in [X] \) and \( a \in [A] \).

(2) the adjunct of a functor \( F : [M] \rightarrow E \) is given by the cell \( \Phi : M \rightarrow (E) \) defined by the composition
\[ \begin{array}{c}
X \xrightarrow{M} A \\
\Phi \xrightarrow{\Phi} A \\
E \xrightarrow{(E)} E
\end{array} \]

of unit cell of \( M \) and the hom of \( F \).

Proof. It is easily verified that the correspondences are inverse to each other. It remains to prove that the family of the unit cells satisfies the naturality condition. For this we need to see that, given a cell
\[ \begin{array}{c}
X \xrightarrow{M} A \\
\phi \xrightarrow{\phi} \phi \\
Y \xrightarrow{N} B
\end{array} \]

, the square
\[ \begin{array}{c}
M \xrightarrow{I_M} ([M]) \\
\Phi \xrightarrow{\Phi} ([\Phi]) \\
N \xrightarrow{I_N} ([N])
\end{array} \]

commutes, i.e. the compositions
\[ \begin{array}{c}
X \xrightarrow{X} A \\
\phi \xrightarrow{\phi} \phi \\
Y \xrightarrow{Y} B
\end{array} \]

yield the same cell. First note that \( P \circ N_0 = M_0 \circ [\Phi] \) and \( N_1 \circ Q = [\Phi] \circ M_1 \) by the construction of the collage functor \([\Phi] \). Since \( I_N \) and \( I_M \) are defined by the identity module morphisms \( \mathbb{N} \rightarrow \mathbb{N} \) and \( M \rightarrow M \), the cells \( \Phi \circ I_N \) and \( I_M \circ [\Phi] \) are given by the module morphisms \( \Phi \) and \( M_0 ([\Phi]) M_1 \). But \( \Phi = M_0 ([\Phi]) M_1 \) by Theorem 3.1.14. \(\Box\)

Remark 3.1.17. The functor \([\_] : \text{MOD} \rightarrow \text{CAT} \) is thus a left adjoint of the embedding \((-) : \text{CAT} \rightarrow \text{MOD} \).

3.2. Commas
Definition 3.2.1.

- A right comma \( \mathcal{K} \) over a category \( X \), written \( \mathcal{K} : X \to * \), consists of a category \( [\mathcal{K}] \), "comma category", and a right comma fibration\(^1\) \( \Phi : [\mathcal{K}] \to X \), i.e. a functor satisfying the following condition: for every object \( k \) in \( [\mathcal{K}] \) and every \( X \)-arrow \( g : y \to [\mathcal{K}(k)] \), there is a unique \( [\mathcal{K}]-\)arrow \( \Phi(k^*:s \to k) \), the lift of \( g \) at \( k \), such that \( \mathcal{K}(\Phi(k^*)) = g \).

- A left comma \( \mathcal{K} \) over a category \( A \), written \( \mathcal{K} : * \to A \), consists of a category \( [\mathcal{K}] \), "comma category", and a left comma fibration \( \Phi : [\mathcal{K}] \to A \), i.e. a functor satisfying the following condition: for every object \( k \) in \( [\mathcal{K}] \) and every \( A \)-arrow \( f : \mathcal{K}(k) \to b \), there is a unique \( [\mathcal{K}]-\)arrow \( \Phi(k^*) : k \to t \), the lift of \( f \) at \( k \), such that \( \mathcal{K}(\Phi(k^*)) = f \).

Remark 3.2.2.

1. A left comma over a category \( A \) is the same thing as a right comma over the opposite category \( A^\sim \).

2. The fibre of a right comma \( \mathcal{K} : X \to * \) at \( x \in \|X\| \), written as \( \mathcal{K} \downarrow x \), is the subcategory of \( [\mathcal{K}] \) consisting of all objects \( k \) with \( \mathcal{K}(k) = x \) and all arrows \( h \) with \( \mathcal{K}(h) = 1_x \). The fibre of a left comma \( \mathcal{K} : * \to A \) at \( a \in \|A\| \) is defined dually and written as \( \mathcal{K} \uparrow a \).

3. A right comma \( \mathcal{K} : X \to * \) is called small (resp. locally small) if \( X \) is small (resp. locally small) and all its fibres are small, and similarly a left comma \( \mathcal{K} : * \to A \) is called small (resp. locally small) if \( A \) is small (resp. locally small) and all its fibres are small.

Definition 3.2.3.

- Given a pair of right commas \( \mathcal{K}, \mathcal{L} : X \to * \), a morphism \( \Phi \) from \( \mathcal{K} \) to \( \mathcal{L} \), written \( \Phi : \mathcal{K} \to \mathcal{L} : X \to * \), is defined by a functor \( [\Phi] : [\mathcal{K}] \to [\mathcal{L}] \), "comma functor", making the triangle

\[
\begin{array}{ccc}
X & \xleftarrow{[\mathcal{K}]} & [\mathcal{K}] \\
\downarrow \mathcal{L} & \downarrow [\Phi] & \downarrow [\mathcal{L}] \\
[A] & \xrightarrow{[\Phi]} & [A]
\end{array}
\]

commute.

- Given a pair of left commas \( \mathcal{K}, \mathcal{L} : * \to A \), a morphism \( \Phi \) from \( \mathcal{K} \) to \( \mathcal{L} \), written \( \Phi : \mathcal{K} \to \mathcal{L} : * \to A \), is defined by a functor \( [\Phi] : [\mathcal{K}] \to [\mathcal{L}] \), "comma functor", making the triangle

\[
\begin{array}{ccc}
[\mathcal{K}] & \xleftarrow{\mathcal{K}} & [\mathcal{K}] \\
\downarrow [\Phi] & \downarrow & \downarrow [\Phi] \\
[\mathcal{L}] \downarrow & \xrightarrow{[\mathcal{L}]} & [A]
\end{array}
\]

commute.

Remark 3.2.4.

1. Given a category \( X \), all locally small right commas over \( X \) and morphisms among them define the category \( [X \downarrow] \) with the obvious identities and the composition. Indeed, \( [X \downarrow] \) is fully embedded into the slice category over \( X \). Dually, given a category \( A \), all locally small left commas over \( A \) and morphisms among them define the category \( [\downarrow A] \).

2. The image of \( k \in \|\mathcal{K}\| \) under the comma functor \( [\Phi] \) is denoted by \( \Phi(k) \).

\(^1\)A comma fibration is called a discrete fibration in the literature.
Proposition 3.2.5.

- A right comma morphism $\Phi : K \to L : X \to A$ sends each $[K]$-arrow $h : s \to t$ to the lift of $K(h)$ at $\Phi(t)$ as indicated in

$$
\begin{align*}
\Phi(s) \xrightarrow{\Phi(h)=K(h)^s(t)} \Phi(t) \\
K(s) \xrightarrow{K(h)} K(t)
\end{align*}
$$

- A left comma morphism $\Phi : K \to L : * \to A$ sends each $[K]$-arrow $h : s \to t$ to the lift of $K(h)$ at $\Phi(s)$ as indicated in

$$
\begin{align*}
\Phi(s) \xrightarrow{\Phi(h)=K(h)^g(s)} \Phi(t) \\
K(s) \xrightarrow{K(h)} K(t)
\end{align*}
$$

Proof. By the definition of a comma morphism,

$$K(h) = L(\Phi(h))$$

that is, $\Phi(h)$ is a lift of $K(h)$ along $L$. The assertion thus follows from the uniqueness of the lift. \qed

Remark 3.2.6. A right (resp. left) comma morphism $\Phi$ is thus determined by the object function of the comma functor $[\Phi]$.

Definition 3.2.7. A [two-sided] comma $K$ from a category $X$ to a category $A$, written $K : X \to A$, consists of a category $[K]$, "comma category", and a comma fibration $K : [K] \to X \times A$, i.e. a pair of functors $X \xleftarrow{K_0} [K] \xrightarrow{K_1} A$, satisfying the following conditions:

1. for every object $k$ in $[K]$ and every $X$-arrow $g : y \to K_0(k)$, there is a unique $[K]$-arrow $g^k : s \to k$, the lift of $g$ at $k$, such that $K_0(g^k) = g$ and $K_1(g^k) = 1_{K_1(k)}$;
2. for every object $k$ in $[K]$ and every $A$-arrow $f : K_1(k) \to b$, there is a unique $[K]$-arrow $f^k : k \to t$, the lift of $f$ at $k$, such that $K_1(f^k) = f$ and $K_0(f^k) = 1_{K_0(k)}$;
3. for every $[K]$-arrow $h : s \to t$, the domain of the lift $K_0(h)^t$ equals the codomain of the lift $K_1(h)^s$ and the triangle

$$
\begin{array}{ccc}
t & \xrightarrow{h} & s \\
\downarrow & & \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
K_0(t) & \xleftarrow{K_0(h)^t} & K_1(s) \\
\end{array}
$$

commutes.

Remark 3.2.8.

1. The fibres of a comma $K : X \to A$ are defined as follows:
   a) the fibre of $K$ at $a \in [A]$, written $K(a)$, is the subcategory of $[K]$ consisting of all objects $k$ with $K_1(k) = a$ and all arrows $h$ with $K_1(h) = 1_a$.
   b) the fibre of $K$ at $x \in [X]$, written $x(K)$, is the subcategory of $[K]$ consisting of all objects $k$ with $K_0(k) = x$ and all arrows $h$ with $K_0(h) = 1_x$.
   c) the fibre of $K$ at $(x,a) \in [X \times A]$, written $x(A)$, is the subcategory of $[K]$ consisting of all objects $k$ with $K(k) = (x,a)$ and all arrows $h$ with $K(h) = 1_{(x,a)}$.
2. With the notion of fibres, the conditions (1) and (2) in Definition 3.2.7 are restated as follows:
a) for every \( a \in \|A\| \), \( K_0 \) restricted to \( (K) a \) is a right comma fibration;
b) for every \( x \in \|X\| \), \( K_1 \) restricted to \( x(K) \) is a left comma fibration.

(3) A comma \( K : X \to A \) is called small (resp. locally small) if \( X \) and \( A \) are small (resp. locally small) and all its fibres are small.

**Definition 3.2.9.** Given a pair of commas \( K, L : X \to A \), a morphism \( \Phi \) from \( K \) to \( L \), written \( \Phi : K \to L : X \to A \), is defined by a functor \([\Phi] : [K] \to [L] \), “comma functor”, making the triangle

\[
\begin{array}{ccc}
[K] & \xrightarrow{[\Phi]} & [L] \\
\downarrow & & \downarrow \\
X \times A & \xrightarrow{L} & A
\end{array}
\]

commute or, equivalently, making the two triangles

\[
\begin{array}{ccc}
X & \xleftarrow{K_0} & [K] \\
\downarrow & & \downarrow \\
X & \xleftarrow{K_1} & A
\end{array}
\]

commute.

**Remark 3.2.10.**

(1) Given a pair of categories \( X \) and \( A \), all locally small commas \( X \to A \) and morphisms among them define the category \([X \downarrow A]\) with the obvious identities and the composition. Indeed, \([X \downarrow A]\) is fully embedded into the slice category over \( X \times A \).

(2) There are obvious isomorphisms

\[
[X \downarrow] \cong [X \down \ast] \quad \quad [\down A] \cong [\ast \down A]
\]

, by which a right comma over \( X \) is identified with a two-sided comma from \( X \) to the terminal category, and a left comma over \( A \) is identified with a two-sided comma from the terminal category to \( A \).

(3) The fibre of a comma morphism \( \Phi : K \to L : X \to A \) at \((x, a) \in \|X \times A\|\), written as

\[
x(\Phi) a : x(K) a \to x(L) a
\]

, is the restriction of the comma functor \([\Phi] : [K] \to [L] \) to the fibre of \( K \) at \((x, a)\).

(4) The image of \( k \in \|K\| \) under the comma functor \([\Phi] \) is denoted by \( \Phi(k) \).

**Proposition 3.2.11.** A comma morphism \( \Phi : K \to L : X \to A \) sends each \([K]\)-arrow \( h : s \to t \) to the composite of the lift of \( K_1(h) \) at \( \Phi(s) \) and the lift of \( K_0(h) \) at \( \Phi(t) \) as indicated in

\[
\begin{array}{ccc}
\Phi(h) & \xrightarrow{K_1(h) \Phi(s)} & K_1(s) \\
\Phi(t) & \xleftarrow{K_0(h) \Phi(t)} & K_0(t)
\end{array}
\]

. The assertion thus follows from the condition (3) in Definition 3.2.7.
Remark 3.2.12. A comma morphism $\Phi$ is thus determined by the object function of the comma functor $[\Phi]$.

Definition 3.2.13. Given a pair of commas $\mathbb{K}: X \rightarrow A$ and $L: Y \rightarrow B$, and given a pair of functors $P: X \rightarrow Y$ and $Q: A \rightarrow B$, a comma cell $\Phi: P \Rightarrow Q: \mathbb{K} \rightarrow L$ is defined by a functor $[\Phi]: [\mathbb{K}] \rightarrow [L]$, “comma functor”, making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{K_0} & [\mathbb{K}] & \xrightarrow{K_1} & A \\
P \downarrow & & \downarrow \Phi & & \downarrow Q \\
Y & \xleftarrow{L_0} & [L] & \xleftarrow{L_1} & B
\end{array}
$$

commute.

Remark 3.2.14.
(1) All locally small commas and cells among them define the category $\text{COM}$ with the obvious identities and the composition.
(2) There is an obvious forgetful functor $[-]: \text{COM} \rightarrow \text{CAT}$, which sends each comma $\mathbb{K}$ to the comma category $[\mathbb{K}]$ and sends each comma cell $\Phi$ to the comma functor $[\Phi]$.
(3) Given a pair of categories $X$ and $A$, there is a canonical embedding $[X \downarrow A] \hookrightarrow \text{COM}$, identical on objects, defined by the arrow function $\Phi \mapsto (\Phi: 1_X \Rightarrow 1_A)$. The embedding is not, in general, full.
(4) The fibre of a comma cell $\Phi: P \Rightarrow Q: \mathbb{K} \rightarrow L$ at $(x, a) \in \|X \times A\|$, written as $x(\Phi)a: x(\mathbb{K})a \rightarrow (x: P)\langle L \rangle (Q:a)$, is the restriction of the comma functor $[\Phi]: [\mathbb{K}] \rightarrow [L]$ to the fibre of $\mathbb{K}$ at $(x, a)$.

Note. The following construction of a comma category is borrowed from [Ell06].

Definition 3.2.15. The comma category $[\mathcal{M}]$ of a module $\mathcal{M}: X \rightarrow A$ is defined in the following way:
(1) the objects of $[\mathcal{M}]$ are all $\mathcal{M}$-arrows $m: x \Rightarrow a$, to be precise, all triples $(x, m, a)$ with $x \in \|X\|$, $a \in \|A\|$, and $m \in x(\mathcal{M})a$;
(2) an arrow of $[\mathcal{M}]$ from $(m: x \Rightarrow a)$ to $(n: y \Rightarrow b)$ is a pair $(g, f)$ consisting of an $X$-arrow $g: x \Rightarrow y$ and an $A$-arrow $f: a \Rightarrow b$ making the square

$$
\begin{array}{ccc}
x & \xrightarrow{m} & a \\
g \downarrow & & \downarrow f \\
y & \xleftarrow{n} & b
\end{array}
$$

commute;
(3) the composition law of $[\mathcal{M}]$ is that induced by the composition laws of $X$ and $A$.

Remark 3.2.16.
(1) In the literature, a comma category is defined for a pair of functors $E \xrightarrow{S} C \xrightarrow{T} D$ and written as $(S \downarrow T)$. Note that

$$(S \downarrow T) = [S \langle C \rangle T]$$

, where $S \langle C \rangle T$ is the composite module in Example 1.1.29(10), and for a module $\mathcal{M}: X \rightarrow A$,

$[\mathcal{M}] = (\mathcal{M}_0 \downarrow \mathcal{M}_1)$

, where $\mathcal{M}_0$ and $\mathcal{M}_1$ are the inclusions in Remark 3.1.2(1).
(2) The comma category $[\mathcal{M}]$ is identified with the full subcategory of the functor category $[2, [\mathcal{M}]]$ consisting of all sections of the functor $![\mathcal{M}]: [\mathcal{M}] \rightarrow 2$ (see Remark 3.1.2(4)), giving a canonical inclusion $[\mathcal{M}] \hookrightarrow [2, [\mathcal{M}]]$. 


Definition 3.2.17.
(1) Given a module $M : X \to A$, the comma $M^i : X \to A$ is defined by the comma category $[M]$ and the pair of unique functors $M_0^i : [M] \to X$ and $M_1^i : [M] \to A$ making the diagram

![Diagram with arrows $M_0^i$, $M_1^i$, and their compositions]

commute, where $[M] \to [2, [M]]$ is the inclusion described in Remark 3.2.16(2) and $[0, [M]]$ and $[1, [M]]$ are the evaluations at $0 \in 2$ and $1 \in 2$.

(2) Given a module morphism $\Phi : M \to N : X \to A$, the comma morphism $\Phi^i : M^i \to N^i : X \to A$, depicted as

![Diagram with arrows $\Phi^i$, $\Phi$, and their compositions]

, is defined by the unique functor $[\Phi] : [M] \to [N]$, “comma functor”, making the diagram

![Diagram with arrows $[\Phi]$, $[2, [\Phi]]$, and their compositions]

commute, where the horizontal arrows are the inclusions described in Remark 3.2.16(2) and $[2, [\Phi]]$ is the postcomposition with the collage functor $[\Phi] : [M] \to [N]$ (see Definition 3.1.9(2)).

(3) Given a module cell

![Diagram with arrows $P$, $\Phi$, $Q$, and their compositions]

, the comma cell $\Phi^i : P \sim Q : M^i \to N^i$, depicted as

![Diagram with arrows $\Phi^i$, $\Phi$, $Q$, and their compositions]

, is defined by the unique functor $[\Phi] : [M] \to [N]$, “comma functor”, making the diagram

![Diagram with arrows $[\Phi]$, $[2, [\Phi]]$, and their compositions]

commute, where the horizontal arrows are the inclusions described in Remark 3.2.16(2) and $[2, [\Phi]]$ is the postcomposition with the collage functor $[\Phi] : [M] \to [N]$ (see Definition 3.1.9(3)).

Remark 3.2.18. Given an arrow

![Diagram with arrows $\mathbf{a}$, $\mathbf{b}$, $f$]

, the comma cell $\Phi^i : P \sim Q : M^i \to N^i$, depicted as

![Diagram with arrows $\Phi^i$, $\Phi$, $Q$, and their compositions]
of the comma category \([M]\), the decomposition diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{m} & a \\
  \downarrow_{1} & \swarrow_{f} & \downarrow_{f} \\
  x & \xrightarrow{m \circ g \circ n} & b \\
  \downarrow_{g} & \downarrow_{1} & \\
  y & \xrightarrow{1} & b \\
\end{array}
\]

illustrates the way the comma \(M\) satisfies the condition (3) in Definition 3.2.7; note that
the upper square forms the lift of the \(A\)-arrow \(f\) at \(m\) and the lower square forms the lift of the
\(X\)-arrow \(g\) at \(n\).

**Proposition 3.2.19.** The module-to-comma correspondence given in Definition 3.2.17 is functorial
and defines the following functors:

1. \([X : A] \xrightarrow{\downarrow} [X \downarrow A]\) for categories \(X\) and \(A\);
2. \(\text{MOD} \xrightarrow{\downarrow} \text{COM}\);

**Proof.** The functoriality is easily verified. \(\Box\)

**Remark 3.2.20.**

1. For a right (resp. left) module \(M\), the corresponding right (resp. left) comma \(^2\) is constructed
in a similar way to a two-sided module; indeed,
   - given a category \(X\), the functor \([X]: \xrightarrow{\downarrow} [X \downarrow]\) which sends each right module \(M : X \rightarrow \ast\)
to the right comma \(M^\downarrow : X \rightarrow \ast\) is defined as a special case of the functor \([X : A] \xrightarrow{\downarrow} [X \downarrow A]\)
   where \(A\) is the terminal category; to be precise, \([X]: \xrightarrow{\downarrow} [X \downarrow]\) is defined so that the diagram

   \[
   \begin{array}{ccc}
   [X] & \xrightarrow{\downarrow} & [X \downarrow] \\
   \downarrow_{\varepsilon} & \varepsilon \downarrow & \downarrow_{\varepsilon} \\
   [X : \ast] & \xrightarrow{\downarrow} & [X \downarrow \ast] \\
   \end{array}
   \]

   commutes, where \(\varepsilon\) denotes the canonical isomorphisms given in Remark 1.1.14(4) and Re-
mark 3.2.10(2).
   - given a category \(A\), the functor \([: A] \xrightarrow{\downarrow} [\downarrow A]\) which sends each left module \(M : \ast \rightarrow A\) to
the left comma \(M^\downarrow : \ast \rightarrow A\) is defined as a special case of the functor \([X : A] \xrightarrow{\downarrow} [X \downarrow A]\)
where \(X\) is the terminal category; to be precise, \([: A] \xrightarrow{\downarrow} [\downarrow A]\) is defined so that the diagram

   \[
   \begin{array}{ccc}
   [A] & \xrightarrow{\downarrow} & [\downarrow A] \\
   \downarrow_{\varepsilon} & \varepsilon \downarrow & \downarrow_{\varepsilon} \\
   [* : A] & \xrightarrow{\downarrow} & [* \downarrow A] \\
   \end{array}
   \]

   commutes, where \(\varepsilon\) denotes the canonical isomorphisms given in Remark 1.1.14(4) and Re-
mark 3.2.10(2).

2. The functors in Proposition 3.2.19 yield the functors \([X \uparrow A] \xrightarrow{\downarrow} [X \downarrow A]\)
and \(\text{CLG} \xrightarrow{\downarrow} \text{COM}\) via
composition with the isomorphisms in Theorem 3.1.14 as shown in the commutative diagrams

\[
\begin{array}{cc}
[X \uparrow A] \xrightarrow{\downarrow} [X \downarrow A] & \text{CLG} \xrightarrow{\downarrow} \text{COM} \\
\downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \\
[X : A] \xrightarrow{\downarrow} [X \downarrow A] & \text{MOD} \xrightarrow{\downarrow} \text{COM}
\end{array}
\]

\(^2\)The comma category \([M]\) of a left module \(M\) is called the category of elements in the literature.
. The functor \([X \uparrow A] \hookrightarrow [X \downarrow A]\) sends a collage \(M : X \to A\) to the comma \(M^\downarrow : X \to A\) and sends a collage morphism \(\Phi : M \to N : X \to A\) (depicted as
\[
\begin{array}{ccc}
X & \overset{M_0}{\leftarrow} & [M] & \overset{M_1}{\rightarrow} & A \\
\downarrow{\Phi} & & \downarrow{\Phi} & & \\
N_0 & \overset{N}{\leftarrow} & [N] & \overset{N_1}{\rightarrow} & A
\end{array}
\]
) to the comma morphism \(\Phi^\downarrow : M^\downarrow \to N^\downarrow : X \to A\) (depicted as
\[
\begin{array}{ccc}
X & \overset{M_0}{\leftarrow} & [M] & \overset{M_1}{\rightarrow} & A \\
\downarrow{\Phi} & & \downarrow{\Phi} & & \\
N_0 & \overset{N}{\leftarrow} & [N] & \overset{N_1}{\rightarrow} & A
\end{array}
\]
). Similarly, the functor \(\text{CLG} \hookrightarrow \text{COM}\) sends a collage cell \(\Phi : P \to Q : M \to N\) (depicted as
\[
\begin{array}{ccc}
X & \overset{M_0}{\leftarrow} & [M] & \overset{M_1}{\rightarrow} & A \\
\downarrow{\Phi} & & \downarrow{\Phi} & & \\
P & \overset{P}{\leftarrow} & [P] & \overset{P_1}{\rightarrow} & B
\end{array}
\]
) to the comma cell \(\Phi^\downarrow : P \to Q : M^\downarrow \to N^\downarrow\) (depicted as
\[
\begin{array}{ccc}
X & \overset{M_0}{\leftarrow} & [M] & \overset{M_1}{\rightarrow} & A \\
\downarrow{\Phi} & & \downarrow{\Phi} & & \\
Y & \overset{Y}{\leftarrow} & [Y] & \overset{Y_1}{\rightarrow} & B
\end{array}
\]
).

(3) The composition of the functor \(\text{MOD} \hookrightarrow \text{COM}\) and the forgetful functor \([-] : \text{COM} \to \text{CAT}\) in Remark 3.2.14(2) yields the functor \([-] : \text{MOD} \to \text{CAT}\), which sends each module \(M\) to the comma category \([M]\) and sends each cell \(\Phi\) to the comma functor \([\Phi]\).

**Definition 3.2.21.**

(1) Given a module \(K : X \to A\), the module \(K^\downarrow : X \to A\) is defined in the following way:

a) for a pair of objects \(x \in \|X\|\) and \(a \in \|A\|\), the set \(x(K^\downarrow) a\) is defined by
\[
x(K^\downarrow) a = \|x(K) a\|
\]
, the set of objects of the fibre of \(K\) at \((x, a)\).

b) for an \(X\)-arrow \(g : y \to x\) and an object \(a \in \|A\|\), the function \(g(K^\downarrow) a : x(K^\downarrow) a \to y(K^\downarrow) a\) is defined by the assignment
\[
k \mapsto \text{dom}(g^k)
\]
; that is, \(g(K^\downarrow) a\) is defined such that it maps each \(k \in \|x(K) a\|\) to the domain of the lift \(g^k\).

c) for an object \(x \in \|X\|\) and an \(A\)-arrow \(f : a \to b\), the function \(x(K^\downarrow) f : x(K^\downarrow) a \to x(K^\downarrow) b\) is defined by the assignment
\[
k \mapsto \text{cod}(f^k)
\]
; that is, \(x(K^\downarrow) f\) is defined such that it maps each \(k \in \|x(K) a\|\) to the codomain of the lift \(f^k\).
(2) Given a comma morphism $\Phi: \mathbb{K} \to \mathbb{L}: X \to A$, the module morphism $\Phi^\dagger: \mathbb{K}^\dagger \to \mathbb{L}^\dagger: X \to A$ is defined by
\[
\Phi^\dagger = (\|x(\mathbb{K})a\|: \|x(\mathbb{K})a\| \to \|x(\mathbb{L})a\|)_{(x,a) \in X \times A}
\]
where $\|x(\mathbb{K})a\|$ is the object function of the fibre of $\Phi$ at $(x,a) \in X \times A$.
(3) Given a comma cell
\[
\begin{array}{ccc}
X & \xrightarrow{X_0} & \mathbb{K} \\ \downarrow \phi & & \downarrow \psi \\ Y & \xleftarrow{Y_0} & \mathbb{L}
\end{array}
\]
the module cell
\[
\begin{array}{ccc}
X & \xrightarrow{X} & A \\ \downarrow \phi^\dagger & & \downarrow \psi \\ Y & \xleftarrow{Y} & B
\end{array}
\]
is defined by the module morphism $\Phi^\dagger: \mathbb{K}^\dagger \to P(\mathbb{L}^\dagger)Q: X \to A$ given by
\[
\Phi^\dagger = (\|x(\mathbb{K})a\|: \|x(\mathbb{K})a\| \to \|(x:P)\mathbb{L}(Q:a)\|)_{(x,a) \in X \times A}
\]
where $\|x(\mathbb{K})a\|$ is the object function of the fibre of $\Phi$ at $(x,a) \in X \times A$.

**Proposition 3.2.22.**

(1) $\mathbb{K}^\dagger$ defined in Definition 3.2.21(1) is indeed a module.
(2) $\Phi^\dagger$ defined in Definition 3.2.21(2) is indeed a module morphism.

**Proof.**

(1) For each object $a \in \|A\|$ (resp. $x \in \|X\|$), the functoriality of the slice $\langle \mathbb{K}^\dagger \rangle a: X \to \text{Set}$ (resp. $x(\mathbb{K}^\dagger): A \to \text{Set}$) follows from the functoriality of $\mathbb{K}_0$ (resp. $\mathbb{K}_1$). By the bifunctor lemma (see [ML98] p37 Proposition 1), the proof is complete if we show that the square
\[
\begin{array}{ccc}
x(\mathbb{K}^\dagger)a & \xrightarrow{g(\mathbb{K}^\dagger)a} & y(\mathbb{K}^\dagger)a \\ \downarrow x(\mathbb{K}^\dagger)f & & \downarrow y(\mathbb{K}^\dagger)f \\ x(\mathbb{K}^\dagger)b & \xrightarrow{g(\mathbb{K}^\dagger)b} & y(\mathbb{K}^\dagger)b
\end{array}
\]
commutes for any $X$-arrow $g: y \to x$ and any $A$-arrow $f: a \to b$. Given an object $k \in \|x(\mathbb{K}^\dagger)a\|$, consider the diagram
\[
\begin{array}{ccc}
k & \xrightarrow{g^k} & s \\ \downarrow f^k & & \downarrow f \\ t & \xrightarrow{g^t} & b
\end{array}
\]
where $s$ is domain of the lift $g^k$ and $t$ is the codomain of the lift $f^k$. $x(\mathbb{K}^\dagger)f \circ g(\mathbb{K}^\dagger)b$ maps $k$ to the domain of $g^t$, and $g(\mathbb{K}^\dagger)a \circ y(\mathbb{K}^\dagger)f$ maps $k$ to the codomain of $f^s$. But they are equal by the condition (3) in Definition 3.2.7.
(2) By [ML98] p38 Proposition 2, it suffices to show that $x(\Phi^\dagger)a$ is natural in $x$ for each $a \in \|A\|$ and natural in $a$ for each $x \in \|X\|$; that is, the squares
\[
\begin{array}{ccc}
x(\mathbb{K}^\dagger)a & \xrightarrow{x(\Phi^\dagger)a} & x(\mathbb{L}^\dagger)a \\ g(\mathbb{K}^\dagger)a & \xleftarrow{g(\mathbb{L}^\dagger)a} & x(\mathbb{L}^\dagger)a \\
y(\mathbb{K}^\dagger)a & \xrightarrow{y(\Phi^\dagger)a} & y(\mathbb{L}^\dagger)a
\end{array}
\]
\[
\begin{array}{ccc}
x(\mathbb{K}^\dagger)a & \xrightarrow{x(\Phi^\dagger)a} & x(\mathbb{L}^\dagger)a \\ x(\mathbb{K}^\dagger)f & \xleftarrow{x(\mathbb{L}^\dagger)f} & x(\mathbb{L}^\dagger)f \\
y(\mathbb{K}^\dagger)b & \xrightarrow{y(\Phi^\dagger)b} & y(\mathbb{L}^\dagger)b
\end{array}
\]
commute for any $X$-arrow $g : y \to x$ and any $A$-arrow $f : a \to b$. Let $k \in |x(\mathbb{K}) a|$. The composite $g(\mathbb{K}) a \circ y(\Phi) a$ sends $k$ to $\Phi(\text{dom}(g^k))$ and the composite $x(\Phi) a \circ g(\mathbb{L}) a$ sends $k$ to $\Phi(\text{dom}(g^\Phi(k)))$; similarly, the composite $x(\Phi) f \circ x(\Phi) b$ sends $k$ to $\Phi(\text{cod}(f^k))$ and the composite $x(\Phi) a \circ x(\mathbb{L}) f$ sends $k$ to $\Phi(f^\Phi(k))$. But since $\Phi(g^k) = g^\Phi(k)$ and $\Phi(f^k) = f^\Phi(k)$ by Proposition 3.2.5, we have

$$\text{dom}(g^\Phi(k)) = \text{dom}(\Phi(g^k)) = \Phi(\text{dom}(g^k))$$

and

$$\text{cod}(f^\Phi(k)) = \text{cod}(\Phi(f^k)) = \Phi(\text{cod}(f^k))$$

\[ \square \]

**Proposition 3.2.23.** The comma-to-module correspondence given in Definition 3.2.21 is functorial and defines the following functors:

1. $[X \downarrow A] \xrightarrow{\top} [X : A]$ for categories $X$ and $A$;
2. $\text{COM} \xrightarrow{\top} \text{MOD}$.

**Proof.** The functoriality is easily verified. \[ \square \]

**Remark 3.2.24.**

1. For a right (resp. left) comma $\mathbb{K}$, the corresponding right (resp. left) module $\mathbb{K}^\top$ is constructed in a similar way to a two-sided comma; indeed,
   - given a category $X$, the functor $[X \downarrow] \xrightarrow{\top} [X :]$ which sends each right comma $\mathbb{K} : X \to \ast$ to the right module $\mathbb{K}^\top : X \to \ast$ is defined as a special case of the functor $[X \downarrow A] \xrightarrow{\top} [X : A]$ where $A$ is the terminal category; to be precise, $[X \downarrow] \xrightarrow{\top} [X :] = \text{dom}$ is defined so that the diagram
     $$\begin{array}{ccc}
     [X \downarrow] & \xrightarrow{\top} & [X :] \\
     \downarrow \cong & & \downarrow \cong \\
     [X \downarrow \ast] & \xrightarrow{\top} & [X : \ast]
     \end{array}$$

    commutes, where $\cong$ denotes the canonical isomorphisms given in Remark 3.2.10(2) and Remark 1.1.14(4).
   - given a category $A$, the functor $[\downarrow A] \xrightarrow{\top} [: A]$ which sends each left comma $\mathbb{K} : \ast \to A$ to the left module $\mathbb{K}^\top : \ast \to A$ is defined as a special case of the functor $[X \downarrow A] \xrightarrow{\top} [X : A]$ where $X$ is the terminal category; to be precise, $[\downarrow A] \xrightarrow{\top} [: A] = \text{cod}$ is defined so that the diagram
     $$\begin{array}{ccc}
     [\downarrow A] & \xrightarrow{\top} & [: A] \\
     \downarrow \cong & & \downarrow \cong \\
     [\ast \downarrow A] & \xrightarrow{\top} & [\ast : A]
     \end{array}$$

    commutes, where $\cong$ denotes the canonical isomorphisms given in Remark 3.2.10(2) and Remark 1.1.14(4).

2. The functors in Proposition 3.2.23 yield the functors $[X \downarrow A] \xrightarrow{\top} [X \uparrow A]$ and $\text{COM} \xrightarrow{\top} \text{CLG}$ via composition with the isomorphisms in Theorem 3.1.14 as shown in the commutative diagrams

$$\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{\top} & [X \uparrow A] \\
\| & & \| \\
[X \downarrow A] & \xrightarrow{\top} & [X : A] \\
\| & & \| \\
\text{COM} & \xrightarrow{\top} & \text{CLG} \\
\| & & \| \\
\text{COM} & \xrightarrow{\top} & \text{MOD}
\end{array}$$
3.2. Commas

The functor $[X \downarrow A] \rightarrow [X \uparrow A]$ sends a comma $K : X \rightarrow A$ to the collage $K^! : X \rightarrow A$ and sends a comma morphism $\Phi : K \rightarrow L : X \rightarrow A$ to the collage morphism $\Phi^! : K^! \rightarrow L^! : X \rightarrow A$; the collage category of $K^!$ is denoted by $[K]$ and called the collage category of $K$, and the collage functor of $\Phi^!$ is denoted by $[\Phi]$ and called the collage functor of $\Phi$; so if a comma morphism $\Phi : K \rightarrow L : X \rightarrow A$ is depicted as

\[
\begin{array}{ccc}
X & \xleftarrow{K_0} & \xrightarrow{K_1} \ X \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\downarrow{L_0} & & \downarrow{L_1} \\
Y & \xleftarrow{L_0} & \xrightarrow{L_1} \ B
\end{array}
\]

, the corresponding collage morphism $\Phi^! : K^! \rightarrow L^! : X \rightarrow A$ is depicted as

\[
\begin{array}{ccc}
X & \xleftarrow{K_0} & \xrightarrow{K_1} \ X \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\downarrow{L_0} & & \downarrow{L_1} \\
Y & \xleftarrow{L_0} & \xrightarrow{L_1} \ B
\end{array}
\]

. Similarly, the functor $\text{COM}^! \rightarrow \text{CLG}$ sends a comma cell $\Phi : P \sim Q : K \rightarrow L$ to the collage cell $\Phi^! : P \sim Q : K^! \rightarrow L^!$; the collage functor of $\Phi^!$ is denoted by $[\Phi]$ and called the collage functor of $\Phi$, so if a comma cell $\Phi : P \sim Q : K \rightarrow L$ is depicted as

\[
\begin{array}{ccc}
X & \xleftarrow{K_0} & \xrightarrow{K_1} \ X \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\downarrow{L_0} & & \downarrow{L_1} \\
Y & \xleftarrow{L_0} & \xrightarrow{L_1} \ B
\end{array}
\]

, the corresponding collage cell $\Phi^! : P \sim Q : K^! \rightarrow L^!$ is depicted as

\[
\begin{array}{ccc}
X & \xleftarrow{K_0} & \xrightarrow{K_1} \ X \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\downarrow{L_0} & & \downarrow{L_1} \\
Y & \xleftarrow{L_0} & \xrightarrow{L_1} \ B
\end{array}
\]

. Similarly, the functor $\text{COM} \rightarrow \text{CLG}$ and the forgetful functor $[-] : \text{CLG} \rightarrow \text{CAT}$ in Remark 3.1.6(2) yields the functor $[-] : \text{COM} \rightarrow \text{CAT}$, which sends each comma $K$ to the collage category $[K]$ and sends each comma cell $\Phi$ to the collage functor $[\Phi]$.

**Theorem 3.2.25.**

1. For each module $M : X \rightarrow A$, there is a canonical isomorphism

$$\epsilon_M : (M^!)^! \cong M$$

2. For each comma $K : X \rightarrow A$, there is a canonical isomorphism

$$\eta_K : K \cong (K^!)^!$$

**Proof.** By Definition 3.2.17 and Definition 3.2.21,
(1) the module \( (M^!)^\dag \) is obtained from \( M \) by changing each element \( m \in x(M) a \) to the triple \( (x, m, a) \).

(2) the comma \( (K^!)^\dashv \) is obtained from \( K \) by changing each object \( k \in \|x(K)a\| \) to the triple \( (x, k, a) \) and changing each \( [K]-\text{arrow} h : s \to t \) to the pair \((K_0(h), K_1(h))\).

Hence \( \epsilon_M \) is defined by the assignment \( (x, m, a) \mapsto m \) and \( \eta_K \) is given by the functor \([\eta_K] \) consisting of the object function \( k \mapsto (x, k, a) \) and the arrow function \( h \mapsto (K_0(h), K_1(h)) \). The bijectivity of the arrow function of \([\eta_K] \) follows from the condition (3) in Definition 3.2.7.

\[ \tag{3.2.27} \]

**Definition 3.2.27.** Let \( M : X \to A \) be a module.

- The right comma exponential transpose of \( M \) is the functor

\[ \[M^\vee\] : A \to [X \downarrow] \]

given by the composition

\[ A \xrightarrow{M^\vee} [X:] \xrightarrow{\downarrow} [X \downarrow] \]

of the right exponential transpose of \( M \) and the functor in Remark 3.2.20(1).

- The left comma exponential transpose of \( M \) is the functor

\[ \left[ \downarrow M \right] : X \to [\downarrow A]^\sim \]

given by the composition

\[ X \xrightarrow{\left[ \downarrow M \right]} [A]^\sim \xrightarrow{\downarrow} [\downarrow A]^\sim \]

of the left exponential transpose of \( M \) and the functor in Remark 3.2.20(1).

**Remark 3.2.28.**

- The right comma exponential transpose of a module \( M : X \to A \) sends each object \( a \in \|A\| \) to the comma \( \langle (M) a \rangle^\dag : X \to \ast \) of the right module \( (M) a \). This comma is called the right comma of \( M \) at \( a \in \|A\| \) and written as

\[ M \downarrow a := \langle (M) a \rangle^\dag \]

. Each right comma \( M \downarrow a : X \to \ast \) of \( M \) consists of the comma category \( \langle (M) a \rangle \) and the right comma fibration \( M \downarrow a : \langle (M) a \rangle \to X \).

- The left comma exponential transpose of a module \( M : X \to A \) sends each object \( x \in \|X\| \) to the comma \( \langle x(M) \rangle^\dag : \ast \to A \) of the left module \( x(M) \). This comma is called the left comma of \( M \) at \( x \in \|X\| \) and written as

\[ x \downarrow M := \langle x(M) \rangle^\dag \]

. Each left comma \( x \downarrow M : \ast \to A \) of \( M \) consists of the comma category \( [x(M)] \) and the left comma fibration \( x \downarrow M : [x(M)] \to A \).

**Note.** The following definition is a special case of Definition 3.2.27 where \( M \) is given by the hom of a category.
Definition 3.2.29.

- Given a category $X$, the functor 
  \[ X \Rightarrow : X \to [X \downarrow] \]
  is defined by the composition
  \[
  X \xrightarrow{\Rightarrow} [X:] \xrightarrow{\iota} [X \downarrow]
  \]
  of the right Yoneda functor for $X$ and the functor in Remark 3.2.20(1).

- Given a category $A$, the functor 
  \[ \downarrow A : A \to [A]^- \]
  is defined by the composition
  \[
  A \xrightarrow{\downarrow A} [: A]^- \xrightarrow{\iota} [A]^- \]
  of the left Yoneda functor for $A$ and the functor in Remark 3.2.20(1).

Remark 3.2.30.

- The functor $X \Rightarrow$ sends each object $x \in |X|$ to the comma $\langle(X) x \rangle^i : X \to *$ of the representable right module $(X)x$. This comma is called the right comma of $X$ at $x \in |X|$ and written as
  \[ X \downarrow x := \langle(X) x \rangle^i \]
  Each right comma $X \downarrow x : X \to *$ of $X$ consists of the comma category $\langle(X) x \rangle$ and the right comma fibration $X \downarrow x : \langle(X) x \rangle \to X$. Note that the comma category $\langle(X) x \rangle$ is the same thing as the slice category $X$ over $x$.

- The functor $\downarrow A$ sends each object $a \in |A|$ to the comma $\langle a(A) \rangle^i : * \to A$ of the representable left module $a(A)$. This comma is called the left comma of $A$ at $a \in |A|$ and written as
  \[ a \downarrow A := \langle a(A) \rangle^i \]
  Each left comma $a \downarrow A : * \to A$ of $A$ consists of the comma category $\langle a(A) \rangle$ and the left comma fibration $a \downarrow A : \langle a(A) \rangle \to A$. Note that the comma category $\langle a(A) \rangle$ is the same thing as the coslice category of $A$ under $a$. 
4. Frames

4.1. Cylindrical frames

**Definition 4.1.1.** A [cylindrical] frame $\alpha$ of an endomodule $\mathcal{M} : E \to E$ is a family of $\mathcal{M}$-arrows $\alpha_e : e \sim e$, one for each object $e \in \|E\|$, that is natural in the sense that the square

$$
\begin{array}{ccc}
  e & \sim & e \\
  h \downarrow & & \downarrow h \\
  e' & \sim & e'
\end{array}
$$

commutes for every $E$-arrow $h : e \to e'$. The $\mathcal{M}$-arrow $\alpha_e$ is called the component of $\alpha$ at $e$. The set of frames of $\mathcal{M}$ is denoted by $\prod_E \mathcal{M}$.

**Remark 4.1.2.**

1. If $E$ is small, so is $\prod_E \mathcal{M}$.
2. If $E$ is discrete, then $\prod_E \mathcal{M}$ is nothing but the cartesian product $\prod_{e \in \|E\|} e(\mathcal{M})e$.
3. The naturality of a frame $\alpha$ of an endomodule $\mathcal{M} : E \to E$ is also expressed in the extraordinary\footnote{The term "extraordinary (natural)" is borrowed from \cite{Keller}. An alternative term "extranatural" seems more common in the other literature.} form by regarding $\mathcal{M}$ as a left module over $E^\times E$ or a right module over $E \times E^\rightarrow$ (see Remark 1.1.14(3)):

- if $\mathcal{M}$ is regarded as a left module $\mathcal{M} : * \to E^\times E$, the naturality of $\alpha$ is expressed by the commutativity of the square

  $$
  \begin{array}{ccc}
    * & \sim & (e, e) \\
    \downarrow \alpha_e & & \downarrow \alpha_{(e,e)} \\
    (e', e') & \sim & (e, e')
  \end{array}
  $$

  for each $E$-arrow $h : e \to e'$ (see Remark 1.1.20(4)).

- if $\mathcal{M}$ is regarded as a right module $\mathcal{M} : E \times E^\rightarrow \to *$, the naturality of $\alpha$ is expressed by the commutativity of the square

  $$
  \begin{array}{ccc}
    (e, e') & \sim & (e', e') \\
    \downarrow (e, h) & & \downarrow \alpha_{(e, h)} \\
    (e, e) & \sim & (e', e')
  \end{array}
  $$

  for each $E$-arrow $h : e \to e'$ (see Remark 1.1.20(4)).

4. A frame of an endomodule $\mathcal{M} : E \to E$ is the same thing as a frame of the opposite endomodule $\mathcal{M}^\rightarrow : E^\rightarrow \to E^\rightarrow$; that is,

$$
\prod_E \mathcal{M} = \prod_{E^\rightarrow} \mathcal{M}^\rightarrow
$$

**Example 4.1.3.** Let $E$ and $C$ be categories.

1. A natural transformation $\alpha : S \to T : E \to C$ is the same thing as a frame $\alpha$ of the composite endomodule $S(C)T : E \to E$. (Conversely, a frame $\alpha$ of an endomodule $\mathcal{M} : E \to E$ is the same thing as a natural transformation $\alpha : \mathcal{M}_0 \to \mathcal{M}_1 : E \to \|\mathcal{M}\|$ (see Remark 3.1.2(1)).
(2) Given a bifunctor $K : E^* \times E \to C$ and an object $c \in \|C\|$, 

- an extraordinary natural transformation $\alpha$ from $c$ to $K$ is the same thing as a cylindrical frame $\alpha$ of the composite left module $c(\mathcal{C})K : \ast \to E^* \times E$ (note that a left module $\ast \to E^* \times E$ is the same thing as an endomodule $E \to E$).

- an extraordinary natural transformation $\alpha$ from $K$ to $c$ is the same thing as a cylindrical frame $\alpha$ of the composite right module $K(\mathcal{C})c : E^* \times E \to \ast$ (note that a right module $E \times E \to \ast$ is the same thing as an endomodule $E \to E$).

**Theorem 4.1.4.** Let $\mathcal{M} : E \times D \to E \times D$ be an endomodule. A family of $\mathcal{M}$-arrows $\alpha_{(e,d)} : (e,d) \to (e,d)$ indexed by the objects of $E \times D$ is a frame of $\mathcal{M}$ if and only if for each $e \in \|E\|$, $(\alpha_{(e,d)})_{d \in \|D\|}$ is a frame of the endomodule $[e \times D] \mathcal{M} [e \times D] : D \to D$ (see Example 1.1.29(6)) and for each $d \in \|D\|$, $(\alpha_{(e,d)})_{e \in \|E\|}$ is a frame of the endomodule $[E \times d] \mathcal{M} [E \times d] : E \to E$.

**Proof.** See [ML98] p38 Proposition 2. 

**Definition 4.1.5.** If $\Phi : \mathcal{M} \to \mathcal{N} : E \to E$ is a module morphism and $\alpha$ is a frame of $\mathcal{M}$, then their composite $\alpha \hat{\Phi} = \Phi \hat{\alpha}$ is the frame of $\mathcal{N}$ defined by 

$$[\alpha \hat{\Phi}]_e = \alpha_e \cdot \Phi(e)$$

for $e \in \|E\|$. The function 

$$\prod_{E} \Phi : \prod_{E} \mathcal{M} \to \prod_{E} \mathcal{N}$$

is defined by 

$$\alpha : \prod_{E} \Phi = \alpha \hat{\Phi}$$


**Remark 4.1.6.** The composite $\alpha \hat{\Phi}$ so defined does form a frame of $\mathcal{N}$. Indeed, for each $E$-arrow $h : e \to e'$, the commutativity of the square

$$
\begin{array}{ccc}
E & \xrightarrow{\alpha_e \Phi} & E \\
\downarrow h & & \downarrow h \\
E' & \xrightarrow{\alpha_{e'} \Phi} & E'
\end{array}
$$

follows from the commutativity of the square

$$
\begin{array}{ccc}
e & \xrightarrow{\alpha_e} & e \\
\downarrow h & & \downarrow h \\
e' & \xrightarrow{\alpha_{e'}} & e'
\end{array}
$$

by the naturality of $\Phi$.

**Proposition 4.1.7.** The assignment $\Phi \mapsto \prod_{E} \Phi$ is functorial and defines the left module

$$\prod_{E} : \ast \to [E : E]$$


**Proof.** The functoriality is easily verified. 

\[\square\]
Definition 4.1.8. If $H : D \to E$ is a functor and $\alpha$ is a frame of an endomodule $M : E \to E$, then their composite $H \circ \alpha = \alpha \circ H$ is the frame of the endomodule $H \langle M \rangle H : D \to D$ defined by

$$[H \circ \alpha]_d = \alpha_{[H,d]}$$

for $d \in \|D\|$. The function

$$\prod_H M : \prod_E M \to \prod_D H \langle M \rangle H$$

is defined by

$$\alpha : \prod_H M = H \circ \alpha$$

Remark 4.1.9. The composite $H \circ \alpha$ so defined does form a frame of $H \langle M \rangle H$. Indeed, by the naturality of $\alpha$, the square

$$\begin{array}{ccc}
d & \xrightarrow{\alpha_{[H,d]}} & H \cdot d \\
\downarrow h & & \downarrow H \\
d' & \xrightarrow{\alpha_{[H,d']}} & H \cdot d'
\end{array}$$

commutes for every $D$-arrow $h : d \to d'$.

Proposition 4.1.10. Given functors $C \xrightarrow{G} D \xrightarrow{H} E$ and a frame $\alpha$ of an endomodule $M : E \to E$, the associative law

$$[G \circ H] \circ \alpha = G \circ [H \circ \alpha]$$

holds.

Proof. For any $c \in \|C\|$, 

$$[[G \circ H] \circ \alpha]_c = \alpha_{[H : G \cdot c]} = [H \circ \alpha]_{(G \cdot c)} = [G \circ [H \circ \alpha]]_c$$

Proposition 4.1.11. The function $\prod_H M$ is natural in $M$; that is, for every module morphism $\Phi : M \to N : E \to E$, the square

$$\begin{array}{ccc}
\prod_E M & \xrightarrow{\prod_H M} & \prod_D H \langle M \rangle H \\
\downarrow \prod_E \Phi & & \downarrow \prod_D H \langle \Phi \rangle H \\
\prod_E N & \xrightarrow{\prod_H N} & \prod_D H \langle N \rangle H
\end{array}$$

commutes.

Proof. For any frame $\alpha$ of $M$,

$$\alpha : \prod_H \Phi : \prod_H N = H \circ [\alpha \circ \Phi]$$

and

$$\alpha : \prod_H M : \prod_H H \langle \Phi \rangle H = [H \circ \alpha] \circ H \langle \Phi \rangle H$$

Hence we need to verify that

$$H \circ [\alpha \circ \Phi] = [H \circ \alpha] \circ H \langle \Phi \rangle H$$
4.2. Conical frames

Proposition 4.1.12. The family of functions $\prod H M : \prod E M \to \prod D H (M) H$, one for each endo-module $M : E \to E$, defines a left module cell

$\begin{align*}
\ast & \to \prod E [E : E] \\
1 & \to \prod H [H H] \\
\ast & \to \prod D [D : D]
\end{align*}$

Moreover, the assignment $H \mapsto \prod H$ is contravariant functorial.

Proof. The first assertion is immediate from Proposition 4.1.11. The functoriality of the assignment is easily verified using Proposition 4.1.10.

4.2. Conical frames

Definition 4.2.1.

- A [conical] frame $\alpha$ of a left module $M : * \to E$ is a family of $M$-arrows $\alpha_e : * \to e$, one for each object $e \in [E]$, that is natural in the sense that the triangle

$\begin{align*}
\ast & \to e \\
\ast & \to e' \\
h & \to h
\end{align*}$

commutes for every $E$-arrow $h : e \to e'$. The $M$-arrow $\alpha_e$ is called the component of $\alpha$ at $e$. The set of frames of $M$ is denoted by $\prod_{E} E M$.

- A [conical] frame $\alpha$ of a right module $M : E \to *$ is a family of $M$-arrows $\alpha_e : e \to *$, one for each object $e \in [E]$, that is natural in the sense that the triangle

$\begin{align*}
e & \to \ast \\
\ast & \to \ast \\
h & \to h
\end{align*}$

commutes for every $E$-arrow $h : e \to e'$. The $M$-arrow $\alpha_e$ is called the component of $\alpha$ at $e$. The set of frames of $M$ is denoted by $\prod_{E} E M$.

Remark 4.2.2.

1. If $E$ is small, so is $\prod_{E} E M$ (resp. $\prod_{E} E M$).
2. If $E$ is discrete, then $\prod_{E} E M$ (resp. $\prod_{E} E M$) is just the cartesian product $\prod_{e \in [E]} (M) e$ (resp. $\prod_{e \in [E]} (M) e (M)$).
3. A frame of a left module $M : * \to E$ is the same thing as a frame of the opposite right module $M^\sim : E^\sim \to *$, and a frame of a right module $M : E \to *$ is the same thing as a frame of the opposite left module $M^\sim : * \to E^\sim$; hence

$\prod_{E} M = \prod_{E^\sim} M^\sim$
Example 4.2.3. Given a functor $K: E \to C$ and an object $c \in |C|$, 
- a cone $\alpha$ from $c$ to $K$ is the same thing as a frame $\alpha$ of the composite left module $c(C) K : * \to E$.
- a cone $\alpha$ from $K$ to $c$ is the same thing as a frame $\alpha$ of the composite right module $K(C) c : E \to *$.

Proposition 4.2.4.
- A frame of a left module $M : * \to E$ is the same thing as a frame of the endomodule $[!E](M) : E \to E$ (see Example 1.1.29(7)); that is,
  $$\prod_{\ast \in E} M = \prod_{E} [!E](M)$$
- A frame of a right module $M : E \to *$ is the same thing as a frame of the endomodule $(M)[!E] : E \to E$ (see Example 1.1.29(7)); that is,
  $$\prod_{E \ast} M = \prod_{E} (M)[!E]$$

Proof. Consider a family of $M$-arrows $\alpha_e : * \to e$ indexed by the objects $e \in |E|$. Since the functor $!E : E \to *$ sends each $E$-arrow $h : e \to e'$ to the identity $* \to *$, the triangle

![Diagram](image)

commutes iff the square

![Square](image)

in $[!E](M)$ commutes. Hence $\alpha$ forms a frame of $M$ iff it forms a frame of $[!E](M)$. $\square$

Proposition 4.2.5.
- A frame of a left module $M : * \to E$ is the same thing as a cylindrical frame of the left module $(M)[!E \times E] : * \to E \times E$ (see Example 1.1.29(8)); that is,
  $$\prod_{\ast \in E} M = \prod_{E \ast} (M)[!E \times E]$$
- A frame of a right module $M : E \to *$ is the same thing as a cylindrical frame of the right module $[!E \times E](M) : E \times E \to *$ (see Example 1.1.29(8)); that is,
  $$\prod_{E \ast} M = \prod_{E \ast} ([!E \times E](M))$$

Proof. Consider a family of $M$-arrows $\alpha_e : * \to e$ indexed by the objects $e \in |E|$. Since $(M)[!E \times E]$ is dummy in its first variable, for each $E$-arrow $h : e \to e'$, the triangle

![Diagram](image)

commutes iff the square

![Square](image)

in $(M)[!E \times E]$ commutes. Hence $\alpha$ forms a frame of $M$ iff it forms a cylindrical frame of $(M)[!E \times E]$. $\square$
Proposition 4.2.9. See Proposition 4.2.4 for the identities.

Proof. We need to verify that \( \alpha \circ \Phi = \alpha \circ [1]_E \{ \Phi \} \) for any frame \( \alpha \in \Pi_{\ast E} \mathcal{M} = \Pi_{\ast E} [1]_E \{ \mathcal{M} \} \). But since \( \Phi = e \{ [1]_E \{ \Phi \} \} e \) for \( e \in \| E \| \), we have
\[
[\alpha \circ \Phi]_e = \alpha_e \cdot \langle \Phi \rangle e = \alpha_e \cdot e \{ [1]_E \{ \Phi \} \} e = [\alpha \circ [1]_E \{ \Phi \}]_e
\]
Proposition 4.2.10.

The assignment $\Phi \mapsto \prod_{E} \Phi$ is functorial and defines the left module
\[
\prod_{E} : \ast \to [E : E]
\]
. In fact, $\prod_{E}$ is obtained from the left module $\prod_{E}$ in Proposition 4.1.7 by the composition
\[
* \xrightarrow{-\cdot \prod_{E}} [E : E] \xrightarrow{[E]_{E}} [E : E]
\]
.

The assignment $\Phi \mapsto \prod_{E} \Phi$ is functorial and defines the left module
\[
\prod_{E} : \ast \to [E : E]
\]
. In fact, $\prod_{E}$ is obtained from the left module $\prod_{E}$ in Proposition 4.1.7 by the composition
\[
* \xrightarrow{-\cdot \prod_{E}} [E : E] \xrightarrow{[E]_{E}} [E : E]
\]
.

Proof. The second assertion is immediate from Proposition 4.2.4 and Proposition 4.2.9. The first assertion follows from the second. \qed

Definition 4.2.11.

If $H : D \to E$ is a functor and $\alpha$ is a frame of a left module $M : \ast \to E$, then their composite $H \circ \alpha = \alpha \circ H$ is the frame of the left module $\langle M \rangle H : \ast \to D$ defined by
\[
[H \circ \alpha]_d = \alpha_{(H : d)}
\]
for $d \in \|D\|$. The function
\[
\prod_{H} M : \prod_{E} M \to \prod_{D} \langle M \rangle H
\]
is defined by
\[
\alpha : \prod_{H} M = H \circ \alpha
\]
.

If $H : D \to E$ is a functor and $\alpha$ is a frame of a right module $M : E \to \ast$, then their composite $H \circ \alpha = \alpha \circ H$ is the frame of the right module $H(M) : D \to \ast$ defined by
\[
[H \circ \alpha]_d = \alpha_{(H : d)}
\]
for $d \in \|D\|$. The function
\[
\prod_{H} M : \prod_{E} M \to \prod_{D} H(M)
\]
is defined by
\[
\alpha : \prod_{H} M = H \circ \alpha
\]
.

Remark 4.2.12. The composite $H \circ \alpha$ so defined does form a frame of $\langle M \rangle H$ (resp. $H(M)$). In fact, we have the following.

Proposition 4.2.13. Let $H : D \to E$ be a functor.
For any left module $M : * \to E$, the diagram
\[
\begin{array}{ccc}
\prod_{*} E M & -\longrightarrow & \prod_{E} [!E] \langle M \rangle \\
\downarrow \prod_{H} E M & & \downarrow \prod_{H} [!E] \langle M \rangle \\
\prod_{D} \langle M \rangle H & = & \prod_{D} [!D] \langle M \rangle H = \prod_{D} H \langle \langle M \rangle \langle !E \rangle \rangle H
\end{array}
\]
commutes; that is,
\[
\prod_{H} M = \prod_{H} [!E] \langle M \rangle
\]

For any right module $M : E \to *$, the diagram
\[
\begin{array}{ccc}
\prod_{E} E M & -\longrightarrow & \prod_{E} \langle M \rangle [!E] \\
\downarrow \prod_{H} E M & & \downarrow \prod_{H} \langle M \rangle [!E] \\
\prod_{D} \langle M \rangle H & = & \prod_{D} H \langle \langle M \rangle \langle !E \rangle \rangle H
\end{array}
\]
commutes; that is,
\[
\prod_{H} M = \prod_{H} \langle M \rangle [!E]
\]

Proof. See Proposition 4.2.4 for the identities $\prod_{*} E M = \prod_{E} [!E] \langle M \rangle$ and $\prod_{*} D \langle M \rangle H = \prod_{D} [!D] \langle M \rangle H$, and observe that the identity
\[
H \langle \langle M \rangle \langle !E \rangle \rangle H = [H \circ !E] \langle M \rangle H = [!D] \langle M \rangle H
\]
holds. It is now clear that for any frame $\alpha \in \prod_{*} E M = \prod_{E} [!E] \langle M \rangle$, the composition $H \circ \alpha$ in Definition 4.2.11 and Definition 4.1.8 yields the same frame.

Proposition 4.2.14. Let $H : D \to E$ be a functor.

- The family of functions $\prod_{H} M : \prod_{*} E M \to \prod_{*} D \langle M \rangle H$, one for each left module $M : * \to E$, defines a left module cell
\[
\begin{array}{cc}
* & \xrightarrow{\prod_{H} E} [E] \\
{1} & \downarrow \quad \downarrow \langle H \rangle \\
* & \xrightarrow{\prod_{D} E} [D]
\end{array}
\]
In fact, the cell $\prod_{*} E$ is obtained from the cell $\prod_{H}$ in Proposition 4.1.12 by the pasting composition
\[
\begin{array}{cc}
* & \xrightarrow{\prod_{E} E} [E : E] \xrightarrow{[!E : E]} [E] \\
{1} & \downarrow \langle H \rangle \quad \downarrow \langle !H \rangle \\
* & \xrightarrow{\prod_{D} E} [D : D] \xrightarrow{[!D : D]} [D]
\end{array}
\]
Moreover, the assignment $H \mapsto \prod_{H}$ is contravariant functorial.

- The family of functions $\prod_{H} M : \prod_{E} E M \to \prod_{D} H \langle M \rangle$, one for each right module $M : E \to *$, defines a left module cell
\[
\begin{array}{cc}
* & \xrightarrow{\prod_{E} E} [E : E] \\
{1} & \downarrow \quad \downarrow \langle H \rangle \\
* & \xrightarrow{\prod_{D} E} [D : D]
\end{array}
\]
In fact, the cell $\prod_{*} E$ is obtained from the cell $\prod_{H}$ in Proposition 4.1.12 by the pasting composition
\[
\begin{array}{cc}
* & \xrightarrow{\prod_{E} E} [E : E] \xrightarrow{[!E : E]} [E] \\
{1} & \downarrow \langle H \rangle \quad \downarrow \langle !H \rangle \\
* & \xrightarrow{\prod_{D} E} [D : D] \xrightarrow{[!D : D]} [D]
\end{array}
\]
Moreover, the assignment $H \mapsto \prod_{H}$ is contravariant functorial.
4.3. Cylinders

Definition 4.3.1. Let $\mathcal{M} : X \to A$ be a module.

- Given a functor $G : A \to X$, a right cylinder $\alpha$ from $G$ to $\mathcal{M}$, written $\alpha : G \sim \mathcal{M}$ or diagrammatically as $X \xrightarrow{\alpha} A$, is defined by a frame $\alpha$ of the composite endomodule $G(\mathcal{M}) : A \to A$.

- Given a functor $F : X \to A$, a left cylinder $\alpha$ from $\mathcal{M}$ to $F$, written $\alpha : \mathcal{M} \sim F$ or diagrammatically as $X \xleftarrow{\alpha} A$, is defined by a frame $\alpha$ of the composite endomodule $(\mathcal{M}) F : X \to X$.

Remark 4.3.2.

1. The naturality of a right cylinder $\alpha : G \sim \mathcal{M}$ is expressed by the commutativity of the square

$$
\begin{array}{ccc}
s : G & \sim & \mathcal{M} \\
\downarrow f & & \downarrow f \\
t : G & \sim & \mathcal{M}
\end{array}
$$

for each $A$-arrow $f : s \to t$; and the naturality of a left cylinder $\alpha : \mathcal{M} \sim F$ is expressed by the commutativity of the square

$$
\begin{array}{ccc}
s : \mathcal{M} & \sim & F \cdot s \\
\downarrow f & & \downarrow f \cdot F \\
t : \mathcal{M} & \sim & F \cdot t
\end{array}
$$

for each $X$-arrow $f : s \to t$.

2. If $A$ is the terminal category in Definition 4.3.1, a right cylinder $X \xrightarrow{\alpha} *$ is identified with an arrow of the right module $\mathcal{M} : X \to *$; and if $X$ is the terminal category, a left cylinder $* \xleftarrow{\alpha} A$ is identified with an arrow of the left module $\mathcal{M} : * \to A$.

3. Consider a pair of functors $X \xrightarrow{G} F \xrightarrow{F} A$. Since

$$[G \circ F](A)[1_A] = G(F(A))$$

a natural transformation $\epsilon : G \circ F \to 1_A : A \to A$ is the same thing as a right cylinder $X \xrightarrow{\epsilon} A$ from $G$ to the representable module $F(A)$; and since

$$[1_X](X)[G \circ F] = (X)(F)(G)$$

a natural transformation $\eta : 1_X \to G \circ F : X \to X$ is the same thing as a left cylinder $X \xleftarrow{\eta} A$ from the corepresentable module $(X) G$ to $F$.

4. By Remark 4.1.2(4) and Remark 1.1.35(4),

$$
\prod_X (\mathcal{M}) F = \prod_X F (\mathcal{M}^{-})
$$

hence a left cylinder $\alpha : \mathcal{M} \sim F$ is the same thing as a right cylinder $\alpha : F \sim \mathcal{M}^{-}$.
Definition 4.3.3. Let $\mathcal{E}$ be a category and $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ be a module. Given a pair of functors $S : \mathcal{E} \to \mathcal{X}$ and $T : \mathcal{E} \to \mathcal{A}$, a [two-sided] cylinder $\alpha$ from $S$ to $T$ along $\mathcal{M}$, written $\alpha : S \Rightarrow T : E \Rightarrow \mathcal{M}$ or diagrammatically as $\begin{array}{c} S \xrightarrow{\alpha} E \xrightarrow{T} \mathcal{M} \xrightarrow{M} \mathcal{A} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ X \xrightarrow{\alpha} A \end{array}$, is defined by a frame $\alpha$ of the composite endomodule $S \cup \mathcal{M} \cup T : \mathcal{E} \Rightarrow \mathcal{A}$.

Remark 4.3.4.

(1) The naturality of a cylinder $\alpha : S \Rightarrow T : E \Rightarrow \mathcal{M}$ is expressed by the commutativity of the square $\begin{array}{ccc} e \circ S & \sim & T \circ e \\ h \downarrow & & \downarrow T \circ h \\ e' \circ S & \sim & T \circ e' \end{array}$ for each $E$-arrow $h : e \to e'$.

(2) A right cylinder $\begin{array}{c} X \xrightarrow{G} \mathcal{A} \\ \downarrow \downarrow \\ \mathcal{M} \xrightarrow{\alpha} \mathcal{A} \\ \downarrow \downarrow \\ X \xrightarrow{\alpha} \mathcal{A} \end{array}$ from $\mathcal{G}$ to $\mathcal{M}$ can be depicted as a two-sided cylinder $\begin{array}{c} \mathcal{G} \xrightarrow{\alpha} \mathcal{A} \\ \downarrow \downarrow \\ \mathcal{X} \xrightarrow{\alpha} \mathcal{A} \end{array}$ from $\mathcal{G}$ to the identity $\mathcal{1}_\mathcal{A}$ along $\mathcal{M}$; and a left cylinder $\begin{array}{c} X \xrightarrow{\mathcal{M}} \mathcal{A} \\ \downarrow \downarrow \\ \mathcal{F} \xrightarrow{\alpha} \mathcal{A} \\ \downarrow \downarrow \\ X \xrightarrow{\alpha} \mathcal{A} \end{array}$ from $\mathcal{M}$ to $\mathcal{F}$ can be depicted as a two-sided cylinder $\begin{array}{c} 1 \xrightarrow{\alpha} \mathcal{F} \\ \downarrow \downarrow \\ \mathcal{X} \xrightarrow{\alpha} \mathcal{A} \end{array}$ from the identity $\mathcal{1}_X$ to $\mathcal{F}$ along $\mathcal{M}$.

(3) Since $S(\mathcal{M}) \cup T = S(\mathcal{M}) \cup (S(\mathcal{M}) \cup T)$, the following right, two-sided, and left cylinders are the same thing:

$\begin{array}{c} X \xrightarrow{S} \mathcal{E} \xrightarrow{T} \mathcal{M} \xrightarrow{S(\mathcal{M})} \mathcal{A} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ X \xrightarrow{S} \mathcal{E} \xrightarrow{T} \mathcal{M} \xrightarrow{S(\mathcal{M})} \mathcal{A} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ X \xrightarrow{S} \mathcal{E} \xrightarrow{T} \mathcal{M} \xrightarrow{S(\mathcal{M})} \mathcal{A} \end{array}$

(4) By Example 4.1.3(1), a natural transformation $\alpha : S \Rightarrow T : E \Rightarrow C$ is the same thing as a cylinder $\alpha : S \Rightarrow T : E \Rightarrow (\mathcal{C})$ along the hom of $\mathcal{C}$. Conversely, since $\mathcal{M} = [\mathcal{M}_0][\mathcal{M}][\mathcal{M}_1]$ for any module $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ (see Remark 3.1.15(2)), a cylinder $\begin{array}{c} \mathcal{X} \xrightarrow{S} \mathcal{E} \xrightarrow{T} \mathcal{M} \xrightarrow{\mathcal{M}_0} \mathcal{A} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ \mathcal{X} \xrightarrow{S} \mathcal{E} \xrightarrow{T} \mathcal{M} \xrightarrow{\mathcal{M}_0} \mathcal{A} \end{array}$ from $S$ to $T$ along $\mathcal{M}$ is the same thing as a natural transformation $\begin{array}{c} \mathcal{X} \xrightarrow{S} \mathcal{E} \xrightarrow{T} \mathcal{M} \xrightarrow{\mathcal{M}_0} \mathcal{A} \end{array}$ from $S \circ \mathcal{M}_0$ to $\mathcal{M}_1 \circ T$ in the collage category $[\mathcal{M}]$. 
4.3. Cylinders

Since

\[ S \circ F \mapsto (E) T = S \circ (F \mapsto E) T \]

, a natural transformation

\[ \begin{array}{ccc}
  \text{S} & \text{C} & \text{T} \\
  \text{D} & \text{E} & \text{F(F(E))} \\
\end{array} \]

from \( S \circ F \) to \( T \) is the same thing as a cylinder

\[ \begin{array}{ccc}
  \text{S} & \text{C} & \text{T} \\
  \text{D} & \text{E} & \text{F(E)} \\
\end{array} \]

from \( S \) to \( T \) along the representable module \( F \mapsto E \); and since

\[ (S \mapsto E) [F \circ T] = T \mapsto (F \mapsto E) T \]

, a natural transformation

\[ \begin{array}{ccc}
  \text{S} & \text{C} & \text{T} \\
  \text{E} & \text{D} & \text{F} \\
\end{array} \]

from \( S \) to \( F \circ T \) is the same thing as a cylinder

\[ \begin{array}{ccc}
  \text{S} & \text{C} & \text{T} \\
  \text{E} & \text{D} & \text{(E)F} \\
\end{array} \]

from \( S \) to \( T \) along the corepresentable module \( \langle E \rangle F \).

**Definition 4.3.5.** Given a category \( E \) and a module \( M : X \to A \), the module of cylinders \( E \mapsto M \),

\[ \langle E, M \rangle : [E, X] \to [E, A] \]

, is defined by

\[ (S) \langle E, M \rangle (T) = \prod_{E} S \langle M \rangle T \]

for \( S \in [E, X] \) and \( T \in [E, X] \).

**Remark 4.3.6.**

(1) For a pair of functors \( S : E \to X \) and \( T : E \to A \), the set \( (S) \langle E, M \rangle (T) \) consists of all cylinders \( S \mapsto T : E \mapsto M \).

(2) Given a cylinder \( \alpha : S \mapsto T \) and natural transformations \( \sigma : S \mapsto S' \) and \( \tau : T \mapsto T' \) as in

\[ \begin{array}{ccc}
  \text{S} & \text{E} & \text{T'} \\
  \text{X} & \text{M} & \text{A} \\
\end{array} \]

, their composite in the module \( \langle E, M \rangle \) is the cylinder

\[ \begin{array}{ccc}
  \text{S'} & \text{E} & \text{T'} \\
  \text{X} & \text{M} & \text{A} \\
\end{array} \]
4.3. Cylinders

defined by
\[ \sigma \circ \alpha \circ \tau = \alpha : \prod E \langle M \rangle \tau \]
the image of a cylindrical frame \( \alpha \in \prod E S \langle M \rangle T \) under the function
\[ \prod E \langle M \rangle \tau : \prod E S \langle M \rangle T \to \prod E S' \langle M \rangle T' \]
By Example 1.1.29(3), each component of the cylinder \( \sigma \circ \alpha \circ \tau \) is given by
\[ [\sigma \circ \alpha \circ \tau]_e = \sigma_e \circ \alpha_e \circ \tau_e \]

(3) If \( M \) is given by the hom of a category \( C \), then the composition in (2) above is just the usual composition of natural transformations. The module of cylinders \( E \sim (C) \) (i.e. natural transformations \( E \to C \)) is thus the same thing as the hom of the functor category \( [E, C] \):
\[ \langle E, (C) \rangle = \langle E, C \rangle \]

(4) If \( E \) is small and \( M \) is locally small, then the module \( \langle E, M \rangle \) is locally small.

Proposition 4.3.7. Given a category \( E \) and a composite module \( P \langle N \rangle Q \) as in
\[
\begin{array}{ccc}
X \xrightarrow{PQ} A \\
\downarrow_{P} & 1 & \downarrow_{Q} \\
Y \xrightarrow{N} B
\end{array}
\]
the identity
\[
\begin{array}{ccc}
[E, X] \xrightarrow{[E, P]Q} [E, A] \\
\downarrow_{[E, P]} & 1 & \downarrow_{[E, Q]} \\
[E, Y] \xrightarrow{[E, N]} [E, B]
\end{array}
\]
\[ \langle E, P \langle N \rangle Q \rangle = [E, P] \langle E, N \rangle [E, Q] \]
holds.

Proof. For any \( G \in [E, X] \) and \( F \in [E, A] \),
\[
\begin{align*}
(G) \langle E, P \langle N \rangle Q \rangle (F) &= \prod E \langle P \langle N \rangle Q \rangle F \\
&= \prod E \langle G \circ P \langle N \rangle [Q \circ F] \\
&= (G \circ P) \langle E, N \rangle (Q \circ F) \\
&= (G : [E, P]) \langle E, N \rangle ([E, Q] : F) \\
&= (G) ([E, P] \langle E, N \rangle [E, Q]) (F)
\end{align*}
\]

Remark 4.3.8. The cell \( \langle E, P \langle N \rangle Q \rangle \xrightarrow{1} \langle E, N \rangle \) above sends each cylinder
\[
\begin{array}{ccc}
\xymatrix{ X \ar[r]^S \ar[dr]_P & E \ar[r]^T \ar[dl]_\alpha & A \ar[d]_Q } 
\end{array}
\]
to the cylinder

\[
\begin{array}{c}
\text{\(S \circ P\)} \\
\text{\(\alpha\)} \\
\text{\(Q \circ T\)} \\
\text{\(Y\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(B\)}
\end{array}
\]

defined by the same frame.

**Definition 4.3.9.** If \(\alpha : E \twoheadrightarrow M\) is a cylinder and \(\Phi : M \rightarrow N\) is a module morphism as in

\[
\begin{array}{c}
\text{\(E\)} \\
\text{\(\alpha\)} \\
\text{\(M\)} \\
\text{\(\alpha \circ \Phi\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(A\)}
\end{array}
\]

, then their composite \(\alpha \circ \Phi = \Phi \circ \alpha\) is the cylinder

\[
\begin{array}{c}
\text{\(E\)} \\
\text{\(\alpha \circ \Phi\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(\rightarrow\)} \\
\text{\(A\)}
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ \Phi \circ \langle \Phi \rangle \circ T = \alpha : \prod_E \langle \Phi \rangle \circ T
\]

, the image of a cylindrical frame \(\alpha \in \prod_E \langle M \rangle\) under the function

\[
\prod_E \langle \Phi \rangle \circ T : \prod_E \langle M \rangle\) \circ T \rightarrow \prod_E \langle N \rangle\) \circ T
\]

.

**Remark 4.3.10.** By Example 1.1.29(2), each component of the cylinder \(\alpha \circ \Phi\) is given by

\[
\left[\alpha \circ \Phi\right]_e = \alpha_e : (e : \langle \Phi \rangle \circ (T : e))
\]

.

**Definition 4.3.11.** Given a category \(E\) and a module morphism \(\Phi : M \rightarrow N : X \\rightarrow A\), the module morphism

\[
\langle E, \Phi \rangle : \langle E, M \rangle \rightarrow \langle E, N \rangle : [E, X] \rightarrow [E, A]
\]

, “postcomposition with \(\Phi\)”, is defined by

\[
(S \langle E, \Phi \rangle \circ T) = \prod_E S \langle \Phi \rangle \circ T
\]

for each pair of functors \(S : E \rightarrow X\) and \(T : E \rightarrow A\).

**Remark 4.3.12.**
(1) The module morphism \(\langle E, \Phi \rangle\) maps each cylinder \(\alpha : S \twoheadrightarrow T : E \twoheadrightarrow M\) to the cylinder \(\alpha \circ \Phi : S \twoheadrightarrow T : E \twoheadrightarrow N\) defined in Definition 4.3.9.
(2) The assignment \(\Phi \mapsto \langle E, \Phi \rangle\) is functorial; indeed, the functor

\[
\langle E, - \rangle : [X : A] \rightarrow [[E, X] : [E, A]]
\]

is defined by

\[
(S \langle E, M \rangle \circ T) = \prod_E S \langle M \rangle \circ T
\]

for \(S \in [E, X]\), \(T \in [E, A]\), and \(M \in [X : A]\).
Note. By Remark 1.2.2(3), the following definition is regarded as a special case of Definition 4.3.9, and vice versa.

**Definition 4.3.13.** Given a cylinder $\alpha$ and a cell $\Phi$ as in

\[
\begin{array}{c}
\text{E} \\
\text{X} \\
\text{P} \\
\text{Y} \\
\text{M} \\
\text{A} \\
\text{T} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{E} \\
\text{X} \\
\text{P} \\
\text{Y} \\
\text{M} \\
\text{A} \\
\text{T} \\
\end{array}
\]

their composite $\alpha \circ \Phi = \Phi \circ \alpha$ is the cylinder

\[
\begin{array}{c}
\text{E} \\
\text{X} \\
\text{P} \\
\text{Y} \\
\text{M} \\
\text{A} \\
\text{T} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{E} \\
\text{X} \\
\text{P} \\
\text{Y} \\
\text{M} \\
\text{A} \\
\text{T} \\
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ S(\Phi) T = \alpha : \prod_E S(\Phi) T
\]

, the image of a cylindrical frame $\alpha \in \prod_E S(M) T$ under the function

\[
\prod_E S(\Phi) T : \prod_E S(M) T \to \prod_E S(N) Q T = \prod_E [S \circ P] Q T
\]

.*

**Remark 4.3.14.**

(1) Each component of the cylinder $\alpha \circ \Phi$ is given by

\[
[\alpha \circ \Phi]_e = \alpha_e : (e \circ S)(\Phi)(T \circ e)
\]

(cf. Remark 4.3.10).

(2) If a cell is given by the hom of a functor $\mathcal{H}$ as in

\[
\begin{array}{c}
\text{E} \\
\text{C} \\
\text{H} \\
\text{B} \\
\text{M} \\
\text{C} \\
\text{T} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{E} \\
\text{C} \\
\text{H} \\
\text{B} \\
\text{M} \\
\text{C} \\
\text{T} \\
\end{array}
\]

, then the composite $\alpha \circ (\mathcal{H})$ is just the usual composite of a natural transformation and a functor; that is,

\[
\alpha \circ (\mathcal{H}) = \alpha \circ \mathcal{H}
\]

.*

**Note.** The postcomposition in Definition 4.3.11 and the identity in Proposition 4.3.7 allow the following definition.

**Definition 4.3.15.** Given a category $\text{E}$ and a cell

\[
\begin{array}{c}
\text{X} \\
\text{P} \\
\text{Y} \\
\text{M} \\
\text{A} \\
\text{T} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{X} \\
\text{P} \\
\text{Y} \\
\text{M} \\
\text{A} \\
\text{T} \\
\end{array}
\]


4.3. Cylinders

, the cell

\[
\begin{array}{c}
[E, X] \xrightarrow{(E, M)} [E, A] \\
[E, P] \xrightarrow{(E, \Phi)} [E, Q] \\
[E, Y] \xrightarrow{(E, N)} [E, B]
\end{array}
\]

, “postcomposition with \(\Phi\)”, is defined by the module morphism

\[
(E, M) \xrightarrow{(E, \Phi)} (E, P \langle N \rangle \ Q) = [E, P \langle E, N \rangle \ Q]
\]

, postcomposition with \(\Phi : M \rightarrow P \langle N \rangle \ Q\).

**Remark 4.3.16.**

1. The cell \((E, \Phi)\) sends each cylinder \(\alpha : S \leadsto T : E \leadsto M\) to the cylinder \(\alpha \circ \Phi : S \circ P \leadsto Q \circ T : E \leadsto N\) defined in Definition 4.3.13.

2. If \(\Phi\) is given by the hom of a functor \(H : C \rightarrow B\), the postcomposition cell

\[
\begin{array}{c}
[E, C] \xrightarrow{(E, (C))} [E, C] \\
[E, H] \xrightarrow{(E, (H))} [E, H] \\
[E, B] \xrightarrow{(E, (B))} [E, B]
\end{array}
\]

sends each cylinder \(\alpha : E \leadsto (C)\) to the cylinder \(\alpha \circ (H) : E \leadsto (B)\); that is, sends each natural transformation \(\alpha : E \rightarrow C\) to the natural transformation \(\alpha \circ H : E \rightarrow B\), the usual composite of a natural transformation and a functor (see Remark 4.3.14(2)). Hence the postcomposition cell \((E, (H))\) is the same thing as the hom

\[
\begin{array}{c}
[E, C] \xrightarrow{(E, C)} [E, C] \\
[E, H] \xrightarrow{(E, H)} [E, H] \\
[E, B] \xrightarrow{(E, B)} [E, B]
\end{array}
\]

of the postcomposition functor \([E, H]\); that is,

\[
(E, (H)) = (E, H)
\]

**Proposition 4.3.17.** If a cell \(\Phi\) is fully faithful, so is the postcomposition cell \((E, \Phi)\) for any category \(E\).

**Proof.** Since the postcomposition operation \((E, -)\) is functorial (see Remark 4.3.12(2)), it preserves isomorphisms. \(\square\)

**Note.** Proposition 4.3.18 is analogous to Proposition 1.2.25. The proofs given for them are almost identical.

**Proposition 4.3.18.** The assignment \(\Phi \mapsto (E, \Phi)\) of the postcomposition cell is functorial.

**Proof.** Clearly, the assignment \(\Phi \mapsto (E, \Phi)\) preserves the identities. To verify that it preserves the composition, let \(\Phi\) and \(\Psi\) be a composable pair of cells and consider the cells \((E, \Phi), (E, \Psi),\) and
\( \langle E, \Phi \circ \Psi \rangle \) depicted in

\[
\begin{align*}
X - \xrightarrow{M} & \to A \\
\to P & \phi \to Q \\
Y - \xrightarrow{N} & \to B \\
\to P' & \psi \to Q' \\
Z - \xrightarrow{\xi} & \to C \\
\end{align*}
\]

\[ [E, X] \xrightarrow{(E,M)} [E, A] \]

\[
\begin{align*}
\to [E,P] & \xrightarrow{(E,\Phi)} [E, Q] \\
\to [E,Y] & \xrightarrow{(E,N)} [E, B] \\
\to [E,P'] & \xrightarrow{(E,\Psi)} [E, Q'] \\
\to [E,Z] & \xrightarrow{(E,\xi)} [E, C] \\
\end{align*}
\]

We need to verify that the composition of the cells \( \langle E, \Phi \rangle \) and \( \langle E, \Psi \rangle \) yields the cell \( \langle E, \Phi \circ \Psi \rangle \). First note that \( [E, P' \circ P'] = [E, P] \circ [E, P'] \) and \( [E, Q' \circ Q] = [E, Q'] \circ [E, Q] \) by the functoriality of the operation \( [E, -] \). The cell \( \langle E, \Phi \rangle \circ \langle E, \Psi \rangle \) is defined by the module morphism \( (E, \Phi) \circ [E, P] \langle E, \Psi \rangle [E, Q] \) and the cell \( \langle E, \Phi \circ \Psi \rangle \) is defined by the module morphism \( (E, \Phi \circ P \circ \Psi) Q \). But by the functoriality of the operation \( \langle E, - \rangle \) (see Remark 4.3.12(2)) and Proposition 4.3.7,

\[ \langle E, \Phi \circ P \circ \Psi \rangle Q = \langle E, \Phi \rangle \circ \langle E, P \circ \Psi \rangle Q = \langle E, \Phi \rangle \circ [E, P] \langle E, \Psi \rangle [E, Q] \]

\[ \square \]

**Remark 4.3.19.**

(1) Given a small category \( E \), the functor

\[ \langle E, - \rangle : \text{MOD} \to \text{MOD} \]

is defined by the object function \( M \mapsto \langle E, M \rangle \) and the arrow function \( \Phi \mapsto \langle E, \Phi \rangle \), extending the functor \( \langle E, - \rangle \) in Remark 4.3.12(2) as shown in

\[
\begin{align*}
[X : A] \xrightarrow{(E,-)} [E, X] : [E, A] \\
\downarrow & \\
\text{MOD} & \xrightarrow{(E,-)} \text{MOD}
\end{align*}
\]

where \( \hookrightarrow \) denotes the canonical embedding in Remark 1.2.19(2).

(2) For \( E \) small, the identity \( \langle E, (H) \rangle = \langle E, H \rangle \) in Remark 4.3.16(2) is now expressed by the commutativity of

\[
\begin{align*}
\text{CAT} & \xrightarrow{(E,-)} \text{CAT} \\
\downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{MOD} & \xrightarrow{(E,-)} \text{MOD}
\end{align*}
\]

**Example 4.3.20.** The postcomposition

\[
\begin{align*}
[E, X] \xrightarrow{(E,M)} [E, A] \\
\downarrow & \\
[E, [M]] \xrightarrow{(E,[M])} [E, [M]]
\end{align*}
\]

with the unit cell in Remark 3.1.15(2) sends each cylinder \( \alpha : S \to T : E \to M \) to the natural transformation \( \alpha : S \circ M_0 \to M_1 \circ T : E \to [M] \) (see Remark 4.3.4(4)).
**Definition 4.3.21.** Given a category $\mathcal{E}$ and a cell morphisms $\tau : \Phi \to \Psi : \mathcal{M} \to \mathcal{N}$, the cell morphism

$$(\mathcal{E}, \tau) : (\mathcal{E}, \Phi) \to (\mathcal{E}, \Psi) : (\mathcal{E}, \mathcal{M}) \to (\mathcal{E}, \mathcal{N})$$

, “postcomposition with $\tau$”, is defined by the pair of postcomposition natural transformations

$$[\mathcal{E}, \tau_0] : [\mathcal{E}, \Phi_0] \to [\mathcal{E}, \Phi] \quad [\mathcal{E}, \tau_1] : [\mathcal{E}, \Phi_1] \to [\mathcal{E}, \Psi]$$

(see Preliminary 9), where $\tau_0 : \Phi_0 \to \Psi_0$ and $\tau_1 : \Phi_1 \to \Psi_1$ are the left and right components of $\tau$.

**Remark 4.3.22.**
(1) $(\mathcal{E}, \tau)$ so defined does form a cell morphism. Indeed, given a cylinder $\alpha : S \to T : \mathcal{E} \to \mathcal{M}$, the commutativity of

\[
\begin{array}{ccc}
\mathcal{S} : [\mathcal{E}, \Phi_0] & \overset{\alpha \circ (\mathcal{E}, \Phi)}{\longrightarrow} & [\mathcal{E}, \Phi_1] \overset{T}{\longrightarrow} \\
\mathcal{S} : [\mathcal{E}, \Psi_0] & \overset{\alpha \circ (\mathcal{E}, \Psi)}{\longrightarrow} & [\mathcal{E}, \Psi_1] \overset{T}{\longrightarrow} \\
\mathcal{S} : [\mathcal{E}, \tau_0] & \downarrow & \downarrow \tau_1 \circ T \\
\mathcal{S} : [\mathcal{E}, \tau_1] & \downarrow & \downarrow \tau_1 \circ T \\
\end{array}
\]

, i.e.

$\mathcal{S} \circ \Phi_0 \overset{\alpha \circ \Phi}{\longrightarrow} \Phi_1 \circ \mathcal{T}$

$\mathcal{S} \circ \tau_0 \downarrow \downarrow \tau_1 \circ \mathcal{T}$

$\mathcal{S} \circ \Psi_0 \overset{\alpha \circ \Psi}{\longrightarrow} \Psi_1 \circ \mathcal{T}$

follows from the commutativity of

\[
\begin{array}{ccc}
\mathcal{S} : [\mathcal{E}, \Phi_0] & \overset{\alpha \circ \Phi}{\longrightarrow} & [\mathcal{E}, \Phi_1] \overset{T}{\longrightarrow} \\
\mathcal{S} : [\mathcal{E}, \Psi_0] & \overset{\alpha \circ \Psi}{\longrightarrow} & [\mathcal{E}, \Psi_1] \overset{T}{\longrightarrow} \\
\mathcal{S} : [\mathcal{E}, \tau_0] & \downarrow & \downarrow \tau_1 \circ T \\
\mathcal{S} : [\mathcal{E}, \tau_1] & \downarrow & \downarrow \tau_1 \circ T \\
\end{array}
\]

for each object $e \in [\mathcal{E}]$.

(2) The assignment $\tau \mapsto (\mathcal{E}, \tau)$ defines the functor

$$(\mathcal{E}, -) : [\mathcal{M} : \mathcal{N}] \to [(\mathcal{E}, \mathcal{M}) : (\mathcal{E}, \mathcal{N})]$$

; indeed, by the definition of the cell morphism $(\mathcal{E}, \tau)$, the functoriality of $(\mathcal{E}, -)$ is reduced to that of $[\mathcal{E}, -] : [\mathcal{M}_i, \mathcal{N}_i] \to [[\mathcal{E}, \mathcal{M}_i], [\mathcal{E}, \mathcal{N}_i]]$ for $i = 0, 1$.

**Definition 4.3.23.** If $\mathcal{H}$ is a functor and $\alpha$ is a cylinder as in

\[
\begin{array}{ccc}
\mathcal{D} & \overset{\mathcal{H}}{\longrightarrow} & \mathcal{E} \\
\mathcal{S} & \overset{\alpha}{\longrightarrow} & \mathcal{T} \\
\mathcal{X} & \overset{\mathcal{M}}{\longrightarrow} & \mathcal{A} \\
\end{array}
\]

, then their composite $\mathcal{H} \circ \alpha = \alpha \circ \mathcal{H}$ is the cylinder

\[
\begin{array}{ccc}
\mathcal{D} & \overset{\mathcal{H} \circ \alpha}{\longrightarrow} & \mathcal{E} \\
\mathcal{H} \circ \mathcal{S} & \overset{\mathcal{T} \circ \mathcal{H}}{\longrightarrow} & \mathcal{T} \\
\mathcal{X} & \overset{\mathcal{M}}{\longrightarrow} & \mathcal{A} \\
\end{array}
\]

defined by

$\mathcal{H} \circ \alpha = \alpha : \prod_{\mathcal{H}} \mathcal{S}(\mathcal{M}) \mathcal{T}$

, the image of a cylindrical frame $\alpha \in \prod_{\mathcal{E}} \mathcal{S}(\mathcal{M}) \mathcal{T}$ under the function

$$\prod_{\mathcal{H}} \mathcal{S}(\mathcal{M}) \mathcal{T} : \prod_{\mathcal{E}} \mathcal{S}(\mathcal{M}) \mathcal{T} \to \prod_{\mathcal{D}} \mathcal{H}(\mathcal{S}(\mathcal{M}) \mathcal{T}) \mathcal{H} = \prod_{\mathcal{D}} [\mathcal{H} \circ \mathcal{S}](\mathcal{M}) [\mathcal{T} \circ \mathcal{H}]$$

.
Remark 4.3.24.
(1) Each component of the cylinder $H \circ \alpha$ is given by

$$[H \circ \alpha]_d = \alpha_{(H \cdot d)}$$

(see Definition 4.1.8).
(2) If $D$ is a subcategory of $E$, then the composition of the inclusion $D \hookrightarrow E$ with $\alpha$ yields a cylinder $D \circ \alpha$, the restriction of $\alpha$ to $D$.
(3) If $\mathcal{M}$ is given by the hom of a category, then the composite $H \circ \alpha$ is just the usual composite of a functor and a natural transformation.

Note. By Remark 4.3.4(2), the following definition is a special case of Definition 4.3.23 (and vice versa by Remark 4.3.4(3)).

Definition 4.3.25.

- If $K$ is a functor and $\alpha$ is a right cylinder as in

$$\begin{array}{c}
\xymatrix{
X \ar[r]^G & A \ar[l]^{K} \\
\ar[r]_M & \\
\mathcal{M} & \\
} 
\end{array}$$

, then their composite $K \circ \alpha = \alpha \circ K$ is the two-sided cylinder

$$\begin{array}{c}
\xymatrix{
X \ar[r]^{K \circ \alpha} & A \ar[l]^{K} \\
\ar[r]_M & \\
\mathcal{M} & \\
} 
\end{array}$$

defined by

$$K \circ \alpha = \alpha \cdot \prod_K G(\mathcal{M})$$

, the image of a cylindrical frame $\alpha \in \prod_A G(\mathcal{M})$ under the function

$$\prod_K G(\mathcal{M}) : \prod_K G(\mathcal{M}) \rightarrow \prod_E K(G(\mathcal{M})) K = \prod_E [K \circ G](\mathcal{M}) K$$

- If $K$ is a functor and $\alpha$ is a left cylinder as in

$$\begin{array}{c}
\xymatrix{
E \ar[r]^K & X \ar[l]_{M} & A \ar[l]_{F} \\
\ar[r]_\alpha & \\
\mathcal{M} & \\
} 
\end{array}$$

, then their composite $K \circ \alpha = \alpha \circ K$ is the two-sided cylinder

$$\begin{array}{c}
\xymatrix{
X \ar[r]^{K \circ \alpha} & A \ar[l]^{F} \\
\ar[r]_M & \\
\mathcal{M} & \\
} 
\end{array}$$

defined by

$$K \circ \alpha = \alpha \cdot \prod_K (\mathcal{M}) F$$

, the image of a cylindrical frame $\alpha \in \prod_X (\mathcal{M}) F$ under the function

$$\prod_K (\mathcal{M}) F : \prod_K (\mathcal{M}) F \rightarrow \prod_E K(\mathcal{M}) F K = \prod_E K(\mathcal{M}) [F \circ K]$$
Remark 4.3.26. Each component of the cylinder $K \circ \alpha$ is given by

$$[K \circ \alpha]_e = \alpha_{(K:e)}$$

(see Definition 4.1.8).

Definition 4.3.27. Given a functor $H : D \to E$ and a module $M : X \to A$, the cell

$$\begin{array}{ccc}
[E, X] & \xrightarrow{(E,M)} & [E, A] \\
[H,X] & \downarrow & \downarrow [H,A] \\
[D, X] & \xrightarrow{(D,M)} & [D, A]
\end{array}$$

“precomposition with $H$”, is defined by

$$(S) \{H,M\}(T) = \prod_H S(M) T$$

for each pair of functors $S : E \to X$ and $T : E \to A$.

Remark 4.3.28.
(1) The cell $\{H,M\}$ sends each cylinder $\alpha : S \leadsto T : E \leadsto M$ to the cylinder $H \circ \alpha : H \circ S \leadsto T \circ H : D \leadsto M$ defined in Definition 4.3.23.
(2) The precomposition cell $\{H,M\}$ is obtained from the left module cell in Proposition 4.1.12 by the pasting composition

$$\begin{array}{ccc}
\times & \xrightarrow{(M)_-} & [E, X] \times [E, A] \\
\downarrow \quad \quad \downarrow [H,H] & & \downarrow [H,X] \times [H,A] \\
\times & \xrightarrow{(M)_-} & [D, X] \times [D, A]
\end{array}$$

, where $\times (M)_-$ denotes the functor given by the assignment $(S,T) \mapsto S(M) T$. (See Remark 1.2.5.)
(3) If $M$ is given by the hom of a category $C$, then the precomposition cell

$$\begin{array}{ccc}
[E, C] & \xrightarrow{(E,C)} & [E, C] \\
[H,C] & \downarrow & \downarrow [H,C] \\
[D, C] & \xrightarrow{(D,C)} & [D, C]
\end{array}$$

sends each cylinder $\alpha : E \leadsto C$ to the cylinder $H \circ \alpha : D \leadsto C$; that is, sends each natural transformation $\alpha : E \to C$ to the natural transformation $H \circ \alpha : D \to C$, the usual composite of a functor and a natural transformation. Hence the precomposition cell $\{H, C\}$ is the same thing as the hom

$$\begin{array}{ccc}
[E, C] & \xrightarrow{(E,C)} & [E, C] \\
[H,C] & \downarrow & \downarrow [H,C] \\
[D, C] & \xrightarrow{(D,C)} & [D, C]
\end{array}$$

of the precomposition functor $[H,C]$; that is,

$$\{H, C\} = \{H, C\}$$
Proposition 4.3.29. Given a module \( \mathcal{M} \), the assignment \( H \mapsto (H, \mathcal{M}) \) defines the contravariant functor
\[
\langle -, \mathcal{M} \rangle : \text{Cat}^\to \to \text{MOD}
\].

Proof. Since the cell \( (H, \mathcal{M}) \) is obtained from the cell \( \prod_H \) by the pasting composition in Remark 4.3.28(2), the functoriality of the assignment \( H \mapsto (H, \mathcal{M}) \) is reduced to that of the assignment \( H \mapsto \prod_H \) (see Proposition 4.1.12) by virtue of Proposition 1.2.34.
\[ \square \]

Example 4.3.30. Let \( \mathbf{E} \) be a category and \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) be a module. Given an object \( e \in |\mathbf{E}| \), precomposition with the functor \( e : * \to \mathbf{E} \) yields the cell
\[
\begin{array}{c}
\left[ \mathbf{E}, \mathbf{X} \right]_{\mathcal{M}} \\
\left[ e, \mathbf{X} \right] \\
\end{array}
\begin{array}{c}
\left[ \mathbf{E}, \mathbf{A} \right] \\
\left[ e, \mathbf{A} \right] \\
\end{array}
\]

\[ \mathbf{X} \to \mathbf{A} \]

“evaluation at \( e \)”, which sends each cylinder \( \alpha : \mathbf{S} \to \mathbf{T} : \mathbf{E} \to \mathcal{M} \) to the \( \mathcal{M} \)-arrow \( \alpha_e : e : \mathbf{S} \to \mathbf{T} : e \), the component of \( \alpha \) at \( e \).

Remark 4.3.31. If \( \mathcal{M} \) in the example above is given by the hom of a category \( \mathbf{C} \), we have a cell
\[
\begin{array}{c}
\left[ \mathbf{E}, \mathbf{C} \right]_{\mathcal{M}} \\
\left[ e, \mathbf{C} \right] \\
\end{array}
\begin{array}{c}
\left[ \mathbf{E}, \mathbf{C} \right] \\
\left[ e, \mathbf{C} \right] \\
\end{array}
\]

\[ \mathbf{C} \to \mathbf{C} \]

; by Remark 4.3.28(3), this cell is the same thing as the hom
\[
\begin{array}{c}
\left[ \mathbf{E}, \mathbf{C} \right]_{\mathcal{M}} \\
\left[ e, \mathbf{C} \right] \\
\end{array}
\begin{array}{c}
\left[ \mathbf{E}, \mathbf{C} \right] \\
\left[ e, \mathbf{C} \right] \\
\end{array}
\]

\[ \mathbf{C} \to \mathbf{C} \]

of the precomposition functor \( [e, \mathbf{C}] : [\mathbf{E}, \mathbf{C}] \to \mathbf{C} \), “evaluation at \( e \)” (see Preliminary 14).

Theorem 4.3.32. There is a functor
\[
\langle -, -, \rangle : \text{Cat}^\to \times \text{MOD} \to \text{MOD}
\]
such that

(1) for each small category \( \mathbf{E} \), \( \langle E, -, \rangle : \text{MOD} \to \text{MOD} \) coincides with the functor in Remark 4.3.19(1).

(2) for each locally small module \( \mathcal{M} \), \( \langle -, \mathcal{M} \rangle : \text{Cat}^\to \to \text{MOD} \) coincides with the functor in Proposition 4.3.29.

Proof. By the bifunctor lemma (see [ML98], p37 Proposition 1), it suffices to show that the square
\[
\begin{array}{c}
\langle \mathbf{E}, \mathcal{M} \rangle \\
\left( \mathbf{H}, \mathcal{M} \right) \\
\end{array}
\begin{array}{c}
\langle \mathbf{E}, \mathcal{N} \rangle \\
\left( \mathbf{H}, \mathcal{N} \right) \\
\end{array}
\begin{array}{c}
\langle \mathbf{D}, \mathcal{M} \rangle \\
\left( \mathbf{D}, \mathcal{N} \right) \\
\end{array}
\]

commutes for any functor \( H : \mathbf{D} \to \mathbf{E} \) and any cell \( \Phi : \mathcal{M} \to \mathcal{N} \); that is,
\[
\alpha : \langle \mathbf{E}, \Phi \rangle : \langle \mathbf{H}, \mathcal{N} \rangle = \alpha : \langle \mathbf{H}, \mathcal{M} \rangle : \langle \mathbf{D}, \Phi \rangle
\]

for any cylinder \( \alpha : \mathbf{S} \to \mathbf{T} : \mathbf{E} \to \mathcal{M} \). But by Remark 4.3.16(1) and Remark 4.3.28(1),
\[
\alpha : \langle \mathbf{E}, \Phi \rangle : \langle \mathbf{H}, \mathcal{N} \rangle = \mathbf{H} \circ (\alpha \circ \Phi) = (\mathbf{H} \circ \alpha) \circ \Phi = \alpha : \langle \mathbf{H}, \mathcal{M} \rangle : \langle \mathbf{D}, \Phi \rangle
\]

\[ \square \]
**Definition 4.3.33.** Let $\mathbf{D}$ be a category and $\mathcal{M}: \mathbf{E} \to \mathbf{E}$ be an endomodule. For any frame $\alpha$ of $\mathcal{M}$, the frame $[\mathbf{D}, \alpha]$ of the endomodule $\langle \mathbf{D}, \mathcal{M} \rangle: [\mathbf{D}, \mathbf{E}] \to [\mathbf{D}, \mathbf{E}]$ is defined by

$$[\mathbf{D}, \alpha]_H = H \circ \alpha : H \Rightarrow H : \mathbf{D} \Rightarrow \mathcal{M}$$

for $H$ a functor $\mathbf{D} \to \mathbf{E}$.

**Remark 4.3.34.**

1. $[\mathbf{D}, \alpha]$ so defined does form a frame of $\langle \mathbf{D}, \mathcal{M} \rangle$. Indeed, given a natural transformation $\tau: H \Rightarrow H': \mathbf{D} \Rightarrow \mathbf{E}$, the commutativity of the square

$$\begin{array}{ccc}
H & \xrightarrow{[\mathbf{D}, \alpha]_H} & H' \\
\downarrow \tau & & \downarrow \tau' \\
H' & \xrightarrow{[\mathbf{D}, \alpha]_{H'}} & H'
\end{array}$$

is reduced to the commutativity of the square

$$\begin{array}{ccc}
d \cdot H & \xrightarrow{[\mathbf{D}, \alpha]_d} & H \cdot d' \\
\downarrow d & & \downarrow d' \\
d \cdot H' & \xrightarrow{[\mathbf{D}, \alpha]_{d'}} & H' \cdot d'
\end{array}$$

for each object $d \in \parallel \mathbf{D}$, i.e., the naturality of $\alpha$.

2. If $\mathbf{D}$ is a category and $\alpha$ is a cylinder

$$\begin{array}{ccc}
S & \xrightarrow{\alpha} & \mathbf{E} \\
\downarrow \tau & & \downarrow \tau \\
X & \xrightarrow{\alpha} & \mathbf{A}
\end{array}$$

, i.e. a frame of the endomodule $\langle S \mathcal{M} \rangle T: \mathbf{E} \Rightarrow \mathbf{E}$, then $[\mathbf{D}, \alpha]$ is a frame of the endomodule $\langle \mathbf{D}, S \mathcal{M} \rangle T = [\mathbf{D}, S] \langle \mathbf{D}, \mathcal{M} \rangle [\mathbf{D}, T]$ (see Proposition 4.3.7), i.e. a cylinder

$$\begin{array}{ccc}
[D, S] & \xrightarrow{[\mathbf{D}, \alpha]} & [D, E] \\
\downarrow & & \downarrow \\
[D, X] & \xrightarrow{[\mathbf{D}, T]} & [D, A]
\end{array}$$

. The component of $[\mathbf{D}, \alpha]$ at a functor $H: \mathbf{D} \Rightarrow \mathbf{E}$ is the cylinder $H \circ \alpha$ defined in Definition 4.3.23. If $\mathcal{M}$ is given by the hom of a category $\mathbf{C}$, then $[\mathbf{D}, \alpha]$ is the same thing as the postcomposition natural transformation

$$[\mathbf{D}, \alpha]: [\mathbf{D}, S] \Rightarrow [\mathbf{D}, T]: [\mathbf{D}, \mathbf{E}] \Rightarrow [\mathbf{D}, \mathbf{C}]$$

(see Preliminary 9).

3. Let $\mathbf{E}$ be a category.

   - If $\alpha$ is a right cylinder

     $$\begin{array}{ccc}
     X & \xrightarrow{\alpha} & \mathbf{A} \\
     \downarrow \tau & & \downarrow \tau \\
     \mathbf{M} & \xrightarrow{\alpha} & \mathbf{A}
     \end{array}$$

     , i.e. a frame of the endomodule $\langle G \mathcal{M} \rangle : \mathbf{A} \Rightarrow \mathbf{A}$, then $[\mathbf{E}, \alpha]$ is a frame of the endomodule $\langle \mathbf{E}, G \mathcal{M} \rangle = [\mathbf{E}, G] \langle \mathbf{E}, \mathcal{M} \rangle$ (see Proposition 4.3.7), i.e. a right cylinder

     $$\begin{array}{ccc}
     [\mathbf{E}, X] & \xrightarrow{[\mathbf{E}, \alpha]} & [\mathbf{E}, \mathbf{A}] \\
     \downarrow & & \downarrow \\
     [\mathbf{E}, \mathbf{M}] & \xrightarrow{[\mathbf{E}, \alpha]} & [\mathbf{E}, \mathbf{A}]
     \end{array}$$
4.4. Extraordinary cylinders

This section is largely analogous to Section 4.3.

**Definition 4.4.1.** Let E be a category and M : X → A be a module.

- Given an object x ∈ ||X|| and a bifunctor K : E×E → A, an [extraordinary] cylinder α from x to K along M, written α : x ∼ K : E ∼ M or diagrammatically as

\[ x \xrightarrow{\alpha} X \xrightarrow{\ M \ } A \]

a cylindrical frame α of the composite left module x(⟨M⟩)K : * → E×E (note that a left module * → E×E is the same thing as an endomodule E → E).

- Given an object a ∈ ||A|| and a bifunctor K : E×E → X, an [extraordinary] cylinder α from K to a along M, written α : K ∼ a : E ∼ M or diagrammatically as

\[ K \xrightarrow{\alpha} A \]  

a cylindrical frame α of the composite right module K(⟨M⟩)a : E×E → * K(C)c : E×E → * (note that a right module E×E → * is the same thing as an endomodule E→E×E).

**Remark 4.4.2.**

1. The naturality of an extraordinary cylinder α : x ∼ K : E ∼ M (resp. α : K ∼ a : E ∼ M) is expressed by the commutativity of the square

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & K(e,e) \\
\downarrow & & \downarrow K(e,h) \\
K(e',e') & \xrightarrow{K(h,e')} & K(e,e')
\end{array}
\]

for each E-arrow h : e → e' (cf. Remark 4.1.2(3)).

2. Any of the cylinders we saw in Section 4.3 has the naturality expressed in the “ordinary” form, and called ordinary to distinguish it from extraordinary cylinders. However, the adjectives “ordinary” and “extraordinary” are often omitted when the context makes it clear which type of cylinder is being talked about. In fact, some bicylinders (see Section 4.7) have both ordinary and extraordinary naturalities.

**Notation 4.4.3.** Given a bifunctor K : E×E → C and an object c ∈ ||C||, an extraordinary natural transformation α from c to K (resp. K to c) is denoted by

\[ \alpha : c \sim K : E \rightarrow C \quad \alpha : K \sim c : E \rightarrow C \]
Remark 4.4.4. Noting Example 4.1.3(2), we see that an extraordinary natural transformation in a category $C$ is the same thing as an extraordinary cylinder along the hom of $C$. Conversely, an extraordinary cylinder along a module $M : X \to A$ is the same thing as an extraordinary natural transformation in the collage category $[M]$ (cf. Remark 4.3.4(4)).

Definition 4.4.5. Given a category $E$ and a module $M : X \to A$,

- the module of extraordinary cylinders $E \Rightarrow M$,

$$ (E, M) : X \to [E^\ast \times E, A] $$

, is defined by

$$ (x) (E, M) (K) = \prod_{E} x(M) K $$

for $x \in X$ and $K \in [E^\ast \times E, A]$.

- the module of extraordinary cylinders $E \Rightarrow M$,

$$ (E, M) : [E^\ast \times E, X] \to A $$

, is defined by

$$ (K) (E, M) (a) = \prod_{E^\ast} K(M) a $$

for $a \in A$ and $K \in [E^\ast \times E, X]$.

Remark 4.4.6.

(1) For an object $x \in [X]$ and a bifunctor $K : E^\ast \times E \to A$, the set $(\alpha) (E, M) (K)$ consists of all cylinders $x \Rightarrow K : E \Rightarrow M$; and for an object $a \in [A]$ and a bifunctor $K : E^\ast \times E \to X$, the set $(K) (E, M) (a)$ consists of all cylinders $K \Rightarrow a : E \Rightarrow M$.

(2) We use the same symbol $(E, M)$ for the module of extraordinary cylinders and the module of ordinary cylinders (see Definition 4.3.5), and let the context determine which is meant.

(3) Given an $X$-arrow $f : x' \Rightarrow x$, a cylinder $\alpha : x \Rightarrow K$, and a natural transformation $\tau : K \Rightarrow K'$, all as in

$$ \xymatrix{ & E^\ast \times E \ar[ld]^{f} \ar@{=}[r] \ar[rd]_{\alpha} & K \ar[d]^{\tau'} \ar[l]_{K} \ar[r]^{f'} & \cr x' & X & M & A } $$

, their composite in the module $(E, M)$ is the cylinder

$$ \xymatrix{ & E^\ast \times E \ar[ld]^{f} \ar@{=}[r] \ar[rd]_{\alpha} & K \ar[d]^{\tau'} \ar[l]_{K} \ar[r]^{f'} & \cr x' & X & M & A } $$

defined by

$$ f \circ \alpha \circ \tau = \alpha \circ f(M) \circ \tau = \alpha : \prod_{E} f(M) \tau $$

, the image of a cylindrical frame $\alpha \in \prod_{E} x(M) K$ under the function

$$ \prod_{E} f(M) \tau : \prod_{E} x(M) K \to \prod_{E} x'(M) K' $$

. Each component of the cylinder $\alpha \circ \tau$ is given by

$$ [f \circ \alpha \circ \tau]_{e} = f \circ \alpha_{e} \circ \tau_{(e,e)} $$
4.4. Extraordinary cylinders 87

(cf. Remark 4.3.6(2)).

- Given an A-arrow $f : a \to a'$, a cylinder $\alpha : K \sim a$, and a natural transformation $\tau : K' \to K$, all as in

$$
\begin{array}{c}
\begin{array}{cc}
E^- \times E & \ast \\
\downarrow \tau & \downarrow \alpha \\
X & \sim M & \to A
\end{array}
\end{array}
$$

, their composite in the module $\langle E, M \rangle$ is the cylinder

$$
\begin{array}{c}
\begin{array}{cc}
E^- \times E & \ast \\
\downarrow \tau \circ \alpha & \downarrow \alpha' \\
X & \sim M & \to A
\end{array}
\end{array}
$$

defined by

$$
\tau \circ \alpha \circ f = \alpha \circ \tau \langle M \rangle f = \alpha : \prod_{E^-} \tau \langle M \rangle f
$$

, the image of a cylindrical frame $\alpha \in \prod_{E^-} K \langle M \rangle a$ under the function

$$
\prod_{E^-} \tau \langle M \rangle f : \prod_{E^-} K \langle M \rangle a \to \prod_{E^-} K' \langle M \rangle a'
$$

. Each component of the cylinder $\tau \circ \alpha$ is given by

$$
[\tau \circ \alpha \circ f]_e = \tau_{(e,e)} \circ \alpha_e \circ f
$$

(cf. Remark 4.3.6(2)).

(4) Definition 4.4.5 has a special version where $M$ is given by the hom of a category: Let $E$ and $C$ be categories.

- The module of extraordinary natural transformations $E \to C$,

$$
\langle E, C \rangle : C \to [E^- \times E, C]
$$

, is defined by

$$
(c) \langle E, C \rangle (K) = \prod_E c \langle C \rangle K
$$

for $c \in C$ and $K \in [E^- \times E, C]$; that is,

$$
\langle E, C \rangle := \langle E, \langle C \rangle \rangle
$$

.

- The module of extraordinary natural transformations $E \to C$,

$$
\langle E, C \rangle : [E^- \times E, C] \to C
$$

, is defined by

$$
(K) \langle E, C \rangle (c) = \prod_E K \langle C \rangle c
$$

for $c \in C$ and $K \in [E^- \times E, C]$; that is,

$$
\langle E, C \rangle := \langle E, \langle C \rangle \rangle
$$

.

For an object $c \in \|C\|$ and a bifunctor $K : E^- \times E \to C$,

- the set $(c) \langle E, C \rangle (K)$ consists of all extraordinary natural transformations $c \sim K : E \to C$.
- the set $(K) \langle E, C \rangle (c)$ consists of all extraordinary natural transformations $K \sim c : E \to C$. 
We use the same symbol \( /uni27E8 \), \( /uni27E9 \), for the module of extraordinary natural transformations and the hom of the functor category \([E, C]\) (see Remark 4.3.6(3)), and let the context determine which is meant.

**Proposition 4.4.7.** Given a category \( E \) and a composite module \( P \langle N \rangle Q \) as in

\[
\begin{array}{c c c c c}
X & \overset{P}{\longrightarrow} & A \\
\downarrow P & & \downarrow Q \\
Y & \overset{\mathcal{N}}{\longrightarrow} & B
\end{array}
\]

\( \cdot \) the identity

\[
\begin{array}{c c c c c}
X & \overset{\langle E, P \langle N \rangle Q \rangle}{\longrightarrow} & [E^* \times E, A] \\
\downarrow P & & \downarrow [E^* \times E, Q] \\
Y & \overset{\langle E, \mathcal{N} \rangle}{\longrightarrow} & [E^* \times E, B]
\end{array}
\]

, i.e.

\[
\langle E, P \langle N \rangle Q \rangle = P \langle E, \mathcal{N} \rangle [E^* \times E, Q]
\]

\( \cdot \) holds.

\( \cdot \) the identity

\[
\begin{array}{c c c c c}
[E^* \times E, X] & \overset{\langle E, P \langle N \rangle Q \rangle}{\longrightarrow} & A \\
\downarrow [E^* \times E, P] & & \downarrow Q \\
[E^* \times E, Y] & \overset{\langle E, \mathcal{N} \rangle}{\longrightarrow} & B
\end{array}
\]

, i.e.

\[
\langle E, P \langle N \rangle Q \rangle = [E^* \times E, P] \langle E, \mathcal{N} \rangle Q
\]

, holds.

**Proof.** For any \( x \in X \) and \( K \in [E^* \times E, A] \),

\[
(x) \langle E, P \langle N \rangle Q \rangle (K) = \prod_E x \langle P \langle N \rangle Q \rangle K
\]

\[
= \prod_E (x : P) \langle N \rangle [Q \circ K]
\]

\[
= (x : P) \langle E, \mathcal{N} \rangle (Q \circ K)
\]

\[
= (x : P) \langle E, \mathcal{N} \rangle ([E^* \times E, Q] \circ K)
\]

\[
= (x) \langle P \langle E, \mathcal{N} \rangle [E^* \times E, Q] \rangle (K)
\]

. \[ \square \]

**Remark 4.4.8.** The cell

\[
\begin{array}{c c c c c}
X & \overset{\langle E, P \langle N \rangle Q \rangle}{\longrightarrow} & [E^* \times E, A] \\
\downarrow P & & \downarrow [E^* \times E, Q] \\
Y & \overset{\langle E, \mathcal{N} \rangle}{\longrightarrow} & [E^* \times E, B]
\end{array}
\]

\( \cdot \) resp.

\[
\begin{array}{c c c c c}
[E^* \times E, X] & \overset{\langle E, P \langle N \rangle Q \rangle}{\longrightarrow} & A \\
\downarrow [E^* \times E, P] & & \downarrow Q \\
[E^* \times E, Y] & \overset{\langle E, \mathcal{N} \rangle}{\longrightarrow} & B
\end{array}
\]

\( \cdot \) sends each cylinder

\[
\begin{array}{c c c c c}
\ast & E^* \times E & \overset{\alpha}{\longrightarrow} & \ast \\
\downarrow \alpha & \downarrow \alpha & & \downarrow \alpha
\end{array}
\]

\( \cdot \) resp.

\[
\begin{array}{c c c c c}
\ast & E^* \times E & \overset{\alpha}{\longrightarrow} & \ast \\
\downarrow \alpha & \downarrow \alpha & & \downarrow \alpha
\end{array}
\]
4.4. Extraordinary cylinders

to the cylinder

\[
\begin{array}{c}
\xymatrix{ \ast & E \times E \\
\downarrow^x \downarrow^\alpha & \downarrow^K \\
X \ar[r]_-{\frac{M}{\Phi}} & A }
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{ \ast & E \times E \\
\downarrow^\kappa \downarrow^\alpha & \downarrow^{\Phi \circ \alpha} \\
Y \ar[r]_-{\frac{N}{\Phi}} & B }
\end{array}
\]

defined by the same frame.

**Definition 4.4.9.**

- If \( \alpha : E \rightarrow M \) is a cylinder and \( \Phi : M \rightarrow N \) is a module morphism as in

\[
\begin{array}{c}
\xymatrix{ \ast & E \times E \\
\downarrow^x \downarrow^\alpha & \downarrow^K \\
X \ar[r]_-{\frac{M}{\Phi}} & A }
\end{array}
\]

, then their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cylinder

\[
\begin{array}{c}
\xymatrix{ \ast & E \times E \\
\downarrow^x \downarrow^\alpha \circ \Phi & \downarrow^K \\
X \ar[r]_-{\frac{N}{\Phi}} & A }
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ x(\Phi) K = \alpha \cdot \prod_E x(\Phi) K
\]

, the image of a cylindrical frame \( \alpha \in \prod_E x(M) K \) under the function

\[
\prod_E x(\Phi) K : \prod_E x(M) K \rightarrow \prod_E x(N) K
\]

- If \( \alpha : E \rightarrow M \) is a cylinder and \( \Phi : M \rightarrow N \) is a module morphism as in

\[
\begin{array}{c}
\xymatrix{ E \times E & \ast \\
\downarrow^K \downarrow^\alpha & \downarrow^a \\
X \ar[r]_-{\frac{M}{\Phi}} & A }
\end{array}
\]

, then their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cylinder

\[
\begin{array}{c}
\xymatrix{ E \times E & \ast \\
\downarrow^K \downarrow^{\alpha \circ \Phi} & \downarrow^a \\
X \ar[r]_-{\frac{N}{\Phi}} & A }
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ K(\Phi) a = \alpha \cdot \prod_{E^\circ} K(\Phi) a
\]

, the image of a cylindrical frame \( \alpha \in \prod_{E^\circ} K(M) a \) under the function

\[
\prod_{E^\circ} K(\Phi) a : \prod_{E^\circ} K(M) a \rightarrow \prod_{E^\circ} K(N) a
\]

**Remark 4.4.10.** Each component of the cylinder \( \alpha \circ \Phi : x \rightarrow K \) (resp. \( \alpha \circ \Phi : K \rightarrow a \)) is given by

\[
[\alpha \circ \Phi]_e = \alpha_{e^\circ} (x)(\Phi)(K(e,e)) \quad \text{resp.} \quad [\alpha \circ \Phi]_e = \alpha_{e^\circ} (K(e,e))(\Phi)(a)
\]

(cf. Remark 4.3.10).
**Definition 4.4.11.** Given a category $E$ and a module morphism $\Phi : M \to N : X \to A$,

$\triangleright$ the module morphism $$(E, \Phi) : (E, M) \to (E, N) : X \to [E^\sim \times E, A]$$

, "postcomposition with $\Phi$", is defined by

$$(x) (E, \Phi)(K) = \prod_{E^\sim} x (\Phi) K$$

for each pair of an object $x \in \|X\|$ and a bifunctor $K : E^\sim \times E \to A$.

$\triangleright$ the module morphism $$(E, \Phi) : (E, M) \to (E, N) : [E^\sim \times E, X] \to A$$

, "postcomposition with $\Phi$", is defined by

$$(K) (E, \Phi)(a) = \prod_{E^\sim} K (\Phi)a$$

for each pair of an object $a \in \|A\|$ and a bifunctor $K : E^\sim \times E \to X$.

**Remark 4.4.12.**

1. The module morphism $$(E, \Phi) : (E, M) \to (E, N) : X \to [E^\sim \times E, A]$$ maps each cylinder $\alpha : x \to K : E \to M$ to the cylinder $\alpha \circ \Phi : x \to K : E \to N$ defined in Definition 4.4.9; and the module morphism $$(E, \Phi) : (E, M) \to (E, N) : [E^\sim \times E, X] \to A$$ maps each cylinder $\alpha : K \to a : E \to M$ to the cylinder $\alpha \circ \Phi : K \to a : E \to X$.

2. The assignment $\Phi \mapsto (E, \Phi)$ is functorial; indeed, the functor $$(E, -) : [X : A] \to [X : [E^\sim \times E, A]]$$ is defined by

$$(x) (E, M)(K) = \prod_{E^\sim} x (M) K$$

for $x \in X, K \in [E^\sim \times E, A]$, and $M \in [X : A]$; and the functor $$(E, -) : [X : A] \to [[E^\sim \times E, X] : A]$$ is defined by

$$(K) (E, M)(a) = \prod_{E^\sim} K (M)a$$

for $a \in A, K \in [E^\sim \times E, X]$, and $M \in [X : A]$.

**Note.** By Remark 1.2.2(3), the following definition is regarded as a special case of Definition 4.4.9, and vice versa.

**Definition 4.4.13.**

$\triangleright$ Given a cylinder $\alpha$ and a cell $\Phi$ as in

$$
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow^x
\end{array}
& \\
E^\sim \times E

\begin{array}{c}
\downarrow^\alpha
\end{array}
& \\
X

\begin{array}{c}
\downarrow^M
\end{array}
& \\
\rightarrow

\begin{array}{c}
\downarrow^K
\end{array}
& \\
\rightarrow

\begin{array}{c}
\Phi
\end{array}
& \\
\downarrow^Q
\begin{array}{c}
\rightarrow
\end{array}
& \\
Y

\begin{array}{c}
\downarrow^N
\end{array}
& \\
\rightarrow

\begin{array}{c}
\rightarrow
\end{array}
& \\
\rightarrow

\begin{array}{c}
\rightarrow
\end{array}
& \\
B
\end{array}
$$
, their composite $\alpha \circ \Phi = \Phi \circ \alpha$ is the cylinder}

\[
\begin{array}{c}
\text{E}^{-} \times \text{E} \\
\xrightarrow{\alpha \circ \Phi} \\
\text{Y} \rightarrow \text{B}
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ \Phi \cdot \Phi \cdot \alpha
\]

, the image of a cylindrical frame $\alpha \in \prod_{\text{E}} x(\Phi)K$ under the function

\[
\prod_{\text{E}} x(\Phi)K : \prod_{\text{E}} x(\text{M})K \rightarrow \prod_{\text{E}} x(\text{P} \langle \text{N} \rangle)K = \prod_{\text{E}} (x : \text{P} ) (\langle \text{N} \rangle [Q \circ \Phi])
\]

* Given a cylinder $\alpha$ and a cell $\Phi$ as in

\[
\begin{array}{c}
\text{E}^{-} \times \text{E} \\
\xrightarrow{\alpha \circ \Phi} \\
\text{Y} \rightarrow \text{B}
\end{array}
\]

, their composite $\alpha \circ \Phi = \Phi \circ \alpha$ is the cylinder

\[
\begin{array}{c}
\text{E}^{-} \times \text{E} \\
\xrightarrow{\alpha \circ \Phi} \\
\text{Y} \rightarrow \text{B}
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ \Phi \cdot \Phi \cdot \alpha
\]

, the image of a cylindrical frame $\alpha \in \prod_{\text{E}} x(\Phi)K$ under the function

\[
\prod_{\text{E}} K(\Phi) : \prod_{\text{E}} (\text{M})K \rightarrow \prod_{\text{E}} (\text{P} \langle \text{N} \rangle)K = \prod_{\text{E}} (K \circ \text{P} ) (\langle \text{N} \rangle [Q \circ \Phi])
\]

\[\text{Remark 4.4.14.}\]

(1) Each component of the cylinder $\alpha \circ \Phi : x : \text{P} \rightarrow Q \circ \Phi$ (resp. $\alpha \circ \Phi : K \circ \text{P} \rightarrow Q : \Phi$) is given by

\[\alpha \circ \Phi \}_{x} = \alpha \circ \Phi \cdot (x) (\Phi) (K(e,e)) \quad \text{resp.} \quad \alpha \circ \Phi \}_{e} = \alpha \circ \Phi \cdot (K(e,e)) (\Phi) \circ \alpha\]

(cf. Remark 4.4.10).

(2) If a cell is given by the hom of a functor $H$, then the composite $\alpha \circ \langle H \rangle$ is just the usual composite of an extraordinary natural transformation and a functor; that is,

\[\alpha \circ \langle H \rangle = \alpha \circ \Phi\]

\[\text{Note. The postcomposition in Definition 4.4.11 and the identity in Proposition 4.4.7 allow the following definition.}\]

\[\text{Definition 4.4.15. Given a category E and a cell} \quad \begin{array}{c}
\text{X} \rightarrow \text{M} \rightarrow \text{A} \\
\text{P} \rightarrow \text{Q} \\
\text{Y} \rightarrow \text{N} \rightarrow \text{B}
\end{array}\]


the cell
\[
\begin{array}{c}
X \xrightarrow{(E,M)} [E^- \times E, A] \\
p \\
Y \xrightarrow{(E,N)} [E^- \times E, B]
\end{array}
\]

“postcomposition with \( \Phi \)”, is defined by the module morphism
\[
\langle E, M \rangle \xrightarrow{(E,\Phi)} \langle E, P \langle N \rangle Q \rangle = P \langle E, N \rangle [E^- \times E, Q]
\]

, postcomposition with \( \Phi : M \to P \langle N \rangle Q \).

the cell
\[
\begin{array}{c}
[E^- \times E, X] \xrightarrow{(E,M)} A \\
[E^- \times E, P] \xrightarrow{(E,\Phi)} Q \\
[E^- \times E, Y] \xrightarrow{(E,N)} B
\end{array}
\]

, “postcomposition with \( \Phi \)”, is defined by the module morphism
\[
\langle E, M \rangle \xrightarrow{(E,\Phi)} \langle E, P \langle N \rangle Q \rangle = [E^- \times E, P] \langle E, N \rangle Q
\]

, postcomposition with \( \Phi : M \to P \langle N \rangle Q \).

Remark 4.4.16. The cell \( \langle E, \Phi \rangle : P \sim [E^- \times E, Q] : \langle E, M \rangle \to \langle E, N \rangle \) sends each cylinder \( \alpha : x \sim K : E \sim M \) to the cylinder \( \alpha \circ \Phi : x : P \sim Q \circ K : E \sim N \) defined in Definition 4.4.13; and the cell
\[
\langle E, \Phi \rangle : [E^- \times E, P] \sim Q : \langle E, M \rangle \to \langle E, N \rangle \)
\]

sends each cylinder \( \alpha : K : E \sim M \) to the cylinder \( \alpha \circ \Phi : K \circ P \sim Q \circ a : E \sim N \).

Proposition 4.4.17. The assignment \( \Phi \mapsto \langle E, \Phi \rangle \) is functorial.

Proof. The functoriality is verified similarly to Proposition 4.3.18 using Remark 4.4.12(2) and Proposition 4.4.7 in place of Remark 4.3.12(2) and Proposition 4.3.7.

Remark 4.4.18. Given a small category \( E \), the functor
\[
\langle E, - \rangle : \text{MOD} \to \text{MOD}
\]
is defined by the object function \( M \mapsto \langle E, M \rangle \) and the arrow function \( \Phi \mapsto \langle E, \Phi \rangle \), extending the functor \( \langle E, - \rangle \) in Remark 4.4.12(2) as shown in
\[
\begin{array}{c}
[X : A] \xrightarrow{(E,-)} [X : [E^- \times E, A]] \\
\text{MOD} \xrightarrow{(E,-)} \text{MOD}
\end{array}
\]
\[
\begin{array}{c}
[X : A] \xrightarrow{(E,-)} [[E^- \times E, X] : A] \\
\text{MOD} \xrightarrow{(E,-)} \text{MOD}
\end{array}
\]

, where \( \hookrightarrow \) denotes the canonical embedding in Remark 1.2.19(2).

4.5. Weighted cylinders
Definition 4.5.1.
- A cylinder $\xymatrix{E \ar[r]^-F & D}$ from $F \circ S$ to $T$ along $\mathcal{M}$ is said to be right weighted by $F$ (or right $F$-weighted).
- A cylinder $\xymatrix{D \ar[r]^-F & E}$ from $S$ to $T \circ F$ along $\mathcal{M}$ is said to be left weighted by $F$ (or left $F$-weighted).

Remark 4.5.2.
(1) Since $\xymatrix{[F \circ S] \ar[r]^-\mathcal{M} & [S \ar[r]^-\mathcal{M} & T]} = [S \ar[r]^-\mathcal{M} & T]$ $\xymatrix{\ar[r]^-F & }$, a right $F$-weighted cylinder $\xymatrix{E \ar[r]^-F & D}$ is the same thing as a right cylinder $\xymatrix{E \ar[r]^-{\alpha \ar[r]^-F & } & D}$; and since $\xymatrix{[S \ar[r]^-\mathcal{M} & T]} = [S \ar[r]^-\mathcal{M} & T]$ $\xymatrix{\ar[r]^-F & }$, a left $F$-weighted cylinder $\xymatrix{D \ar[r]^-F & E}$ is the same thing as a left cylinder $\xymatrix{D \ar[r]^-{\alpha \ar[r]^-F & } & E}$.

(2) Conversely, a right (resp. left) cylinder $\xymatrix{X \ar[r]^-{\alpha \ar[r]^-M & } & A}$ (resp. $\xymatrix{\ar[r]^-F & X \ar[r]^-{\alpha \ar[r]^-M & } & A}$) may be depicted as a right (resp. left) weighted cylinder $\xymatrix{X \ar[r]^-{\alpha \ar[r]^-G & } & A}$ (resp. $\xymatrix{\ar[r]^-M & X \ar[r]^-{\alpha \ar[r]^-G & } & A}$).

(3) A cylinder $\xymatrix{\ar[r]^-{\alpha \ar[r]^-M & } & A}$ is regarded to be weighted by the identity $\xymatrix{E \ar[r]^-{\alpha \ar[r]^-M & } & E}$ and sometimes depicted as $\xymatrix{E \ar[r]^-{\alpha \ar[r]^-1 & } & E}$ or $\xymatrix{\ar[r]^-{\alpha \ar[r]^-1 & } & E}$.

Definition 4.5.3. Given a functor $F : D \to E$ and a module $\mathcal{M} : X \to A$,
- the module of right $F$-weighted cylinders along $\mathcal{M}$, $\langle F \circ \mathcal{M} : [E, X] \to [D, A] \rangle$ , is defined by the composition $\xymatrix{[E, X] \ar[r]^-{[F, X]} & [D, X] \ar[r]^-{(D, \mathcal{M})} & [D, A] \ar[r]^-{F, A} & [E, A]}$.
- the module of left $F$-weighted cylinders along $\mathcal{M}$, $\langle F \circ \mathcal{M} : [D, X] \to [E, A] \rangle$ , is defined by the composition $\xymatrix{[D, X] \ar[r]^-{(D, \mathcal{M})} & [D, A] \ar[r]^-{F, A} & [E, A]}$. 
Remark 4.5.4.

1. For any pair of functors $S : E \to X$ and $T : D \to A$, the set

$$(S \langle F \rtimes M \rangle)(T) = (S \langle [F, X] \langle D, M \rangle \rangle)(T) = (F \circ S \langle D, M \rangle)(T)$$

consists of all cylinders $F \circ S \dashv T : D \to M$ right weighted by $F$; and for any pair of functors $S : D \to X$ and $T : E \to A$, the set

$$(S \langle F \ltimes M \rangle)(T) = (S \langle [D, M] [F, A] \rangle)(T) = (S \langle D, M \rangle)(T \circ F)$$

consists of all cylinders $S \dashv T \circ F : D \to M$ left weighted by $F$.

2. If, as a special case, $F$ is given by the identity $E \to E$, we have

$$(1_E \ltimes M) = (E, M) = (1_E \rtimes M)$$

immediately from the definition (cf. Remark 4.5.2(3)).

3. As another special case where $M$ is given by the hom of a category $C$, we have the following definition.

- The module

$$(F \ltimes C) : [E, C] \to [D, C]$$

is given by the composition

$$[E, C] \xrightarrow{[F, C]} [D, C] \xrightarrow{[D, C]} [D, C]$$

, i.e. by the representable module of the precomposition functor $[F, C]$. An $(F \ltimes C)$-arrow $\alpha : S \dashv T$ is a right $F$-weighted natural transformation $E \xrightarrow{\alpha} D$.

The identity

$$(F \ltimes C) = (F \ltimes \langle C \rangle)$$

follows from the identity $(D, C) = (D, \langle C \rangle)$ (see Remark 4.3.6(3)).

- The module

$$(F \ltimes C) : [D, C] \to [E, C]$$

is given by the composition

$$[D, C] \xrightarrow{[D, C]} [D, C] \xrightarrow{[F, C]} [E, C]$$

, i.e. by the corepresentable module of the precomposition functor $[F, C]$. An $(F \ltimes C)$-arrow $\alpha : S \dashv T$ is a left $F$-weighted natural transformation $D \xleftarrow{\alpha} E$.

The identity

$$(F \ltimes C) = (F \ltimes \langle C \rangle)$$

follows from the identity $(D, C) = (D, \langle C \rangle)$ (see Remark 4.3.6(3)).

Definition 4.5.5. Given a functor $F : D \to E$ and a module morphism $\Phi : M \to N : X \to A$,
4.5. Weighted cylinders

- the module morphism
\[ \langle F \circ \Phi \rangle : \langle F \circ M \rangle \to \langle F \circ N \rangle : [E, X] \to [D, A] \]

"postcomposition with \( \Phi \)" is defined by the pasting composition
\[
\begin{array}{ccc}
[E, X] & \xrightarrow{[F, X]} & [D, X] \\
\xrightarrow{[E, P]} & & \xleftarrow{\langle D, M \rangle} \\
[E, Y] & \xrightarrow{[F, Y]} & [D, Y]
\end{array}
\]

- the module morphism
\[ \langle F \circ \Phi \rangle : \langle F \circ M \rangle \to \langle F \circ N \rangle : [D, X] \to [E, A] \]

"postcomposition with \( \Phi \)" is defined by the pasting composition
\[
\begin{array}{ccc}
[D, X] & \xrightarrow{\langle D, M \rangle} & [D, A] \\
\xleftarrow{\langle D, N \rangle} & & \xrightarrow{[F, A]} \\
[D, Y] & \xleftarrow{\langle F, N \rangle}
\end{array}
\]

Remark 4.5.6.
1. The module morphism \( \langle F \circ \Phi \rangle \) maps each cylinder \( \alpha : F \circ S \to T : D \to M \) to the cylinder
\( \alpha \circ \Phi : F \circ S \to T : D \to N \) (cf. Remark 4.3.12(1)); and the module morphism \( \langle F \circ \Phi \rangle \) maps each cylinder \( \alpha : S \to T \circ F : D \to M \) to the cylinder \( \alpha \circ \Phi : S \to T \circ F : D \to N \).
2. The assignments \( \Phi \mapsto \langle F \circ \Phi \rangle \) and \( \Phi \mapsto \langle F \circ \Phi \rangle \) are functorial; indeed, the functors
\[
\langle F \circ - \rangle : [X : A] \to [[E, X] : [D, A]]
\]
\[
\langle F \circ - \rangle : [X : A] \to [[D, X] : [E, A]]
\]
are defined by
\[
\langle F \circ M \rangle = [F, X] \langle D, M \rangle \quad \langle F \circ M \rangle = \langle D, M \rangle [F, A]
\]

Definition 4.5.7. Given a functor \( F : D \to E \) and a cell
\[
X \xrightarrow{\Phi} A \quad \overset{p}{\Rightarrow} \quad Y \xrightarrow{\Phi} B
\]

- the cell
\[
\begin{array}{ccc}
[E, X] & \xrightarrow{[F, X]} & [D, X] \\
\xrightarrow{[E, P]} & & \xleftarrow{\langle D, M \rangle} \\
[E, Y] & \xrightarrow{[F, Y]} & [D, Y]
\end{array}
\]

"postcomposition with \( \Phi \)" is defined by the pasting composition
\[
\begin{array}{ccc}
[E, X] & \xrightarrow{[F, X]} & [D, X] \\
\xrightarrow{[E, P]} & & \xleftarrow{\langle D, M \rangle} \\
[E, Y] & \xrightarrow{[F, Y]} & [D, Y]
\end{array}
\]

- the cell
\[
\begin{array}{ccc}
[D, X] & \xrightarrow{[F, D \circ M]} & [E, A] \\
\xrightarrow{[D, P]} & & \xleftarrow{\langle F, N \rangle} \\
[D, Y] & \xrightarrow{[F, D \circ N]} & [E, B]
\end{array}
\]
Remark 4.5.8.
(1) The cell $(F \circ \Phi)$ sends each cylinder $\alpha : F \circ S \to T : D \to \mathcal{M}$ to the cylinder $\alpha \circ \Phi : F \circ S \circ P \to Q \circ T : D \to \mathcal{N}$ given by the composition

\[
\begin{array}{c}
E \\ F \\ D \\
\downarrow \alpha \\
X \\
\downarrow \phi \\
Y \\
\end{array}
\]

(cf. Remark 4.3.16(1)); and the cell $(F \circ \Phi)$ sends each cylinder $\alpha : S \to T \circ F : D \to \mathcal{M}$ to the cylinder $\alpha \circ \Phi : S \circ P \circ Q \circ T \circ F : D \to \mathcal{N}$ given by the composition

\[
\begin{array}{c}
D \\
\downarrow \alpha \\
X \\
\downarrow \phi \\
Y \\
\end{array}
\]

(2) If, as a special case, $F$ is given by the identity $E \to E$, we have

\[
\langle 1_E \circ \Phi \rangle = \langle E,\Phi \rangle = \langle 1_E \circ \Phi \rangle
\]

immediately from the definition.

(3) As another special case where $\Phi$ is given by the hom of a functor $H : C \to B$, the cell

\[
\begin{array}{c}
[E, C] \\
\downarrow [D, C] \\
[E, H] \\
\downarrow [D, H] \\
[E, B] \\
\downarrow [D, B] \\
\end{array}
\]

, “postcomposition with $H$”, is defined by the pasting composition

\[
\begin{array}{c}
[E, C] \\
\downarrow [D, C] \\
[E, H] \\
\downarrow [D, H] \\
[E, B] \\
\downarrow [D, B] \\
\end{array}
\]

. The cell $(F \circ H)$ sends each natural transformation $\alpha : F \circ S \to T : D \to C$ to the natural transformation $\alpha \circ H : F \circ S \circ H \to H \circ T : D \to B$, the usual composite of a natural transformation and a functor. The identity

\[
\langle F \circ H \rangle = \langle F \circ (H) \rangle
\]

follows from the identity $\langle D, H \rangle = \langle D, (H) \rangle$ (see Remark 4.3.16(2)).
Proposition 4.5.9. The assignment \( \Phi \mapsto \langle F \circ \Phi \rangle \) (resp. \( \Phi \mapsto \langle F \circ \Phi \rangle \)) is functorial.

Proof. Since the cell \( \langle F \circ \Phi \rangle \) is obtained from the cell \( \langle D, \Phi \rangle \) by the pasting composition as in Definition 4.5.7, the functoriality of the assignment \( \Phi \mapsto \langle F \circ \Phi \rangle \) follows from the identity \( \langle F \circ \Phi \rangle = \langle F \circ \langle H \rangle \rangle \) (see Remark 4.3.16(2)).

Remark 4.5.10. Given a functor \( F : D \to E \) with \( D \) small, the functor

\[ \langle F \circ - \rangle : \text{MOD} \to \text{MOD} \]

is defined by the object function \( M \mapsto \langle F \circ M \rangle \) and the arrow function \( \Phi \mapsto \langle F \circ \Phi \rangle \), extending the functor \( \langle F \circ - \rangle \) in Remark 4.5.6(2) as shown in

\[
\begin{array}{c}
[X : A] \xrightarrow{(F \circ -)} [[E, X] : [D, A]] \\
\downarrow \quad \downarrow \\
\text{MOD} \xrightarrow{(F \circ -)} \text{MOD}
\end{array}
\]

, where \( \hookrightarrow \) denotes the canonical embedding in Remark 1.2.19(2). Dually, the functor

\[ \langle F \circ - \rangle : \text{MOD} \to \text{MOD} \]

is defined by the object function \( M \mapsto \langle F \circ M \rangle \) and the arrow function \( \Phi \mapsto \langle F \circ \Phi \rangle \), extending the functor \( \langle F \circ - \rangle \) in Remark 4.5.6(2) as shown in

\[
\begin{array}{c}
[X : A] \xrightarrow{(F \circ -)} [[D, X] : [E, A]] \\
\downarrow \quad \downarrow \\
\text{MOD} \xrightarrow{(F \circ -)} \text{MOD}
\end{array}
\]

.
4.6. Cones

This section is largely analogous to Section 4.3 (resp. Section 4.4); in fact, cones may be regarded as special sorts of ordinary (resp. extraordinary) cylinders.

Definition 4.6.1.
- Let \( \mathcal{M} : * \to A \) be a left module. Given a functor \( K : \mathcal{E} \to A \), a cone \( \alpha \) to \( K \) along \( \mathcal{M} \), written \( \alpha : * \to K : *E \to \mathcal{M} \), is defined by a frame \( \alpha \) of the composite left module \( (\mathcal{M})K : * \to \mathcal{E} \).
- Let \( \mathcal{M} : X \to * \) be a right module. Given a functor \( K : \mathcal{E} \to X \), a cone \( \alpha \) from \( K \) along \( \mathcal{M} \), written \( \alpha : K \to * : E* \to \mathcal{M} \), is defined by a frame \( \alpha \) of the composite right module \( K(\mathcal{M}) : \mathcal{E} \to * \).

Remark 4.6.2. By Proposition 4.2.4,
- a frame \( \alpha \) of the left module \( (\mathcal{M})K \) is the same thing as a frame of the endomodule \( [!_E](\mathcal{M})K : \mathcal{E} \to \mathcal{E} \). Hence a cone \( \alpha : * \to K : *E \to \mathcal{M} \) is the same thing as a cylinder \( \alpha : [!_E] \to K : \mathcal{E} \to \mathcal{M} \) and depicted as

\[
\begin{array}{c}
\xymatrix{E \ar[rr]^\alpha \ar[dr]_* & & \mathcal{M} \ar[dl]_K \ar[dr]_* & & A \\
& \ast & & \ast & \}
\end{array}
\]

- a frame \( \alpha \) of the right module \( K(\mathcal{M}) \) is the same thing as a frame of the endomodule \( K(\mathcal{M})[!_E] : \mathcal{E} \to \mathcal{E} \). Hence a cone \( \alpha : K \to * : E* \to \mathcal{M} \) is the same thing as a cylinder \( \alpha : K \to [!_E] : \mathcal{E} \to \mathcal{M} \) and depicted as

\[
\begin{array}{c}
\xymatrix{K \ar[rr]^\alpha \ar[dr]_{\mathcal{E}} & & \mathcal{M} \ar[dl]_{E} \ar[dr]_{\mathcal{E}} & & \ast \\
& \ast & & \ast & \}
\end{array}
\]

Definition 4.6.3. Let \( \mathcal{E} \) be a category and \( \mathcal{M} : X \to A \) be a module.
- Given an object \( x \in [\mathcal{X}] \) and a functor \( K : \mathcal{E} \to A \), a cone \( \alpha \) from \( x \) to \( K \) along \( \mathcal{M} \), written \( \alpha : x \to K : xE \to \mathcal{M} \), is defined by a frame \( \alpha \) of the composite left module \( x(\mathcal{M})K : x \to \mathcal{E} \).
- Given an object \( a \in [\mathcal{A}] \) and a functor \( K : \mathcal{E} \to X \), a cone \( \alpha \) from \( K \) to \( a \) along \( \mathcal{M} \), written \( \alpha : K \to a : E* \to \mathcal{M} \), is defined by a frame \( \alpha \) of the composite right module \( K(\mathcal{M})a : \mathcal{E} \to * \).

Remark 4.6.4.
1. By Proposition 4.2.4,
   - a frame \( \alpha \) of the left module \( x(\mathcal{M})K \) is the same thing as a frame of the endomodule \( [!_E]x(\mathcal{M})K = [!_E]\delta x(\mathcal{M})K \). Hence a cone \( \alpha : x \to K : xE \to \mathcal{M} \) is the same thing as a cylinder \( \alpha : [!_E] \to \delta x : \mathcal{E} \to \mathcal{M} \) right weighted by the unique functor \( \mathcal{E} \to * \), and thus depicted as

\[
\begin{array}{c}
\xymatrix{E \ar[rr]^\alpha \ar[d]_x & & \mathcal{M} \ar[d]_{\delta x} \ar[dr]_K \ar[d]_x & & A \\
X & & \ast & & \ast \}
\end{array}
\]

   - a frame \( \alpha \) of the right module \( K(\mathcal{M})a \) is the same thing as a frame of the endomodule \( (K(\mathcal{M})a)[!_E] \). Hence a cone \( \alpha : K \to a : E* \to \mathcal{M} \) is the same thing as a cylinder \( \alpha : K \to a : [!_E] : \mathcal{E} \to \mathcal{M} \) left weighted by the unique functor \( \mathcal{E} \to * \), and thus depicted as

\[
\begin{array}{c}
\xymatrix{E \ar[rr]^\alpha \ar[d]_{\delta a} & & \mathcal{M} \ar[d]_a \ar[dr]_K \ar[d]_{\delta a} & & A \\
X & & \ast & & \ast \}
\end{array}
\]

2. By Proposition 4.2.5,
   - a frame \( \alpha \) of the left module \( x(\mathcal{M})K \) is the same thing as a cylindrical frame of the left module \( x(\mathcal{M})K [!_E \times E] = x(\mathcal{M})K \delta[!_E \times E] \). Hence a cone \( \alpha : x \to K : xE \to \mathcal{M} \) is the same thing as an extraordinary cylinder \( \alpha : x \to K \delta[!_E \times E] : \mathcal{E} \to \mathcal{M} \); a cone is thus regarded as an extraordinary cylinder \( \alpha : x \to K : \mathcal{E} \to \mathcal{M} \) with \( K : E^{-} \times E \to A \) dummy in the first variable.
   - a frame \( \alpha \) of the right module \( K(\mathcal{M})a \) is the same thing as a cylindrical frame of the right module \( [!_E \times E](K(\mathcal{M})a) = [[!_E \times E] \delta K(\mathcal{M})a] \). Hence a cone \( \alpha : K \to a : E* \to \mathcal{M} \) is the same thing as an extraordinary cylinder \( \alpha : [!_E \times E] \delta K \to a : \mathcal{E} \to \mathcal{M} \); a cone is thus regarded as an extraordinary cylinder \( \alpha : K \to a : \mathcal{E} \to \mathcal{M} \) with \( K : E^{-} \times E \to X \) dummy in the first variable.
(3) A cone defined in Definition 4.6.1 is a special instance of a cone defined in Definition 4.6.3 where $X$ (resp. $A$) is the terminal category. Conversely,
- a cone $\alpha : x \to K$ along a module $M : X \to A$ is the same thing as a cone $\alpha : * \to K$ along the left module $x(M) : * \to X$.
- a cone $\alpha : K \to a$ along a module $M : X \to A$ is the same thing as a cone $\alpha : K \to *$ along the right module $(M) a : X \to *$.

(4) If $E$ is discrete, a cone $\alpha : x \to K : *E \to M$ (resp. $\alpha : K \to a : E^* \to M$) is called discrete as well.
- Given an object $x \in |X|$ and a family of objects $a_i \in |A|$ indexed by some set $I$, a discrete cone $\alpha$ from $x$ to $(a_i)_{i \in I}$ along $M$ is defined by a family of $M$-arrows $\alpha_i : x \to a_i$ indexed by $I$; conversely, a family of $M$-arrows $\alpha_i : x_i \to a_i$ indexed by $I$ defines a discrete cone from $x$ to $(a_i)_{i \in I}$ along $M$.
- Given an object $a \in |A|$ and a family of objects $x_i \in |X|$ indexed by some set $I$, a discrete cone $\alpha$ from $(x_i)_{i \in I}$ to $a$ along $M$ is defined by a family of $M$-arrows $\alpha_i : x_i \to a$ indexed by $I$; conversely, a family of $M$-arrows $\alpha_i : x_i \to a$ indexed by $I$ defines a discrete cone from $(x_i)_{i \in I}$ to $a$ along $M$.

(5) By Remark 4.2.2(3), a frame of a left module $x(M) K : * \to E$ is the same thing as a frame of the opposite right module $K(M^*) x : E^* \to *$ (see Remark 1.1.35(4)), and a frame of a right module $K(M) a : E \to *$ is the same thing as a cone $\alpha : K \to a : E^* \to M$.

**Definition 4.6.5.** Given a category $E$ and a module $M : X \to A$,
- the module of cones $*E \to M$,

$$\langle *E, M \rangle : X \to [E, A]$$

is defined by

$$(x) \langle *E, M \rangle (K) = \prod_{*E} x(M) K$$

for $x \in X$ and $K \in [E, A]$.

- the module of cones $E^* \to M$,

$$\langle E^*, M \rangle : [E, X] \to A$$

is defined by

$$(K) \langle E^*, M \rangle (a) = \prod_{E^*} K(M) a$$

for $a \in A$ and $K \in [E, X]$.

**Remark 4.6.6.**

(1) For an object $x \in |X|$ and a functor $K : E \to A$, the set $(x) \langle *E, M \rangle (K)$ consists of all cones $x \to K : *E \to M$; and for an object $a \in |A|$ and a functor $K : E \to X$, the set $(K) \langle E^*, M \rangle (a)$ consists of all cones $K \to a : E^* \to M$.

(2) Given a $X$-arrow $f : x' \to x$, a cone $\alpha : x \to K$, and a natural transformation $\tau : K \Rightarrow K'$, all as in

$$\begin{array}{ccc}
E & \xrightarrow{!} & X \\
\xrightarrow{x} & \downarrow & \xrightarrow{\alpha} \\
& \xrightarrow{f \circ \alpha \circ \tau} & \xrightarrow{K'} \\
& \xrightarrow{\alpha} & \xrightarrow{\tau} \\
& \xrightarrow{K} & \xrightarrow{A} \\
\end{array}$$

, their composite in the module $(*E, M)$ is the cone

$$\begin{array}{ccc}
E & \xrightarrow{!} & X \\
\xrightarrow{x} & \downarrow & \xrightarrow{f \circ \alpha \circ \tau} \\
& \xrightarrow{\alpha} & \xrightarrow{K'} \\
& \xrightarrow{\alpha} & \xrightarrow{\tau} \\
& \xrightarrow{K} & \xrightarrow{A} \\
\end{array}$$
defined by

\[ f \circ \alpha \circ \tau = \alpha \circ f (\mathcal{M}) \tau = \alpha : \prod_{\mathcal{E}} f (\mathcal{M}) \tau \]

, the image of a conical frame \( \alpha \in \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) K \) under the function

\[ \prod_{\mathcal{E}} f (\mathcal{M}) \tau : \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) K \to \prod_{\mathcal{E}} \mathbf{x}' (\mathcal{M}) K' \]

\[ \mathbf{x} (\mathcal{M}) K \rightarrow \mathbf{x}' (\mathcal{M}) K' \]

\[ \tau \circ \alpha \circ f = \alpha \circ f (\mathcal{M}) \tau = \alpha : \prod_{\mathcal{E}} f (\mathcal{M}) f \]

, the image of a conical frame \( \alpha \in \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) a \) under the function

\[ \prod_{\mathcal{E}} f (\mathcal{M}) f : \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) a \to \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) a' \]

\[ \mathbf{x} (\mathcal{M}) a \rightarrow \mathbf{x} (\mathcal{M}) a' \]

\( \tau \circ \alpha \circ f = \alpha \circ \tau (\mathcal{M}) f = \alpha : \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) a \)

\[ \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) a \rightarrow \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) a' \]

\( \mathbf{x} (\mathcal{M}) a \rightarrow \mathbf{x} (\mathcal{M}) a' \)

(3) If \( \mathbf{E} \) is small and \( \mathcal{M} \) is locally small, then the module \( \langle \ast \mathbf{E}, \mathcal{M} \rangle \) (resp. \( \langle \mathbf{E} \ast, \mathcal{M} \rangle \)) is locally small.

(4) The identities

\[ \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) K = \prod_{\mathcal{E}^{-}} \mathbf{x} (\mathcal{M}^{-}) \quad \prod_{\mathcal{E}} \mathbf{x} (\mathcal{M}) a = \prod_{\mathcal{E}^{-}} \mathbf{x} (\mathcal{M}^{-}) K \]

hold by Remark 4.2.2(3) and Remark 1.1.35(4), and translate into the identities

\[ [\mathbf{E}, \mathbf{X}]^{-} \cong \langle \mathbf{E}, \mathbf{X} \rangle^{-} \]

\[ [\mathbf{E}, \mathbf{A}]^{-} \cong \langle \mathbf{E}, \mathbf{A} \rangle^{-} \]

by the definitions of \( \langle \ast \mathbf{E}, \mathcal{M} \rangle \) and \( \langle \mathbf{E} \ast, \mathcal{M} \rangle \), giving canonical isomorphisms

\[ \langle \ast \mathbf{E}, \mathcal{M} \rangle^{-} \cong \langle \mathbf{E}^{-} \ast, \mathcal{M}^{-} \rangle \]

\[ \langle \mathbf{E} \ast, \mathcal{M} \rangle^{-} \cong \langle \ast \mathbf{E}^{-}, \mathcal{M}^{-} \rangle \]

Proposition 4.6.7. Given a category \( \mathbf{E} \) and a composite module \( P \mathcal{N} Q \) as in

\[ \mathbf{X} \xrightarrow{P} \mathbf{A} \]

\[ \mathbf{Y} \xrightarrow{Q} \mathbf{B} \]

\[ \mathbf{X} \xrightarrow{P} \mathbf{A} \]

\[ \mathbf{Y} \xrightarrow{Q} \mathbf{B} \]
4.6. Cones 101

- the identity

\[
\begin{array}{c}
X \xrightarrow{\scriptscriptstyle \{E, A\}} [E, A] \\
p \downarrow 1 \downarrow [E, Q] \\
Y \xrightarrow{\scriptscriptstyle \{E, N\}} [E, B]
\end{array}
\]

, i.e.

\[(E, P \langle N \rangle) Q) = P \langle E, N \rangle [E, Q] \]

, holds.

- the identity

\[
\begin{array}{c}
[E, X] \xrightarrow{\scriptscriptstyle \{E, P \langle N \rangle Q\}} A \\
[E, P] \downarrow 1 \downarrow Q \\
[E, Y] \xrightarrow{\scriptscriptstyle \{E, N\}} B
\end{array}
\]

, i.e.

\[(E, P \langle N \rangle) Q) = [E, P] \{E, N\} Q \]

, holds.

**Proof.** For any \(x \in X\) and \(K \in [E, A]\),

\[
(x) \langle E, P \langle N \rangle Q) \rangle (K) = \prod_{E} x \langle P \langle N \rangle Q) K
\]

\[
= \prod_{E} (x : P) \langle N \rangle [Q \delta K]
\]

\[
= (x : P) \langle E, N \rangle (Q \delta K)
\]

\[
= (x : P) \langle E, N \rangle ([E, Q] : K)
\]

\[
= (x) \langle P \langle E, N \rangle [E, Q] \rangle (K)
\].

\[\square\]

**Remark 4.6.8.** The cell

\[
\begin{array}{c}
X \xrightarrow{\scriptscriptstyle \{E, P \langle N \rangle Q\}} [E, A] \\
p \downarrow 1 \downarrow [E, Q] \\
Y \xrightarrow{\scriptscriptstyle \{E, N\}} [E, B]
\end{array}
\]

resp.

\[
\begin{array}{c}
[E, X] \xrightarrow{\scriptscriptstyle \{E, P \langle N \rangle Q\}} A \\
[E, P] \downarrow 1 \downarrow Q \\
[E, Y] \xrightarrow{\scriptscriptstyle \{E, N\}} B
\end{array}
\]

sends each cone

\[
\begin{array}{c}
\xrightarrow{\scriptscriptstyle \{E, P \langle N \rangle Q\}} \alpha \xrightarrow{\scriptscriptstyle K} \bar{X}
\end{array}
\]

resp.

\[
\begin{array}{c}
E \xrightarrow{\scriptscriptstyle \{E, P \langle N \rangle Q\}} \alpha \xrightarrow{\scriptscriptstyle K} E
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\scriptscriptstyle \{E, P \langle N \rangle Q\}} \alpha \xrightarrow{\scriptscriptstyle Q \delta K} \bar{X}
\end{array}
\]

resp.

\[
\begin{array}{c}
E \xrightarrow{\scriptscriptstyle \{E, P \langle N \rangle Q\}} \alpha \xrightarrow{\scriptscriptstyle Q \delta a} E
\end{array}
\]

defined by the same frame.
Definition 4.6.9.

- If \( \alpha : \ast \to \mathcal{M} \) is a cone and \( \Phi : \mathcal{M} \to \mathcal{N} \) is a module morphism as in

\[
\begin{array}{ccc}
\ast & \to & E \\
\xrightarrow{\alpha} & \downarrow \Phi \\
X & \to & \mathcal{M} \to \mathcal{N} \\
\end{array}
\]

, then their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cone

\[
\begin{array}{ccc}
\ast & \to & E \\
\xrightarrow{\alpha \circ \Phi} & \downarrow \Phi \\
X & \to & \mathcal{N} \to \mathcal{N} \\
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ \Phi (\ast) K = \alpha \circ \prod_{\ast \to \mathcal{M}} \mathcal{X} (\Phi) K
\]

, the image of a conical frame \( \alpha \in \prod_{\ast \to \mathcal{M}} \mathcal{X} (\mathcal{M}) K \) under the function

\[
\prod_{\ast \to \mathcal{M}} \mathcal{X} (\Phi) K : \prod_{\ast \to \mathcal{M}} \mathcal{X} (\mathcal{M}) K \to \prod_{\ast \to \mathcal{M}} \mathcal{X} (\mathcal{N}) K
\]

- If \( \alpha : \mathcal{E}^* \to \mathcal{M} \) is a cone and \( \Phi : \mathcal{M} \to \mathcal{N} \) is a module morphism as in

\[
\begin{array}{ccc}
\mathcal{E}^* & \to & \ast \\
\xrightarrow{\alpha} & \downarrow \Phi \\
X & \to & \mathcal{M} \to \mathcal{N} \\
\end{array}
\]

, then their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cone

\[
\begin{array}{ccc}
\mathcal{E}^* & \to & \ast \\
\xrightarrow{\alpha \circ \Phi} & \downarrow \Phi \\
X & \to & \mathcal{N} \to \mathcal{N} \\
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ \Phi (\mathcal{E}^*) a = \alpha \circ \prod_{\mathcal{E}^* \to \mathcal{M}} K (\Phi) a
\]

, the image of a conical frame \( \alpha \in \prod_{\mathcal{E}^* \to \mathcal{M}} K (\mathcal{M}) a \) under the function

\[
\prod_{\mathcal{E}^* \to \mathcal{M}} K (\Phi) a : \prod_{\mathcal{E}^* \to \mathcal{M}} K (\mathcal{M}) a \to \prod_{\mathcal{E}^* \to \mathcal{M}} K (\mathcal{N}) a
\]

Remark 4.6.10. Each component of the cone \( \alpha \circ \Phi : \mathcal{X} \to K \) (resp. \( \alpha \circ \Phi : K \to a \)) is given by

\[
[\alpha \circ \Phi]_e = \alpha_e : \mathcal{X} (\Phi) (K \mathcal{e}) \quad \text{resp.} \quad [\alpha \circ \Phi]_e = \alpha_e : (\mathcal{K} \mathcal{e}) (\Phi) a
\]

(cf. Remark 4.3.10).

Definition 4.6.11. Given a category \( \mathcal{E} \) and a module morphism \( \Phi : \mathcal{M} \to \mathcal{N} : \mathcal{X} \to A \),
the module morphism
\[ \langle \ast E, \Phi \rangle : \langle \ast E, M \rangle \to \langle \ast E, N \rangle : X \to [E, A] \]

“postcomposition with \( \Phi \)”, is defined by
\[ (x) \langle \ast E, \Phi \rangle (K) = \prod_{x \in E} x \langle \Phi \rangle K \]

for each pair of an object \( x \in \|X\| \) and a functor \( K : E \to A \).

the module morphism
\[ \langle E_\ast, \Phi \rangle : \langle E_\ast, M \rangle \to \langle E_\ast, N \rangle : [E, X] \to A \]

“postcomposition with \( \Phi \)”, is defined by
\[ (K) \langle E_\ast, \Phi \rangle (a) = \prod_{E_\ast} K \langle \Phi \rangle a \]

for each pair of an object \( a \in \|A\| \) and a functor \( K : E \to X \).

Remark 4.6.12.
(1) The module morphism \( \langle \ast E, \Phi \rangle \) maps each cone \( \alpha : x \rightrightarrows K : \ast E \rightrightarrows M \) to the cone \( \alpha \circ \Phi : x \rightrightarrows K : \ast E \rightrightarrows N \) defined in Definition 4.6.9; and the module morphism \( \langle E_\ast, \Phi \rangle \) maps each cone \( \alpha : K \rightrightarrows a : E_\ast \rightrightarrows M \) to the cone \( \alpha \circ \Phi : K \rightrightarrows a : E_\ast \rightrightarrows N \).

(2) The assignments \( \Phi \mapsto \langle \ast E, \Phi \rangle \) and \( \Phi \mapsto \langle E_\ast, \Phi \rangle \) are functorial; indeed, the functor
\[ \langle \ast E, - \rangle : [X : A] \to [X : [E, A]] \]

is defined by
\[ (x) \langle \ast E, M \rangle (K) = \prod_{x \in E} x \langle M \rangle K \]

for \( x \in X \), \( K \in [E, A] \), and \( M \in [X : A] \), and the functor
\[ \langle E_\ast, - \rangle : [X : A] \to [[E, X] : A] \]

is defined by
\[ (K) \langle E_\ast, M \rangle (a) = \prod_{E_\ast} K \langle M \rangle a \]

for \( a \in A \), \( K \in [E, X] \), and \( M \in [X : A] \).

Note. By Remark 1.2.2(3), the following definition is regarded as a special case of Definition 4.6.9, and vice versa.

Definition 4.6.13.
> Given a cone \( \alpha \) and cell \( \Phi \) as in

\[
\begin{array}{ccc}
\ast & \leftarrow & E \\
\xrightarrow{\alpha} & & \xrightarrow{K} \\
X & \rightrightarrows & A \\
\phi & \rightrightarrows & Q \\
Y & \rightrightarrows & B
\end{array}
\]

, their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cone

\[
\begin{array}{ccc}
\ast & \leftarrow & E \\
\xrightarrow{x : \phi} & & \xrightarrow{Q \circ K} \\
Y & \rightrightarrows & B
\end{array}
\]


defined by
\[ \alpha \circ \Phi = \alpha \circ \mathbf{x}(\Phi) \mathbf{K} = \alpha \colon \prod_{x \in \mathcal{E}} \mathbf{x}(\Phi) \mathbf{K} \]
, the image of a conical frame \( \alpha \in \prod_{x \in \mathcal{E}} \mathbf{x}(\mathcal{M}) \mathbf{K} \) under the function
\[ \prod_{x \in \mathcal{E}} \mathbf{x}(\Phi) \mathbf{K} : \prod_{x \in \mathcal{E}} \mathbf{x}(\mathcal{M}) \mathbf{K} \to \prod_{x \in \mathcal{E}} \mathbf{x}(\mathcal{P}(\mathcal{N}) \mathbf{Q}) \mathbf{K} = \prod_{x \in \mathcal{E}} (\mathbf{x}(\mathcal{P}) \mathcal{N} \mathbf{Q}) \mathbf{K} \]

- Given a cone \( \alpha \) and cell \( \Phi \) as in

\[
\begin{array}{c}
\mathcal{E} \xrightarrow{!} \ast \\
\mathbf{K} \downarrow \alpha \downarrow \mathbf{a} \\
\mathcal{X} \xrightarrow{\mathcal{M}} \mathcal{A} \\
\mathcal{P} \downarrow \Phi \downarrow \mathcal{Q} \\
\mathcal{Y} \xrightarrow{\mathcal{N}} \mathcal{B}
\end{array}
\]

, their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cone

\[
\begin{array}{c}
\mathcal{E} \xrightarrow{!} \ast \\
\mathbf{K} \circ \mathbf{P} \downarrow \alpha \circ \Phi \downarrow \mathbf{Q} \circ \mathbf{a} \\
\mathcal{X} \xrightarrow{\mathcal{N}} \mathcal{A}
\end{array}
\]

defined by
\[ \alpha \circ \Phi = \alpha \circ \mathbf{K}(\Phi) \mathbf{a} = \alpha \colon \prod_{x \in \mathcal{E}^*} \mathbf{K}(\Phi) \mathbf{a} \]
, the image of a conical frame \( \alpha \in \prod_{x \in \mathcal{E}^*} \mathbf{K}(\mathcal{M}) \mathbf{a} \) under the function
\[ \prod_{x \in \mathcal{E}^*} \mathbf{K}(\Phi) \mathbf{a} : \prod_{x \in \mathcal{E}^*} \mathbf{K}(\mathcal{M}) \mathbf{a} \to \prod_{x \in \mathcal{E}^*} \mathbf{K}(\mathcal{P}(\mathcal{N}) \mathbf{Q}) \mathbf{a} = \prod_{x \in \mathcal{E}^*} [\mathbf{K} \circ \mathbf{P}] (\mathcal{N}) (\mathbf{Q} \circ \mathbf{a}) \]

Remark 4.6.14. Each component of the cone \( \alpha \circ \Phi : \mathbf{x} : \mathcal{P} \to Q \circ \mathbf{K} \) (resp. \( \alpha \circ \Phi : \mathbf{K} \circ \mathbf{P} \to Q \circ \mathbf{a} \)) is given by
\[ [\alpha \circ \Phi]_e = \alpha_e : \mathbf{x}(\Phi) (\mathbf{K} \cdot e) \quad \text{resp.} \quad [\alpha \circ \Phi]_e = \alpha_e : (\mathbf{e} \cdot \mathbf{K}) (\Phi) \mathbf{a} \]
(cf. Remark 4.6.10).

Note. The postcomposition in Definition 4.6.11 and the identity in Proposition 4.6.7 allow the following definition.

Definition 4.6.15. Given a category \( \mathcal{E} \) and a cell \( \mathcal{X} \xrightarrow{\mathcal{M}} \mathcal{A} \),

\[
\begin{array}{c}
\mathcal{P} \downarrow \Phi \downarrow \mathcal{Q} \\
\mathcal{Y} \xrightarrow{\mathcal{N}} \mathcal{B}
\end{array}
\]

the cell

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{\mathbf{E} \mathcal{M}} \mathbf{[E, A]} \\
\mathcal{P} \downarrow (\mathbf{E} \Phi) \downarrow \mathbf{[E, Q]} \\
\mathcal{Y} \xrightarrow{\mathbf{E} \mathcal{N}} \mathbf{[E, B]}
\end{array}
\]

, “postcomposition with \( \Phi \)”, is defined by the module morphism
\[ \langle \mathbf{E} \mathcal{M} \rangle \xrightarrow{(\mathbf{E} \Phi)} \langle \mathbf{E} \mathcal{P}(\mathcal{N}) \mathbf{Q} \rangle = \mathbf{P} \langle \mathbf{E} \mathcal{N} \rangle \mathbf{[E, Q]} \]
, postcomposition with \( \Phi : \mathcal{M} \to \mathcal{P}(\mathcal{N}) \mathbf{Q} \).
the cell

\[
\begin{align*}
\left[ E, X \right] & \xrightarrow{(E*, M)} A \\
\left[ E, P \right] & \xrightarrow{(E*, \Phi)} Q \\
\left[ E, Y \right] & \xrightarrow{(E*, N)} B
\end{align*}
\]

"postcomposition with \( \Phi \)", is defined by the module morphism

\[
\langle E*, M \rangle \xrightarrow{(E*, \Phi)} \langle E*, P \rangle \langle N \rangle \ Q = [E, P] \langle E*, N \rangle \ Q
\]

Remark 4.6.16. The cell \( \langle *E, \Phi \rangle \) sends each cone \( \alpha : x \rightsquigarrow K : *E \rightsquigarrow M \) to the cone \( \alpha \circ \Phi : x' : P \rightsquigarrow Q \circ K : *E \rightsquigarrow N \) defined in Definition 4.6.13; and the cell \( \langle E*, \Phi \rangle \) sends each cone \( \alpha : K \rightsquigarrow a : E* \rightsquigarrow M \) to the cone \( \alpha \circ \Phi : K \circ P \rightsquigarrow Q : a : E* \rightsquigarrow N \).

Proposition 4.6.17. If a cell \( \Phi \) is fully faithful, so is the postcomposition cell \( \langle *E, \Phi \rangle \) (resp. \( \langle E*, \Phi \rangle \)) for any category \( E \).

Proof. Since the postcomposition operation \( \langle *E, - \rangle \) is functorial (see Remark 4.6.12(2)), it preserves isomorphisms.

Proposition 4.6.18. The assignment \( \Phi \mapsto \langle *E, \Phi \rangle \) (resp. \( \Phi \mapsto \langle E*, \Phi \rangle \)) of the postcomposition cell is functorial.

Proof. The functoriality is verified similarly to Proposition 4.3.18 using Remark 4.6.12(2) and Proposition 4.6.7 in place of Remark 4.3.12(2) and Proposition 4.3.7.

Remark 4.6.19.
1. In fact, the functoriality of the assignment \( \Phi \mapsto \langle *E, \Phi \rangle \) (resp. \( \Phi \mapsto \langle E*, \Phi \rangle \)) is reduced to that of the assignment \( \Phi \mapsto \langle E, \Phi \rangle \) by Corollary 4.6.21 below.
2. Given a small category \( E \), the functor

\[
\langle *E, - \rangle : \text{MOD} \to \text{MOD}
\]

is defined by the object function \( M \mapsto \langle *E, M \rangle \) and the arrow function \( \Phi \mapsto \langle *E, \Phi \rangle \), extending the functor \( \langle *E, - \rangle \) in Remark 4.6.12(2) as shown in

\[
\begin{align*}
\left[ X : A \right] & \xrightarrow{(E*, -)} \left[ X : [E, A] \right] \\
\text{MOD} & \xrightarrow{(E*, -)} \text{MOD}
\end{align*}
\]

, where \( \rightarrow \) denotes the canonical embedding in Remark 1.2.19(2). Dually, the functor

\[
\langle E*, - \rangle : \text{MOD} \to \text{MOD}
\]

is defined by the object function \( M \mapsto \langle E*, M \rangle \) and the arrow function \( \Phi \mapsto \langle E*, \Phi \rangle \), extending the functor \( \langle E*, - \rangle \) in Remark 4.6.12(2) as shown in

\[
\begin{align*}
\left[ X : A \right] & \xrightarrow{(E*, -)} [[E, X] : A] \\
\text{MOD} & \xrightarrow{(E*, -)} \text{MOD}
\end{align*}
\]
Note. In Remark 4.6.4(1) we saw that a cone \(* E \rightsquigarrow M\) is regarded as a cylinder \(E \rightsquigarrow M\) right weighted by the functor \(!_E\). This fact is restated below showing that a module of cones is given by a module of cylinders weighted by \(!_E\) (see Definition 4.5.3).

**Theorem 4.6.20.** For a module \(M : X \to A\),
- the module \(\langle * E, M \rangle : X \to [E, A]\) is given by the composition

\[
X \xrightarrow{[!_E, X]} [E, X] \xrightarrow{(E, M)} [E, A]
\]

; that is,

\[
\langle * E, M \rangle = \langle !_E \cdot M \rangle
\]

- the module \(\langle E^*, M \rangle : [E, X] \to A\) is given by the composition

\[
[E, X] \xrightarrow{(E, A)} [E, A] \xleftarrow{[!_E, A]} A
\]

; that is,

\[
\langle E^*, M \rangle = \langle !_E \cdot M \rangle
\]

Similarly, for a module morphism \(\Phi : M \to N : X \to A\),
- the module morphism \(\langle * E, \Phi \rangle : \langle * E, M \rangle \to \langle * E, N \rangle : X \to [E, A]\) is given by the composition

\[
X \xrightarrow{[!_E, X]} [E, X] \xrightarrow{(E, \Phi)} [E, N] \xrightarrow{(E, A)} [E, A]
\]

; that is,

\[
\langle * E, \Phi \rangle = \langle !_E \cdot \Phi \rangle
\]

- the module morphism \(\langle E^*, \Phi \rangle : \langle E^*, M \rangle \to \langle E^*, N \rangle : [E, X] \to A\) is given by the composition

\[
[E, X] \xrightarrow{(E, \Phi)} [E, N] \xrightarrow{(E, A)} [E, A] \xleftarrow{[!_E, A]} A
\]

; that is,

\[
\langle E^*, \Phi \rangle = \langle !_E \cdot \Phi \rangle
\]

**Proof.** For any \(x \in X\) and \(K \in [E, A]\),

\[
(x) \langle * E, M \rangle (K) = \prod_{x \in E} x (M) K
\]

\[
= \prod_{x \in E} \langle !_E \cdot x \rangle (M) K
\]

\[
= \prod_{x \in E} \langle !_E \cdot x \rangle (E, M) (K)
\]

\[
= (x; \langle !_E \cdot X \rangle) (E, M) (K)
\]

\[
= (x) \langle [!_E, X] \cdot (E, M) \rangle (K)
\]

\((\ast 1)\) by Proposition 4.2.4).
Corollary 4.6.21.

- the postcomposition cell

\[
\begin{array}{c}
X \xrightarrow{(\ast E, M)} [E, A] \\
\downarrow \Phi \\
Y \xrightarrow{(\ast E, N)} [E, B]
\end{array}
\]

in Definition 4.6.15 is obtained from the postcomposition cell in Definition 4.3.15 by the pasting composition

\[
\begin{array}{c}
X \xrightarrow{[E, X]} [E, X] \xrightarrow{(E, M)} [E, A] \\
\downarrow [E, P] \downarrow [E, \Phi] \downarrow [E, Q]
\end{array}
\]

\[
\begin{array}{c}
Y \xrightarrow{[E, Y]} [E, Y] \xrightarrow{(E, N)} [E, B]
\end{array}
\]

; that is,

\[
\langle \ast E, \Phi \rangle = \langle \! \! E \, \ast \, \Phi \rangle
\]

(see Definition 4.5.7).

- the postcomposition cell

\[
\begin{array}{c}
[X, E] \xrightarrow{(E, M)} A \\
\downarrow [E, P] \downarrow [E, \Phi] \downarrow [E, Q]
\end{array}
\]

\[
\begin{array}{c}
[Y, E] \xrightarrow{(E, N)} B
\end{array}
\]

in Definition 4.6.15 is obtained from the postcomposition cell in Definition 4.3.15 by the pasting composition

\[
\begin{array}{c}
[X, E] \xrightarrow{(E, M)} [E, A] \xrightarrow{[E, A]} A \\
\downarrow [E, P] \downarrow [E, \Phi] \downarrow [E, Q]
\end{array}
\]

\[
\begin{array}{c}
[Y, E] \xrightarrow{(E, N)} [E, B] \xrightarrow{[E, B]} B
\end{array}
\]

; that is,

\[
\langle E \ast, \Phi \rangle = \langle \! \! E \, \ast \, \Phi \rangle
\]

(see Definition 4.5.7).

Proof. Immediate from Theorem 4.6.20, recalling the definition of the postcomposition cell \(\langle E, \Phi \rangle\) and the definition of pasting composition. \(\square\)

Remark 4.6.22. By Theorem 4.6.20 and Corollary 4.6.21, the compositions in Remark 4.6.12(2), Definition 4.6.9, and Definition 4.6.13 are in fact the same thing as those in Remark 4.3.6(2), Definition 4.3.9, and Definition 4.3.13 with the cone \(\alpha\) regarded as the cylinder weighted by the functor \(\! \! E\).

Definition 4.6.23. Given a category \(E\) and a cell morphisms \(\tau : \Phi \to \Psi : \mathcal{M} \to \mathcal{N}\),

- the cell morphism

\[
\langle \ast E, \tau \rangle : \langle \ast E, \Phi \rangle \to \langle \ast E, \Psi \rangle : \langle \ast E, \mathcal{M} \rangle \to \langle \ast E, \mathcal{N} \rangle
\]

"postcomposition with \(\tau\)", is defined by the pair of natural transformations

\[
\tau_0 : \Phi_0 \to \Psi_0 \quad [E, \tau_1] : [E, \Phi_1] \to [E, \Psi_1]
\]

; the left component of \(\tau\) and the postcomposition with the right component of \(\tau\) (see Preliminary 9).
the cell morphism
\[(\ast E, \tau) : (\ast E, \Phi) \to (\ast E, \Psi) : (\ast E, \mathcal{M}) \to (\ast E, \mathcal{N})\]

"postcomposition with \(\tau\)" is defined by the pair of natural transformations
\[\tau_0 : [E, \Phi] \to [E, \Psi] \quad \tau_1 : \Phi_0 \to \Psi_1\]

of the postcomposition with the left component of \(\tau\) (see Preliminary 9) and the right component of \(\tau\).

**Remark 4.6.24.**

(1) \((\ast E, \tau)\) (resp. \((\ast E, \tau)\)) so defined does form a cell morphism. Indeed, given a cone \(\alpha : x \to K : \ast E \to \mathcal{M}\) (resp. \(\alpha : K \to a : E \to \mathcal{M}\)), the commutativity of
\[
\begin{array}{c}
\xymatrix{ x : \Phi_0 \ar[r]^-{\alpha \circ (\ast E, \Phi)} & [E, \Phi_1] \ar[d]^-{\Phi_1} : K \\
\ar@{=>}[r]_{\tau_0} & \ar@{=>}[r]_{\Phi_1} & [E, \tau_1] : K \ar[d]^-{\tau_1} \\
x : \Psi_0 \ar[r]^-{\alpha \circ (\ast E, \Psi)} & [E, \Psi_1] \ar[d]^-{\Psi_1} : K}
\end{array}
\]

resp.
\[
\begin{array}{c}
\xymatrix{ K : [E, \Phi_0] \ar[r]^-{\alpha \circ (E, \Psi)} & \Phi_1 : a \\
\ar@{=>}[r]_{\tau_0} & \ar@{=>}[r]_{\Phi_1} & [E, \tau_1] : K \ar[d]^-{\tau_1} \\
K : [E, \Psi_0] \ar[r]^-{\alpha \circ (E, \Psi)} & \Psi_1 : a}
\end{array}
\]

, i.e.
\[
\begin{array}{c}
\xymatrix{ x : \Phi_0 \ar[r]^-{\alpha \circ (\ast E, \Phi)} & \Phi_1 \circ K \\
\ar@{=>}[r]_{\tau_0} & \ar@{=>}[r]_{\Phi_1 \circ K} & [E, \tau_1] : K \ar[d]^-{\tau_1} \\
x : \Psi_0 \ar[r]^-{\alpha \circ (\ast E, \Psi)} & \Psi_1 \circ K}
\end{array}
\]

follows from the commutativity of
\[
\begin{array}{c}
\xymatrix{ x : \Phi_0 \ar[r]^-{\alpha \circ (\ast E, \Phi)} & K : e \\
\ar@{=>}[r]_{\tau_0} & \ar@{=>}[r]_{K \circ \tau_0} & \ar@{=>}[r]_{\Phi_1 \circ \tau_0} & [E, \tau_1] : K \ar[d]^-{\tau_1} \\
x : \Psi_0 \ar[r]^-{\alpha \circ (\ast E, \Psi)} & \Psi_1 \circ K}
\end{array}
\]

for each object \(e \in [E]\).

(2) The assignment \(\tau \mapsto (\ast E, \tau)\) defines the functor
\[\ast E, \tau : [\mathcal{M} : \mathcal{N}] \to [(\ast E, \mathcal{M}) : (\ast E, \mathcal{N})]\]

; indeed, by the definition of the cell morphism \((\ast E, \tau)\), the functoriality of \((\ast E, \tau)\) is reduced to that of \([E, \tau] : [\mathcal{M}_1, \mathcal{N}_1] \to [(\mathcal{E}, \mathcal{M}_1), (\mathcal{E}, \mathcal{N}_1)]\). Dually, the assignment \(\tau \mapsto (\ast E, \tau)\) defines the functor
\[(\ast E, \tau) : [\mathcal{M} : \mathcal{N}] \to [(\mathcal{E}, \mathcal{M}), (\ast E, \mathcal{N})]\]

. 

**Definition 4.6.25.**

- If \(H\) is a functor and \(\alpha\) is a cone as in

\[
\begin{array}{c}
\xymatrix{ \ast \ar[d]_-{X} \ar[r]^-{\alpha} & E \ar[d]_-{K} & D \ar[l]_-{H} \\
\ar@{-->}[r]_-{X} & \ar@{-->}[r]_-{\alpha} & \ar@{-->}[r]_-{A} }
\end{array}
\]

, then their composite \(H \circ \alpha = \alpha \circ H\) is the cone

\[
\begin{array}{c}
\xymatrix{ \ast \ar[d]_-{X} \ar[r]^-{\alpha} & D \ar[d]_-{K \circ H} \\
\ar@{-->}[r]_-{X} & \ar@{-->}[r]_-{H \circ \alpha} & \ar@{-->}[r]_-{A} }
\end{array}
\]
defined by
\[ H \circ \alpha = \alpha : \prod_{x \in \mathcal{M}} x \langle \mathcal{M} \rangle K \]
, the image of a conical frame \( \alpha \in \prod_{x \in \mathcal{E}} x \langle \mathcal{M} \rangle K \) under the function
\[ \prod_{x \in \mathcal{H}} x \langle \mathcal{M} \rangle K : \prod_{x \in \mathcal{E}} x \langle \mathcal{M} \rangle K \to \prod_{x \in \mathcal{D}} x \langle \mathcal{M} \rangle K = \prod_{x \in \mathcal{D}} x \langle \mathcal{M} \rangle [K \circ H] \]

- If \( H \) is a functor and \( \alpha \) is a cone as in
\[ \begin{array}{ccc}
  \mathcal{D} & \xrightarrow{H} & \mathcal{E} \\
  \downarrow{\mathcal{K}} & \searrow{\alpha} & \downarrow{a} \\
  \mathcal{X} & \to & \mathcal{M} \to \mathcal{A}
\end{array} \]
, then their composite \( H \circ \alpha = \alpha \circ H \) is the cone
\[ \begin{array}{ccc}
  \mathcal{D} & \to & * \\
  \downarrow{\mathcal{H} \circ \mathcal{K}} & \downarrow{\mathcal{H} \circ \alpha} & \downarrow{a} \\
  \mathcal{X} & \to & \mathcal{M} \to \mathcal{A}
\end{array} \]
defined by
\[ H \circ \alpha = \alpha : \prod_{x \in \mathcal{H}^+} K \langle \mathcal{M} \rangle a \]
, the image of a conical frame \( \alpha \in \prod_{E^*} K \langle \mathcal{M} \rangle a \) under the function
\[ \prod_{H^*} K \langle \mathcal{M} \rangle a : \prod_{E^*} K \langle \mathcal{M} \rangle a \to \prod_{D^*} H(K \langle \mathcal{M} \rangle a) = \prod_{D^*} [H \circ K] \langle \mathcal{M} \rangle a \]

Remark 4.6.26. Each component of the cone \( H \circ \alpha \) is given by
\[ [H \circ \alpha]_d = \alpha(x \cdot d) \]
(see Definition 4.2.11).

Definition 4.6.27. Given a functor \( H : \mathcal{D} \to \mathcal{E} \) and a module \( \mathcal{M} : \mathcal{X} \to \mathcal{A} \),
- the cell
\[ \begin{array}{ccc}
  \mathcal{X} & \xrightarrow{(x, \mathcal{M})} & [E, \mathcal{A}] \\
  \downarrow{1} & \downarrow{(H, \mathcal{M})} & \downarrow{[H, \mathcal{A}]} \\
  \mathcal{X} & \xrightarrow{(x, \mathcal{M})} & [D, \mathcal{A}]
\end{array} \]
, "precomposition with \( H \)" , is defined by
\[ (x) \langle x, \mathcal{M} \rangle (K) = \prod_{x \in \mathcal{H}} x \langle \mathcal{M} \rangle K \]
for \( x \in \| \mathcal{X} \| \) and \( K \) a functor \( \mathcal{E} \to \mathcal{A} \).
- the cell
\[ \begin{array}{ccc}
  [E, \mathcal{X}] & \xrightarrow{(x, \mathcal{M})} & \mathcal{A} \\
  \downarrow{[H, \mathcal{X}]} & \downarrow{(H, \mathcal{M})} & \downarrow{1} \\
  [D, \mathcal{X}] & \xrightarrow{(x, \mathcal{M})} & \mathcal{A}
\end{array} \]
, "precomposition with \( H \)" , is defined by
\[ (K) \langle x, \mathcal{M} \rangle (a) = \prod_{x \in \mathcal{H}^+} K \langle \mathcal{M} \rangle a \]
for \( a \in \| \mathcal{A} \| \) and \( K \) a functor \( \mathcal{E} \to \mathcal{X} \).
Proposition 4.6.30. The cell \( \langle *H, M \rangle \) is obtained from the cell \( \langle H, M \rangle \) in Definition 4.3.27 by the pasting composition
\[
\begin{array}{c}
\text{X} \xrightarrow{[E,X]} [E, X] \xrightarrow{(E,M)} [E, A] \\
\downarrow [H,X] \quad \downarrow [H,A] \\
\text{D} \xrightarrow{[D,X]} [D, X] \xrightarrow{(D,M)} [D, A]
\end{array}
\]

Example 4.6.29.
1. Let \( M : X \rightarrow A \) be a module and \( E \) be a category. Given a subcategory \( D \) of \( E \), precomposition with the inclusion \( D \rightarrow E \) yields
   - the cell
   \[
   \begin{array}{c}
   \text{X} \xrightarrow{([E,M],)} [E, A] \\
   \downarrow [D,X] \quad \downarrow [D,A] \\
   \text{D} \xrightarrow{([D,M],)} [D, A]
   \end{array}
   \]
   , "restriction to \( D \)", which sends each cone \( \alpha : x \rightarrow K : *E \rightarrow M \) to the cone \( D \circ \alpha : x \rightarrow K \circ D : *D \rightarrow M \), \( \alpha \) restricted to \( D \).
   - the cell
   \[
   \begin{array}{c}
   [E, X] \xrightarrow{(E*,M)} A \\
   [D,X] \xrightarrow{(D*,M)} A
   \end{array}
   \]
   , "restriction to \( D \)", which sends each cone \( \alpha : K \rightarrow a : E* \rightarrow M \) to the cone \( D \circ \alpha : D \circ K \rightarrow a : D* \rightarrow M \), \( \alpha \) restricted to \( D \).
2. Let \( M : X \rightarrow A \) be a module and \( E \) be a category. Given an object \( e \in \|E\| \), precomposition with the functor \( e : * \rightarrow E \) yields
   - the cell
   \[
   \begin{array}{c}
   \text{X} \xrightarrow{([E,M],)} [E, A] \\
   \downarrow [e,X] \quad \downarrow [e,A] \\
   \text{X} \xrightarrow{(e*,M)} A
   \end{array}
   \]
   , "evaluation at \( e \)", which sends each cone \( \alpha : x \rightarrow K : *E \rightarrow M \) to the \( M \)-arrow \( \alpha_e : x \rightarrow K \cdot e \), the component of \( \alpha \) at \( e \).
   - the cell
   \[
   \begin{array}{c}
   [E, X] \xrightarrow{(E*,M)} A \\
   [e,X] \xrightarrow{(e*,M)} A
   \end{array}
   \]
   , "evaluation at \( e \)", which sends each cone \( \alpha : K \rightarrow a : E* \rightarrow M \) to the \( M \)-arrow \( \alpha_e : e \cdot K \rightarrow a \), the component of \( \alpha \) at \( e \).
Proof. We need to verify that $(\ast H, M) = [1_\mathcal{E}, \mathcal{X}](H, M)$. But by Proposition 4.2.13, for any object $x \in \parallel \mathcal{X} \parallel$ and any functor $K : \mathcal{E} \to \mathcal{A}$,

$$(x)(\ast H, M)(K) = \prod_{x \in \ast H} \mathcal{X}(\mathcal{M}) \ K$$

$$(x)(\ast H, M)(K) = \prod_{x \in \ast H} [1_\mathcal{E}] \mathcal{X}(\mathcal{M}) \ K$$

$$(x)(\ast H, M)(K) = \prod_{x \in \ast H} [1_\mathcal{E} \ast x] \mathcal{M} \ K$$

$$(x)(\ast H, M)(K) = [1_\mathcal{E} \ast x] (H, M)(K)$$

$$(x)(\ast H, M)(K) = (x) ([1_\mathcal{E}, \mathcal{X}](H, M))(K)$$

($^1$ by Proposition 4.2.13).

\[\square\]

**Proposition 4.6.31.** If $\mathcal{M}$ is a locally small module, then
- the assignment $H \mapsto \{\ast H, M\}$ defines the contravariant functor

$$\{\ast -, M\} : \text{Cat}^\sim \to \text{MOD}$$

- the assignment $H \mapsto \{H \ast, M\}$ defines the contravariant functor

$$\{- \ast, M\} : \text{Cat}^\sim \to \text{MOD}$$

Proof. Since the cell $(\ast H, M)$ is obtained from the cell $(H, M)$ by the pasting composition in Proposition 4.6.30, the functoriality of the assignment $H \mapsto \{\ast H, M\}$ is reduced to that of the assignment $H \mapsto \{H, M\}$ (see Proposition 4.3.29) by virtue of Proposition 1.2.34.

\[\square\]

**Theorem 4.6.32.**

- There is a functor

$$\{\ast -, -\} : \text{Cat}^\sim \times \text{MOD} \to \text{MOD}$$

such that

1. for each small category $\mathcal{E}$, $\{\ast \mathcal{E}, -\} : \text{MOD} \to \text{MOD}$ coincides with the functor in Remark 4.6.19(2).
2. for each locally small module $\mathcal{M}$, $\{\ast -, \mathcal{M}\} : \text{Cat}^\sim \to \text{MOD}$ coincides with the functor in Proposition 4.6.31.

- There is a functor

$$\{- *, -\} : \text{Cat}^\sim \times \text{MOD} \to \text{MOD}$$

such that

1. for each small category $\mathcal{E}$, $\{\mathcal{E} *, -\} : \text{MOD} \to \text{MOD}$ coincides with the functor in Remark 4.6.19(2).
2. for each locally small module $\mathcal{M}$, $\{- *, \mathcal{M}\} : \text{Cat}^\sim \to \text{MOD}$ coincides with the functor in Proposition 4.6.31.

Proof. Similar to the proof of Theorem 4.3.32.

\[\square\]

Note. In Remark 4.6.4(2) we saw that a cone is a special instance of an extraordinary cylinder. This fact is restated below showing that a module of cones is given by a module of extraordinary cylinders composed with a diagonal functor.
Theorem 4.6.33. For a module $M : X \to A$,

- the module $(E, M) : X \to [E, A]$ is given by the composition

$$X \xrightarrow{(E, M)} [E^{-} \times E, A] \xrightarrow{[E^{-} \times E, A]} [E, A]$$

that is,

$$(E, M) = [E^{-} \times E, A]$$

- the module $(E^{*}, M) : [E, X] \to A$ is given by the composition

$$[E, X] \xrightarrow{[E^{-} \times E, X]} [E^{-} \times E, X] \xrightarrow{(E, M)} A$$

that is,

$$(E^{*}, M) = [E^{-} \times E, X](E, M)$$

Similarly, for a module $\Phi : M \to N : X \to A$,

- the module morphism $(E^{*}, \Phi) : (E^{*}, M) \to (E^{*}, N) : X \to [E, A]$ is given by the composition

$$[E, X] \xrightarrow{[E^{-} \times E, X]} [E^{-} \times E, X] \xrightarrow{(E, M)} [E, A]$$

that is,

$$(E^{*}, \Phi) = (E, \Phi)[E^{-} \times E, A]$$

- the module morphism $(E^{*}, \Phi) : (E^{*}, M) \to (E^{*}, N) : [E, X] \to A$ is given by the composition

$$[E, X] \xrightarrow{[E^{-} \times E, X]} [E^{-} \times E, X] \xrightarrow{(E, M)} [E, A]$$

that is,

$$(E^{*}, \Phi) = [E^{-} \times E, X](E, \Phi)$$

Proof. For any $x \in X$ and $K \in [E, A]$,

$$(x) (E, M)(K) = \prod_{E} x\{M\} K$$

$$= \prod_{E} x\{M\} [E^{-} \times E]$$

$$(x) (E, M)(K \circ [E^{-} \times E])$$

$$(x) (E, M)(\{E^{-} \times E, A\} : K)$$

$$(x) \{(E, M)[E^{-} \times E, A]\} (K)$$

(*1 by Proposition 4.2.5). □
Corollary 4.6.34. The postcomposition cell

\[
\begin{align*}
X & \overset{\ast}{\gets} \coprod [E, X] \\
\coprod & [E, \phi] \quad [E, \delta] \\
Y & \overset{\ast}{\gets} \coprod [E, B]
\end{align*}
\]

in Definition 4.6.15 is obtained from the postcomposition cell in Definition 4.4.15 by the pasting composition

\[
\begin{align*}
X & \overset{(E,M)}{\gets} \coprod [E \times E, A] \quad [E \times E, A] \\
\coprod & [E \times E, Q] \quad [E, \delta] \\
Y & \overset{(E,N)}{\gets} \coprod [E \times E, B] \quad [E, \beta]
\end{align*}
\]

Proof. Immediate from Theorem 4.6.33, recalling the definition of the postcomposition cell \((E, \phi)\) and the definition of pasting composition.

4.7. Bicylinders

Definition 4.7.1. A cylinder from a product category is called a bicylinder. Given a pair of categories \(E\) and \(D\), and a module \(M : X \to A\), the following types of bicylinders \(E \times D \to M\) are considered: ordinary

\[
\begin{align*}
S & \quad E \times D \\
\alpha & \quad T
\end{align*}
\]

, extraordinary

\[
\begin{align*}
\ast & \quad [E \times D]^\ast \times [E \times D] \\
& \quad [E \times D]^\ast \times [E \times D] \\
\quad & \quad [E \times D]^\ast \times [E \times D] \\
& \quad [E \times D]^\ast \times [E \times D] \\
X & \quad \alpha \\
\quad & \quad \beta \\
\quad & \quad \gamma
\end{align*}
\]

, and complex

\[
\begin{align*}
E & \quad E \times D \times D \\
\quad & \quad E \times D \times D \\
S & \quad \alpha \\
\quad & \quad \beta \\
\quad & \quad \gamma
\end{align*}
\]
Remark 4.7.2.

(1) The two complex bicylinders in Definition 4.7.1 are defined by a cylindrical frame $\alpha$ of the composite module $S(\mathcal{M}) K : E \to E \times D^* \times D$ (regarded as an endomodule $E \times D \to E \times D$) and by a cylindrical frame $\alpha$ of the composite module $K(\mathcal{M}) T : E \times D^* \times D \to E$ (regarded as an endomodule $E \times D \to E \times D$). The two modules of complex bicylinders $E \times D \to \mathcal{M}$,

\[
(E \times D, \mathcal{M}) : [E, X] \to [E \times D^* \times D, A] \quad (E \times D, \mathcal{M}) : [E \times D^* \times D, X] \to [E, A]
\]

are defined by

\[
(S) (E \times D, \mathcal{M})(K) = \prod_{E \times D} S(\mathcal{M}) K \quad (K)(E \times D, \mathcal{M})(T) = \prod_{E \times D} K(\mathcal{M}) T
\]

respectively.

(2) Each of the bicylinders in Definition 4.7.1 is thus given by a cylindrical frame of an endomodule $E \times D \to E \times D$. The same notation $(E \times D, \mathcal{M})$ is used to denote the following modules of bicylinders:

<table>
<thead>
<tr>
<th>Module</th>
<th>Bicylinder</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(E \times D, \mathcal{M}) : [E \times D, X] \to [E \times D, A]$</td>
<td>Ordinary</td>
</tr>
<tr>
<td>$(E \times D, \mathcal{M}) : X \to [[E \times D]^* \times [E \times D], A]$</td>
<td>Extraordinary (left)</td>
</tr>
<tr>
<td>$(E \times D, \mathcal{M}) : [[E \times D]^* \times [E \times D], X] \to A$</td>
<td>Extraordinary (right)</td>
</tr>
<tr>
<td>$(E \times D, \mathcal{M}) : [E \times D^* \times D, A]$</td>
<td>Complex (left)</td>
</tr>
<tr>
<td>$(E \times D, \mathcal{M}) : [E \times D^* \times D, X] \to [E, A]$</td>
<td>Complex (right)</td>
</tr>
</tbody>
</table>

(3) An ordinary bicylinder along the hom of a category is just a natural transformation between bifunctors.

Definition 4.7.3. The right and left exponential transposes of a bicylinder

\[
\begin{array}{ccc}
S & \alpha & T \\
\leftarrow & \otimes & \rightarrow \\
E \times D & \alpha & A \\
\end{array}
\]

are the cylinders

\[
\begin{array}{ccc}
\alpha^- & T^- & \alpha^- \\
\leftarrow & \rightarrow & \rightarrow \\
[E, X] & \rightarrow & [E, A] \\
\end{array} \quad \begin{array}{ccc}
\alpha^- & T^- & \alpha^- \\
\leftarrow & \rightarrow & \rightarrow \\
[D, X] & \rightarrow & [D, A] \\
\end{array}
\]

defined by

\[
[\alpha_e]^d = \alpha(e, d) = [\alpha^e]^d
\]

for $e \in \|E\|$ and $d \in \|D\|$.

Remark 4.7.4.

(1) By Theorem 4.1.4, $\alpha^-$ and $\alpha^+$ and all of their components do form cylinders, and the right and left exponential transpositions form the iso cells

\[
\begin{array}{ccc}
[E \times D, X] & \otimes & [E \times D, A] \\
\simeq & \simeq & \simeq \\
[D, [E, X]] & \simeq & [D, [E, A]] \\
\end{array} \quad \begin{array}{ccc}
[E \times D, X] & \otimes & [E \times D, A] \\
\simeq & \simeq & \simeq \\
[E, [D, X]] & \simeq & [E, [D, A]] \\
\end{array}
\]

, natural in $E$, $D$, and $\mathcal{M}$. 
(2) The right and left exponential transposes of an extraordinary bicylinder is defined in the same way. The right and left exponential transposition for extraordinary bicylinders $E \times D \leadsto \mathcal{M}$ form the iso cells

$$
\begin{align*}
X \xrightarrow{(E \times D, \mathcal{M})} \left[ \left( \left( E \times D \right)^\sim \times \left( E \times D \right), A \right) \right] \quad & \quad \left[ \left( E \times D \right)^\sim \times \left( E \times D \right), X \right] \xrightarrow{(E \times D, \mathcal{M})} A \\
\xrightarrow{1} \quad & \quad \left[ D^\sim \times D, \left[ E^\sim \times E, A \right] \right] \quad & \quad \left[ D^\sim \times D, \left[ E^\sim \times E, X \right] \right] \xrightarrow{(D, (E, \mathcal{M}))} A
\end{align*}
$$

and

$$
\begin{align*}
X \xrightarrow{(E \times D, \mathcal{M})} \left[ \left( \left( E \times D \right)^\sim \times \left( E \times D \right), A \right) \right] \quad & \quad \left[ \left( E \times D \right)^\sim \times \left( E \times D \right), X \right] \xrightarrow{(E \times D, \mathcal{M})} A \\
\xrightarrow{1} \quad & \quad \left[ E^\sim \times E, \left[ D^\sim \times D, A \right] \right] \quad & \quad \left[ E^\sim \times E, \left[ D^\sim \times D, X \right] \right] \xrightarrow{(E, (D, \mathcal{M}))} A
\end{align*}
$$

(3) The right and left exponential transposes of a complex bicylinder is also defined in the same way. Since they play an important role in Section 12.2, we treat them separately below.

**Definition 4.7.5.**

- The right and left exponential transposes of a bicylinder

$$
\begin{align*}
& E \\
& \xrightarrow{s} \alpha \\
& \xrightarrow{\kappa} X
\end{align*}
$$

are the cylinders

$$
\begin{align*}
& \left[ E, X \right] \xrightarrow{(E, \mathcal{M})} \left[ E, A \right] \quad & \quad \left[ E \right] \xrightarrow{(E, \mathcal{M})} \left[ E, A \right]
\end{align*}
$$

defined by

$$
\left[ \alpha^\sim_{\mathcal{M}} \right]_e = \alpha(e, d) = \left[ \alpha^\sim_e \right]_d
$$

for $e \in [E]$ and $d \in [D]$. The left slice of $\alpha$ at $e \in [E]$ is the cylinder

$$
\begin{align*}
& S(e) \quad \left[ \alpha^\sim_e \right] \quad \kappa^\sim(e) \\
& \left[ E, X \right] \xrightarrow{(E, \mathcal{M})} \left[ E, A \right]
\end{align*}
$$

given by the component of the cylinder $\alpha^\sim$ at $e$.

- The right and left exponential transposes of a bicylinder

$$
\begin{align*}
& E \times D^\sim \times D \\
& \kappa \xrightarrow{\alpha} \left[ E \right] \\
& \xrightarrow{T} X
\end{align*}
$$

are the cylinders

$$
\begin{align*}
& \left[ E, X \right] \xrightarrow{(E, \mathcal{M})} \left[ E, A \right] \quad & \quad \left[ D^\sim \times D, X \right] \xrightarrow{(D, \mathcal{M})} A
\end{align*}
$$
defined by

\[ [\alpha_d]_e = \alpha_{(e,d)} = [\alpha_e]_d \]

for \( e \in \|E\| \) and \( d \in \|D\| \). The left slice of \( \alpha \) at \( e \in \|E\| \) is the cylinder

\[
\begin{array}{c}
\xrightarrow{\kappa'(e)} \\
\xrightarrow{\alpha^\circ}
\end{array}
\]

\[ X \xrightarrow{\alpha} A \]

given by the component of the cylinder \( \alpha^\circ \) at \( e \).

**Remark 4.7.6.** The right and left exponential transpositions form the iso cells

\[
\begin{array}{c}
\xrightarrow{(E \times D,M)} \\
\xrightarrow{1}
\end{array}
\]

\[ E, X \xrightarrow{-} E, \ A \]

\[
\begin{array}{c}
\xrightarrow{(E \times D,M)} \\
\xrightarrow{1}
\end{array}
\]

\[ E, X \xrightarrow{-} [E \times D, A] \]

and

\[
\begin{array}{c}
\xrightarrow{(E \times D,M)} \\
\xrightarrow{1}
\end{array}
\]

\[ E, X \xrightarrow{-} E, D, A \]

\[
\begin{array}{c}
\xrightarrow{(E \times D,M)} \\
\xrightarrow{1}
\end{array}
\]

\[ E, X \xrightarrow{-} E, D, A \]

, natural in \( E, D, \) and \( M \).

**Definition 4.7.7.** The transpose of a cylinder

\[
\begin{array}{c}
\xrightarrow{s} \\
\xrightarrow{\alpha}
\end{array}
\]

\[ E \xrightarrow{-} E \]

\[
\begin{array}{c}
\xrightarrow{\kappa}
\end{array}
\]

\[ X \xrightarrow{-} D, A \]

is the cylinder

\[
\begin{array}{c}
\xrightarrow{s} \\
\xrightarrow{\alpha^\top}
\end{array}
\]

\[ E \xrightarrow{-} E \]

\[
\begin{array}{c}
\xrightarrow{\kappa^\top}
\end{array}
\]

\[ X \xrightarrow{-} D, A \]

defined by

\[ [\alpha^\top]_d = [\alpha_d]_e \]

for \( e \in \|E\| \) and \( d \in \|D\| \). The slice of \( \alpha \) at \( e \in \|E\| \) is the cylinder

\[
\begin{array}{c}
\xrightarrow{S(e)} \\
\xrightarrow{\alpha^\circ_\top}
\end{array}
\]

\[ X \xrightarrow{-} A \]

\[
\begin{array}{c}
\xrightarrow{\kappa'(e)} \\
\xrightarrow{\alpha^\circ_\top}
\end{array}
\]

\[ X \xrightarrow{-} A \]

given by the component of the cylinder \( \alpha^\top \) at \( e \).

**Remark 4.7.8.**

1. The transposition \( \alpha \leftrightarrow \alpha^\top \) forms the iso cell

\[
\begin{array}{c}
\xrightarrow{1} \\
\xrightarrow{\tau}
\end{array}
\]

\[ E, X \xrightarrow{-} E, X \]

\[
\begin{array}{c}
\xrightarrow{\tau}
\end{array}
\]

\[ E, [D \times D, E, X] \]

\[
\begin{array}{c}
\xrightarrow{1}
\end{array}
\]

\[ E, [D \times D, X] \]
, natural in $E$, $D$ and $M$, making the diagram

\[
\begin{array}{c}
\xymatrix{
\langle E \times D, M \rangle \ar[r]^-{\iota} & \langle D, \langle E, M \rangle \rangle \\
\langle D, \langle E, M \rangle \rangle \ar[u] \ar[r]^-{\iota} & \langle E, \langle D, M \rangle \rangle \ar[u]
}
\end{array}
\]

commute, where $\iota$ and $\iota$ are the cells in Remark 4.7.6.

(2) The diagram

\[
\begin{array}{c}
\xymatrix{
\langle D, \langle E, M \rangle \rangle \ar[r]^-{\iota} & \langle E, \langle D, M \rangle \rangle \\
\langle D, M \rangle \ar[u] \ar[r]^-{\iota} & \langle E, \langle D, M \rangle \rangle \ar[u]
}
\end{array}
\]

commutes, where $\langle e, M \rangle$ and $\langle e, \langle D, M \rangle \rangle$ are evaluations at $e \in \|E\|$ (see Example 4.3.30). Hence the postcomposition

\[
\begin{array}{c}
\xymatrix{
\ast \ar[d]^s & D^\times D \ar[r]^-{\alpha} & D^\times D \ar[d]^K \\
[E, X] \ar[r]^-{(E, M)} & [E, A] \\
\{e, X\} \ar[r]^-{(e, M)} & \{e, A\}
}
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
X \ar[r]^-{\iota_M} & A \\
X \ar[r]^-{\iota_M} & A
}
\end{array}
\]

of $\alpha$ with the evaluation $\langle e, M \rangle$ yields the same cylinder as the one given by the component of $\alpha^T$ at $e$, i.e. the slice of $\alpha$ at $e$ (cf. Preliminary 16).

### 4.8. Wedges

**Note.** A combination of a cone and a cylinder is called a wedge\(^2\). Defined below are representative instances of wedges.

**Definition 4.8.1.** Let $E$ and $D$ be categories and $M : X \to A$ be a module.

- Given a functor $S : E \to X$ and a bifunctor $K : E \times D \to A$, a wedge $\alpha$ from $S$ to $K$ along $M$,

    written $\alpha : S \sim K : E \times D \sim M$, is a cylinder $E \xrightarrow{\alpha} E \times D$ right weighted by the projection $\xymatrix{\ast \ar[d]^s & D \ar[r]^-{\alpha} & \ast \ar[d]^T \\
\{e, X\} & \{e, A\} & \{e, A\}
}$

    \[
    \begin{array}{c}
    \xymatrix{\ast \ar[d]^s & D \ar[r]^-{\alpha} & \ast \ar[d]^T \\
    \{e, X\} & \{e, A\} & \{e, A\}
    }
    \end{array}
    \]

    with a dummy variable varying over $D\sim$ introduced to $K$.

\[
\begin{array}{c}
\xymatrix{E \ar[d]^s & E \times D \ar[r]^-{\alpha} & \ast \ar[d]^K \\
\{e, X\} & \{e, A\} & \{e, A\}
}
\end{array}
\]

- Given a functor $T : E \to A$ and a bifunctor $K : E \times D \to X$, a wedge $\alpha$ from $K$ to $T$ along $M$,

    written $\alpha : K \sim T : E \times D \sim M$, is a cylinder $E \xrightarrow{\alpha} E \times D$ left weighted by the projection $\xymatrix{\ast \ar[d]^s & E \ar[r]^-{\alpha} & \ast \ar[d]^T \\
\{e, X\} & \{e, A\} & \{e, A\}
}$

    \[
    \begin{array}{c}
    \xymatrix{\ast \ar[d]^s & E \ar[r]^-{\alpha} & \ast \ar[d]^T \\
    \{e, X\} & \{e, A\} & \{e, A\}
    }
    \end{array}
    \]

**Remark 4.8.2.**

(1) Just like a cone is a special instance of a cylinder, a wedge is a special instance of a bicylinder.

(2) A wedge $\alpha : S \sim K : E \times D \sim M$ is also defined by a bicylinder

\[
\begin{array}{c}
\xymatrix{E \ar[d]^s & E \times D \ar[r]^-{\alpha} & \ast \ar[d]^K \\
\{e, X\} & \{e, A\} & \{e, A\}
}
\end{array}
\]

\[\text{This terminology differs from the literature, which uses the term "wedge" to refer to what this book calls an extraordinary cylinder.}\]
(3) A wedge also arises as a combination of a cone and an extraordinary cylinder. For example, given an object \( x \in \| X \| \) and a bifunctor \( K : E^- \times E \times D \to A \), an extraordinary wedge

\[
\alpha : x \mapsto K : E \times *D \to M
\]

is defined by one of the following bicylinders:

\[
\begin{array}{ccc}
D & E^- \times E \times D & * \quad E^- \times D^- \times E \times D \\
\downarrow & \quad \downarrow & \quad \downarrow \\
X & * \quad \alpha \quad K & E^- \times E \times D \\
\end{array}
\]

\[
\begin{array}{ccc}
X & * \quad \alpha \quad K & E^- \times E \times D \\
\downarrow & \quad \downarrow & \quad \downarrow \\
X & * \quad \alpha \quad K & E^- \times E \times D \\
\end{array}
\]

**Definition 4.8.3.** Let \( E \) and \( D \) be categories and \( M : X \to A \) be a module.

- The module of wedges \( E \times *D \to M \),

\[
\left( E \times *D, M \right) : [E, X] \to [E \times D, A]
\]

is defined by the composition

\[
[E, X] \xrightarrow{[E \times *D, X]} [E \times D, X] \xrightarrow{[E \times D, M]} [E \times D, A]
\]

; that is,

\[
\left( E \times *D, M \right) := \left( E \times *D \dashv M \right)
\]

(see Definition 4.5.3).

- The module of wedges \( E \times D^* \to M \),

\[
\left( E \times D^*, M \right) : [E \times D, X] \to [E, A]
\]

is defined by the composition

\[
[E \times D, X] \xrightarrow{[E \times D, M]} [E \times D, A] \xrightarrow{[E \times *D, A]} [E, A]
\]

; that is,

\[
\left( E \times D^*, M \right) := \left( E \times D^* \dashv M \right)
\]

(see Definition 4.5.3).

**Remark 4.8.4.**

(1) For a functor \( S : E \to X \) and a bifunctor \( K : E \times D \to A \), the set \( \left( S \right) \left( E \times *D, M \right) (K) \) consists of all wedges \( S \sim K : E \times *D \to M \); and for a functor \( T : E \to A \) and a bifunctor \( K : E \times D \to X \), the set \( \left( K \right) \left( E \times D^*, M \right) (T) \) consists of all wedges \( K \sim T : E \times D^* \to M \).

(2) The module \( \left( E \times *D, M \right) \) is alternatively defined by the composition

\[
[E, X] \xrightarrow{[E \times *D, M]} [E \times D^* \times D, A] \xrightarrow{[E \times *D, A]} [E \times D, A]
\]

(see Remark 4.8.2(2)).

(3) The module of extraordinary wedges

\[
\left( E \times *D, M \right) : X \to [E^\circ \times E \times D, A]
\]

(see Remark 4.8.2(3)) is defined by the composition

\[
X \xrightarrow{[\cdot D, X]} [D, X] \xrightarrow{[E \times D, M]} [E^\circ \times E \times D, A]
\]

, or by the composition

\[
X \xrightarrow{[E \times *D, M]} [E^\circ \times D^* \times E \times D, A] \xrightarrow{[E^\circ \times *D, A]} [E^\circ \times E \times D, A]
\]

**Note.** The following definition is analogous to Definition 4.7.5.
Definition 4.8.5.

- The right and left exponential transposes of a wedge

\[
\begin{array}{c}
E \xrightarrow{\text{Exl}} E \times D \\
s \downarrow \alpha \downarrow \kappa \\
X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\alpha} A
\end{array}
\]

are the cone and the cylinder

\[
\begin{array}{c}
s \downarrow \alpha' \downarrow \kappa' \\
[X, X] \xrightarrow{\alpha'} [E, A] \\
X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\alpha} A
\end{array}
\]

defined by

\[
[\alpha_d]_e = \alpha(e, d) = [\alpha[e]]_d
\]

for \( e \in \|E\| \) and \( d \in \|D\| \). The left slice of \( \alpha \) at \( e \in \|E\| \) is the cone

\[
\begin{array}{c}
s(e) \downarrow \alpha'_e \downarrow \kappa'(e) \\
[X, X] \xrightarrow{\alpha'_e} [E, A] \\
X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\alpha} A
\end{array}
\]

given by the component of the cylinder \( \alpha' \) at \( e \).

- The right and left exponential transposes of a wedge

\[
\begin{array}{c}
E \times D \xrightarrow{\text{Exl}} E \\
\kappa \downarrow \alpha \downarrow \tau \\
X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\alpha} A
\end{array}
\]

are the cone and the cylinder

\[
\begin{array}{c}
\kappa' \downarrow \alpha' \downarrow \tau \\
[E, X] \xrightarrow{\alpha'} [E, A] \\
X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\alpha} A
\end{array}
\]

defined by

\[
[\alpha_d']_e = \alpha(e, d) = [\alpha[e]]_d
\]

for \( e \in \|E\| \) and \( d \in \|D\| \). The left slice of \( \alpha \) at \( e \in \|E\| \) is the cone

\[
\begin{array}{c}
\kappa'(e) \downarrow \alpha'_e \downarrow \tau(e) \\
[X, X] \xrightarrow{\alpha'_e} [E, A] \\
X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\alpha} A
\end{array}
\]

given by the component of the cylinder \( \alpha' \) at \( e \).

Remark 4.8.6.

- The right and left exponential transpositions of wedges \( E \times \ast D \twoheadrightarrow \mathcal{M} \) form the iso cells

\[
\begin{array}{c}
[E, X] \xrightarrow{\text{Exl}(\ast D, \mathcal{M})} [E \times D, A] \\
1 \downarrow \alpha \downarrow 1 \\
[E, X] \xrightarrow{\alpha} [D, [E, A]]
\end{array}
\]

\[
\begin{array}{c}
[E, X] \xrightarrow{\text{Exr}(\ast D, \mathcal{M})} [E \times D, A] \\
1 \downarrow \alpha \downarrow 1 \\
[E, X] \xrightarrow{\alpha} [E, [D, A]]
\end{array}
\]
, natural in \( E, D, \) and \( \mathcal{M} \). In fact, these iso cells are obtained from the iso cells in Remark 4.7.4(1) by pasting a commutative diagram of diagonal functors (see Preliminary 18) as shown in

![Diagram](image)

- The right and left exponential transpositions of wedges \( E \times D \rightarrowtail \mathcal{M} \) form the iso cells

\[
\begin{align*}
[ E \times D, X ] & \xrightarrow{(E \times D, M)_X} [ E \times D, A ] & \xrightarrow{(E \times D, M)_X} [ E \times D, A ] \\
[ D, [ E, X ] ] & \xrightarrow{(D, (E, M)_X)} [ E, A ] & \xrightarrow{(D, (E, M)_X)} [ E, A ] \\
[ E, [ D, X ] ] & \xrightarrow{(E, (D, M)_X)} [ D, \mathcal{M} ] & \xrightarrow{(E, (D, M)_X)} [ E, \mathcal{A} ]
\end{align*}
\]

, natural in \( E, D, \) and \( \mathcal{M} \). In fact, these iso cells are obtained from the iso cells in Remark 4.7.4(1) by pasting a commutative diagram of diagonal functors (see Preliminary 18) as shown in

![Diagram](image)

Note. The following definition is analogous to Definition 4.7.7.

**Definition 4.8.7.** The transpose of a cone

\[
\begin{array}{c}
\ast \xrightarrow{1} D \\
\ast \xrightarrow{1} D
\end{array}
\]

is the cylinder

\[
\begin{array}{c}
E \xrightarrow{\alpha^\top} E \\
\ast \xrightarrow{1} D
\end{array}
\]

defined by

\[
[ \alpha^\top ]_{\mathcal{M}} = [ \alpha ]_{\mathcal{E}}
\]

for \( e \in [ E ] \) and \( d \in [ D ] \). The slice of \( \alpha \) at \( e \in [ E ] \) is the cone

\[
\begin{array}{c}
\ast \xrightarrow{1} D \\
\ast \xrightarrow{1} D
\end{array}
\]

given by the component of the cylinder \( \alpha^\top \) at \( e \).
Remark 4.8.8.

1. The transposition $\alpha \mapsto \alpha^\top$ forms the iso cell

$$\begin{array}{ccc}
\{E, X\} & \xrightarrow{\mathsf{D}(E, M)} & \{D, [E, A]\} \\
\downarrow^1 & & \downarrow^1 \\
\{E, X\} & \xrightarrow{\{E, \mathsf{D}(E, M)\}} & \{E, [D, A]\}
\end{array}$$

resp.

$$\begin{array}{ccc}
\{D, [E, X]\} & \xrightarrow{\mathsf{D}(E, M)} & \{E, [E, A]\} \\
\downarrow^1 & & \downarrow^1 \\
\{E, [D, X]\} & \xrightarrow{\{E, \mathsf{D}(E, M)\}} & \{E, [E, A]\}
\end{array}$$

, natural in $E$, $D$, and $M$, making the diagram

$$\begin{array}{ccc}
\{E \times \mathsf{D}(E, M)\} & \xrightarrow{\mathsf{D}(E, M)} & \{E, \{E, \mathsf{D}(E, M)\}\} \\
\downarrow^\top & & \downarrow^\top \\
\{E \times \mathsf{D}(E, M)\} & \xrightarrow{\mathsf{D}(E, M)} & \{E, \{E, \mathsf{D}(E, M)\}\}
\end{array}$$

and

$$\begin{array}{ccc}
\{E \times \mathsf{D}(E, M)\} & \xrightarrow{\mathsf{D}(E, M)} & \{E, \{E, \mathsf{D}(E, M)\}\} \\
\downarrow^\top & & \downarrow^\top \\
\{E \times \mathsf{D}(E, M)\} & \xrightarrow{\mathsf{D}(E, M)} & \{E, \{E, \mathsf{D}(E, M)\}\}
\end{array}$$

commute, where $\vdash$ and $\dashv$ are the cells in Remark 4.8.6.

2. The diagram

$$\begin{array}{ccc}
\{E \times \mathsf{D}(E, M)\} & \xrightarrow{\mathsf{D}(E, M)} & \{E, \{E, \mathsf{D}(E, M)\}\} \\
\downarrow^\top & & \downarrow^\top \\
\{E \times \mathsf{D}(E, M)\} & \xrightarrow{\mathsf{D}(E, M)} & \{E, \{E, \mathsf{D}(E, M)\}\}
\end{array}$$

commutes, where $\{e, M\}$ and $\{e, \{E, \mathsf{D}(E, M)\}\}$ (resp. $\{e, \{E, \mathsf{D}(E, M)\}\}$) are evaluations at $e \in \|E\|$ (see Example 4.3.30); the postcomposition

$$\begin{array}{ccc}
\star & \xrightarrow{\mathsf{D}} & D \\
\downarrow^\alpha & & \downarrow^\mathsf{K} \\
\{E, X\} & \xrightarrow{\{E, \mathsf{M}\}} & \{E, A\}
\end{array}$$

resp.

$$\begin{array}{ccc}
\mathsf{D} & \xrightarrow{\star} & \star \\
\downarrow^\mathsf{K} & & \downarrow^\alpha \\
\{E, X\} & \xrightarrow{\{E, \mathsf{M}\}} & \{E, A\}
\end{array}$$

of $\alpha$ with the evaluation $\{e, M\}$ thus yields the same cone as the one given by the component of $\alpha^\top$ at $e$, i.e. the slice of $\alpha$ at $e$ (cf. Preliminary 16).

4.9. Cones and wedges in a category

Notation 4.9.1. Given a functor $K : E \to C$ and an object $c \in \|C\|$, a cone $\alpha$ from $c$ to $K$ is denoted by

$$\alpha : c \to K : \star E \to C$$

a cone $\alpha$ from $K$ to $c$ is denoted by

$$\alpha : K \Rightarrow c : \mathsf{E} \to C$$

Remark 4.9.2.

1. By Example 4.2.3, a cone in a category $C$ is just a special instance of a cone in Definition 4.6.3 where $\mathcal{M}$ is the hom of $C$. Conversely, a cone along a module $\mathcal{M}$ is the same thing as a cone in the collage category $[\mathcal{M}]$ (cf. Remark 4.3.4(4)).
4.9. Cones and wedges in a category

(2) By Remark 4.6.4(1),

- a cone $\alpha : c \to K : E \to C$ is the same thing as a natural transformation $\begin{array}{c} * \xrightarrow{\alpha} E \xrightarrow{\kappa} C \\ \downarrow \alpha \downarrow \kappa \end{array}$

weighted by the unique functor $E \to *$.

- a cone $\alpha : K \to c : E * \to C$ is the same thing as a natural transformation $\begin{array}{c} E \xrightarrow{\kappa} * \xleftarrow{\alpha} C \\ \downarrow \kappa \downarrow \alpha \end{array}$

weighted by the unique functor $E \to *$.

Note. The module of cones in a category is defined as below as a special case of Definition 4.6.5 where $\mathcal{M}$ is given by the hom of a category.

**Definition 4.9.3.** Given categories $E$ and $C$,

- the module of cones $\ast E \to C$,

$$(\ast E, C) : C \to [E, C]$$

is defined by

$$(c)(\ast E, C)(K) = \prod_{E} c(C) K$$

for $c \in C$ and $K \in [E, C]$; that is,

$$(\ast E, C) := (\ast E, \{C\})$$

- the module of cones $E * \to C$,

$$(E *, C) : [E, C] \to C$$

is defined by

$$(K)(E *, C)(c) = \prod_{E} K(C) c$$

for $c \in C$ and $K \in [E, C]$; that is,

$$(E *, C) := (E *, \{C\})$$

**Remark 4.9.4.**

(1) For an object $c \in \|C\|$ and a functor $K : E \to C$,

- the set $(c)(\ast E, C)(K)$ consists of all cones $c \to K : E \to C$.

- the set $(K)(E *, C)(c)$ consists of all cones $K \to c : E * \to C$.

(2) By applying Theorem 4.6.20 to the hom endomodule $(C)$ and noting Remark 4.5.4(3), we have

$$(\ast E, C) = (\ast E \uparrow C)$$

resp. $(E *, C) = (E * \uparrow C)$

hence

- the module $(\ast E, C) : C \to [E, C]$ is given by the composition

$$C \xrightarrow{[1_{E}, C]} [E, C] \xrightarrow{E, C} [E, C]$$

i.e. by the representable module of the diagonal functor $[1_{E}, C]$.

- the module $(E *, C) : [E, C] \to C$ is given by the composition

$$[E, C] \xrightarrow{E, C} [E, C] \xrightarrow{[1_{E}, C]} C$$

i.e. by the corepresentable module of the diagonal functor $[1_{E}, C]$. 
Note. The following definition is a special case of Definition 4.6.15 where \( \Phi \) is given by the hom of a functor.

**Definition 4.9.5.** Given a category \( E \) and a functor \( H : C \to B \),

- the cell

\[
\begin{align*}
C &\xrightarrow{(\cdot E, C)} [E, C] \\
H &\downarrow \quad (\cdot E, H) \quad \downarrow [E, H] \\
B &\xrightarrow{(\cdot E, B)} [E, B]
\end{align*}
\]

, “postcomposition with \( H \)”, is defined by the module morphism

\[
\langle \ast E, \langle C \rangle \rangle \xrightarrow{(\cdot E, \langle H \rangle)} \langle \ast E, H \langle B \rangle \rangle \ast H = [H] \langle \ast E, \langle B \rangle \rangle [E, H]
\]

, postcomposition (see Definition 4.6.11) with the hom of \( H \); that is,

\[
\langle \ast E, H \rangle := \langle \ast E, \langle H \rangle \rangle
\]

- the cell

\[
\begin{align*}
[E, C] &\xrightarrow{(E_\ast, C)} C \\
[E, H] &\downarrow \quad (E_\ast, H) \quad \downarrow H \\
[E, B] &\xrightarrow{(E_\ast, B)} B
\end{align*}
\]

, “postcomposition with \( H \)”, is defined by the module morphism

\[
\langle E_\ast, \langle C \rangle \rangle \xrightarrow{(E_\ast, \langle H \rangle)} \langle E_\ast, H \langle B \rangle \rangle \ast H = [E, H] \langle E_\ast, \langle B \rangle \rangle [H]
\]

, postcomposition (see Definition 4.6.11) with the hom of \( H \); that is,

\[
\langle E_\ast, H \rangle := \langle E_\ast, \langle H \rangle \rangle
\]

**Remark 4.9.6.**

1. The cell \( \langle \ast E, H \rangle \) sends each cone \( \alpha : c \to K : \ast E \to C \) to the cone \( \alpha \circ H : c \circ H \to H \circ K : \ast E \to B \), the usual composite of a cone and a functor; and the cell \( \langle E_\ast, H \rangle \) sends each cone \( \alpha : K \to c : E_\ast \to C \) to the cone \( \alpha \circ H : K \circ H \to H \circ c : E_\ast \to B \).

2. By replacing \( \Phi \) in Corollary 4.6.21 with the hom cell \( \langle H \rangle \) and noting Remark 4.5.8(3), we have

\[
\langle \ast E, H \rangle = \langle !_E \circ H \rangle \quad \langle E_\ast, H \rangle = \langle !_E \ast H \rangle
\]

; hence \( \langle \ast E, H \rangle \) and \( \langle E_\ast, H \rangle \) are obtained from the hom of the postcomposition functor \([E, C]\) by the pasting compositions

\[
\begin{align*}
C &\xrightarrow{[E, C]} [E, C] \\
H &\downarrow \quad \downarrow [E, H] \\
B &\xrightarrow{[E, B]} [E, B]
\end{align*}
\]

\[
\begin{align*}
[E, C] &\xrightarrow{(E, C)} [E, C] \\
[E, H] &\downarrow \quad \downarrow [E, H] \\
[E, B] &\xrightarrow{(E, B)} [E, B]
\end{align*}
\]

\[
\begin{align*}
C &\xrightarrow{[E, C]} [E, C] \\
H &\downarrow \quad \downarrow [E, H] \\
B &\xrightarrow{[E, B]} [E, B]
\end{align*}
\]

\[
\begin{align*}
[E, C] &\xrightarrow{(E, C)} [E, C] \\
[E, H] &\downarrow \quad \downarrow [E, H] \\
[E, B] &\xrightarrow{(E, B)} [E, B]
\end{align*}
\]

\[
\begin{align*}
C &\xrightarrow{[E, C]} [E, C] \\
H &\downarrow \quad \downarrow [E, H] \\
B &\xrightarrow{[E, B]} [E, B]
\end{align*}
\]


Note. The following definition is a special case of Definition 4.6.27 where \( \mathcal{M} \) is given by the hom of a category.
Definition 4.9.7. Given a functor $H : D \to E$ and a category $C$,

- the cell

\[
\begin{array}{c}
C \xrightarrow{(E, C)} [E, C] \\
\downarrow (H, C) \\
C \xrightarrow{(D, C)} [D, C]
\end{array}
\]

, "precomposition with $H"$, is defined by

\((c) \langle H, C \rangle (K) = \prod_{H} c \langle C \rangle K \)

for $c \in \|C\|$ and $K$ a functor $E \to C$; that is,

\(\langle H, C \rangle := \langle H, \{C\} \rangle \)

- the cell

\[
\begin{array}{c}
[E, C] \xrightarrow{(E, C)} C \\
[H, C] \downarrow (H, C) \\
[D, C] \xrightarrow{(D, C)} C
\end{array}
\]

, "precomposition with $H"$, is defined by

\((K) \langle H\ast, C \rangle (c) = \prod_{H\ast} K \langle C \rangle c \)

for $c \in \|C\|$ and $K$ a functor $E \to C$; that is,

\(\langle H\ast, C \rangle := \langle H\ast, \{C\} \rangle \)

Remark 4.9.8.

1. The cell $\langle H, C \rangle$ sends each cone $\alpha : x \sim K : *E \to C$ to the cone $H \circ \alpha : x \sim K \circ H : *D \to C$, the usual composite of a functor and a cone; and the cell $\langle H\ast, C \rangle$ sends each cone $\alpha : K \sim a : *E \to C$ to the cone $H \circ \alpha : H \circ K \sim a : *D \to C$.

2. Replacing $M$ in Proposition 4.6.30 with the hom endomodule $\{C\}$ and noting that $\{H, \{C\}\} = \{H, C\}$ (see Remark 4.3.28(3)), we see that $\langle H, C \rangle$ and $\langle H\ast, C \rangle$ are obtained from the hom of the precomposition functor $[H, C]$ by the pasting compositions

\[
\begin{array}{ccc}
C & \xrightarrow{[E, C]} & [E, C] \xrightarrow{(E, C)} [E, C] \\
\downarrow & \downarrow & \downarrow \\
C & \xrightarrow{[D, C]} & [D, C] \xrightarrow{(D, C)} [D, C]
\end{array}
\]

Example 4.9.9.

1. Let $C$ and $E$ be categories. Given a subcategory $D$ of $E$, precomposition with the inclusion $D \hookrightarrow E$ yields

- the cell

\[
\begin{array}{c}
C \xrightarrow{(E, C)} [E, C] \\
\downarrow (D, C) \\
C \xrightarrow{(D, C)} [D, C]
\end{array}
\]

, "restriction to $D"", which sends each cone $\alpha : c \sim K : *E \to C$ to the cone $D \circ \alpha : C \sim K \circ D : *D \to C$, $\alpha$ restricted to $D$. 


the cell

\[
\begin{array}{ccc}
[C, C] & \stackrel{(E \times C)}{\longrightarrow} & C \\
\downarrow & & \downarrow \\
[D, C] & \stackrel{(D \times C)}{\longrightarrow} & C
\end{array}
\]

Note. A wedge in a category is defined as below as a special case of Definition 4.8.1. Where

\[\begin{array}{ccc}
\text{C} & \downarrow & \text{D} \\
\end{array}\]

Definition 4.9.10. Given a functor \(F : E \to C\) and a bifunctor \(K : E \times D \to C\),

- a wedge \(\alpha\) from \(F\) to \(K\), written \(\alpha : F \to K : E \times D \to C\), is a natural transformation

\[\begin{array}{ccc}
E & \overset{\text{\(E\) to \(E\)}}{\longrightarrow} & E \times D \\
\downarrow & \alpha & \downarrow \Phi \\
C & \longrightarrow & C
\end{array}\]

- a wedge \(\alpha\) from \(K\) to \(F\), written \(\alpha : K \to F : E \times D \to C\), is a natural transformation

\[\begin{array}{ccc}
E \times D & \overset{\text{\(E\) to \(E\)}}{\longrightarrow} & E \\
\downarrow & \alpha & \downarrow \Phi \\
C & \longrightarrow & C
\end{array}\]

Remark 4.9.11. A wedge in a category \(C\) defined above is just a special instance of a wedge in

Definition 4.8.1 where \(M\) is the hom of \(C\). Conversely, a wedge along a module \(M\) is the same thing
as a wedge in the collage category \([M]\) (cf. Remark 4.3.4(4)).

Note. The module of wedges in a category is defined as below as a special case of Definition 4.8.3 where \(M\) is given by the hom of a category.

Definition 4.9.12. Let \(E\), \(D\), and \(C\) be categories.

- The module of wedges \(E \times D \to C\),

\[\langle E \times D, C \rangle : [E, C] \to [E \times D, C]\]

is defined by

\[\langle E \times D, C \rangle = \langle E \times D \uparrow C \rangle = \langle E \times D \uparrow \langle C \rangle \rangle\]
(see Remark 4.5.4(3)); that is,
\[ \langle E \times D, C \rangle := \langle E \times \ast, D \rangle, \]

- The module of wedges \( E \times D \ast \rightarrow C \),
\[ \langle E \times D \ast, C \rangle : [E \times D, C] \rightarrow [E, C] \]
, is defined by
\[ \langle E \times D \ast, C \rangle = \langle E \times !_D \ast, C \rangle = \langle E \times !_D \ast, C \rangle \]
(see Remark 4.5.4(3)); that is,
\[ \langle E \times D \ast, C \rangle := \langle E \times D \ast, C \rangle \]

Remark 4.9.13.

(1) For a functor \( F : E \rightarrow C \) and a bifunctor \( K : E \times D \rightarrow C \),
- the set \( (F) \langle E \times \ast, D, C \rangle, (K) \langle E \times \ast, D, C \rangle \) consists of all wedges \( F \Rightarrow K : E \times \ast \rightarrow C \).
- the set \( (K) \langle E \times D \ast, C \rangle, (F) \langle E \times D \ast, C \rangle \) consists of all wedges \( K \Rightarrow F : E \times D \ast \rightarrow C \).

(2) By Remark 4.5.4(3),
- the module \( \langle E \times \ast, D, C \rangle : [E, C] \rightarrow [E \times D, C] \) is given by the composition
\[ [E, C] \xrightarrow{[E \times !_D, C]} [E \times D, C] \xrightarrow{\langle E, D \rangle, C} [E \times D, C] \]
; that is, \( \langle E \times \ast, D, C \rangle \) is the representable module of the diagonal functor \([E \times !_D, C] \).
- the module \( \langle E \times D \ast, C \rangle : [E \times D, C] \rightarrow [E, C] \) is given by the composition
\[ [E \times D, C] \xrightarrow{\langle E, D \rangle, C} [E \times D, C] \xrightarrow{[E \times !D, C]} [E, C] \]
; that is, \( \langle E \times D \ast, C \rangle \) is the corepresentable module of the diagonal functor \([E \times !D, C] \).

(3) The exponential transpositions of a wedge in a category is defined as a special case of Definition 4.8.5 where \( M \) is given by the hom of a category.
- The right and left exponential transpositions of wedges \( E \times \ast D \rightarrow C \) form the iso cells
\[ [E, C] \xrightarrow{\langle E, D \rangle, C} [E \times D, C] \xrightarrow{\langle E, D \rangle, C} [E \times D, C] \xrightarrow{\langle E, D \rangle, C} [E \times D, C] \]
, natural in \( E, D, \) and \( C \). In fact, these iso cells are obtained from the homs of the functors
\[ [E \times D, C] \xrightarrow{\langle D, [E, C] \rangle} [D, [E, C]] \]
and \( [E \times D, C] \xrightarrow{\langle E, [D, C] \rangle} [E, [D, C]] \) by pasting a commutative diagram of diagonal functors (see Preliminary 18) as shown in
\[ [E, C] \xrightarrow{\langle E \times !D, C \rangle} [E \times D, C] \xrightarrow{\langle E \times !D, C \rangle} [E \times D, C] \]
\[ [E, C] \xrightarrow{\langle E \times !D, C \rangle} [E \times D, C] \xrightarrow{\langle E \times !D, C \rangle} [E \times D, C] \]
4.10. Cones and wedges in Set

Notation 4.10.1.

(1) A cone $\alpha$ from a set $S$ to a left module $M : \ast \to E$ (i.e. a functor $M : E \to \text{Set}$) is denoted by

$$\alpha : S \to M : \ast \to \ast E$$

rather than $\alpha : S \Rightarrow M : \ast E \to \text{Set}$, and the component at $e \in |E|$ is written as $\langle \alpha \rangle e : S \to \langle M \rangle e$.

The module of cones $\ast E \to \text{Set}$ is denoted by

$$\langle \ast E \rangle : \text{Set} \to [\ast : E]$$

rather than $\langle \ast E, \text{Set} \rangle : \text{Set} \to [E, \text{Set}]$.

(2) A wedge $\alpha$ from a right module $L : X \to \ast$ (i.e. a functor $L : X^\ast \to \text{Set}$) to a module $M : X \to E$ (i.e. a functor $M : X^\ast \times E \to \text{Set}$) is denoted by

$$\alpha : L \Rightarrow M : X \to \ast E$$

rather than $\alpha : L \Rightarrow M : X^\ast \times \ast E \to \text{Set}$, and the component at $(x, e) \in |X \times E|$ is written as $x \langle \alpha \rangle e : x(L) \to x(M) e$. The module of wedges $X^\ast \times \ast E \to \text{Set}$ is denoted by

$$\langle X \times \ast E : [x : X] \to [x : E]$$
rather than \( (X^{-} \times ^*E, \text{Set}) : [X^{-}, \text{Set}] \to [X^{-} \times E, \text{Set}] \).

\( \triangleright \) A wedge \( \alpha \) from a left module \( \mathcal{L} : \ast \to A \) (i.e. a functor \( \mathcal{L} : A \to \text{Set} \)) to a module \( \mathcal{M} : E \to A \) (i.e. a functor \( \mathcal{M} : E^{-} \times A \to \text{Set} \)) is denoted by

\[
\alpha : \mathcal{L} \to \mathcal{M} : E \to A
\]

rather than \( \alpha : \mathcal{L} \to \mathcal{M} : *E^{-} \times A \to \text{Set} \), and the component at \( (e, a) \in \|E \times A\| \) is written as \( e(\alpha) a : (\mathcal{L}) a \to e(\mathcal{M}) a \). The module of wedges \( *E^{-} \times A \to \text{Set} \) is denoted by

\[
(E^* : A) : [: A] \to [E : A]
\]

rather than \( (\ast E^{-} \times A, \text{Set}) : [A, \text{Set}] \to [E^{-} \times A, \text{Set}] \).

**Remark 4.10.2.**

(1) By Remark 4.9.4(2),

\( \triangleright \) \( (\ast E) : \text{Set} \to [: E] \) is the representable module of the diagonal functor \( [: !_E] : \text{Set} \to [: E] \) (see Example 1.1.29(7)); that is, \( (\ast E) \) is given by the composition

\[
\text{Set} \xrightarrow{[\cdot |_E]} [: E] \xrightarrow{(\ast E)} [: E]
\]

, and a cone \( \alpha : \mathcal{S} \to \mathcal{M} : * \to \ast E \) is the same thing as a left module morphism \( \alpha : (\mathcal{S}) [\cdot |_E] \to \mathcal{M} : * \to E \).

\( \triangleright \) \( (E^* :) : \text{Set} \to [E:] \) is the representable module of the diagonal functor \( !_E : \text{Set} \to [E:] \) (see Example 1.1.29(7)); that is, \( (E^* :) \) is given by the composition

\[
\text{Set} \xrightarrow{[\cdot |_E]} [E:] \xrightarrow{(E^*)} [E:]
\]

, and a cone \( \alpha : \mathcal{S} \to \mathcal{M} : E^* \to * \) is the same thing as a right module morphism \( \alpha : [\cdot |_E] (\mathcal{S}) \to \mathcal{M} : E \to * \).

(2) By Remark 4.9.13(2),

\( \triangleright \) \( (X : \ast E) : [X :] \to [X : E] \) is the representable module of the diagonal functor \( [X : !_E] : [X :] \to [X : E] \) (see Example 1.1.29(7)); that is, \( (X : \ast E) \) is given by the composition

\[
[X :] \xrightarrow{[X : |_E]} [X : E] \xrightarrow{(X^*E)} [X : E]
\]

, and a wedge \( \alpha : \mathcal{L} \to \mathcal{M} : X \to \ast E \) is the same thing as a module morphism \( \alpha : (\mathcal{L}) [\cdot |_E] \to \mathcal{M} : X \to E \).

\( \triangleright \) \( (E^* : A) : [: A] \to [E : A] \) is the representable module of the diagonal functor \( ![E] : [A] \to [E : A] \) (see Example 1.1.29(7)); that is, \( (E^* : A) \) is given by the composition

\[
[: A] \xrightarrow{[\cdot |_E]} [E : A] \xrightarrow{(E^*_A)} [E : A]
\]

, and a wedge \( \alpha : \mathcal{L} \to \mathcal{M} : E^* \to A \) is the same thing as a module morphism \( \alpha : ![E] (\mathcal{L}) \to \mathcal{M} : E \to A \).

**Notation 4.10.3.**

\( \triangleright \) The right and left exponential transpositions of wedges \( X^{-} \times \ast E \to \text{Set} \) are denoted by

\[
\begin{align*}
[X :] & \xrightarrow{(X^*E)} [X : E] \\
\downarrow & \quad \downarrow \\
[X :] & \xrightarrow{(\ast E,[X :])} [E,[X :]]
\end{align*}
\]

\[
\begin{align*}
[X :] & \xrightarrow{(X^*E)} [X : E] \\
\downarrow & \quad \downarrow \\
[X :] & \xrightarrow{([\cdot |_E]^E,[X :])} [X^{-},[: E]]
\end{align*}
\]
rather than

\[
\begin{array}{c}
\mathbf{[X^-, \text{Set}]} \xrightarrow{(X^\times \ast \text{Set})} \mathbf{[X^\times \text{E, Set}]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

\[
\begin{array}{c}
\mathbf{[X^-, \text{Set}]} \xrightarrow{(\ast \text{E, Set})} \mathbf{[\text{E}, [X^-, \text{Set}]]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

- The right and left exponential transpositions of wedges \( *\text{E} \times A \rightarrow \text{Set} \) are denoted by

\[
\begin{array}{c}
\mathbf{[A]} \xrightarrow{(\ast \text{E} \times A, \text{Set})} \mathbf{[\text{E} \times A, \text{Set}]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

\[
\begin{array}{c}
\mathbf{[A]} \xrightarrow{(\ast \text{E}, \text{Set})} \mathbf{[\text{E} \times \ast, \text{Set}]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

\[
\begin{array}{c}
\mathbf{[A]} \xrightarrow{(\ast \text{E} \times A, \text{Set})} \mathbf{[\text{E}, [A], \text{Set}]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

\[
\begin{array}{c}
\mathbf{[A]} \xrightarrow{(\ast \text{E}, \text{Set})} \mathbf{[(\ast \text{E}, A)^\times \text{Set}]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

. The cell \( (\ast \text{E}^-, \{A\}) \rightarrow (\ast \text{E}^-, \{A\})^{-} \) often appears in the opposite form

\[
\begin{array}{c}
\mathbf{[E]} \xrightarrow{\{E, \{A\}\}} \mathbf{[\{A\}]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

(note that \( (\ast \text{E}^-, \{A\})^{-} \cong (\ast \text{E}^-, \{A\})^{-} \) by Remark 4.6.6(4)).

Remark 4.10.4. By Remark 4.9.13(3),

- the iso cell \( \langle \mathbf{X} : \ast \text{E} \rangle \rightarrow \langle \ast \text{E}, \langle \mathbf{X} \rangle \rangle \) is obtained from the hom of the functor \( \mathbf{[X : E]} \xrightarrow{\sim} \mathbf{[E, [X ]]}' \) by pasting a commutative diagram of diagonal functors in Remark 2.1.2(5) as shown in

\[
\begin{array}{c}
\mathbf{[X : E]} \xrightarrow{\langle \mathbf{X} \rangle} \mathbf{[X : E]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

\[
\begin{array}{c}
\mathbf{[X : E]} \xrightarrow{\langle \mathbf{X} \rangle} \mathbf{[E, [X ]]}' \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

. Likewise, the iso cell \( \langle \mathbf{X} : \ast \text{E} \rangle \rightarrow \langle \ast \text{E}^-, \{\ast \text{E}\} \rangle \) is obtained from the hom of the functor \( \mathbf{[X : E]} \xrightarrow{\sim} \mathbf{[X^-, \{E\}]} \) by the pasting composition

\[
\begin{array}{c}
\mathbf{[X : E]} \xrightarrow{\langle \mathbf{X} \rangle} \mathbf{[X : E]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

\[
\begin{array}{c}
\mathbf{[X : E]} \xrightarrow{\langle \mathbf{X} \rangle} \mathbf{[X^-, \{E\}]} \xrightarrow{\langle \mathbf{X} \rangle} \mathbf{[X^-, \{E\}]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

- the iso cell \( \langle \ast \text{E}^+, \{A\} \rangle \rightarrow \langle \text{E}^+, \{\ast \text{E}^+, \{A\}\} \rangle \) is obtained from the hom of the functor \( \mathbf{[E : A]} \xrightarrow{\sim} \mathbf{[A, [E ]]}' \) by pasting a commutative diagram of diagonal functors in Remark 2.1.2(5) as shown in

\[
\begin{array}{c}
\mathbf{[E : A]} \xrightarrow{\langle \mathbf{E} \rangle} \mathbf{[E : A]} \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]

\[
\begin{array}{c}
\mathbf{[E : A]} \xrightarrow{\langle \mathbf{E} \rangle} \mathbf{[A, [E ]]}' \\
\downarrow 1 \quad \downarrow 1
\end{array}
\]
Likewise, the iso cell \( (E^* : A)^- \xrightarrow{\sim} (E^*, :: A)^- \) is obtained from the hom of the functor \([E : A]^- \xrightarrow{\sim} [E, :: A]^-\) by the pasting composition

\[
\begin{array}{ccc}
\downarrow & & \downarrow \cong \\
[E, :: A]^- & \xrightarrow{\sim} & [E, :: A]^- \end{array}
\]
5. Yoneda Lemma

5.1. Yoneda modules

Definition 5.1.1.

- The right Yoneda module for a category $X$ is the module
  $$\langle X, \cdot \rangle : X \to [X:]$$
  given by the evaluation
  $$(x, M) \mapsto x(M) : X^\sim \times [X:] \to \text{Set}$$
  ; that is,
  $$(x)(X, \cdot)(M) := x(M)$$
  for $x \in X$ and $M \in [X:]$.

- The left Yoneda module for a category $A$ is the module
  $$\langle \cdot, A \rangle : [\cdot : A]^{-} \to A$$
  given by the evaluation
  $$(M, a) \mapsto (M)a : [\cdot : A] \times A \to \text{Set}$$
  ; that is,
  $$(M)(\cdot, A)(a) := (M)a$$
  for $a \in A$ and $M \in [\cdot : A]$.

Remark 5.1.2.

1. The module $X, \cdot$ (resp. $\cdot, A$) is called the Yoneda module just because it is represented (resp. corepresented) by the Yoneda functor $X \downarrow$ (resp. $\downarrow A$) (see Theorem 5.2.17).

2. For a right module $M : X \to \ast$ and an object $x \in [X]$, the set $(x)(X, \cdot)(M)$ consists of all $M$-arrows $x \leadsto \ast$, and for a left module $M : \ast \to A$ and an object $a \in [\cdot A]$, the set $(M)(\cdot, A)(a)$ consists of all $M$-arrows $\ast \leadsto a$.

Proposition 5.1.3.

- The right exponential transpose of the right Yoneda module $\langle X, \cdot \rangle : X \to [X:]$ is the identity $[X:] \to [X:]$; that is,
  $$[X:] \xrightarrow{(X, \cdot)^\sim = 1} [X:]$$
  ; the right slice of $X, \cdot$ at a right module $M : X \to \ast$ is $M$ itself:
  $$\langle X, \cdot \rangle(M) = M$$

- The left exponential transpose of the left Yoneda module $\langle \cdot, A \rangle : [\cdot : A]^{-} \to A$ is the identity $[\cdot : A] \to [\cdot : A]$; that is,
  $$[\cdot : A] \xrightarrow{\langle \cdot, A \rangle = 1} [\cdot : A]$$
  ; the left slice of $\cdot, A$ at a left module $M : \ast \to A$ is $M$ itself:
  $$(M)(\cdot, A) = M$$

Proof. Immediate from the definition.
Proposition 5.1.4. Given a module (resp. module morphism) $M : X \to A$, the identity
\[ M = (X, \ast) [M \ast] \]
holds; that is, $M$ is recovered from its right exponential transpose by the composition
\[ X - \xrightarrow{X, \ast} [X:] \xleftarrow{M, \ast} A \]
; hence the right action of the right Yoneda module $X, \ast$ on the functor category $[A, [X:]]$ yields the inverse of the right exponential transposition $[X:A] \xrightarrow{\ast} [A, [X:]]$; that is,
\[ [X:A] \xrightarrow{\sim} (X, \ast) A = \ast \]

the identity
\[ M = [\ast M, \ast] (\ast, A) \]
holds; that is, $M$ is recovered from its left exponential transpose by the composition
\[ X \xleftarrow{\ast M} [:A:] - \xrightarrow{\ast} A \]
; hence the left action of the left Yoneda module $\ast, A$ on the functor category $[X, [:A:]]$ yields the inverse of the left exponential transposition $[X:A]^{-1} \xrightarrow{\ast} [X, [:A:]^{-1}]$; that is,
\[ [X:A]^{-1} \xrightarrow{\sim} X \xleftarrow{\ast} [A]^{-1} \]

Proof. By the bijectiveness of exponential transposition, it suffices to show that
\[ M \ast = ((X, \ast) [M \ast]) \ast \]
But by Proposition 2.1.6 and Proposition 5.1.3,
\[ ((X, \ast) [M \ast]) \ast = [\langle X, \ast \rangle [M \ast]] \ast = M \ast \]

Remark 5.1.5. For a module $M : X \to A$, the identities in Proposition 5.1.4 are expressed by
\[ X \xrightarrow{\sim} [X:] \xrightarrow{\ast M} A \]
\[ X \xrightarrow{\sim} X, \ast [X:] \xrightarrow{\sim} [:A:] \xrightarrow{\ast} A \]
These fully faithful cells identify each $M$-arrow $m : x \sim a$ with the $\langle M \rangle a$-arrow $m : x \sim \ast$ and with the $x (M)$-arrow $m : \ast \sim a$ respectively (see Remark 5.1.2(2)).

Definition 5.1.6. Given categories $X$ and $A$,

the right general Yoneda module for $[A, X]$, 
\[ \langle X, \ast A \rangle : [A, X] \to [X : A] \]
is defined by
\[ (G \langle X, \ast A \rangle) (M) = \prod_A G \langle M \rangle \]
for $G \in [A, X]$ and $M \in [X : A]$. 

• the left general Yoneda module for $[X, A]$,

$$(X^*, A) : [X : A]^\sim \to [X, A]$$

is defined by

$$(M) (X^*, A) (F) = \prod_X (M) F$$

for $M \in [X : A]$ and $F \in [X, A]$.

**Remark 5.1.7.**
(1) The module $X^* A$ (resp. $X^\ast A$) is called the general Yoneda module just because it is represented (resp. corepresented) by the general Yoneda functor $X^* A$ (resp. $X^\ast A$) (see Theorem 5.3.17).
(2) For a module $M : X \to A$ and a functor $G : A \to X$, the set $(G) (X^* A) (M)$ consists of all right cylinders $G \Rightarrow M$, and for a module $M : X \to A$ and a functor $F : X \to A$, the set $(M) (X^\ast A) (F)$ consists of all left cylinders $M \Rightarrow F$.
(3) The identification in Remark 4.3.2(2) yields canonical isomorphisms

$$\langle X^* \rangle \cong \langle X^\ast \rangle$$

and

$$\langle \ast, A \rangle \cong \langle \ast^\ast, A \rangle$$

. Yoneda modules are thus special instances of general Yoneda modules.

**Theorem 5.1.8.** Given a category $E$ and a module $M : X \to A$, the identities

$$[E, X] \langle E, M \rangle \xrightarrow{\alpha} [E, A]$$

$$(E, M)$$

$$(E, M)$$

$$(E, M)$$

$$(E, M)$$

$$(E, M)$$

hold.

**Proof.** Immediate from Definition 4.3.5 and Definition 5.1.6; indeed, for any $S \in [E, X]$ and any $T \in [E, A]$,

$$S (E, M) T = \prod_S \langle (E, M) T \rangle$$

$$= \prod_E \langle (E, M) T \rangle$$

$$(S (E, M) (E, A) T)$$

$$(S (E, M) (E, A) T)$$

$$(S (E, M) (E, A) T)$$

$$(S (E, M) (E, A) T)$$

$$(S (E, M) (E, A) T)$$

$$S \langle [E, M] \rangle (E, A) T$$

$$S \langle [E, M] \rangle (E, A) T$$

$$S \langle [E, M] \rangle (E, A) T$$

$$S \langle [E, M] \rangle (E, A) T$$

Remark 5.1.9.
(1) This is the identity stated in Remark 4.3.4(3): the fully faithful cells in Theorem 5.1.8 identify each two-sided cylinder

$$S \xrightarrow{\alpha} T \xleftarrow{\varepsilon} A$$

\[\square\]
with the right cylinder

\[
\begin{array}{c}
X \\[-s]
\end{array}
\begin{array}{c}
\downarrow \alpha \\
\downarrow \beta \\
(M,T)
\end{array}
\begin{array}{c}
E \\
\end{array}
\]

and with the left cylinder

\[
\begin{array}{c}
E \\
\end{array}
\begin{array}{c}
\downarrow \alpha \\
\downarrow \beta \\
(M,T)
\end{array}
\begin{array}{c}
A \\
\end{array}
\]

respectively.

(2) If we replace \(E\) with the terminal category, then we obtain the cells in Remark 5.1.5.

**Corollary 5.1.10.** Given categories \(X\) and \(A\), the identities

\[
\begin{array}{c}
[A,X] - \xrightarrow{\alpha} - [X:A] \\
\downarrow \downarrow \\
A \uparrow \uparrow \\
\end{array}
\begin{array}{c}
[X:A] - \xleftarrow{\alpha} - [X,A] \\
\downarrow \downarrow \\
A \downarrow \downarrow \\
\end{array}
\]

hold.

**Proof.** Replacing \(M\) in Theorem 5.1.8 with the right and left Yoneda modules, we have the identities

\[
\begin{array}{c}
[A,X] - \xrightarrow{\alpha} - [A,[X:]]) \\
\downarrow \downarrow \\
A \uparrow \uparrow \\
\end{array}
\begin{array}{c}
[X,[\cdot:A]] - \xleftarrow{\alpha} - [X,A] \\
\downarrow \downarrow \\
A \downarrow \downarrow \\
\end{array}
\]

Since \((X,\cdot)\) \(\cdot\) and \(X\) \((\cdot,\cdot)\) by Proposition 5.1.4, the assertion follows by taking the inverses. \(\square\)

**Remark 5.1.11.**

- The iso cell \(((X,\cdot)) \xrightarrow{\alpha} (A,\langle X,\cdot \rangle)\) identifies each right cylinder

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
\downarrow \alpha \\
\downarrow \beta \\
(M,T)
\end{array}
\begin{array}{c}
A \\
\end{array}
\]

with the two-sided cylinder

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
\downarrow \alpha \\
\downarrow \beta \\
(M,T)
\end{array}
\begin{array}{c}
A \\
\end{array}
\]

- The iso cell \((X,\cdot) \xrightarrow{\alpha} (X,\langle X,\cdot \rangle)\) identifies each left cylinder

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
\downarrow \alpha \\
\downarrow \beta \\
(M,T)
\end{array}
\begin{array}{c}
A \\
\end{array}
\]

with the two-sided cylinder

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
\downarrow \alpha \\
\downarrow \beta \\
(M,T)
\end{array}
\begin{array}{c}
A \\
\end{array}
\]

**Theorem 5.1.12.** Given categories \(X\) and \(A\),
the composition
\[
[A, X] \xrightarrow{\times \cdot A} [X : A] \xleftarrow{\times \cdot A} [X, A]^*
\]
of the right general Yoneda module for $[A, X]$ and the left general Yoneda functor for $[X, A]$ yields the same module as the composition
\[
[A, X] \times [X, A] \xrightarrow{[- \cdot -]} [A, A] \xrightarrow{(A, A)} [A, A] \xleftarrow{1_A} *
\]
.(Recall from Remark 1.1.14(3) that a right module $[A, X] \times [X, A] \rightarrow *$ is the same thing as a two-sided module $[A, X] ightarrow [X, A]^*$.)

- the composition
\[
[X, A]^* \xrightarrow{X \cdot A} [X : A]^* \xleftarrow{X \cdot A} [X, A]
\]
of the left general Yoneda module for $[X, A]$ and the right general Yoneda functor for $[A, X]$ yields the same module as the composition
\[
* \xrightarrow{1_X} [X, X] \xrightarrow{(X \cdot X)} [X, X] \xrightarrow{[- \cdot -]} [A, X] \times [X, A]
\]
.(Recall from Remark 1.1.14(3) that a left module $* \rightarrow [A, X] \times [X, A]$ is the same thing as a two-sided module $[A, X]^* \rightarrow [X, A]$.)

Proof. For any $G \in [A, X]$ and any $F \in [X, A]$, we have
\[
(G) (X \cdot A) (F \langle A \rangle) = \prod_A G (F \langle A \rangle) = \prod_A [G \circ F] \langle A \rangle [1_A] = (G \circ F) \langle A, A \rangle (1_A)
\]

Remark 5.1.13. This is the identity in Remark 4.3.2(3): given a pair of functors $G : A \rightarrow X$ and $F : X \rightarrow A$, the equation
\[
(G) (X \cdot A) (F \langle A \rangle) = (G \circ F) \langle A, A \rangle (1_A) \quad \text{resp.} \quad ((X) G) (X \cdot A) (F) = (1_X) (X, X) (G \circ F)
\]
identifies a right cylinder $X \xrightarrow{\varepsilon \in F(A)} A$ (resp. a left cylinder $X \xrightarrow{(X)^G \eta \in F} A$) with a natural transformation $\varepsilon : G \circ F \rightarrow 1_A$ (resp. $\eta : 1_X \rightarrow G \circ F$).

5.2. Yoneda morphisms

Definition 5.2.1. Let $\mathcal{M} : X \rightarrow A$ be a module.

- The right hom of $\mathcal{M}$ is the module
\[
\langle X \upharpoonright \mathcal{M} \rangle : X \rightarrow [\mathcal{M}]
\]
from $X$ to the collage category of $\mathcal{M}$, given by the composition
\[
X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{- \cdot [\mathcal{M}]} [\mathcal{M}]
\]
; that is, $\langle X \upharpoonright \mathcal{M} \rangle$ is the representable module of the inclusion $\mathcal{M}_0 : X \rightarrow [\mathcal{M}]$; in short,
\[
\langle X \upharpoonright \mathcal{M} \rangle := \mathcal{M}_0 ([\mathcal{M}])
\]
The left hom of $\mathcal{M}$ is the module

$$(\mathcal{M} \mathcal{A}) : [\mathcal{M}] \to \mathcal{A}$$

from the collage category of $\mathcal{M}$ to $\mathcal{A}$, given by the composition

$$\mathcal{M} \xrightarrow{\text{left hom}} \mathcal{A} \xrightarrow{\mathcal{M}_1} \mathcal{A} ;$$

that is, $(\mathcal{M} \mathcal{A})$ is the corepresentable module of the inclusion $\mathcal{M}_1 : \mathcal{A} \to [\mathcal{M}]$; in short,

$$(\mathcal{M} \mathcal{A}) := ([\mathcal{M}]) \mathcal{M}_1$$

Remark 5.2.2.

- The composition

$$X \underbrace{\mathcal{M}_0} \mathcal{M} \underbrace{\mathcal{M}_1} \xrightarrow{\text{left hom}} X$$

, i.e.

$$X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{\text{left hom}} [\mathcal{M}] \xrightarrow{\mathcal{M}_1} X$$

, yields $(X)$, the hom of $X$, and the composition

$$X \underbrace{\mathcal{M}_0} \mathcal{M} \underbrace{\mathcal{M}_1} \xrightarrow{\text{left hom}} A$$

, i.e.

$$X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{\text{left hom}} [\mathcal{M}] \xrightarrow{\mathcal{M}_1} A$$

, yields $\mathcal{M}$ (see Remark 3.1.15(2)).

- The composition

$$A \underbrace{\mathcal{M}_1} \mathcal{M} \underbrace{\mathcal{M}_0} \xrightarrow{\text{right hom}} A$$

, i.e.

$$A \xrightarrow{\mathcal{M}_1} [\mathcal{M}] \xrightarrow{\text{right hom}} [\mathcal{M}] \xrightarrow{\mathcal{M}_0} A$$

, yields $(A)$, the hom of $A$, and the composition

$$X \underbrace{\mathcal{M}_0} \mathcal{M} \underbrace{\mathcal{M}_1} \xrightarrow{\text{right hom}} A$$

, i.e.

$$X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{\text{right hom}} [\mathcal{M}] \xrightarrow{\mathcal{M}_1} A$$

, yields $\mathcal{M}$ (see Remark 3.1.15(2)).

Definition 5.2.3. Let $\mathcal{M} : X \to \mathcal{A}$ be a module.

- The right exponential transpose

$$[(X \mathcal{M})] : [\mathcal{M}] \to [X :]$$

of the right hom of $\mathcal{M}$ is called the right Yoneda functor for $\mathcal{M}$.

- The left exponential transpose

$$[\mathcal{M} \mathcal{A}] : [\mathcal{M}] \to [:\mathcal{A}]$$

of the left hom of $\mathcal{M}$ is called the left Yoneda functor for $\mathcal{M}$.
Remark 5.2.4.

- By Remark 5.2.2, the right slices of \( (X \downarrow M) \) at \( s \in \| X \| \) and \( t \in \| A \| \) are given by
  \[
  (X \downarrow M) s = (X \downarrow M) (M_0 \cdot s) = \langle (X \downarrow M) M_0 \rangle s = (X) s
  \]
  and
  \[
  (X \downarrow M) t = (X \downarrow M) (M_1 \cdot t) = \langle (X \downarrow M) M_1 \rangle t = (M) t
  \]
  hence the right Yoneda functor \( (X \downarrow M) \) sends each \( M \)-arrow \( m : s \to t \) to the right module morphism
  \[
  (X \downarrow M) m : (X) s \to (M) t : X \to *
  \]
  which maps each \( X \)-arrow \( h : x \to s \) to the \( M \)-arrow \( h \circ m : x \to t \) as indicated in
  \[
  \begin{array}{c}
  \circ \circ \\
  x \downarrow \end{array} \\
  \begin{array}{c}
  \circ \circ \\
  h \circ (X \downarrow M) m
  \end{array}
  \]
  (cf. Remark 2.1.2(1)). When \( M \) is understood, \( (X \downarrow M) m \) is also written as
  \[
  X \downarrow m : (X) s \to (M) t : X \to *
  \]
  and called the right module morphism generated by \( X \) direct along \( m \).

- By Remark 5.2.2, the left slices of \( (M \downarrow A) \) at \( s \in \| X \| \) and \( t \in \| A \| \) are given by
  \[
  s (M \downarrow A) = (s \downarrow M_0) (M \downarrow A) = s (M_0 (M \downarrow A)) = s (M)
  \]
  and
  \[
  t (M \downarrow A) = (t \downarrow M_1) (M \downarrow A) = t (M_1 (M \downarrow A)) = t (A)
  \]
  hence the left Yoneda functor \( \backslash (M \downarrow A) \) sends each \( M \)-arrow \( m : s \to t \) to the left module morphism
  \[
  m (M \downarrow A) : t (A) \to s (M) : * \to A
  \]
  which maps each \( A \)-arrow \( h : t \to a \) to the \( M \)-arrow \( m \circ h : s \to a \) as indicated in
  \[
  \begin{array}{c}
  \circ \circ \\
  m \downarrow (M \downarrow A) \downarrow
  \begin{array}{c}
  \circ \circ \\
  h
  \end{array}
  \end{array}
  \]
  (cf. Remark 2.1.2(1)). When \( M \) is understood, \( m (M \downarrow A) \) is also written as
  \[
  m \downarrow A : t (A) \to s (M) : * \to A
  \]
  and called the left module morphism generated by \( A \) inverse along \( m \).

Definition 5.2.5. Let \( M : X \to A \) be a module.

- The right Yoneda morphism for \( M \) is the cell
  \[
  \begin{array}{c}
  \circ \circ \\
  X \downarrow M
  \end{array} \\
  \begin{array}{c}
  \circ \circ \\
  M \downarrow
  \end{array} \\
  \begin{array}{c}
  \circ \circ \\
  [X] \downarrow
  \end{array}
  \]
  sending each \( M \)-arrow \( m : s \to t \) to the right module morphism
  \[
  X \downarrow m = (X \downarrow M) m : (X) s \to (M) t : X \to *
  \]
The left Yoneda morphism for $\mathcal{M}$ is the cell
\[
\begin{array}{c}
\text{X} \leftarrow \mathcal{M} \rightarrow \text{A} \\
\downarrow \text{x}_{\mathcal{M}} \downarrow \\
[\text{X}] \leftarrow \mathcal{M} \rightarrow [\text{A}]
\end{array}
\]
sending each $\mathcal{M}$-arrow $m : s \rightarrow t$ to the left module morphism
\[
m\upharpoonright \text{A} = m (\mathcal{M} \upharpoonright \text{A}) : t (\text{A}) \rightarrow s (\mathcal{M}) : * \rightarrow \text{A}
\]

Remark 5.2.6.
1. The cell $(\text{X} \upharpoonright \mathcal{M}) \rightarrow$ is formally given by the adjunct (see Theorem 3.1.16) of the right Yoneda functor for $\mathcal{M}$, i.e. by the composition
\[
\begin{array}{c}
\text{X} \leftarrow \mathcal{M} \rightarrow \text{A} \\
\downarrow \text{X}_{\mathcal{M}} \downarrow \\
[\text{X}] \leftarrow \mathcal{M} \rightarrow [\text{A}] \\
\downarrow \text{x}_{\mathcal{M}} \downarrow \\
\text{X} \uparrow \mathcal{M} \rightarrow [\text{X}] \leftarrow \text{X}
\end{array}
\]
of the unit cell of $\mathcal{M}$ and the hom of the right Yoneda functor for $\mathcal{M}$; and the cell $\nearrow (\mathcal{M} \upharpoonright \text{A})$ is given by the adjunct of the left Yoneda functor for $\mathcal{M}$, i.e. by the composition
\[
\begin{array}{c}
\text{X} \leftarrow \mathcal{M} \rightarrow \text{A} \\
\downarrow \text{X}_{\mathcal{M}} \downarrow \\
[\text{X}] \leftarrow \mathcal{M} \rightarrow [\text{A}] \\
\downarrow \text{x}_{\mathcal{M}} \downarrow \\
\text{X} \downarrow \text{X}
\end{array}
\]
of the unit cell of $\mathcal{M}$ and the hom of the left Yoneda functor for $\mathcal{M}$.
2. The Yoneda morphism
\[
\begin{array}{c}
\text{X} \leftarrow \mathcal{M} \rightarrow * \\
\downarrow \text{x}_{\mathcal{M}} \downarrow \\
[\text{X}] \leftarrow \mathcal{M} \rightarrow [\text{X}]
\end{array}
\]
for a right module $\mathcal{M} : \text{X} \rightarrow *$ is defined as a special case of Definition 5.2.5 where $\text{A}$ is the terminal category under the identification $[\text{X}] \equiv [\text{X} : *]$. This conical cell sends each $\mathcal{M}$-arrow $m : r \rightarrow *$ to the right module morphism
\[
\text{X} \uparrow m : (\text{X}) r \rightarrow \mathcal{M} : \text{X} \rightarrow *
\]
which maps each $\text{X}$-arrow $h : x \rightarrow r$ to the $\mathcal{M}$-arrow $h \circ m : x \rightarrow *$ as indicated in
\[
\begin{array}{c}
r \xleftarrow{m} \rightarrow * \\
\downarrow h \downarrow \\
\text{x} \xrightarrow{h : (\text{X} \uparrow m)}
\end{array}
\]
Conversely, given an arrow $m : s \rightarrow t$ of a two-sided module $\mathcal{M} : \text{X} \rightarrow \text{A}$, the right module morphism
\[
\text{X} \uparrow m : (\text{X}) s \rightarrow (\mathcal{M}) t : \text{X} \rightarrow *
\]
coincides with that generated by \( X \) direct along the arrow \( m : s \rightarrow \ast \) of the right module \( \langle M \rangle t : X \rightarrow \ast \).

- The Yoneda morphism

\[
\begin{array}{ccc}
\ast & \longrightarrow & \langle M \rangle \\
\downarrow & & \downarrow \\
\langle \langle M \rangle A \rangle & \longrightarrow & \langle A \rangle \\
\langle [A] \rangle & \longrightarrow & \langle [A] \rangle
\end{array}
\]

for a left module \( M : \ast \rightarrow A \) is defined as a special case of Definition 5.2.5 where \( X \) is the terminal category under the identification \([A] \cong \langle [\ast : A] \rangle\). This conical cell sends each \( \langle M \rangle \)-arrow \( m : \ast \rightarrow r \) to the left module morphism

\[ m \uparrow A : r(A) \rightarrow M : \ast \rightarrow A \]

which maps each \( A \)-arrow \( h : r \rightarrow a \) to the \( M \)-arrow \( m \circ h : \ast \rightarrow a \) as indicated in

\[
\begin{array}{ccc}
\ast & \longrightarrow & r \\
\downarrow & & \downarrow \\
\langle m \uparrow A \rangle : h & \longrightarrow & a \\
\end{array}
\]

. Conversely, given an arrow \( m : s \rightarrow t \) of a two-sided module \( M : X \rightarrow A \), the left module morphism

\[ m \uparrow A : t(A) \rightarrow s(\langle M \rangle) : \ast \rightarrow A \]

coincides with that generated by \( A \) inverse along the arrow \( m : \ast \rightarrow t \) of the left module \( s(\langle M \rangle) : \ast \rightarrow A \).

**Proposition 5.2.7.** Let \( M : X \rightarrow A \) be a module.

- The right slice (see Definition 2.1.8)

\[
\begin{array}{ccc}
X & \longrightarrow & \ast \\
\downarrow & & \downarrow \\
\langle X \rangle : & \longrightarrow & [X :] \\
\langle [A] \rangle & \longrightarrow & \langle [A] \rangle
\end{array}
\]

at \( t \in \|A\| \) of the right Yoneda morphism for \( M \) is given by the Yoneda morphism

\[
\begin{array}{ccc}
X & \longrightarrow & \langle M \rangle t \\
\downarrow & & \downarrow \\
\langle \langle X \rangle \rangle : & \longrightarrow & [X :] \\
\langle [A] \rangle & \longrightarrow & \langle [A] \rangle
\end{array}
\]

for the right module \( \langle M \rangle t \), the right slice of \( M \) at \( t \).

- The left slice (see Definition 2.1.8)

\[
\begin{array}{ccc}
\ast & \longrightarrow & A \\
\downarrow & & \downarrow \\
\langle s(\langle M \rangle) \rangle : & \longrightarrow & [A] \\
\langle [A] \rangle & \longrightarrow & \langle [A] \rangle
\end{array}
\]

at \( s \in \|X\| \) of the left Yoneda morphism for \( M \) is given by the Yoneda morphism

\[
\begin{array}{ccc}
\ast & \longrightarrow & s(\langle M \rangle) \\
\downarrow & & \downarrow \\
\langle \langle s(\langle M \rangle) \rangle \rangle : & \longrightarrow & [A] \\
\langle [A] \rangle & \longrightarrow & \langle [A] \rangle
\end{array}
\]

for the left module \( s(\langle M \rangle) \), the left slice of \( M \) at \( s \).

**Proof.** Immediate from Remark 5.2.6(2).
Proposition 5.2.8.

- Given a category $X$, the right Yoneda morphism

$$X \to \text{Hom}(X,X)$$

for the hom of $X$ is the same thing as the hom

$$X \to \text{Hom}(X,X)$$

of the right Yoneda functor for $X$; that is, for any $X$-arrow $f : s \to t$, $Xf$ and $\langle X \rangle f$ yield the same right module morphism

$$Xf = \langle X \rangle f : \langle X \rangle s \to \langle X \rangle t : X \to *$$

- Given a category $A$, the left Yoneda morphism

$$A \to \text{Hom}(A,A)$$

for the hom of $A$ is the same thing as the hom

$$A \to \text{Hom}(A,A)$$

of the left Yoneda functor for $A$; that is, for any $A$-arrow $f : s \to t$, $fA$ and $\langle A \rangle f$ yield the same left module morphism

$$fA = f\langle A \rangle : t\langle A \rangle \to s\langle A \rangle : * \to A$$

Proof. Both $Xf$ and $\langle X \rangle f$ map each $X$-arrow $h : x \to s$ to the $X$-arrow $h \circ f : x \to t$ (see Remark 2.3.2).

Example 5.2.9.

1. Consider a module and functors as in

$$E \xrightarrow{S} X \xrightarrow{M} A \xrightarrow{T} D$$

- The right Yoneda morphism for the composite module $S\langle M \rangle T : E \to D$ sends each $S\langle M \rangle T$-arrow $m : s \to t$ to the right module morphism

$$E \xrightarrow{m : \langle E \rangle s} \langle S\langle M \rangle T \rangle t : E \to *$$
which maps each \( \mathbf{E} \)-arrow \( h : e \to s \) to the \( \mathcal{S}(\mathcal{M}) \) \( \mathbf{T} \)-arrow \( h \circ m : e \to t \) as indicated in

\[
\begin{array}{ccc}
  s & \xrightarrow{m} & T \circ t \\
  h \uparrow & & \downarrow h \\
  e & \xrightarrow{\mathcal{S}(\mathcal{M}) \circ m} & e \circ (\mathbf{E} \circ m)
\end{array}
\]

(cf. Example 1.1.29(1)). The commutativity of the triangle above for every \( h \in \mathcal{S}(\mathcal{E}) \) \( s \) translates into the commutativity of

\[
\begin{array}{ccc}
  \mathcal{S}(\mathcal{M}) \ T & \xrightarrow{T \circ t} & \mathcal{S}(\mathcal{M}) \ T \\
  (\mathcal{S}(\mathcal{M}) \ T \circ m) \uparrow & & \downarrow (\mathcal{S}(\mathcal{M}) \ T \circ m) \\
  \mathcal{S}(\mathcal{M}) \ T & \xrightarrow{S(\mathcal{M}) \ T \circ m} & \mathcal{S}(\mathcal{M}) \ T \circ m
\end{array}
\]

- The left Yoneda morphism for the composite module \( \mathcal{S}(\mathcal{M}) \ T : \mathcal{M} \to \mathcal{D} \) sends each \( \mathcal{S}(\mathcal{M}) \ T \)-arrow \( m : s \to t \) to the left module morphism

\[
m \uparrow \mathcal{D} : t(D) \to s(\mathcal{S}(\mathcal{M}) \ T) : \mathcal{S}(\mathcal{M}) \ T \to \mathcal{D}
\]

which maps each \( \mathcal{D} \)-arrow \( h : t \to d \) to the \( \mathcal{S}(\mathcal{M}) \ T \)-arrow \( m \circ h : s \to d \) as indicated in

\[
\begin{array}{ccc}
  s \circ T \circ t & \xrightarrow{T \circ t} & T \circ t \\
  (m \circ h) \uparrow & & \downarrow h \\
  T \circ t & \xrightarrow{m \circ h \circ T} & m \circ h \circ T
\end{array}
\]

(cf. Example 1.1.29(1)). The commutativity of the triangle above for every \( h \in \mathcal{S}(\mathcal{D}) \) \( t \) translates into the commutativity of

\[
\begin{array}{ccc}
  t(\mathcal{D}) & \xrightarrow{t(\mathcal{M})} & t(\mathcal{D}) \\
  m \downarrow & & \downarrow (m \circ \mathcal{E}) \circ \mathcal{A}^{-1} \\
  s(\mathcal{S}(\mathcal{M}) \ T) & \xrightarrow{s(\mathcal{S}(\mathcal{M}) \ T \circ m)} & s(\mathcal{S}(\mathcal{M}) \ T \circ m)
\end{array}
\]

(2) As a special case of (1) above, for any functor \( \mathcal{F} : \mathcal{D} \to \mathcal{E} \), consider its representable and corepresentable modules (see Definition 2.3.5).

- The right Yoneda morphism for the representable module \( \mathcal{F}(\mathcal{E}) : \mathcal{D} \to \mathcal{E} \) sends each \( \mathcal{F}(\mathcal{E}) \)-arrow \( f : r \to e \) (i.e. \( \mathcal{E} \)-arrow \( f : r \to e \)) to the right module morphism

\[
\mathcal{D} \uparrow f : (\mathcal{D}) \to \mathcal{F}(\mathcal{E}) \ e : \mathcal{D} \to \mathcal{E}
\]

which maps each \( \mathcal{D} \)-arrow \( h : d \to r \) to the \( \mathcal{F}(\mathcal{E}) \)-arrow \( h \circ f : d \to e \) as indicated in

\[
\begin{array}{ccc}
  r & \xrightarrow{r \circ f} & e \\
  h \uparrow & & \downarrow h \circ f \\
  d & \xrightarrow{h \circ f} & d \circ (\mathcal{D} \circ f)
\end{array}
\]

- The commutativity of the triangle above for every \( h \in \mathcal{D} \) \( r \) translates into the commutativity of

\[
\begin{array}{ccc}
  \mathcal{D} \uparrow f : (\mathcal{D}) \to \mathcal{F}(\mathcal{E}) \ e : \mathcal{D} \to \mathcal{E}
\end{array}
\]

- The left Yoneda morphism for the corepresentable module \( \mathcal{F}(\mathcal{E}) : \mathcal{E} \to \mathcal{D} \) sends each \( \mathcal{F}(\mathcal{E}) \)-arrow \( f : e \to r \) (i.e. \( \mathcal{E} \)-arrow \( f : e \to \mathcal{F}(\mathcal{E}) \)) to the left module morphism

\[
f \uparrow \mathcal{D} : r(\mathcal{D}) \to e(\mathcal{E}) \ f : \mathcal{D} \to \mathcal{E}
\]
which maps each D-arrow \( h : r \to d \) to the \( (E) \) F-arrow \( f \circ h : e \to d \) as indicated in

\[
\begin{array}{ccc}
    e & \xrightarrow{f} & F \cdot r \\
\downarrow{(f \cdot D)^\circ h} & & \downarrow{F \cdot h} \\
    F \cdot d & \to & d
\end{array}
\]

. The commutativity of the triangle above for every \( h \in r(D) \) translates into the commutativity of

\[
\begin{array}{ccc}
    r(D) & \xrightarrow{r(F)} & r(F(E)F) \\
\downarrow{r \cdot (E)F} & & \downarrow{r \cdot (E)F} \\
    e & \xrightarrow{(E)F} & e \cdot (E)F
\end{array}
\]

.\)

**Theorem 5.2.10.** (Yoneda Lemma : Part one).

- The Yoneda morphism for a right module \( M : X \to * \) (see Remark 5.2.6(2)) is fully faithful. Specifically, for each object \( r \in \|X\| \), the assignment \( m \mapsto X \uparrow m \) yields a bijection

\[
\{M\} r \cong \langle (X) r \rangle \langle (X \cdot) (M) \rangle
\]

from the set of \( M \)-arrows \( r \to * \) to the set of right module morphisms \( \langle X \rangle r : M : X \to * \), whose inverse sends each right module morphism \( \Phi : \langle X \rangle r \to M \) to the \( M \)-arrow \( 1_r : \Phi : r \to * \), the image of the identity \( r \to r \) under the function \( r(\Phi) : r(\langle X \rangle r) \to r(\{M\} r) \).

- The Yoneda morphism for a left module \( M : * \to A \) (see Remark 5.2.6(2)) is fully faithful. Specifically, for each object \( r \in \|A\| \), the assignment \( m \mapsto m \uparrow A \) yields a bijection

\[
\{M\} r \cong \langle r(A) \rangle \langle (A \cdot) (M) \rangle
\]

from the set of \( M \)-arrows \( * \to r \) to the set of left module morphisms \( r(A) \to M : * \to A \), whose inverse sends each left module morphism \( \Phi : r(A) \to M \) to the \( M \)-arrow \( \Phi \cdot 1_r : * \to r \), the image of the identity \( r \to r \) under the function \( \langle \Phi \rangle r : r(A) r \to r(\{M\} r) \).

**Proof.** Let \( m : r \to * \) be an \( M \)-arrow and \( \Phi : \langle X \rangle r \to M \) be a right module morphism. We need to show that \( m = 1_r : \langle X \uparrow m \rangle \) and \( \Phi = X \uparrow (1_r : \Phi) \). Replacing \( h : x \to r \) with \( 1_r : r \to r \) in the triangle in Remark 5.2.6(2), we have

\[
\begin{array}{ccc}
    r & \xrightarrow{m} & * \\
\downarrow{1_r} & & \downarrow{1_r : (X \uparrow m)} \\
    r & \xrightarrow{1_r} & (X \uparrow m)
\end{array}
\]

, i.e.

\[
m = 1_r : \langle X \uparrow m \rangle
\]

. For any object \( x \in \|X\| \) and any arrow \( h : x \to r \), the commutative triangle

\[
\begin{array}{ccc}
    r & \xrightarrow{1_r} & r \\
\downarrow{h} & & \downarrow{h} \\
    x & \xrightarrow{h} & h
\end{array}
\]

yields a commutative triangle

\[
\begin{array}{ccc}
    r & \xrightarrow{1_r : \Phi} & * \\
\downarrow{h} & & \downarrow{h : \Phi} \\
    x & \xrightarrow{h} & h \cdot \Phi
\end{array}
\]

by the naturality of \( \Phi \). Comparing this triangle with that in Remark 5.2.6(2), we have

\[
\Phi = X \uparrow (1_r : \Phi)
\]

. \qed
Theorem 5.2.12. Let $\mathcal{M} : X \to A$ be a module.

- The right Yoneda morphism

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & A \\
\cdot\mathcal{M} & \downarrow & \cdot\mathcal{A} \\
[X] & \xrightarrow{\phi} & [A] \\
\end{array}
\]

for $\mathcal{M}$ is fully faithful. Specifically, for each pair of objects $s \in \|X\|$ and $t \in \|A\|$, the assignment $m \mapsto X|_{m}$ yields a bijection

\[
s(\mathcal{M}) t \cong ((X)s) (X:) ((\mathcal{M}) t)
\]

from the set of $\mathcal{M}$-arrows $s \to t$ to the set of right module morphisms $(X)s \to (\mathcal{M}) t : X \to *$, whose inverse sends each right module morphism $\Phi : (X)s \to (\mathcal{M}) t$ to the $\mathcal{M}$-arrow $1s : \Phi : s \to t$, the image of the identity $s \to s$ under the function $s(\Phi) : s(X)s \to s(\mathcal{M}) t$.

- The left Yoneda morphism

\[
\begin{array}{ccc}
X & \xleftarrow{\mathcal{M}} & A \\
\cdot\mathcal{M} & \downarrow & \cdot\mathcal{A} \\
[A]^{-} & \xleftarrow{\phi} & [A]^{-} \\
\end{array}
\]

for $\mathcal{M}$ is fully faithful. Specifically, for each pair of objects $s \in \|X\|$ and $t \in \|A\|$, the assignment $m \mapsto m|_{A}$ yields a bijection

\[
s(\mathcal{M}) t \cong (t(A)) (A) (s(\mathcal{M}))
\]

from the set of $\mathcal{M}$-arrows $s \to t$ to the set of left module morphisms $t(A) \to s(\mathcal{M}) : * \to A$, whose inverse sends each left module morphism $\Phi : t(A) \to s(\mathcal{M})$ to the $\mathcal{M}$-arrow $1t : \Phi : t \to t$, the image of the identity $t \to t$ under the function $\Phi (t) : t(A)t \to s(\mathcal{M}) t$.

Proof. Since a cell is fully faithful iff so is each right slice (see Proposition 2.1.9), by Proposition 5.2.7, the assertion is reduced to Theorem 5.2.10.

Remark 5.2.13. As we have just seen, Theorem 5.2.12 follows from the Yoneda lemma (Theorem 5.2.10). Conversely, the Yoneda lemma is a special case of Theorem 5.2.12 where $A$ (resp. $X$) is the terminal category. We will see below that Theorem 5.2.12 is also a generalization of the Yoneda embedding.

Corollary 5.2.14. (Yoneda Embedding).

- For any category $X$, the right Yoneda functor $[X :] : X \to [X :]$ is fully faithful. Specifically, for each pair of objects $s, t \in \|X\|$, the assignment $f \mapsto (X)f$ yields a bijection

\[
s(X)t \cong ((X)s) (X:) ((X)t)
\]

from the set of $X$-arrows $s \to t$ to the set of right module morphisms $(X)s \to (X)t : X \to *$, whose inverse sends each module morphism $\Phi : (X)s \to (X)t$ to the $X$-arrow $1s : \Phi : s \to t$, the image of the identity $s \to s$ under the function $s(\Phi) : s(X)s \to s(X)t$.

- For any category $A$, the left Yoneda functor $[\cdot A] : A \to [\cdot A]$ is fully faithful. Specifically, for each pair of objects $s, t \in \|A\|$, the assignment $f \mapsto f(A)$ yields a bijection

\[
s(A)t \cong (t(A)) (A) (s(A))
\]

from the set of $A$-arrows $s \to t$ to the set of left module morphisms $t(A) \to s(A) : * \to A$, whose inverse sends each module morphism $\Phi : t(A) \to s(A)$ to the $A$-arrow $1t : \Phi : s \to t$, the image of the identity $t \to t$ under the function $\Phi (t) : t(A)t \to s(A)t$. 

Remark 5.2.11. The bijection $r(\mathcal{M}) \cong ((X)r)(X:)(\mathcal{M})$ is natural in $r$ because it is the component at $r$ of the Yoneda morphism. We will see in Corollary 5.2.18 that the bijection is also natural in $\mathcal{M}$. 

For any category $\mathcal{C}$, the right Yoneda functor $\mathcal{C} \to [\mathcal{C} :]$ is fully faithful. Specifically, for each pair of objects $s, t \in \|\mathcal{C}\|$, the assignment $f \mapsto (\mathcal{C})f$ yields a bijection

\[
s(\mathcal{C})t \cong ((\mathcal{C})s) (\mathcal{C}:) ((\mathcal{C})t)
\]

from the set of $\mathcal{C}$-arrows $s \to t$ to the set of right module morphisms $(\mathcal{C})s \to (\mathcal{C})t : \mathcal{C} \to *$, whose inverse sends each module morphism $\Phi : (\mathcal{C})s \to (\mathcal{C})t$ to the $\mathcal{C}$-arrow $1s : \Phi : s \to t$, the image of the identity $s \to s$ under the function $s(\Phi) : s(\mathcal{C})s \to s(\mathcal{C})t$. 

Remark 5.2.13. As we have just seen, Theorem 5.2.12 follows from the Yoneda lemma (Theorem 5.2.10). Conversely, the Yoneda lemma is a special case of Theorem 5.2.12 where $A$ (resp. $X$) is the terminal category. We will see below that Theorem 5.2.12 is also a generalization of the Yoneda embedding.
Proof. Recalling Proposition 5.2.8, we see that the assertion is a special case of Theorem 5.2.12 where \( \mathcal{M} \) is given by the hom of \( X \).

\[ \square \]

Note. If \( \mathcal{M} \) in Theorem 5.2.12 is given by a representable module as in Example 5.2.9(2), the theorem reads as follows.

**Corollary 5.2.15.** Let \( F : D \to E \) be a functor.

- The assignment \( f \mapsto D \upharpoonright f \) yields a bijection
  \[ (r : F)(E)(e) \cong ((D)r)(D):(F(E)e) \]
  natural in \( r \in \| D \| \) and \( e \in \| E \| \), from the set of \( E \)-arrows \( r : F \to e \) to the set of right module morphisms \( (D)r \to F(E)e : D \to * \), whose inverse sends each right module morphism \( \Phi : (D)r \to F(E)e \) to the \( E \)-arrow \( 1_r : \Phi : r : F \to e \), the image of the identity \( r \to r \) under the function \( (\Phi) : r(D)r \to (r : F)(E)e \).

- The assignment \( f \mapsto f \upharpoonright D \) yields a bijection
  \[ (e)(E)(F \upharpoonright r) \cong (r)(D) : (e)(E)(F) \]
  natural in \( r \in \| D \| \) and \( e \in \| E \| \), from the set of \( E \)-arrows \( e \to F \upharpoonright r \) to the set of left module morphisms \( r(D) \to e(F)F : * \to D \), whose inverse sends each left module morphism \( \Phi : r(D) \to e(F)F \) to the \( E \)-arrow \( \Phi : 1_r : e \to F \upharpoonright r \), the image of the identity \( r \to r \) under the function \( (\Phi) : r(r(D)r \to e(F)(F \upharpoonright r)) \).

**Proof.** By replacing \( \mathcal{M} : X \to A \) in Theorem 5.2.12 with the representable module \( F(E) : D \to E \), we have the bijection

\[ (r : F)(E)(e) \cong r(F(E))e \cong ((D)r)(D):(F(E)e) \]

natural in \( r \in \| D \| \) and \( e \in \| E \| \).

\[ \square \]

Remark 5.2.16. Corollary 5.2.15 is a special case of Theorem 5.2.12 where \( \mathcal{M} \) is representable, and the Yoneda embedding (Corollary 5.2.14) is a special case of Theorem 5.2.12 where \( \mathcal{M} \) is given by the hom of a category. The Yoneda embedding may also be seen as a special case of Corollary 5.2.15 where \( F \) is an identity functor.

**Theorem 5.2.17.**

- Given a category \( X \), the right Yoneda morphism

\[ X \to X^\flat \]
\[ X \downarrow \]
\[ (X[(X^\flat)])^\flat \]
\[ (X^\flat)^\flat = 1 \]
\[ [X:] \downarrow \]
\[ (X^\flat)^\flat \]

for the right Yoneda module \( X^\flat \) yields a representation

\[ \langle X,^\flat \rangle \cong [X^\flat][X:] : X \to [X:] \]

of the right Yoneda module \( X^\flat \) by the right Yoneda functor \( X^\flat \). The right slice of the representation cell \( \langle X^\flat \rangle^\flat \) at a right module \( \mathcal{M} : X \to * \) is nothing but the Yoneda morphism for \( \mathcal{M} \) (see Remark 5.2.6(2)); the cell thus sends each \( (X^\flat)^\flat \)-arrow \( m : r \to \mathcal{M} \) (i.e. \( \mathcal{M} \)-arrow \( m : r \to * \)) to the right module morphism \( X \upharpoonright m : (X)r \to \mathcal{M} \).
5.3. Yoneda morphisms for cylinders

- Given a category $\mathbf{A}$, the left Yoneda morphism

$$M : \ast \to \mathbf{A}$$

for the left Yoneda module $\ast \mathbf{A}$ yields a corepresentation

$$\langle \ast, \mathbf{A} \rangle \cong \{ \ast : \mathbf{A} \} \setminus (\ast \mathbf{A}) : \mathbf{A} \to \mathbf{A}$$

of the left Yoneda module $\ast \mathbf{A}$ by the left Yoneda functor $\ast \mathbf{A}$. The left slice of the corepresentation cell $\ast \left\langle \begin{array}{c} \ast \mathbf{A} \end{array} \right\rangle \mathbf{A}$ at a left module $M : \ast \to \mathbf{A}$ is nothing but the Yoneda morphism for $M$ (see Remark 5.2.6(iii)); the cell thus sends each $\ast \mathbf{A}$-arrow $m : M \to r$ (i.e. $\mathbf{A}$-arrow $m : \ast \to r$) to the left module morphism $m \downarrow \mathbf{A} : r(\mathbf{A}) \to \mathbf{M}$.

Proof. See Proposition 5.1.3 for the identity $\langle (\mathbf{X} \ast), \mathbf{A} \rangle \ast = 1$. The first assertion now follows from the fully faithfulness of the Yoneda morphism (Theorem 5.2.12). By Proposition 5.2.7 and Proposition 5.1.3, the right slice of $\langle (\mathbf{X} \ast), \mathbf{A} \rangle \mathbf{A}$ at $M$ is given by

$$\langle \mathbf{X} \rangle \langle \mathbf{X} \ast \rangle \mathbf{A} = (\mathbf{X} \langle \mathbf{X} \ast \rangle (\mathbf{M})) \ast = (\mathbf{X} \langle \mathbf{M} \rangle) \ast$$

, i.e. by the Yoneda morphism for $\mathbf{M}$.

Corollary 5.2.18. (Yoneda Lemma : Part two). The bijection in Theorem 5.2.10 is natural in $\mathbf{r}$ and $\mathbf{M}$.

Proof. This is a restatement of Theorem 5.2.17.

Remark 5.2.19. The representation (resp. corepresentation) in Theorem 5.2.17 is called the Yoneda representation (resp. corepresentation) and denoted by $\mathbf{X} \downarrow$ (resp. $\downarrow \mathbf{A}$) as in

$$\mathbf{X} \downarrow \mathbf{A} \leftarrow \mathbf{X} \downarrow \mathbf{A} \\
\downarrow \mathbf{X} \leftarrow \downarrow \mathbf{X}$$

; with this notation, the bijections of the Yoneda lemma are written as

$$(\mathbf{r}) \langle \mathbf{X} \rangle \langle \mathbf{M} \rangle : (\mathbf{r}) \langle \mathbf{X} \ast \rangle (\mathbf{M}) \cong ((\mathbf{X}) \mathbf{r}) \langle \mathbf{X} \rangle (\mathbf{M})$$

and

$$(\mathbf{M}) \langle \downarrow \mathbf{A} \rangle \langle \mathbf{r} \rangle : (\mathbf{M}) \langle \ast \mathbf{A} \rangle (\mathbf{r}) \cong (\mathbf{r} \langle \mathbf{A} \rangle) \langle \mathbf{A} \rangle (\mathbf{M})$$

.

5.3. Yoneda morphisms for cylinders

Definition 5.3.1. Let $\mathbf{E}$ be a category and $\mathbf{M} : \mathbf{X} \to \mathbf{A}$ be a module.

- The right action

$$\langle \mathbf{X} \rangle \mathbf{M} \ast \mathbf{E} : [\mathbf{E}, [\mathbf{M}]] \to [\mathbf{X} : \mathbf{E}]$$

of the right hom of $\mathbf{M}$ on the functor category $[\mathbf{E}, [\mathbf{M}]]$ is called the right general Yoneda functor for $\langle \mathbf{E}, \mathbf{M} \rangle$. 
5.3. Yoneda morphisms for cylinders

\textbf{Remark 5.3.2.} Consider a cylinder \( \xymatrix{ \mathbf{E} \ar[r]^-{S} & \mathbf{T} \ar[r]^-{\alpha} & \mathbf{X} \ar[r]^-{\gamma} & \mathbf{A} } \), i.e. a natural transformation \( \alpha : S \circ \mathcal{M}_0 \to \mathcal{M}_1 \circ \mathbf{T} : \mathbf{E} \to [\mathcal{M}] \) (see Remark 4.3.4(4)).

By Remark 5.2.2, the module \( \langle \mathbf{X} \uparrow \mathcal{M} \rangle \) acts on \( \mathcal{M}_0 \circ S \) and \( \mathcal{M}_1 \circ \mathbf{T} \) and yields
\[
\langle \mathbf{X} \uparrow \mathcal{M} \rangle \circ \mathcal{M}_0 = \langle \langle \mathbf{X} \uparrow \mathcal{M} \rangle \circ \mathcal{M}_0 \rangle = \langle \mathbf{X} \rangle \circ S
\]
and
\[
\langle \mathbf{X} \uparrow \mathcal{M} \rangle \circ \mathcal{M}_1 = \langle \langle \mathbf{X} \uparrow \mathcal{M} \rangle \circ \mathcal{M}_1 \rangle \circ \mathcal{T} = \langle \mathcal{M} \rangle \circ \mathcal{T}
\]; hence \( \langle \mathbf{X} \uparrow \mathcal{M} \rangle \) acts on \( \alpha \) and gives the module morphism
\[
\langle \mathbf{X} \uparrow \mathcal{M} \rangle \alpha : \langle \mathbf{X} \rangle \circ \mathcal{M} \to \langle \mathcal{M} \rangle \circ \mathbf{T} \circ \mathbf{X} \to \mathbf{E}
\]
which maps each \( \langle \mathbf{X} \rangle \mathcal{M} \)-arrow \( h : x \rightsquigarrow e \) to the \( \mathcal{M} \)-arrow \( \mathcal{M}_1 \circ \mathbf{T} \circ \mathbf{X} \)-arrow \( h \circ \alpha_x : x \rightsquigarrow e \) as indicated in
\[
\begin{array}{c}
\xymatrix{ e : S & \ar[l]_-{\alpha_x} \mathbf{T} \cdot e \ar[l]^-{h} \\
\mathbf{X} \ar[ur]^-{h : (\mathbf{X} \uparrow \mathcal{M}) \alpha} & }
\end{array}
\]
(cf. Remark 2.2.2(1)). When \( \mathcal{M} \) is understood, \( \langle \mathbf{X} \uparrow \mathcal{M} \rangle \alpha \) is also written as
\[
\mathbf{X} \uparrow \alpha : \langle \mathbf{X} \rangle \circ \mathcal{M} \to \langle \mathcal{M} \rangle \circ \mathbf{T} \circ \mathbf{X} \to \mathbf{E}
\]
and called the module morphism generated by \( \mathbf{X} \) direct along \( \alpha \).

\textbf{Remark 5.2.2.} The left action
\[
\mathbf{E} \ltimes \langle \mathcal{M} \uparrow \mathbf{A} \rangle : [\mathbf{E}, [\mathcal{M}]] \to [\mathbf{E} : \mathbf{A}]
\]
of the left hom of \( \mathcal{M} \) on the functor category \([\mathbf{E}, [\mathcal{M}]]\) is called the left general Yoneda functor for \( \langle \mathbf{E}, \mathcal{M} \rangle \).

\textbf{Definition 5.3.3.} Let \( \mathbf{E} \) be a category and \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) be a module.
The right general Yoneda morphism for \( (E, \mathcal{M}) \) (see Definition 4.3.5) is the cell

\[
[E, X] \xrightarrow{(E, M)} [E, A] \\
X \xrightarrow{\alpha} E \xrightarrow{(X, M)} \mathcal{M} \xrightarrow{E} E
\]

sending each cylinder \( \alpha : S \to T : E \to \mathcal{M} \) to the module morphism

\[
X \mid \alpha = (X \mid \alpha) \quad \alpha : (X) S \to (\mathcal{M}) T : X \to E
\]

The left general Yoneda morphism for \( (E, \mathcal{M}) \) (see Definition 4.3.5) is the cell

\[
[E, X] \xrightarrow{(E, M)} [E, A] \\
E \xrightarrow{\alpha} E \xrightarrow{(E, \mathcal{M})} (E, \mathcal{M}) \xrightarrow{E \times \mathcal{M}} [E, \mathcal{M}] \\
\xrightarrow{((X, \mathcal{M}) \times E)} (X, \mathcal{M}) \xrightarrow{E} E
\]

sending each cylinder \( \alpha : S \to T : E \to \mathcal{M} \) to the module morphism

\[
\alpha \mid A = \alpha (\mathcal{M} \mid A) : T(A) \to S(\mathcal{M}) : E \to A
\]

**Remark 5.3.4.**

1. The cell \( (X \mid \mathcal{M}) \times E \) is formally given by the composition

\[
[E, X] \xrightarrow{(E, M)} [E, A] \\
[E, \mathcal{M}] \xrightarrow{E \times \mathcal{M}} (E, \mathcal{M}) \xrightarrow{E \times \mathcal{M}} [E, \mathcal{M}] \\
\xrightarrow{((X, \mathcal{M}) \times E)} (X, \mathcal{M}) \xrightarrow{E} E
\]

of the postcomposition cell in Example 4.3.20 and the hom of the right general Yoneda functor; and the cell \( E \times (\mathcal{M} \mid A) \) is given by the composition

\[
[E, X] \xrightarrow{(E, M)} [E, A] \\
[E, \mathcal{M}] \xrightarrow{E \times \mathcal{M}} (E, \mathcal{M}) \xrightarrow{E \times \mathcal{M}} [E, \mathcal{M}] \\
E \times (\mathcal{M} \mid A) \xrightarrow{E \times (\mathcal{M} \mid A)} (E \times (\mathcal{M} \mid A)) \xrightarrow{E \times (\mathcal{M} \mid A)} [E, \mathcal{M}]
\]

of the postcomposition cell and the hom of the left general Yoneda functor.

2. The Yoneda morphism for \( \mathcal{M} \) is identified with the special instance of the general Yoneda morphism for \( (E, \mathcal{M}) \) where \( E \) is the terminal category.

**Example 5.3.5.**

1. Given a right cylinder \( X \xrightarrow{\alpha} \mathcal{M} \to A \), i.e. a two-sided cylinder

\[
X \xrightarrow{G} A \xrightarrow{M} A
\]

the category \( X \) acts on \( \alpha \) and generates a module morphism

\[
X \xrightarrow{\alpha} (X \mid \mathcal{M}) \alpha : (X) \xrightarrow{G} \mathcal{M} \to (X) \mathcal{M} \to A
\]
direct along \( \alpha \), mapping each \( \langle X \rangle \) \( G \)-arrow \( h : x \rightarrow a \) to the \( \mathcal{M} \)-arrow \( h \circ \alpha : x \rightarrow a \) as indicated in

\[
\begin{array}{c}
a : G \\
\downarrow h \\
x
\end{array} \\
\begin{array}{c}
a : G
\end{array}
\]

, and the category \( A \) acts on \( \alpha \) and generates a module morphism

\[
\alpha \uparrow A = \alpha \langle \mathcal{M} \uparrow A \rangle : \langle A \rangle \rightarrow G \langle \mathcal{M} \rangle : A \rightarrow A
\]

inverse along \( \alpha \), mapping each \( A \)-arrow \( h : a \rightarrow b \) to the \( G \langle \mathcal{M} \rangle \)-arrow \( \alpha \circ h : a \rightarrow b \) as indicated in

\[
\begin{array}{c}
a : G \\
\downarrow h \\
b
\end{array} \\
\begin{array}{c}
a : G
\end{array}
\]

, and the category \( A \) acts on \( \alpha \) and generates a module morphism

\[
\alpha \uparrow A = \alpha \langle \mathcal{M} \uparrow A \rangle : \langle A \rangle \rightarrow \mathcal{M} : X \rightarrow A
\]

inverse along \( \alpha \), mapping each \( \mathcal{M} \)-arrow \( \alpha \circ h : x \rightarrow a \) as indicated in

\[
\begin{array}{c}
x \\
\downarrow h \\
a
\end{array} \\
\begin{array}{c}
x
\end{array}
\]

, and the category \( A \) acts on \( \alpha \) and generates a module morphism

\[
\alpha \uparrow A = \alpha \langle \mathcal{M} \uparrow A \rangle : \langle A \rangle \rightarrow \mathcal{M} : X \rightarrow A
\]

direct along \( \alpha \), mapping each \( X \)-arrow \( h : y \rightarrow x \) to the \( \langle \mathcal{M} \rangle \)-arrow \( h \circ \alpha : y \rightarrow x \) as indicated in

\[
\begin{array}{c}
x \\
\downarrow h \\
y
\end{array} \\
\begin{array}{c}
x
\end{array}
\]

, and the category \( A \) acts on \( \alpha \) and generates a module morphism

\[
\alpha \uparrow A = \alpha \langle \mathcal{M} \uparrow A \rangle : \langle A \rangle \rightarrow \mathcal{M} : X \rightarrow A
\]

(2) Consider a pair of functors \( X \xrightarrow{G} F \xleftarrow{F} A \).

- Given a natural transformation \( \epsilon : G \circ F \rightarrow 1_A : A \rightarrow A \), i.e. a right cylinder \( X \xrightarrow{G} F \xleftarrow{F} A \) (see Remark 4.3.2(3)), the category \( X \) acts on \( \epsilon \) and generates a module morphism

\[
X \uparrow \epsilon = \langle X \uparrow \langle F \langle A \rangle \rangle \rangle \epsilon : \langle X \rangle G \rightarrow \langle F \langle A \rangle \rangle : X \rightarrow A
\]

direct along \( \epsilon \), mapping each \( \langle X \rangle \) \( G \)-arrow \( h : x \rightarrow a \) to the \( \langle F \langle A \rangle \rangle \)-arrow \( h \circ \epsilon : x \rightarrow a \) as indicated in

\[
\begin{array}{c}
a : G \\
\downarrow h \\
x
\end{array} \\
\begin{array}{c}
a : G
\end{array}
\]

, and the category \( A \) acts on \( \epsilon \) and generates a module morphism

\[
\epsilon \uparrow A = \epsilon \langle \langle F \langle A \rangle \rangle \uparrow A \rangle : \langle A \rangle \rightarrow G \langle F \langle A \rangle \rangle : A \rightarrow A
\]
inverse along \( \epsilon \), mapping each \( A \)-arrow \( h : a \to b \) to the \( G(F(A)) \)-arrow \( \epsilon_a \circ h : a \to b \) as indicated in

\[
a : G \cdot F \quad \xrightarrow{\epsilon_a \circ h} \quad a
\]

\[
\downarrow^{h}
\]

(this action \( \epsilon \mid A \) of \( A \) on a right cylinder \( X \xrightarrow{\epsilon_{\mid F(A)}} A \) is the same thing as the action \( \epsilon \langle A \rangle \) (see Remark 2.3.8(1)) of \( A \) on a natural transformation \( \epsilon : G \circ F \to 1_A : A \to A \): the diagram

\[
\begin{array}{c}
G(F(A)) \\
\xrightarrow{\epsilon \mid A} \langle A \rangle
\end{array}
\]

\[
\xrightarrow{\epsilon \langle A \rangle}
\]

commutes).

- Given a natural transformation \( \eta : 1_X \to G \circ F : X \to X \), i.e. a left cylinder \( X \xleftarrow{\eta} G \cdot F \) (see Remark 4.3.2(3)), the category \( A \) acts on \( \eta \) and generates a module morphism

\[
\eta \mid A = \eta \langle \langle (X) \rangle \rangle \mid A : F(A) \to \langle (X) \rangle G : X \to A
\]

inverse along \( \eta \), mapping each \( F(A) \)-arrow \( h : x \to a \) to the \( \langle (X) \rangle G \)-arrow \( \eta_x \circ h : x \to a \) as indicated in

\[
x \xleftarrow{\eta_x} G \cdot F : x \quad \xrightarrow{\eta} x : F
\]

\[
G : a \quad \xrightarrow{h} a
\]

and the category \( X \) acts on \( \eta \) and generates a module morphism

\[
X \mid \eta = \langle (X) \rangle \mid X : \langle (X) \rangle G : X \to X
\]

direct along \( \eta \), mapping each \( X \)-arrow \( h : y \to x \) to the \( \langle (X) \rangle G \)-arrow \( h \circ \eta_x : y \to x \) as indicated in

\[
x \xrightarrow{\eta_x} G \cdot F : x
\]

\[
\downarrow^{h}
\]

\[
h : \langle (X) \rangle \eta
\]

\[
y
\]

(this action \( X \mid \eta \) of \( X \) on a left cylinder \( X \xleftarrow{\eta \mid F} G \cdot F \) is the same thing as the action \( \langle (X) \rangle \eta \) (see Remark 2.3.8(1)) of \( X \) on a natural transformation \( \eta : 1_X \to G \circ F : X \to X \): the diagram

\[
\begin{array}{c}
\langle (X) \rangle G \\
\xrightarrow{\langle (X) \rangle \eta} \langle (X) \rangle
\end{array}
\]

\[
\xrightarrow{\langle (X) \rangle \eta}
\]

\[
\langle (X) \rangle [G \circ F] = \langle (X) \rangle G
\]

commutes).

(3) - Given a right \( F \)-weighted cylinder \( E \xleftarrow{\alpha} D \), i.e. a right cylinder \( E \xleftarrow{\alpha} D \) (see Remark 4.5.2(1)), the category \( E \) acts on \( \alpha \) and generates a module morphism

\[
E \mid \alpha = \langle E \rangle \langle S(M) \rangle : E \to D
\]
5.3. Yoneda morphisms for cylinders

150

, i.e. a cell

\[ E \xrightarrow{(E) F} D \]
\[ S \xrightarrow{E \alpha} T \]
\[ X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\cdot} A \]

direct along \( \alpha \), mapping each \((E) F\)-arrow \( h : e \rightarrow d \) to the \( \mathcal{M}\)-arrow \((h : S) \circ \alpha_d : e : S \rightarrow T : d \) as indicated in

\[ \begin{array}{c}
   \text{d} : F \\
   h \uparrow \\
   \text{e}
\end{array} \xrightarrow{h : (E | \alpha)} \begin{array}{c}
   \text{d} : F : S \\
   h : S
\end{array} \xrightarrow{\alpha_d} \begin{array}{c}
   T : d \\
   h : (E | \alpha)
\end{array} \]

\[ e \]

Given a left \( F \)-weighted cylinder \( D \xrightarrow{F} E \), i.e. a left cylinder \( \xrightarrow{S(M)T} F \xrightarrow{\alpha} E \) (see Remark 4.5.2(1)), the category \( E \) acts on \( \alpha \) and generates a module morphism

\[ \alpha \uparrow E = \alpha \langle (S \langle M \rangle) T \rangle \uparrow E : F \langle E \rangle \rightarrow S \langle M \rangle) T : D \rightarrow E \]

, i.e. a cell

\[ \begin{array}{c}
   D \\
   \text{e}
\end{array} \xrightarrow{E \alpha} \begin{array}{c}
   E \\
   \text{e}
\end{array} \]

inverse along \( \alpha \), mapping each \( F \langle E \rangle\)-arrow \( h : d \rightarrow e \) to the \( \mathcal{M}\)-arrow \( \alpha_d \circ (T : h) : d : S \rightarrow T : e \) as indicated in

\[ \begin{array}{c}
   d : S \\
   (\alpha | E) \downarrow h
\end{array} \xrightarrow{\alpha_d \circ (T : h)} \begin{array}{c}
   T : d \\
   (T : h)
\end{array} \xrightarrow{h} \begin{array}{c}
   F : d \\
   h
\end{array} \xrightarrow{h} \begin{array}{c}
   e \\
   h \uparrow \text{e}
\end{array} \]

\[ (4) \]

\[ (4) \]

- Given a cone \( r \rightarrow K : \ast E \rightarrow \mathcal{M} \), i.e. a cylinder \( \xrightarrow{r \rightarrow K \alpha} E \), the category \( X \) acts on \( \alpha \) and generates a wedge

\[ X \uparrow \alpha = \langle X \uparrow \mathcal{M} \rangle \alpha : \langle X \rangle r \rightarrow \langle \mathcal{M} \rangle K : X \rightarrow \ast E \]

direct along \( \alpha \), mapping each \( X\)-arrow \( h : x \rightarrow r \) to the \( \mathcal{M}\)-arrow \( h : x \rightarrow e \) as indicated in

\[ \begin{array}{c}
   r \\
   h \uparrow \text{e}
\end{array} \xrightarrow{h \rightarrow (X | \alpha)} \begin{array}{c}
   K \cdot e \\
   h \uparrow \text{e}
\end{array} \]

for each \( e \in \| E \| \).

- Given a cone \( \alpha : K \rightarrow r : E \ast \rightarrow \mathcal{M} \), i.e. a cylinder \( \xrightarrow{K \rightarrow r \alpha} E \ast \rightarrow \mathcal{M} \), the category \( A \) acts on \( \alpha \) and generates a wedge

\[ \alpha \uparrow A = \alpha \langle \mathcal{M} \uparrow A \rangle : r \langle A \rangle \rightarrow K \langle \mathcal{M} \rangle : E \ast \rightarrow A \]

inverse along \( \alpha \), mapping each \( A\)-arrow \( r \rightarrow a \) to the \( \mathcal{M}\)-arrow \( \alpha_e \circ h : e \rightarrow a \) as indicated in

\[ \begin{array}{c}
   e : K \\
   (\alpha | A) \downarrow h
\end{array} \xrightarrow{h \rightarrow (A | \alpha)} \begin{array}{c}
   r \\
   h \uparrow \text{a}
\end{array} \]

for each \( e \in \| E \| \).
5.3. Yoneda morphisms for cylinders

Proposition 5.3.6.

- For any right cylinder \( \xymatrix{ X \ar[r]_-C & M \ar@<0.5ex>[r]^-{\alpha} & A } \), the triangle

\[
\begin{array}{ccc}
G(X) & \xrightarrow{(G)} & A \\
\downarrow \alpha & & \downarrow A \\
\downarrow M & & \downarrow A
\end{array}
\]

commutes.

- For any left cylinder \( \xymatrix{ X \ar[r]^-C & M \ar@<0.5ex>[r]_-{\alpha} & A } \), the triangle

\[
\begin{array}{ccc}
X & \xrightarrow{(F)} & F(A) \\
\downarrow X & & \downarrow (\alpha \cdot A) \\
\downarrow M & & \downarrow A
\end{array}
\]

commutes.

Proof. We need to show that the identity

\( h \cdot G \cdot (X \cdot \alpha) = (\alpha \cdot A) \cdot h \)

holds for any \( A \)-arrow \( h : a \to b \). But since the triangles

\[
\begin{array}{ccc}
a : G & \xrightarrow{\alpha a} & a \\
\downarrow h & & \downarrow h \\
b : G & \xrightarrow{\alpha b} & b
\end{array}
\]

commute (see Example 5.3.5(1)), \( h \cdot G \cdot (X \cdot \alpha) \) and \( (\alpha \cdot A) \cdot h \) are identical, being the diagonal of the naturality square

\[
\begin{array}{ccc}
a : G & \xrightarrow{\alpha a} & a \\
\downarrow h & & \downarrow h \\
b : G & \xrightarrow{\alpha b} & b
\end{array}
\]

Note. The following is a special case of Proposition 5.3.6 where \( M \) is given by a representable (resp. corepresentable) module.

Proposition 5.3.7. Consider a pair of functors \( \xymatrix{ X \ar[r]^-C & F } \).

- Given a natural transformation \( \epsilon : G \cdot \circ F \to 1_A : A \to A \), the diagram

\[
\begin{array}{ccc}
G(X) & \xrightarrow{(G)} & A \\
\downarrow \epsilon \cdot A & & \downarrow \epsilon(A) \\
\downarrow G(F(A)) & & \downarrow [G \cdot \circ F](A)
\end{array}
\]

commutes.

- Given a natural transformation \( \eta : 1_X \to G \cdot \circ F : X \to X \), the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(F)} & F(A) \\
\downarrow (X) \cdot \eta & & \downarrow (\eta \cdot A) \\
\downarrow [G \cdot \circ F] & & \downarrow ([X \cdot G] \cdot F)
\end{array}
\]

commutes.
Proof. In Proposition 5.3.6, replace \( \mathcal{M} \) with the representable module \( F(A) \) and replace \( \alpha \) with \( \epsilon \), and we have the commutative triangle depicted in the upper left-hand side of the diagram. The commutativity of the lower right-hand side triangle was seen in Example 5.3.5(2).

\[ \square \]

Note. A two-sided cylinder \( \frac{S}{M} \frac{E}{T} \) is also depicted as a right cylinder \( \frac{X}{\alpha} \frac{S}{M} \frac{E}{T} \) or as a left cylinder \( \frac{S(M)}{T} \frac{E}{A} \) (see Remark 4.3.4(3)). Proposition 5.3.8 says that \( X \) (resp. \( A \)) generates the same module morphism direct (resp. inverse) along \( \alpha \) irrespective of the way \( \alpha \) is depicted.

**Proposition 5.3.8.** Consider a cylinder as in Note above. Then

- the module morphism
  \[ X \rightarrow \alpha = (X \rightarrow M) \alpha : (X) S \rightarrow (M) T : X \rightarrow E \]
  coincides with
  \[ X \rightarrow \alpha = (X \rightarrow (M) T) \alpha : (X) S \rightarrow (M) T : X \rightarrow E \]

- the module morphism
  \[ \alpha \rightarrow A = \alpha (\alpha \rightarrow A) : (A) M \rightarrow S(M) : E \rightarrow A \]
  coincides with
  \[ \alpha \rightarrow A = \alpha (S(\alpha \rightarrow A)) : (A) M \rightarrow S(M) : E \rightarrow A \]

**Proof.** Both map each \( (X) S \rightarrow \alpha \rightarrow e \) to the \( (M) T \rightarrow \alpha \ightarrow e \) (see Example 5.3.5(1)). \[ \square \]

Note. Given a pair of categories \( X \) and \( A \), we saw in Remark 4.3.6(3) that the hom of the functor category \( [A, X] \) is the same thing as the module \( (A, (X)) \), and will see in Proposition 5.3.9 that the hom of the right general Yoneda functor for \( [A, X] \) is the same thing as the right general Yoneda morphism for \( (A, (X)) \).

**Proposition 5.3.9.** Given a pair of categories \( X \) and \( A \),

- the right general Yoneda morphism
  \[ [A, X] \rightarrow \alpha : [A, X] \]
  \[ (X \rightarrow A) \rightarrow (X \rightarrow A) \rightarrow [X \rightarrow A] \]
  \[ [X : A] \rightarrow (X \rightarrow A) \rightarrow [X : A] \]
  for \( (A, (X)) \) is the same thing as the hom
  \[ [A, X] \rightarrow \alpha : [A, X] \]
  \[ (X \rightarrow A) \rightarrow (X \rightarrow A) \rightarrow [X : A] \]
  \[ [X : A] \rightarrow (X \rightarrow A) \rightarrow [X : A] \]
  of the right general Yoneda functor for \( [A, X] \); that is, for any natural transformation \( \tau : S \rightarrow T : A \rightarrow X \), the module morphism
  \[ X \rightarrow \tau : (X) S \rightarrow (X) T : X \rightarrow A \]
  coincides with
  \[ (X) \tau : (X) S \rightarrow (X) T : X \rightarrow A \]
Remark 2.3.8(1).

Proof. Both Theorem 5.3.10.

\[ \tau : \mathbb{S} \to \mathbb{T} : X \to A, \]

coincides with

\[ \tau \downarrow A : T(A) \to S(A) : X \to A \]

for \((X, (A))\) is the same thing as the hom

\[ \tau \downarrow A : T(A) \to S(A) : X \to A \]

of the left general Yoneda functor for \([X, A] : \text{that is, for any natural transformation } \tau : \mathbb{S} \to \mathbb{T} : X \to A, \]

\[ \tau \downarrow A : T(A) \to S(A) : X \to A \]

Proof. Both \(X \downarrow \tau\) and \((X) \tau\) map each \((X)\) \(S\)-arrow \(h : x \sim a\) to the \((X)\) \(T\)-arrow \(h \circ \tau : x \sim a\) (see Remark 2.3.8(1)). \(\square\)

Theorem 5.3.10. Given a category \(E\) and a module \(M : X \to A,\)

\[ \text{the triangle} \]

\[ \begin{array}{ccc}
(X, M) & \xrightarrow{E} & (E, (X, M)) \\
\downarrow \gamma & & \downarrow \gamma \\
(E, X) & \xrightarrow{(\gamma)} & (E, (X, \gamma)) \\
\end{array} \]

commutes; that is, the composition

\[ \begin{array}{ccc}
[E, X] & \xrightarrow{\gamma} & [E, A] \\
\downarrow \gamma & & \downarrow \gamma \\
[E, (X, \gamma)] & \xrightarrow{(\gamma)} & [E, (X, \gamma)] \\
\end{array} \]

of the right general Yoneda morphism for \((E, M)\) and the right exponential transposition yields the cell

\[ \begin{array}{ccc}
[E, X] & \xrightarrow{\gamma} & [E, A] \\
\downarrow \gamma & & \downarrow \gamma \\
[E, (X, \gamma)] & \xrightarrow{(\gamma)} & [E, (X, \gamma)] \\
\end{array} \]

, postcomposition with the right Yoneda morphism for \(M.\)

\[ \text{the triangle} \]

\[ \begin{array}{ccc}
E \times (M \downarrow A) & \xrightarrow{(E, \downarrow M)} & (E, (M \downarrow A)) \\
\downarrow \gamma & & \downarrow \gamma \\
(E, A) & \xrightarrow{(\gamma)} & (E, (A, \gamma)) \\
\end{array} \]
commutes; that is, the composition

\[
\begin{align*}
\left[ E, X \right] &\rightarrow \left[ E, A \right] \\
\left[ E : A \right] &\rightarrow \left[ E, A \right]
\end{align*}
\]

of the left general Yoneda morphism for \((E, M)\) and the left exponential transposition yields the cell

\[
\begin{align*}
\left[ E, X \right] &\rightarrow \left[ E, A \right] \\
\left[ E, A \right] &\rightarrow \left[ E, [A] \right]
\end{align*}
\]

, postcomposition with the left Yoneda morphism for \(M\).

**Proof.** By Proposition 2.2.3, the diagram

\[
\begin{array}{ccc}
\xymatrix{ (X|_{M}) \ar[r] & \left(E, [M]\right) \\
\left[X : E\right] \ar[r]^\sim & \left[E, \left[X : \right]\right] }
\end{array}
\]

commutes. The hom of this diagram composed with \((E, I_M)\) as shown in

\[
\begin{align*}
\left\langle E, M \right\rangle &\downarrow \left\langle E, I_M \right\rangle \\
\left((X|_{M}) \ar[r]^\sim & \left(E, [M]\right) \ar[r]^{(\cdot)} & \left(E, \left(X : \right)\right) \right)
\end{align*}
\]

yields the desired commutative triangle by Remark 5.3.4(1) and Remark 5.2.6(1).

**Remark 5.3.11.** Theorem 5.3.10 says that, given a cylinder \(\alpha : S \rightarrow T : E \rightarrow M\), the identity

\[
\left\langle (X|_{M}) \alpha \right\rangle \frac{(\cdot)}{=} \left\langle (X|_{M}) \right\rangle \delta \alpha
\]

holds; that is, the right exponential transpose of the module morphism

\[
\left\langle X|_{M} \right\rangle \alpha : (X) S \rightarrow (M) T : X \rightarrow E
\]

is the natural transformation given by the composition

\[
\begin{align*}
\begin{array}{ccc}
\xymatrix{ S \ar[r]^{\alpha} & T \\
\left[X : E\right] \ar[r]^{\sim} & \left[E, \left[X : \right]\right] \ar[r]^{(\cdot)} & \left(E, \left(X \right)\right) }
\end{array}
\end{align*}
\]

of \(\alpha\) and the right Yoneda morphism for \(M\).
5.3. Yoneda morphisms for cylinders

• the identity

\[ \kappa(\alpha(M|A)) = \alpha \circ (\kappa(M|A)) \]

holds; that is, the left exponential transpose of the module morphism

\[ \alpha(M|A) : T(A) \rightarrow S(M) : E \rightarrow A \]

is the natural transformation given by the composition

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & T \\
\downarrow \kappa(M|A) & & \downarrow \kappa(A) \\
X & \xrightarrow{\kappa} & A \\
\end{array}
\]

of \( \alpha \) and the left Yoneda morphism for \( \mathcal{M} \).

**Note.** The following is a pointwise description of Remark 5.3.11.

**Corollary 5.3.12.** For any cylinder

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & T \\
\downarrow M & & \downarrow M \\
X & \xrightarrow{\kappa(M|A)} & A \\
\end{array}
\]

and any object \( e \in \|E\| \),

• the identity

\[ (X|\alpha)e = X|\alpha_e \]

holds; that is, the right slice

\[ (X|\alpha)e : (\langle X \rangle S)e \rightarrow (\langle M \rangle T)e : X \rightarrow * \]

at \( e \) of the module morphism generated by \( X \) direct along \( \alpha \) is given by the right module

\[ X|\alpha_e : \langle X \rangle (S:e) \rightarrow \langle M \rangle (T:e) : X \rightarrow * \]

generated by \( X \) direct along the component of \( \alpha \) at \( e \).

• the identity

\[ e(\alpha|A) = \alpha_e|A \]

holds; that is, the left slice

\[ e(\alpha|A) : e(\langle T \rangle A) \rightarrow e(\langle S \rangle M) : * \rightarrow A \]

at \( e \) of the module morphism generated by \( A \) inverse along \( \alpha \) is given by the left module

\[ \alpha_e|A : (e : \langle T \rangle A) \rightarrow (e : \langle S \rangle M) : * \rightarrow A \]

generated by \( A \) inverse along the component of \( \alpha \) at \( e \).

**Proof.** By Remark 5.3.11, \( (X|\alpha)e = (\langle X|M \rangle \alpha)e \) is the same thing as the component of the cylinder \( [(\langle X|\alpha \rangle e) \circ \alpha] \) at \( e \), which is given by the image of \( \alpha_e \) under the cell \( (X|\alpha) \), i.e. given by \( X|\alpha_e \). \( \square \)

**Theorem 5.3.13.** Let \( E \) be a category and \( \mathcal{M} : X \rightarrow A \) be a module.

• The right general Yoneda morphism

\[
\begin{array}{ccc}
[X,E] & \xrightarrow{(E,M)} & [E,A] \\
\downarrow X\rightarrow E & & \downarrow \rightarrow \mathcal{M}\rightarrow E \\
[X : E] & \xrightarrow{(X|E)} & [X : E] \\
\end{array}
\]
Corollary 5.3.15. For \( (E, M) \) is fully faithful. Specifically, for each pair of functors \( S : E \to X \) and \( T : E \to A \), the assignment \( \alpha \mapsto X|M \) yields a bijection

\[
(S)(E, M)(T) \cong ((X)S)(X : E)((M)T)
\]

from the set of cylinders \( S \sim T : E \sim M \) to the set of module morphisms \( (X)S \to (M)T : X \to E \), whose inverse sends each module morphism \( \Phi : (X)S \to (M)T \) to the cylinder \( X|M : S \sim T : E \sim M \) defined by

\[
[X|M]_{\Phi} = 1_{(e : S)} \cdot \Phi
\]

for \( e \in \parallel E \parallel \), where \( 1_{(e : S)} \cdot \Phi \) is the image of the identity \( X \)-arrow \( e : S \to S \cdot e \) (i.e. \( (X)S \)-arrow \( e : S \sim e \) under the function

\[
(e : S)(X : e) = (e : S)((X)S)e \overset{(e : S)(\Phi)e}{\longrightarrow} (e : S)((M)T)e = (e : S)((M)(T : e))
\]

The left general Yoneda morphism

\[
[E, X] \ar (E, M) \ar [E, A] \ar [E \times (M)A] \ar [E \times A] \ar [E : A] \ar [E : A]
\]

for \( (E, M) \) is fully faithful. Specifically, for each pair of functors \( S : E \to X \) and \( T : E \to A \), the assignment \( \alpha \mapsto \alpha|M \) yields a bijection

\[
(S)(E, M)(T) \cong (T \langle A \rangle)(E : A)(S \langle M \rangle)
\]

from the set of cylinders \( S \sim T : E \sim M \) to the set of module morphisms \( T \langle A \rangle \to S \langle M \rangle : E \to A \), whose inverse sends each module morphism \( \Phi : T \langle A \rangle \to S \langle M \rangle \) to the cylinder \( \Phi|M : S \sim T : E \sim M \) defined by

\[
[\Phi|M]_{e} = \Phi \cdot 1_{(T : e)}
\]

for \( e \in \parallel E \parallel \), where \( 1_{(T : e)} \cdot \Phi \) is the image of the identity \( A \)-arrow \( e : T \to T \cdot e \) (i.e. \( T \langle A \rangle \)-arrow \( e : T \sim T \cdot e \)) under the function

\[
(e : T) \langle A \rangle(T : e) = e \langle T \langle A \rangle \rangle(T : e) \overset{e(\Phi)(T : e)}{\longrightarrow} e \langle S \langle M \rangle \rangle(T : e) = (e : S) \langle M \rangle(T : e)
\]

Proof. By Theorem 5.3.10, the cell \( (X|M) \sim E \) is fully faithful iff so is the cell \( (E, (X|M) \sim E) \). But since the cell \( (X|M) \sim E \) is fully faithful (see Theorem 5.2.12), so is \( (E, (X|M) \sim E) \) by Proposition 4.3.17. For the second assertion, it suffices to show that a cylinder \( \alpha : S \sim T : E \sim M \) is recovered from the module morphism \( (X|M) \alpha \) by \( \alpha_{e} = 1_{(e : S)} \cdot (e : S) \langle X|M \rangle \alpha e \). But by Theorem 5.2.12 and Corollary 5.3.12,

\[
\alpha_{e} = 1_{(e : S)} \cdot (e : S) \langle X|M \rangle \alpha e = 1_{(e : S)} \cdot (e : S) \langle X|M \rangle \alpha e
\]

Remark 5.3.14. Theorem 5.2.12 is regarded as a special case of Theorem 5.3.13 where \( E \) is the terminal category.

Corollary 5.3.15. (General Yoneda embedding). Let \( X \) and \( A \) be categories.
The right general Yoneda functor \([X \circ A] : [A, X] \to [X : A]\) is fully faithful. Specifically, for each pair of functors \(S, T : A \to X\), the assignment \(\tau \mapsto (X) \tau\) yields a bijection
\[
(S)(A, X)(T) \cong (\langle (X) S \rangle (X : A) ((X) T))
\]
from the set of natural transformations \(S \to T : A \to X\) to the set of module morphisms \((X) S \to (X) T : X \to A\), whose inverse sends each module morphism \(\Phi : (X) S \to (X) T\) to the natural transformation \([\Phi] : S \to T\) defined by
\[
[\Phi]_a = 1_{(a ; S)} : \Phi
\]
for \(a \in \| A \|\), where \(1_{(a ; S)} : \Phi\) is the image of the identity \(X\)-arrow \(a : S \to S \cdot a\) (i.e. \((X) S\)-arrow \(a : S \rightsquigarrow a\)) under the function
\[
(a : S) (X) (S \cdot a) = (a : S) ((X) S) a \xrightarrow{(a : S)(\Phi) a} (a : S) ((X) T) a = (a : S) (X) (T \cdot a)
\]
• The left general Yoneda functor \([X \otimes A] : [X, A] \to [X : A]\) is fully faithful. Specifically, for each pair of functors \(S, T : X \to A\), the assignment \(\tau \mapsto \tau (A)\) yields a bijection
\[
(S)(X, A)(T) \cong (T(A))(X : A) (S(A))
\]
from the set of natural transformations \(S \to T : X \to A\) to the set of module morphisms \(T(A) \to S(A) : X \to A\), whose inverse sends each module morphism \(\Phi : T(A) \to S(A)\) to the natural transformation \([\Phi] : S \to T\) defined by
\[
[\Phi]_x = \Phi \cdot 1_{(T \cdot x)}
\]
for \(x \in \| X \|\), where \(\Phi \cdot 1_{(T \cdot x)}\) is the image of the identity \(A\)-arrow \(x : T \to T \cdot x\) (i.e. \(T(A)\)-arrow \(x \rightsquigarrow T \cdot x\)) under the function
\[
(x : T) (A) (T \cdot x) = x (T(A)) (T \cdot x) \xrightarrow{\tau (T \cdot x)} x (S(A)) (T \cdot x) = (x : S) (A) (T \cdot x)
\]

Proof. Since the hom of the right general Yoneda functor \(X \circ A\) is the same thing as the right general Yoneda morphism for \((A, (X))\) (see Proposition 5.3.9), this is a special case of Theorem 5.3.13 where \(M\) is given by the hom of a category.

Remark 5.3.16. Corollary 5.2.14 (Yoneda embedding) is regarded as a special case of Corollary 5.3.15 where \(A\) (resp. \(X\)) is the terminal category.

**Theorem 5.3.17.** Let \(X\) and \(A\) be categories.

• The right general Yoneda morphism for \((A, (X \circ \cdot))\) composed with the iso cell in Corollary 5.1.10 as shown in
\[
\begin{array}{ccc}
[A, X] & \xrightarrow{(X \circ \cdot)} & [X : A] \\
\downarrow & & \downarrow \\
[A, X] & \xrightarrow{\langle (X(X \circ \cdot)) \rangle \cdot A} & [A, [X : ::] \\
\end{array}
\]
\[
\begin{array}{ccc}
X \circ A & \xrightarrow{(X \circ (X \circ \cdot)) \cdot A} & X \circ A \\
\end{array}
\]
\[
\begin{array}{ccc}
[X : A] & \xrightarrow{\langle (X \circ \cdot) \rangle \cdot A} & [X : A] \\
\end{array}
\]
yields a representation
\[
\langle X \circ \cdot A \rangle \cong [X \circ A] (X : A) : [A, X] \to [X : A]
\]
of the right general Yoneda module \(X \circ A\) by the right general Yoneda functor \(X \circ \cdot A\). For a functor \(G : A \to X\) and a module \(M : X \to A\), the representation sends each right cylinder \(\alpha : G \rightsquigarrow M\) to the module morphism \(X | \alpha : (X) G \to M\) in Example 5.3.5(1).
The left general Yoneda morphism for $\{X, \{\cdot, A\}\}$ composed with the iso cell in Corollary 5.1.10 as shown in

$$\begin{align*}
[X : A] \cong (X \cdot A) \rightarrow [X, A] \\
\downarrow \quad 1 \\
[X, [\cdot, A]] \rightarrow [X, A]
\end{align*}$$

yields a corepresentation

$$(X \cdot A) \cong (X : A) \cdot [X : A] : [X : A] \rightarrow [X, A]$$

of the left general Yoneda module $X \cdot A$ by the left general Yoneda functor $X \cdot A$. For a functor $F : X \rightarrow A$ and a module $M : X \rightarrow A$, the representation sends each left cylinder $\alpha : M \sim F$ to the module morphism $\alpha \cdot A : F(A) \rightarrow M$ in Example 5.3.5(1).

**Proof.** See Proposition 5.1.4 for the identity $(X \cdot A) \sim A$. The first assertion now follows from the fully faithfulness of the general Yoneda morphism (Theorem 5.3.13). The second assertion is immediate from Remark 5.1.11 and Proposition 5.3.8. \qed

**Note.** The following is a componentwise description of Theorem 5.3.17.

**Corollary 5.3.18.** *(General Yoneda Lemma).*

- Given a module $M : X \rightarrow A$ and a functor $G : A \rightarrow X$, the assignment $\alpha \mapsto \chi \cdot \alpha$ (see Example 5.3.5(1)) yields a bijection

$$(G)(X \cdot A)(M) \cong ((X)G)(X : A)(M)$$

from the set of right cylinders $G \sim M$ to the set of module morphisms $(X)G \rightarrow M$, whose inverse sends each module morphism $\Phi : (X)G \rightarrow M$ to the right cylinder $X \cdot \Phi : \sim M$ defined by

$$[X \cdot \Phi]_a = 1_{(a \cdot G)} : \Phi$$

for $a \in \|A\|$, where $1_{(a \cdot G)} : \Phi$ is the image of the identity $\chi \cdot a : G \rightarrow G \cdot a$ (i.e. $(X)G$-arrow $a : G \sim a$) under the function

$$(a \cdot G)(X)(G \cdot a) = (a \cdot G)((X)G) a \xrightarrow{(a \cdot G)(\Phi) a} (a \cdot G)(M) a$$

. Moreover, the bijection is natural in $G$ and $M$.

- Given a module $M : X \rightarrow A$ and a functor $F : X \rightarrow A$, the assignment $\alpha \mapsto \alpha \cdot A$ (see Example 5.3.5(1)) yields a bijection

$$(M)(X \cdot A)(F) \cong (F(A))(X : A)(M)$$

from the set of left cylinders $M \sim F$ to the set of module morphisms $F(A) \rightarrow M$, whose inverse sends each module morphism $\Phi : F(A) \rightarrow M$ to the left cylinder $\Phi \cdot A : M \sim F$ defined by

$$[\Phi \cdot A]_x = \Phi : 1_{(F \cdot x)}$$

for $x \in \|X\|$, where $\Phi : 1_{(F \cdot x)}$ is the image of the identity $A$-arrow $x : F \rightarrow F \cdot x$ (i.e. $F(A)$-arrow $x \sim F \cdot x$) under the function

$$(x \cdot F)(A)(F \cdot x) = x(F(A))(F \cdot x) \xrightarrow{x(\Phi)(F \cdot x)} x(M)(F \cdot x)$$

. Moreover, the bijection is natural in $F$ and $M$.

**Proof.** The bijective correspondence follows from Theorem 5.3.13 by identifying right cylinders $G \sim M$ with two-sided cylinders $G \sim 1_A : A \sim M$. The naturality of the bijection in $G$ and $M$ is just a restatement of Theorem 5.3.17. \qed
Remark 5.3.19.

(1) The representation (resp. corepresentation) in Theorem 5.3.17 is called the general Yoneda representation (resp. corepresentation) and denoted by $X\downarrow A$ (resp. $X\uparrow A$) as in

\[
\begin{align*}
\mathsf{[A,X]} & \xrightarrow{\mathsf{X}\cdot A} \mathsf{[X:A]} \quad \text{resp.} \quad \mathsf{[X:A]} \xrightarrow{\mathsf{X\cdot A}} \mathsf{[X:A]} \\
\xmath{X\cdot A} & \xmapsto{\mathsf{X}} \mathsf{X\cdot A} \quad \text{resp.} \quad \mathsf{X\cdot A} \xmapsto{\mathsf{X}} \mathsf{X\cdot A} \quad \mathsf{[X:A]} \xrightarrow{\mathsf{(X,A)}} \mathsf{[X:A]} \quad \mathsf{[X:A]} \xrightarrow{\mathsf{(X,A)}} \mathsf{[X:A]}
\end{align*}
\]

; with this notation, the bijections of the general Yoneda lemma are written respectively as

\[
\begin{align*}
\Theta (X\downarrow A)(M) : (\Theta)(X\cdot A)(M) & \cong ((\Theta)(X)(A))(M) \\
(M)(X\downarrow A)(F) : (M)(X\cdot A)(F) & \cong (F(A))(X\cdot A)(M)
\end{align*}
\]

(2) The Yoneda lemma (Theorem 5.2.10 and Corollary 5.2.18) and the Yoneda representation (Remark 5.2.19) are special cases of the general Yoneda lemma and the general Yoneda representation where $A$ (resp. $X$) is the terminal category; by replacing $A$ (resp. $X$) in the bijection in (1) above with the terminal category, and $\Theta : A \to X$ (resp. $F : X \to A$) with $r : * \to X$ (resp. $r : * \to A$), we have

\[
\begin{align*}
\Theta (X\downarrow *)(M) : (\Theta)(X\cdot *)(M) & \cong ((\Theta)(X)(*)(M) \\
(M)(X\downarrow *)(F) : (M)(X\cdot *)(F) & \cong (F(A))(X\cdot *)(M)
\end{align*}
\]

, which are identified with the bijections in Remark 5.2.19 by the isomorphisms in Remark 5.1.7(3) and Remark 1.1.14(4).

Corollary 5.3.20. Let $\Theta : M \to N : X \to A$ be a module morphism and consider the bijection in Corollary 5.3.18.

\begin{itemize}
\item For any right cylinder $\alpha : G \Rightarrow M$,

\[
X\downarrow[\alpha \circ \Theta] = (X\downarrow \alpha) \circ \Theta
\]

, and for any module morphism $\Phi : (X)G \to M : X \to A$,

\[
X\downarrow(\Phi \circ \Theta) = [X\downarrow \Phi] \circ \Theta
\]

\item For any left cylinder $\alpha : M \Rightarrow F$,

\[
[\alpha \circ \Theta]\uparrow A = (\alpha\uparrow A) \circ \Theta
\]

, and for any module morphism $\Phi : (A)G \to M : X \to A$,

\[
(\Phi \circ \Theta)\uparrow A = [\Phi\uparrow A] \circ \Theta
\]
\end{itemize}

Proof. Immediate by the naturality of the bijection.
Theorem 5.3.21. Given a pair of natural transformations $\beta : P \to F \circ Q : X \to E$ and $\alpha : F \circ S \to T : D \to A$ as in

\[
\begin{array}{c}
\begin{array}{ccc}
P & \xrightarrow{\beta} & F \\
\downarrow & & \downarrow \\
E & \xrightarrow{\alpha} & D \\
\downarrow & & \downarrow \\
S & & A
\end{array}
\end{array}
\]

, their pasting composite $[\beta \circ S] \circ [Q \circ \alpha] : P \circ S \to T \circ Q$ is given by the composition

\[
\begin{array}{cc}
P & \xrightarrow{\beta} & Q \\
\downarrow & & \downarrow \\
E & \xrightarrow{\alpha} & D \\
\downarrow & & \downarrow \\
S & & A
\end{array}
\]

of the cylinder $\beta$ (see Remark 4.3.4(5)) and the cell $E \uparrow \alpha$ (see Example 5.3.5(3)).

- Given a pair of natural transformations $\beta : P \circ F \to Q : X \to E$ and $\alpha : S \to T \circ F : D \to A$ as in

\[
\begin{array}{c}
\begin{array}{ccc}
P & \xrightarrow{\beta} & F \\
\downarrow & & \downarrow \\
D & \xrightarrow{\alpha} & E \\
\downarrow & & \downarrow \\
S & & A
\end{array}
\end{array}
\]

, their pasting composite $[P \circ \alpha] \circ [\beta \circ T] : P \circ S \to T \circ Q$ is given by the composition

\[
\begin{array}{cc}
P & \xrightarrow{\beta} & Q \\
\downarrow & & \downarrow \\
D & \xrightarrow{\alpha} & E \\
\downarrow & & \downarrow \\
S & & A
\end{array}
\]

of the cylinder $\beta$ (see Remark 4.3.4(5)) and the cell $\alpha \uparrow E$ (see Example 5.3.5(3)).

Proof. For any $x \in \|X\|$, 

\[
[[\beta \circ S] \circ [Q \circ \alpha]]_x = [\beta \circ S]_x \circ [Q \circ \alpha]_x = (\beta_x : S) \circ (\alpha : Q)
\]

and 

\[
[\beta \circ (E \uparrow \alpha)]_x = \beta_x : (E \uparrow \alpha)
\]

. We thus need to show that $\beta_x : (E \uparrow \alpha) = (\beta_x : S) \circ (\alpha : Q)$. But by replacing $h : e \to F \cdot d$ with $\beta_x : x : P \to F \cdot Q \cdot x$ in the commutative diagram of Example 5.3.5(3), we have

\[
\begin{array}{cccc}
x : Q : F & x : Q : F : S \xrightarrow{\alpha(x : Q)} & T : Q \cdot x \\
\beta_x \uparrow & \beta_x \uparrow & \beta_x \uparrow (E \uparrow \alpha) \\
x : P & x : P : S
\end{array}
\]

as required. \qed
Corollary 5.3.22.

- Given a pair of natural transformations \( \eta : 1_X \to G \circ F : X \to X \) and \( \epsilon : G \circ H \to 1_A : A \to A \) as in

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & A \\
\downarrow & & \downarrow \\
G & \xrightarrow{\epsilon} & H
\end{array}
\]

, the triangle

\[
\begin{array}{ccc}
F(A) & \xrightarrow{[[\eta \circ H] \circ [F \circ \epsilon]](A)} & H(A) \\
\eta|A & & \downarrow X|\epsilon \\
\langle X \rangle G & & \langle X \rangle H
\end{array}
\]

commutes, where \([\eta \circ H] \circ [F \circ \epsilon] : H \to F\) is the pasting composite of \(\eta\) and \(\epsilon\).

- Given a pair of natural transformations \(\epsilon : G \circ F \to 1_A : A \to A\) and \(\eta : 1_X \to H \circ F : X \to X\) as in

\[
\begin{array}{ccc}
X & \xleftarrow{\epsilon} & A \\
\downarrow & & \downarrow \\
H & \xleftarrow{\eta} & F
\end{array}
\]

, the triangle

\[
\begin{array}{ccc}
\langle X \rangle G & \xrightarrow{\langle X \rangle [[G \circ \eta] \circ [\epsilon \circ H]]} & \langle X \rangle H \\
X|\epsilon & & \downarrow F|\eta \\
\langle X \rangle \downarrow & & \langle X \rangle \downarrow
\end{array}
\]

commutes, where \([G \circ \eta] \circ [\epsilon \circ H] : G \to H\) is the pasting composite of \(\epsilon\) and \(\eta\).

Proof. Indeed,

\[
[[\eta \circ H] \circ [F \circ \epsilon]](A) = [[\eta \circ (X|\epsilon)](A) \quad \star^1 \\
= [\eta \circ (X|\epsilon)]|A \quad \star^2 \\
= (\eta|A) \circ (X|\epsilon) \quad \star^3
\]

\((\star^1 \text{ by Theorem } 5.3.21; \star^2 \text{ by Proposition } 5.3.9; \star^3 \text{ by Corollary } 5.3.20). \qed

5.4. Yoneda morphisms for cones

Definition 5.4.1. Let \(E\) be a category and \(M : X \to A\) be a module.

- The right general Yoneda morphism for \(\langle \ast E, M \rangle\) (see Definition 4.6.5) is the cell

\[
\begin{array}{ccc}
X & \xrightarrow{\langle \ast E, M \rangle} & [E, A] \\
\downarrow & & \downarrow \\
\langle X \rangle M & \xrightarrow{\ast E} & M \circ \ast E \\
\downarrow & & \downarrow \downarrow \\
[X] & \xrightarrow{\langle X \rangle E} & [X : E]
\end{array}
\]

sending each cone \(\alpha : r \leadsto K : \ast E \leadsto M\) to the wedge

\[
X \uparrow \alpha = \langle X \uparrow M \rangle \alpha : \langle X \rangle r \leadsto \langle M \rangle K \leadsto X \leadsto \ast E
\]

(see Example 5.3.5(4)).
The left general Yoneda morphism for \( (E*, M) \) (see Definition 4.6.5) is the cell
\[
\begin{array}{c}
[E, X] \xrightarrow{[E, A]} A \\
\downarrow_{E \otimes M} \quad \downarrow_{E \otimes (M \otimes A)} \quad \downarrow_{\gamma_A} \\
[E : A] \xrightarrow{[E : A]} [X : A]
\end{array}
\]

sending each cone \( \alpha : K \twoheadrightarrow r : E* \twoheadrightarrow M \) to the wedge
\[
\alpha \uparrow A = \alpha (M \otimes A) : r (A) \twoheadrightarrow K (M) : E* \twoheadrightarrow A
\]
(see Example 5.3.5(4)).

Remark 5.4.2. By virtue of Theorem 4.6.20 and Remark 4.10.2(2),

- the cell \( (X \uparrow M) \twoheadrightarrow E \) is formally given by the pasting composition

\[
\begin{array}{c}
X \xrightarrow{[X]} [E, X] \xrightarrow{[E, M]} [E, A] \\
\downarrow_{X \otimes E} \quad \downarrow_{(X \uparrow M) \otimes E} \quad \downarrow_{M \otimes E} \\
[X] \xrightarrow{[X]} [X : E] \xrightarrow{[X : E]} [X : E]
\end{array}
\]

of the right general Yoneda morphism for \( (E, M) \) and the commutative diagram in Proposition 2.3.9.

- the cell \( E* \otimes (M \otimes A) \) is formally given by the pasting composition

\[
\begin{array}{c}
[E, X] \xrightarrow{[E, A]} [E, A] \\
\downarrow_{E \otimes M} \quad \downarrow_{E \otimes (M \otimes A)} \quad \downarrow_{\gamma_A} \\
[E : A] \xrightarrow{[E : A]} [E : A] \xrightarrow{[E : A]} [X : E]
\end{array}
\]

of the left general Yoneda morphism for \( (E, M) \) and the commutative diagram in Proposition 2.3.9.

Proposition 5.4.3. The right (resp. left) general Yoneda morphism for \( (*E, M) \) (resp. \( (E*, M) \)) is fully faithful.

Proof. Since the cell \( (X \uparrow M) \twoheadrightarrow *E \) is obtained from the cell \( (X \uparrow M) \twoheadrightarrow E \) (the right general Yoneda morphism for \( (E, M) \)) by the pasting composition in Remark 5.4.2, the fully faithfulness of \( (X \uparrow M) \twoheadrightarrow *E \) follows from that of \( (X \uparrow M) \twoheadrightarrow E \) (see Theorem 5.3.13) by applying Proposition 1.2.33.

Proposition 5.4.4. Given a category \( E \) and a module \( M : X \twoheadrightarrow A \),

- the triangle

\[
\begin{array}{c}
(X \uparrow M) \twoheadrightarrow *E \quad (E, M) \quad (E, (X \uparrow M) \twoheadrightarrow)
\end{array}
\]

\[
\begin{array}{c}
(X : *E) \xrightarrow{r} (*E, (X : *))
\end{array}
\]

commutes; that is, the composition
\[
\begin{array}{c}
X \xrightarrow{[X]} [E, X] \xrightarrow{[E, A]} A \\
\downarrow_{X \otimes E} \quad \downarrow_{(X \uparrow M) \otimes E} \quad \downarrow_{M \otimes E} \\
[X] \xrightarrow{[X]} [X : E] \xrightarrow{[X : E]} [X : E]
\end{array}
\]

\[
\begin{array}{c}
1 \xrightarrow{[1]} \quad 1 \xrightarrow{r} \quad 1\]
\end{array}
\]

\[
\begin{array}{c}
[X] \xrightarrow{[X]} [E, X] \xrightarrow{[E, A]} A
\end{array}
\]
5.4. Yoneda morphisms for cones

of the right general Yoneda morphism for \((\ast E, \mathcal{M})\) and the right exponential transposition of wedges \(X \rightarrow \ast E\) (see Notation 4.10.3) yields the cell

\[
\begin{align*}
X & \rightarrow \mathcal{E} \mathcal{M} \\
X \times \mathcal{M} & \rightarrow \mathcal{E} \mathcal{M} \\
[X] & \rightarrow \mathcal{E} \mathcal{M} \\
\end{align*}
\]

, postcomposition with the right Yoneda morphism for \(\mathcal{M}\).

- the triangle

\[
\begin{tikzcd}
E \times \mathcal{M} & (E \times \mathcal{M}) \\
(E \times \mathcal{M}) & (E, \mathcal{M}) \\
\end{tikzcd}
\]

commutes; that is, the composition

\[
[\mathcal{E}, X] \rightarrow [\mathcal{E}, \mathcal{M}] \rightarrow A
\]

of the left general Yoneda morphism for \((E, \mathcal{M})\) and the left exponential transposition of wedges \(E \rightarrow A\) (see Notation 4.10.3) yields the cell

\[
\begin{align*}
\mathcal{E} & \rightarrow \mathcal{M} \\
\mathcal{E} \mathcal{M} & \rightarrow \mathcal{M} \\
\mathcal{E} \mathcal{M} & \rightarrow \mathcal{E} \\
\end{align*}
\]

, postcomposition with the left Yoneda morphism for \(\mathcal{M}\).

**Proof.** Consider the cells and commutative diagrams as in

\[
\begin{tikzcd}
X & [\mathcal{E}, X] \\
X \times \mathcal{M} & [\mathcal{E}, X] \\
\end{tikzcd}
\]

. The horizontal composition yields

\[
\begin{align*}
X & \rightarrow \mathcal{E} \mathcal{M} \\
X \times \mathcal{M} & \rightarrow \mathcal{E} \mathcal{M} \\
[X] & \rightarrow \mathcal{E} \mathcal{M} \\
\end{align*}
\]

by Remark 5.4.2 and Remark 4.10.4, and the vertical composition yields

\[
\begin{align*}
X & \rightarrow \mathcal{E} \mathcal{M} \\
X \times \mathcal{M} & \rightarrow \mathcal{E} \mathcal{M} \\
[X] & \rightarrow \mathcal{E} \mathcal{M} \\
\end{align*}
\]
by Theorem 5.3.10, which then produces

\[
\begin{align*}
X \ar[(\ast \ast \ast)_{\mathcal{E}, \mathcal{M}}] & \rightarrow \mathcal{E}, \mathcal{A} \\
X \ar[(\ast \ast \ast)_{\mathcal{E}, (\mathcal{X} : \mathcal{M})}] & \rightarrow \mathcal{E}, \mathcal{M}, \mathcal{X} \\
X & \ar[(\ast \ast \ast)_{\mathcal{E}, (\mathcal{X} : \mathcal{M})}] \rightarrow \mathcal{E}, \mathcal{X} \\
\end{align*}
\]

by Corollary 4.6.21. The assertion now follows from Proposition 1.2.34.

Remark 5.4.5. Proposition 5.4.4 says that

\[\mathcal{E}\] for any cone \(\alpha : r \rightarrow \mathbb{K} : \ast \rightarrow \mathcal{M}\) the identity

\[
\langle (\mathcal{X} : \mathcal{M}) \alpha \rangle \mathcal{X} = \langle (\mathcal{X} : \mathcal{M}) \mathcal{X} \rangle \delta \alpha
\]

holds; that is, the right exponential transpose of the wedge

\[
\langle \mathcal{M} \alpha \rangle (\mathcal{X} r \rightarrow \mathcal{M}) \mathcal{K} : \mathcal{X} \rightarrow \ast \mathcal{E}
\]

is the cone given by the composition

\[
\begin{align*}
\ast & \ar[\mathcal{E}] \\
r & \ar[\mathcal{K}] \\
\mathcal{X} & \ar[\mathcal{A}] \\
\mathcal{M} & \ar[\mathcal{\mathcal{M}}] \\
\mathcal{X} & \ar[\mathcal{\mathcal{X}}] \\
\end{align*}
\]

of \(\alpha\) and the right Yoneda morphism for \(\mathcal{M}\).

\[\mathcal{E}\] for any cone \(\alpha : \mathcal{K} \rightarrow r : \mathcal{E} \rightarrow \ast \mathcal{M}\) the identity

\[
\kappa \langle \mathcal{M} \alpha \rangle \mathcal{A} = \alpha \delta \kappa \langle \mathcal{M} \alpha \rangle \mathcal{A}
\]

holds; that is, the left exponential transpose of the wedge

\[
\alpha (\mathcal{M} \alpha) : r (\mathcal{A}) \rightarrow (\mathcal{K}) (\mathcal{M}) : \mathcal{E} \rightarrow \mathcal{A}
\]

is the cone given by the composition

\[
\begin{align*}
\mathcal{E} & \ar[\mathcal{\mathcal{E}}] \\
\mathcal{K} & \ar[\mathcal{\mathcal{K}}] \\
\mathcal{X} & \ar[\mathcal{\mathcal{A}}] \\
\mathcal{\mathcal{M}} & \ar[\mathcal{\mathcal{\mathcal{M}}}] \\
\mathcal{\mathcal{X}} & \ar[\mathcal{\mathcal{\mathcal{X}}}] \\
\end{align*}
\]

of \(\alpha\) and the left Yoneda morphism for \(\mathcal{M}\).

Note. The following definition is a special case of Definition 5.4.1 where \(\mathcal{M}\) is given by the hom of a category.

Definition 5.4.6. Let \(\mathcal{E}\) and \(\mathcal{C}\) be categories.

\[\mathcal{E}\] The right general Yoneda morphism for \(\ast \mathcal{E}, \mathcal{C}\) (see Definition 4.9.3) is the cell

\[
\begin{align*}
\mathcal{C} \ar[(\ast \ast \ast)_{\mathcal{E}, \mathcal{C}}] & \rightarrow \mathcal{E}, \mathcal{C} \\
\mathcal{C} \ar[(\ast \ast \ast)_{\mathcal{C}, \ast \mathcal{E}}] & \rightarrow \mathcal{C}, \ast \mathcal{E} \\
\mathcal{C} & \ar[(\ast \ast \ast)_{\mathcal{C}, \ast \mathcal{E}}] \rightarrow \mathcal{C}, \ast \mathcal{E}
\end{align*}
\]

sending each cone \(\alpha : r \rightarrow \mathbb{K} : \ast \mathcal{E} \rightarrow \mathcal{C}\) to the wedge \(\langle \mathcal{C} \alpha \rangle (\mathcal{C} r \rightarrow (\mathcal{C}) \mathcal{K} : \mathcal{C} \rightarrow \ast \mathcal{E}\).
The left general Yoneda morphism for \(\langle E^*, C \rangle\) (see Definition 4.9.3) is the cell

\[
\begin{array}{c}
\left[ E, C \right] - \frac{(E, C)}{E \times C} - \left[ E, C \right] \\
\left[ E : C \right] - \frac{(E : C)}{E \times C} - \left[ E : C \right]
\end{array}
\]

sending each cone \(\alpha : K \to r : E^* \to C\) to the wedge \(\alpha \langle C \rangle : r \langle C \rangle \to K \langle C \rangle : E^* \to C\).

Remark 5.4.7.

1. By virtue of Remark 4.9.4(2) and Remark 4.10.2(2),
   - the cell \(\langle C \rtimes E \rangle\) is formally given by the pasting composition
     \[
     \begin{array}{c}
     [C : E] \rightarrow \left[ C : E \right] \\
     [C : E] \rightarrow \left[ C : E \right]
     \end{array}
     \]
     of the hom of the right general Yoneda functor for \([E, C]\) and the commutative diagram in Proposition 2.3.9.
   - the cell \(\langle E^* \ltimes C \rangle\) is formally given by the pasting composition
     \[
     \begin{array}{c}
     [E, C] - \frac{(E, C)}{E \times C} - \left[ E, C \right] \\
     [E : C] - \frac{(E : C)}{E \times C} - \left[ E : C \right]
     \end{array}
     \]
     of the hom of the left general Yoneda functor for \([E, C]\) and the commutative diagram in Proposition 2.3.9.

2. Comparing the compositions above and those in Remark 5.4.2, and noting Proposition 5.3.9, we see that
   - the right general Yoneda morphism for \(\langle *E, C \rangle\) is just a special instance of the right general Yoneda morphism for \(\langle *E, M \rangle\) where \(M\) is given by the hom of \(C\); that is,
     \[
     \langle C \rtimes *E \rangle = \langle C \uparrow \langle C \rangle \rangle \rtimes *E
     \]
   - the left general Yoneda morphism for \(\langle E^*, C \rangle\) is just a special instance of the left general Yoneda morphism for \(\langle E^*, M \rangle\) where \(M\) is given by the hom of \(C\); that is,
     \[
     \langle E^* \ltimes C \rangle = E^* \ltimes \langle \langle C \rangle \downarrow C \rangle
     \]

Note. The following is a special case of Proposition 5.4.3 where \(M\) is given by the hom of a category.

Proposition 5.4.8. The right (resp. left) general Yoneda morphism for \(\langle *E, C \rangle\) (resp. \(\langle E^*, C \rangle\)) is fully faithful.

Proof. Since the cell \(\langle C \rtimes *E \rangle\) is obtained from the cell \(\langle C \rtimes E \rangle\) (the hom of the right general Yoneda functor for \([E, C]\)) by the pasting composition in Remark 5.4.7(1), the fully faithfulness of \(\langle C \rtimes *E \rangle\) follows from that of \(\langle C \rtimes E \rangle\) (see Corollary 5.3.15) by applying Proposition 1.2.33.

Note. The following is a special case of Proposition 5.4.4 where \(M\) is given by the hom of a category.
Proposition 5.4.9. Given categories $E$ and $C$, the triangle

\[
\begin{array}{c}
\text{C} \xrightarrow{\cdot E} \langle *E, C \rangle \\
\downarrow{	ext{C}} \quad \text{C} \xrightarrow{\cdot E} \langle *E, C \rangle \\
\text{[C : ] } \xrightarrow{\cdot (C \cdot E)} \langle C : C \rangle \\
\downarrow{	ext{1}} \quad \downarrow{	ext{1}} \\
\text{[C : ] } \xrightarrow{\cdot (C \cdot (C : ))} \langle E, [C : ] \rangle
\end{array}
\]

commutes; that is, the composition

\[
\begin{array}{c}
\text{C} \xrightarrow{\cdot (E \cdot C)} \langle E, C \rangle \\
\downarrow{	ext{C}} \\
\text{[C : ] } \xrightarrow{\cdot (E \cdot (C : ))} \langle E, [C : ] \rangle \\
\end{array}
\]

of the right general Yoneda morphism for $\langle *E, C \rangle$ and the right exponential transposition of wedges $C \mapsto *E$ (see Notation 4.10.3) yields the cell

\[
\begin{array}{c}
\text{C} \xrightarrow{\cdot (E \cdot C)} \langle E, C \rangle \\
\downarrow{	ext{C}} \\
\text{[C : ] } \xrightarrow{\cdot (E \cdot (C : ))} \langle E, [C : ] \rangle \\
\end{array}
\]

, postcomposition with the right Yoneda functor for $C$.

- the triangle

\[
\begin{array}{c}
\langle *E, C \rangle \\
\downarrow{} \\
\langle E, : C \rangle \\
\end{array}
\]

commutes; that is, the composition

\[
\begin{array}{c}
\text{[E, C]} \xrightarrow{\cdot (E \cdot C)} \langle E, C \rangle \\
\downarrow{	ext{E} \cdot C} \\
\text{[E : C]} \xrightarrow{\cdot (E \cdot (C : ))} \langle : C \rangle \\
\downarrow{} \\
\text{[E, [C : ]] } \xrightarrow{\cdot (E \cdot (C : ))} \langle : C \rangle \\
\end{array}
\]

of the left general Yoneda morphism for $\langle E, C \rangle$ and the left exponential transposition of wedges $E \mapsto C$ (see Notation 4.10.3) yields the cell

\[
\begin{array}{c}
\text{[E, C]} \xrightarrow{\cdot (E \cdot C)} \langle E, C \rangle \\
\downarrow{	ext{[E, C]} } \\
\text{[E, : C]} \xrightarrow{\cdot (E \cdot (C : ))} \langle : C \rangle \\
\end{array}
\]

, postcomposition with the left Yoneda functor for $C$.

Proof. Since $\langle C \mapsto *E \rangle = \langle C \mid (C) \rangle \mapsto *E$ (see Remark 5.4.7(2)) and $\langle C \mapsto \rangle = \langle C \mid (C) \rangle$ (see Proposition 5.2.8), the desired commutative diagram is obtained from that in Proposition 5.4.4 by replacing $\mathcal{M}$ with the hom of $C$. 

Remark 5.4.10. Proposition 5.4.9 says that
The following is a special case of the general Yoneda lemma (Corollary 5.3.18) where $\alpha$ is a representable module.

Note. 5.5. Correspondences between frames and cells

**Theorem 5.5.1.** Consider a pair of functors $X \xrightarrow{G} A$.

- The assignment $\epsilon \mapsto X|\epsilon$ (see Example 5.3.5(2)) yields a bijection

$$\langle (G \circ F) \rangle (A, A) (1_A) = \langle (G \circ X \circ A) (F (A)) \rangle \cong \langle (X) G \rangle (X : A) (F (A))$$

(see Theorem 5.1.12 for the identity on the left) from the set of natural transformations $G \circ F \to 1_A : A \to A$ to the set of module morphisms $(X) G \to F (A) : X \to A$, whose inverse sends each module morphism $\Phi : (X) G \to F (A)$ to the natural transformation $X \xrightarrow{\Phi} G \circ F \to 1_A$ defined by

$$[X | \Phi]_a = 1_{(a \cdot G) \cdot \Phi}$$

for $a \in \|A\|$, where $1_{(a \cdot G) \cdot \Phi}$ is the image of the identity $X$-arrow $a : G \to G \cdot a$ (i.e. $(X) G$-arrow $a : (a \cdot G) \to a$) under the function

$$(a : G) (G \cdot a) = (a : G) \langle (X) G \rangle a \xrightarrow{(a : G) (\Phi) a} (a : G) (F (A)) a = (a : G \cdot F) (A) a$$

Moreover, the bijection is natural in $G$ and $F$. 

5.5. Correspondences between frames and cells

The assignment $\epsilon \mapsto X|\epsilon$ (see Example 5.3.5(2)) yields a bijection

$$\langle (G \circ F) \rangle (A, A) (1_A) = \langle (G \circ X \circ A) (F (A)) \rangle \cong \langle (X) G \rangle (X : A) (F (A))$$

(see Theorem 5.1.12 for the identity on the left) from the set of natural transformations $G \circ F \to 1_A : A \to A$ to the set of module morphisms $(X) G \to F (A) : X \to A$, whose inverse sends each module morphism $\Phi : (X) G \to F (A)$ to the natural transformation $X \xrightarrow{\Phi} G \circ F \to 1_A$ defined by

$$[X | \Phi]_a = 1_{(a \cdot G) \cdot \Phi}$$

for $a \in \|A\|$, where $1_{(a \cdot G) \cdot \Phi}$ is the image of the identity $X$-arrow $a : G \to G \cdot a$ (i.e. $(X) G$-arrow $a : (a \cdot G) \to a$) under the function

$$(a : G) (G \cdot a) = (a : G) \langle (X) G \rangle a \xrightarrow{(a : G) (\Phi) a} (a : G) (F (A)) a = (a : G \cdot F) (A) a$$

Moreover, the bijection is natural in $G$ and $F$. 

- for any cone $\alpha : r \to K : *E \to C$ the identity

$$(\langle C \rangle \alpha) \circ [C \cdot r] \circ \delta \alpha$$

holds; that is, the right exponential transpose of the wedge

$$(\langle C \rangle \alpha : \langle C \rangle K : C \to *E)$$

is the cone given by the composition

$$\begin{array}{c}
\ast \\
\downarrow \bigcirc \alpha \downarrow r K \\
C \downarrow \alpha \downarrow \langle C \rangle C \\
\langle C \cdot r \rangle C \downarrow \alpha \downarrow [\langle C \rangle C] \\
[C :] \downarrow \alpha \downarrow [C :] \\
\end{array}$$

of $\alpha$ and the right Yoneda functor for $C$.

- for any cone $\alpha : r : E * \to C$ the identity

$$\kappa \langle \alpha (C) \rangle = \alpha \circ \delta \kappa [\kappa C]$$

holds; that is, the left exponential transpose of the wedge

$$\alpha \langle C \rangle : r \langle C \rangle \to K \langle C \rangle : E * \to C$$

is the cone given by the composition

$$\begin{array}{c}
E \\
\downarrow \bigcirc \alpha \downarrow r K \\
C \downarrow \alpha \downarrow \langle C \rangle C \\
\langle \kappa C \rangle \downarrow \alpha \downarrow [\kappa C] \\
[C :] \downarrow \alpha \downarrow [C :] \\
\end{array}$$

of $\alpha$ and the left Yoneda functor for $C$. 

5.5. Correspondences between frames and cells

**Note.** The following is a special case of the general Yoneda lemma (Corollary 5.3.18) where $\mathcal{M}$ is a representable module.
The assignment \( \eta \mapsto \eta \mathcal{A} \) (see Example 5.3.5(2)) yields a bijection

\[
(1_X)\langle X, X \rangle (G \diamond F) = ((X) G) \langle X, A \rangle (F) \cong (F \langle A \rangle) \langle X : A \rangle ((X) G)
\]

(see Theorem 5.1.12 for the identity on the left) from the set of natural transformations \( 1_X \to G \diamond F : X \to X \) to the set of module morphisms \( F \langle A \rangle \to (X) G : X \to A \), whose inverse sends each module morphism \( \Phi : F \langle A \rangle \to (X) G \) to the natural transformation \( \Phi \mathcal{A} : 1_X \to G \diamond F \) defined by

\[
[\Phi \mathcal{A}]_x = \Phi : 1_{(F : x)}
\]

for \( x \in \mathcal{X} \), where \( \Phi : 1_{(F : x)} \) is the image of the identity \( A \)-arrow \( x : F \to F : x \) (i.e. \( F \langle A \rangle \)-arrow \( x \sim F : x \)) under the function

\[
(x : F) \langle A \rangle (F : x) = x \langle F \langle A \rangle \rangle (F : x) = (X) G (F : x)
\]

. Moreover, the bijection is natural in \( G \) and \( F \).

**Proof.** Replacing \( \mathcal{M} \) with \( F \langle A \rangle \) in Corollary 5.3.18, we have a bijection

\[
(G) \langle X, \mathcal{A} \rangle (F \langle A \rangle) \cong ((X) G) \langle X : A \rangle (F \langle A \rangle)
\]

, natural in \( G \) and \( F \). \qed

**Remark 5.5.2.** Just like the general Yoneda lemma is a componentwise description of the general Yoneda representation (resp. corepresentation), Theorem 5.5.1 is a componentwise description of the identity in Theorem 5.1.12 and

- the module isomorphism

\[
(X \mathcal{A}) [X, A] : (X, \mathcal{A}) [X, A] \to [X, \mathcal{A}] (X : A) [X, A] : [A, X] \to [X, A]^-
\]

given by the composition

\[
[A, X] \xrightarrow{X : A} [X, \mathcal{A}] \xrightarrow{X \mathcal{A}} [X : A] \leftarrow [X, A] \xrightarrow{X : A} [X, A]^-
\]

of the general Yoneda representation and the left general Yoneda functor.

- the module isomorphism

\[
[X, \mathcal{A}] (X \mathcal{A}) : [X, \mathcal{A}] (X, \mathcal{A}) \to [X, \mathcal{A}] (X : A) [X, A] : [A, X]^- \to [X, A]
\]

given by the composition

\[
[A, X]^- \xrightarrow{X : \mathcal{A}} [X : A]^- \xrightarrow{X : \mathcal{A}} [X, \mathcal{A}] \xrightarrow{X : \mathcal{A}} [X, A]
\]

of the general Yoneda corepresentation and the right general Yoneda functor.

**Note.** The following is a special case of the general Yoneda lemma (Corollary 5.3.18) where \( \mathcal{M} \) is given by the composite module \( S(M) T \).

**Theorem 5.5.3.** Given a functor \( F : D \to E \) and a module \( \mathcal{M} : X \to A \),
there is a canonical module isomorphism

\[ \psi_{F_M}^s : \langle F \triangleright M \rangle \cong \langle (E) F, M \rangle : [E, X] \to [D, A] \]

, natural in \( F \) and \( M \), giving for each pair of functors \( S : E \to X \) and \( T : D \to A \) a bijection

\[ (S) \langle \psi_{F_M}^s \rangle (T) : (S) \langle F \triangleright M \rangle (T) \cong (S) \langle (E) F, M \rangle (T) \]

from the set of cylinders \( F \circ S \rightharpoonup T : D \rightharpoonup M \) to the set of cells \( S \rightharpoonup T : \langle E \rangle F \rightharpoonup M \); the bijection sends each cylinder

\[
\begin{array}{ccc}
E & \xleftarrow{F} & D \\
\downarrow{s} & \alpha & \downarrow{T} \\
X & \xrightarrow{M} & A
\end{array}
\]

to the cell

\[
\begin{array}{ccc}
E & \xleftarrow{(E)F} & D \\
\downarrow{s} & E \circ \alpha & \downarrow{T} \\
X & \xrightarrow{M} & A
\end{array}
\]

in Example 5.3.5(3), and the inverse sends each cell

\[
\begin{array}{ccc}
E & \xleftarrow{(E)F} & D \\
\downarrow{s} & E \circ \Theta & \downarrow{T} \\
X & \xrightarrow{M} & A
\end{array}
\]

to the cylinder

\[
\begin{array}{ccc}
E & \xleftarrow{F} & D \\
\downarrow{s} & E \circ \Theta & \downarrow{T} \\
X & \xrightarrow{M} & A
\end{array}
\]

defined by

\[ [E \circ \Theta]_d = 1_{(d \circ F)} : \Theta \]

for \( d \in \llbracket D \rrbracket \), where \( 1_{(d \circ F)} : \Theta \) is the image of the identity \( E \circ \theta : F \to F \circ d \) (i.e. \( (E) F \circ d : F \rightharpoonup d \)) under the cell \( \Theta \); moreover, \( \psi_{F_M}^s \) is natural in \( M \) with \( M \) varying in \( \text{MOD} \); that is, the square

\[
\begin{array}{ccc}
\langle F \triangleright M \rangle & \xrightarrow{\psi_{F_M}^s} & \langle (E) F, M \rangle \\
\downarrow{(F \circ \Phi)} & & \downarrow{(E) F, \Phi} \\
\langle F \triangleright N \rangle & \xrightarrow{\psi_{F_N}^s} & \langle (E) F, N \rangle
\end{array}
\]

commutes for every cell \( \Phi : M \to N \).

there is a canonical module isomorphism

\[ \psi_{F_M}^s : \langle F \rightharpoonup M \rangle \cong \langle F (E), M \rangle : [D, X] \to [E, A] \]

, natural in \( F \) and \( M \), giving for each pair of functors \( S : D \to X \) and \( T : E \to A \) a bijection

\[ (S) \langle \psi_{F_M}^s \rangle (T) : (S) \langle F \rightharpoonup M \rangle (T) \cong (S) \langle F (E), M \rangle (T) \]

from the set of cylinders \( S \rightharpoonup T \circ F : D \rightharpoonup M \) to the set of cells \( S \rightharpoonup T : F (E) \rightharpoonup M \); the bijection sends each cylinder

\[
\begin{array}{ccc}
D & \xrightarrow{F} & E \\
\downarrow{s} & \alpha & \downarrow{T} \\
X & \xrightarrow{M} & A
\end{array}
\]
to the cell

\[
\begin{array}{c}
D \xrightarrow{F} E \\
S \xrightarrow{\alpha \in E} T \\
X \xrightarrow{\varphi \in M} A
\end{array}
\]

in Example 5.3.5(3), and the inverse sends each cell

\[
\begin{array}{c}
D \xrightarrow{F} E \\
S \xrightarrow{\Theta \in E} T \\
X \xrightarrow{\varphi \in M} A
\end{array}
\]

to the cylinder

\[
\begin{array}{c}
D \xrightarrow{F} E \\
S \xrightarrow{\Theta \in E} T \\
X \xrightarrow{\varphi \in M} A
\end{array}
\]

defined by

\[
[\Theta|E]_d = \Theta \cdot 1_{(F\cdot d)}
\]

for \(d \in |D|\), where \(\Theta \cdot 1_{(F\cdot d)}\) is the image of the identity \(E\)-arrow \(d: F \rightarrow F\cdot d\) \(i.e.\ F(E)\)-arrow \(d \rightarrow F\cdot d\) under the cell \(\Theta\); moreover, \(\Psi^E_M\) is natural in \(M\) with \(M\) varying in \(MOD\); that is, the square

\[
\begin{array}{c}
\langle F \cdot M \rangle \\
\downarrow (\Phi \cdot \Theta)
\end{array}
\xrightarrow{\Psi^E_M} 
\begin{array}{c}
\langle F \langle E \cdot D \rangle \rangle \\
\downarrow (\Phi \cdot \Theta)
\end{array}
\]

commutes for every cell \(\Phi : M \rightarrow N\).

**Proof.** In Corollary 5.3.18, replace \(M : X \rightarrow A\) with \(S(M) : E \rightarrow D\) and replace \(G : A \rightarrow X\) with \(F : D \rightarrow E\). Then we have a bijection

\[
(F) \langle E|D \rangle (S(\langle M \rangle) T) = (F) \langle E \cdot D \rangle (S(\langle M \rangle) T) = (S) \langle F \cdot M \rangle (T)
\]

(see Remark 5.3.19(1) for \(E|D\)), natural in \(F, S, T,\) and \(M\). But

\[
(F) \langle E \cdot D \rangle (S(\langle M \rangle) T) = \prod_D (F) \langle S(\langle M \rangle) T \rangle = (F \cdot S)(D,M)(T) = (S) \langle F \cdot M \rangle (T)
\]

and

\[
(S) \langle \Psi^E_M \rangle (T) = (F) \langle E|D \rangle (S(\langle M \rangle) T)
\]

by the definitions. Hence an isomorphism \(\Psi^E_M : \langle F \cdot M \rangle \cong \langle (E) F, M \rangle\) is given by

\[
(S) \langle \Psi^E_M \rangle (T) = (F) \langle E|D \rangle (S(\langle M \rangle) T)
\]

. The last assertion holds since for any cell

\[
X \xrightarrow{\varphi \in M} A,
\]

the square

\[
\begin{array}{c}
(S) \langle F \cdot M \rangle (T) \\
\downarrow (S)(\Phi \cdot \Theta)(T)
\end{array}
\xrightarrow{(S)\langle \Psi^E_M \rangle (T)}
\begin{array}{c}
(S) \langle (E) F, M \rangle (T) \\
\downarrow (S)(\Phi \cdot \Theta)(T)
\end{array}
\]

\[
\begin{array}{c}
(S \cdot P) \langle F \cdot N \rangle (Q \cdot T) \\
\downarrow (S \cdot P)(\Psi^E_M)(Q \cdot T)
\end{array}
\xrightarrow{(S \cdot P)(\Psi^E_M)(Q \cdot T)}
\begin{array}{c}
(S \cdot P) \langle (E) F, N \rangle (Q \cdot T) \\
\downarrow (S \cdot P)(\Psi^E_M)(Q \cdot T)
\end{array}
\]


5.5. Correspondences between frames and cells

Theorem 5.3.13 is viewed as a special case of Theorem 5.5.3 by depicting a cylinder

\[
(F \cdot E \cdot D) \circ (S \cdot M \cdot T) \xrightarrow{(F \cdot E \cdot D)(S \cdot M \cdot T)} ((E) \cdot F \cdot D) \circ (S \cdot M \cdot T)
\]

\[
(F \cdot E \cdot D)(S \cdot P \cdot Q \cdot T) \xrightarrow{(F \cdot E \cdot D)(S \cdot P \cdot Q \cdot T)} ((E) \cdot F \cdot D) \circ (S \cdot P \cdot Q \cdot T)
\]

commutes by the naturality of \( E \cdot D \).

\[\square\]

Remark 5.5.4.

(1) Although derived as a special case of the general Yoneda lemma, Theorem 5.5.3 is more versatile; Corollary 5.3.18 turns to be a special case of the bijection in Theorem 5.5.3 where \( S \) and \( T \) are the identities.

(2) Theorem 5.3.13 is viewed as a special case of Theorem 5.5.3 by depicting a cylinder

\[
\begin{array}{c}
\xymatrix{X & \ar[l]_\alpha S & \ar[r]^T & A} \\
\end{array}
\]

direct along \( \alpha \), and the assignment \( \alpha \mapsto X \cdot \alpha \) yields a bijection from the set of cylinders \( S \cdot T : E \cdot M \) to the set of cells \( 1_X \cdot T : (X) \cdot S \cdot M \), i.e. the set of module morphisms \( \langle X \rangle \cdot S \cdot \langle M \rangle \cdot T : X \cdot E \).

(3) Theorem 5.5.1 is also viewed as a special case of Theorem 5.5.3 by depicting natural transformations \( \epsilon : G \cdot F \to 1_A : A \to A \) and \( \eta : 1_X \to G \cdot F : X \to X \) as

\[
\begin{array}{c}
\xymatrix{X & \ar[l]_{\epsilon} F & \ar[r]^{G} & A} \\
\end{array}
\]

and

\[
\begin{array}{c}
\xymatrix{X & \ar[l]_{\eta} \ar[r]^{F} & A} \\
\end{array}
\]

respectively.

\[\blacktriangleleft\] The category \( X \) acts on \( \epsilon \) and generates a cell

\[
\begin{array}{c}
\xymatrix{X & \ar[l]_{\epsilon} F & \ar[r]^{G} & A} \\
\end{array}
\]

direct along \( \epsilon \), and the assignment \( \epsilon \mapsto X \cdot \epsilon \) yields a bijection from the set of natural transformations \( G \cdot F \to 1_A : A \to A \) to the set of cells \( F \cdot \langle X \rangle \cdot G \to \langle A \rangle \), i.e. the set of module...
Corollary 5.5.5. Given a category $E$ and a module $M : X \rightarrow A$, there is a canonical module isomorphism

$$
\Psi_{M}^{E} : \langle (E, M) \rangle : [E, X] \rightarrow [E, A]
$$

, giving for each pair of functors $S : E \rightarrow X$ and $T : E \rightarrow A$ a bijection

$$(S) \langle \Psi_{M}^{E} \rangle (T) : (S) \langle (E, M) \rangle (T) \cong (S) \langle \langle (E), (M) \rangle (T)$$

from the set of cylinders $S \sim T : E \sim M$ to the set of cells $S \sim T : \langle E \rangle \rightarrow M$; the bijection sends each cylinder

$$
\begin{array}{c}
\xymatrix{
X \\
\ar_{\alpha} \ar@{-->}[rr]^{S} & & E \\
\ar_{\Theta} \ar@{-->}[rr]^{T} & & A
}
\end{array}
$$

to the cell

$$
\begin{array}{c}
\xymatrix{
E \\
\ar_{\Theta} \ar@{-->}[rr]^{S} & & M \\
\ar_{T} \ar@{-->}[rr]^{\alpha} & & A
}
\end{array}
$$

which maps each $E$-arrow $h : e \rightarrow e'$ to the $M$-arrow

$$(h : S) \circ \alpha_{e^{'}} = \alpha_{e} \circ (T : h) : e \sim T : e'$$

as indicated in

$$
\begin{array}{c}
\xymatrix{
E \\
\ar_{\Theta} \ar@{-->}[rr]^{S} & & M \\
\ar_{T} \ar@{-->}[rr]^{\alpha} & & A
}
\end{array}
$$

, and the inverse sends each cell

$$
\begin{array}{c}
\xymatrix{
E \\
\ar_{\Theta} \ar@{-->}[rr]^{S} & & M \\
\ar_{T} \ar@{-->}[rr]^{\alpha} & & A
}
\end{array}
$$

to the cylinder

$$
\begin{array}{c}
\xymatrix{
X \\
\ar_{\Theta} \ar@{-->}[rr]^{S} & & E \\
\ar_{T} \ar@{-->}[rr]^{\alpha} & & M \\
\ar_{\Theta} \ar@{-->}[rr]^{\alpha} & & A
}
\end{array}
$$

defined by

$$
[\Theta]_{e} = 1_{e} : \Theta
$$

for $e \in |E|$, where $1_{e} : \Theta$ is the image of the identity $e \sim e$ under the function

$$
e (E) e \sim^{(e(\Theta)e)} e (S (M) T) e = (e : S (M) (T : e)
$$
Moreover, $\Psi^E_M$ is natural in $\mathcal{M}$ with $\mathcal{M}$ varying in $\text{MOD}$; that is, the square

$$
\begin{array}{ccc}
\langle E, \mathcal{M} \rangle & \xrightarrow{\Psi^E_M} & \langle \langle E \rangle, \mathcal{M} \rangle \\
(\langle E, \Phi \rangle) & \downarrow & \downarrow \langle(\langle E \rangle, \Phi) \rangle \\
\langle E, \mathcal{N} \rangle & \xrightarrow{\Psi^E_N} & \langle \langle E \rangle, \mathcal{N} \rangle
\end{array}
$$

commutes for every cell $\Phi : \mathcal{M} \rightarrow \mathcal{N}$.

**Proof.** Since the hom $\langle E \rangle$ is represented and corepresented by the identity functor $1_E : E \rightarrow E$, and since $\langle E, \mathcal{M} \rangle = \langle 1_E \triangleright \mathcal{M} \rangle = \langle 1_E \triangleright \mathcal{F} \rangle$ (see Remark 4.5.4(2)) and $\langle E, \Phi \rangle = \langle 1_E \triangleright \Phi \rangle = \langle 1_E \triangleright \Phi \rangle$ (see Remark 4.5.8(2)), the assertion follows as a special case of Theorem 5.5.3 where $\mathcal{F}$ is given by $1_E$; with $\Psi^E_M$ defined by

$$\Psi^E_M = \Psi^E_1 = \Psi^E_{\mathcal{M}}$$

; the cell $\langle \alpha \rangle$ is given by $E\triangleright \alpha = \alpha \triangleright E$ and the cylinder $[\Theta]$ is given by $E\triangleright \Theta = \Theta \triangleright E$. $\square$

**Remark 5.5.6.** If the module $\mathcal{M}$ in Corollary 5.5.5 is replaced by the hom of a category $\mathcal{C}$, we have a bijection between the set of natural transformations $\alpha : S \rightarrow T : E \rightarrow C$ and the set of cells

$$
\begin{array}{c}
E \xrightarrow{E} E \\
S \xrightarrow{\langle \alpha \rangle} S \\
C \xrightarrow{\langle C \rangle} C
\end{array}
$$

A morphism between two paralleling functors $S, T : E \rightarrow C$ is thus defined either by a cell shown above or by a natural transformation $\alpha : S \rightarrow T : E \rightarrow C$, which is a frame of the end modules $S(\langle C \rangle)T : E \rightarrow E$. The relation between cells and frames can be compared with that between linear maps and matrices.

**Corollary 5.5.7.** Given a category $E$ and a module $\mathcal{M} : X \rightarrow A$,

- there is a canonical module isomorphism

$$
\Psi^E_M : \langle * \Delta_E, \mathcal{M} \rangle \cong \langle * \Delta_E, \mathcal{M} \rangle : X \rightarrow [E, A]
$$

(see Example 1.1.29(9) for $* \Delta_E$), giving for each pair of an object $x \in \|X\|$ and a functor $K : E \rightarrow A$ a bijection

$$
(\langle x \rangle) \langle \Psi^E_M \rangle (K) : (\langle x \rangle) \langle * \Delta_E, \mathcal{M} \rangle (K) \cong (\langle x \rangle) \langle * \Delta_E, \mathcal{M} \rangle (K)
$$

from the set of cones $x \rightarrow K : * \rightarrow \mathcal{M}$ to the set of conical cells $x \rightarrow K : * \Delta_E \rightarrow \mathcal{M}$; the bijection sends each cone

$$
\begin{array}{c}
\ast \xrightarrow{\ast \Delta_E} E \\
x \xrightarrow{\alpha} x \\
X \xrightarrow{\mathcal{M}} \rightarrow A
\end{array}
$$

to the conical cell

$$
\begin{array}{c}
* \xrightarrow{\ast \Delta_E} E \\
x \xrightarrow{\langle \alpha \rangle} x \\
X \xrightarrow{\mathcal{M}} \rightarrow A
\end{array}
$$

which maps each $\langle * \Delta_E \rangle$-arrow $\ast : \ast \rightarrow e$ to the component of $\alpha$ at $e$, and the inverse sends each conical cell

$$
\begin{array}{c}
* \xrightarrow{\ast \Delta_E} E \\
x \xrightarrow{\Theta} x \\
X \xrightarrow{\mathcal{M}} \rightarrow A
\end{array}$$
to the cone

\[
\begin{array}{c}
\ast \leftarrow \overset{\eta_E}{\bullet} E \\
x \downarrow \quad [\theta] \downarrow \kappa \\
X - \overset{\phi}{\rightarrow} M - A
\end{array}
\]

defined by

\[ [\theta]_e = \ast \cdot \Theta \]

for \( e \in \|E\| \), where \( \ast \cdot \Theta \) is the image of the \( \ast \Delta_E \)-arrow \( \ast : \ast \sim e \) under the cell \( \Theta \); moreover, \( \Psi^*_{\mathcal{M}} \) is natural in \( \mathcal{M} \) with \( \mathcal{M} \) varying in \( \text{MOD} \); that is, the square

\[
\begin{array}{c}
\{\ast E, \mathcal{M}\} \xrightarrow{\Psi^*_{\mathcal{M}} \Psi^*_E} \{\ast \Delta_E, \mathcal{M}\} \\
\downarrow \phi \quad \downarrow \phi \kappa \\
\{\ast E, \mathcal{N}\} \xrightarrow{\Psi^*_E} \{\ast \Delta_E, \mathcal{N}\}
\end{array}
\]

commutes for every cell \( \phi : \mathcal{M} \rightarrow \mathcal{N} \).

* there is a canonical module isomorphism

\[ \Psi^*_{\mathcal{M}} : \langle E^*, \mathcal{M} \rangle \cong \langle \Delta_E^*, \mathcal{M} \rangle : [E, X] \rightarrow A \]

(see Example 1.1.29(9) for \( \Delta_E^* \)), giving for each pair of an object \( a \in \|A\| \) and a functor \( K : E \rightarrow X \) a bijection

\[
(K) \langle \Psi^*_{\mathcal{M}} \rangle (a) : (K) \langle E^*, \mathcal{M} \rangle (a) \cong (K) \langle \Delta_E^*, \mathcal{M} \rangle (a)
\]

from the set of cones \( K \sim a : E^* \sim \mathcal{M} \) to the set of conical cells \( K \sim a : \Delta_E^* \rightarrow \mathcal{M} \); the bijection sends each cone

\[
\begin{array}{c}
E \xrightarrow{\eta_E} \ast \\
\kappa \downarrow \alpha \downarrow a \\
X - \overset{\phi}{\rightarrow} M - A
\end{array}
\]

to the conical cell

\[
\begin{array}{c}
E - \overset{\Delta_E^*}{\rightarrow} \ast \\
\kappa \downarrow \langle \alpha \rangle \downarrow a \\
X - \overset{\phi}{\rightarrow} M - A
\end{array}
\]

which maps each \( \langle \Delta_E^* \rangle \)-arrow \( \ast : e \sim \ast \) to the component of \( \alpha \) at \( e \), and the inverse sends each conical cell

\[
\begin{array}{c}
E - \overset{\Delta_E^*}{\rightarrow} \ast \\
\kappa \downarrow \Theta \downarrow a \\
X - \overset{\phi}{\rightarrow} M - A
\end{array}
\]

to the cone

\[
\begin{array}{c}
E \xrightarrow{\eta_E} \ast \\
\kappa \downarrow \langle \Theta \rangle \downarrow a \\
X - \overset{\phi}{\rightarrow} M - A
\end{array}
\]

defined by

\[ [\Theta]_e = \Theta \cdot \ast \]

for \( e \in \|E\| \), where \( \Theta \cdot \ast \) is the image of the \( \langle \Delta_E^* \rangle \)-arrow \( \ast : e \sim \ast \) under the cell \( \Theta \); moreover, \( \Psi^*_{\mathcal{M}} \) is natural in \( \mathcal{M} \) with \( \mathcal{M} \) varying in \( \text{MOD} \); that is, the square

\[
\begin{array}{c}
\langle E^*, \mathcal{M}\rangle \xrightarrow{\Psi^*_{\mathcal{M}} \Psi^*_E} \langle \Delta_E^*, \mathcal{M}\rangle \\
\downarrow \phi \quad \downarrow \phi \kappa \\
\langle E^*, \mathcal{N}\rangle \xrightarrow{\Psi^*_E} \langle \Delta_E^*, \mathcal{N}\rangle
\end{array}
\]
commutes for every cell $\Phi : \mathcal{M} \to \mathcal{N}$.

**Proof.** Since $\ast \Delta_\mathcal{E}$ is the corepresentable module of the functor $!_\mathcal{E} : \mathcal{E} \to \ast$, and since $(\ast \mathcal{E}, \mathcal{M}) = \langle !_\mathcal{E}, \mathcal{M} \rangle$ (see Theorem 4.6.20) and $(\ast \mathcal{E}, \Phi) = \langle !_\mathcal{E}, \Phi \rangle$ (see Corollary 4.6.21), the assertion follows as a special case of Theorem 5.5.3 where $\mathcal{F}$ is given by $!_\mathcal{E}$, with $\Psi^\mathcal{E}_\mathcal{M}$ defined by

$$\Psi^\mathcal{E}_\mathcal{M} = \Psi^\mathcal{E}_\mathcal{M},$$

; the conical cell $(\alpha)$ is given by $\ast \alpha$ and the cone $[\Theta]$ is given by $\ast \Theta$. $\square$

**Theorem 5.5.8.** Given an endomodule $\mathcal{M} : \mathcal{E} \to \mathcal{E}$, there is a canonical bijection

$$\Phi^\mathcal{E}_\mathcal{M} : \prod_{\mathcal{E}} \mathcal{M} \cong (\langle \mathcal{E} \rangle \langle \mathcal{E} : \mathcal{E} \rangle \langle \mathcal{M} \rangle)$$

from the set of frames of $\mathcal{M}$ to the set of module morphisms $(\mathcal{E}) \to \mathcal{M} : \mathcal{E} \to \mathcal{E}$. The bijection sends each frame $\alpha$ of $\mathcal{M}$ to the module morphism $(\alpha) : (\mathcal{E}) \to \mathcal{M}$ which maps each $\mathcal{E}$-arrow $h : e \to e'$ to the $\mathcal{M}$-arrow $h \circ \alpha_{e'} = \alpha_e \circ h : e \to e'$ as indicated in

$$e \xrightarrow{e} \alpha_e \xrightarrow{e} e$$

$h \xrightarrow{h \circ (\alpha)} h \xrightarrow{h}$

$$e' \xrightarrow{\alpha_{e'}} \xrightarrow{e'}$$

, and the inverse sends each module morphism $\Phi : (\mathcal{E}) \to \mathcal{M}$ to the frame $[\Phi]$ of $\mathcal{M}$ defined by

$$[\Phi]_e = 1_{\mathcal{E}}, \Phi$$

for $e \in \|\mathcal{E}\|$, where $1_{\mathcal{E}} \cdot \Phi$ is the image of the identity $e \to e$ under the function

$$e \langle \Phi \rangle e : e \langle \mathcal{E} \rangle e \to e \langle \mathcal{M} \rangle e$$

. Moreover, the bijection is natural in $\mathcal{M}$.

**Proof.** This follows from Corollary 5.5.5 by setting $X = A = \mathcal{E}$ and $S = T = 1_{\mathcal{E}}$ and noting that $(1_{\mathcal{E}}) \langle \mathcal{E}, \mathcal{M} \rangle (1_{\mathcal{E}}) = \prod_{\mathcal{E}} \mathcal{M}$ and $(1_{\mathcal{E}}) \langle \langle \mathcal{E} \rangle, \mathcal{M} \rangle (1_{\mathcal{E}}) = (\langle \mathcal{E} \rangle \langle \mathcal{E} : \mathcal{E} \rangle \langle \mathcal{M} \rangle)$ by the definitions. $\square$

**Note.** In Theorem 5.5.9 and Theorem 5.5.10, the hom of a category $\mathcal{E}$ is regarded as a left module $(\mathcal{E}) : \ast \to \mathcal{E} \times \mathcal{E}$ (resp. a right module $(\mathcal{E}) : \mathcal{E} \times \mathcal{E} \to \ast$).

**Theorem 5.5.9.**

*Given a left module $\mathcal{M} : \ast \to \mathcal{E} \times \mathcal{E}$, there is a canonical bijection

$$\Phi^\mathcal{E}_\mathcal{M} : \prod_{\mathcal{E}} \mathcal{M} \cong (\langle \mathcal{E} \rangle \langle \mathcal{E} \times \mathcal{E} \rangle \langle \mathcal{M} \rangle)$$

from the set of cylindrical frames of $\mathcal{M}$ to the set of module morphisms $(\mathcal{E}) \to \mathcal{M} : \ast \to \mathcal{E} \times \mathcal{E}$. The bijection sends each cylindrical frame $\alpha$ of $\mathcal{M}$ to the module morphism $(\alpha) : (\mathcal{E}) \to \mathcal{M}$ which maps each $\mathcal{E}$-arrow $h : e \to e'$ to the $\mathcal{M}$-arrow $\alpha_{e'} \circ (h, e') = \alpha_e \circ (h, e') : \ast \to (e, e')$ as indicated in

$$* \xrightarrow{\ast \alpha_e} (e, e)$$

$h \xrightarrow{h \circ (\alpha)} (e, h)$

$\langle e, e' \rangle \xrightarrow{(h, e')} (e, e')$

, and the inverse sends each module morphism $\Phi : (\mathcal{E}) \to \mathcal{M}$ to the cylindrical frame $[\Phi]$ of $\mathcal{M}$ defined by

$$[\Phi]_e = \Phi : 1_{\mathcal{E}}$$

for $e \in \|\mathcal{E}\|$, where $\Phi : 1_{\mathcal{E}}$ is the image of the identity $e \to e$ under the function

$$\langle \Phi \rangle (e, e) : (\mathcal{E} \langle e, e \rangle) \to (\mathcal{M} \langle e, e \rangle)$$

. Moreover, the bijection is natural in $\mathcal{M}$.**
Given a right module $\mathcal{M} : E \times E^{-} \to *$, there is a canonical bijection

$$\Phi^E_{\mathcal{M}} : \prod_E \mathcal{M} \cong (\langle E \rangle) (E \times E^{-} : \mathcal{M})$$

from the set of cylindrical frames of $\mathcal{M}$ to the set of module morphisms $(E) \to \mathcal{M} : E \times E^{-} \to *$. The bijection sends each cylindrical frame $\alpha$ of $\mathcal{M}$ to the module morphism $(\alpha) : (E) \to \mathcal{M}$ which maps each $E$-arrow $h : e \to e'$ to the $\mathcal{M}$-arrow $(h, e') \circ \alpha_e = (e, h) \circ \alpha_e : (e, e') \to *$ as indicated in

$$(e, e') \xrightarrow{(h, e')} (e', e')$$

$$(e, h) \xrightarrow{h \cdot (\alpha)} \xrightarrow{\alpha_{e'}} (e', e')$$

and the inverse sends each module morphism $\Phi : \langle E \rangle \to \mathcal{M}$ to the cylindrical frame $[\Phi]$ of $\mathcal{M}$ defined by

$$[\Phi]_e = 1_e \cdot \Phi$$

for $e \in \parallel E \parallel$, where $1_e : \Phi$ is the image of the identity $e \to e$ under the function

$$(e, e) (\Phi) : (e, e) \langle E \rangle \to (e, e) \langle \mathcal{M} \rangle$$

Moreover, the bijection is natural in $\mathcal{M}$.

**Proof.** This is a restatement of Theorem 5.5.8 with $\mathcal{M} : E \to E$ regarded as a left module $(E) : * \to E^{-} \times E$.

**Theorem 5.5.10.** Given a category $E$ and a module $\mathcal{M} : X \to A$,

there is a canonical module isomorphism

$$\Psi^E_{\mathcal{M}} : \langle E, \mathcal{M} \rangle \cong \langle \langle E \rangle, \mathcal{M} \rangle : X \to [E^{-} \times E, A]$$

, giving for each pair of an object $x \in \parallel X \parallel$ and a bifunctor $K : E^{-} \times E \to X$ a bijection

$$(x) \left( \Psi^E_{\mathcal{M}} \right) (K) : (x) \langle E, \mathcal{M} \rangle (K) \cong (x) \langle \langle E \rangle, \mathcal{M} \rangle (K)$$

from the set of cylinders $x \sim K : E \to \mathcal{M}$ to the set of cells $x \sim K : \langle E \rangle \to \mathcal{M}$; the bijection sends each cylinder

$$* \xrightarrow{\alpha} E^{-} \times E$$

$$X \xrightarrow{\mathcal{M}} \sim A$$

to the cell

$$* \xrightarrow{\langle E \rangle} E^{-} \times E$$

$$X \xrightarrow{\mathcal{M}} \sim A$$

which maps each $E$-arrow $h : e \to e'$ to the $\mathcal{M}$-arrow

$$\alpha_{e'} \circ K(h, e') = \alpha_e \circ K(e, h) : x \sim K(e, e')$$

as indicated in

$$X \xrightarrow{\alpha_{e'}} K(e, e)$$

$$\xrightarrow{(\alpha) \cdot h} K(e, e)$$

$$K(e', e') \xrightarrow{K(h, e')} K(e, e')$$
5.5. Correspondences between frames and cells

, and the inverse sends each cell

\[ \begin{array}{c}
  \ast \xrightarrow{\Phi} E^\times E \\
  x \downarrow \Theta \\
  X \xrightarrow{\sim} A
\end{array} \]

to the cylinder \([\Theta] : x \sim K : E \sim M\) defined by

\[ [\Theta]_e = \Theta \cdot 1_e \]

for \(e \in \| E \|\), where \(\Theta \cdot 1_e\) is the image of the identity \(e \rightarrow e\) under the function

\[ \langle E \rangle (e, e) \xrightarrow{\langle \Theta \rangle (e,e)} \langle x \langle M \rangle K \rangle (e, e) = x \langle M \rangle (K(e, e)) \]

; moreover, \(\Psi^E_M\) is natural in \(M\) with \(M\) varying in \(\text{MOD}\); that is, the square

\[ \begin{array}{c}
  \langle E, M \rangle \xrightarrow{\Psi^E_M} \langle (E), M \rangle \\
  \langle E, \Phi \rangle \downarrow \langle (E), \Phi \rangle \\
  \langle E, N \rangle \xrightarrow{\Psi^E_N} \langle (E), N \rangle
\end{array} \]

commutes for every cell \(\Phi : M \rightarrow N\).

- there is a canonical module isomorphism

\[ \Psi^E_M : \langle E, M \rangle \cong \langle (E^\sim), M \rangle : [E^\sim \times E, X] \rightarrow A \]

, giving for each pair of an object \(a \in \| A \|\) and a bifunctor \(K : E^\sim \times E \rightarrow X\) a bijection

\[ (K) \langle \Psi^E_M \rangle (a) : (K) \langle E, M \rangle (a) \cong (K) \langle (E^\sim), M \rangle (a) \]

from the set of cylinders \(K \sim a : E \sim M\) to the set of cells \(K \sim a : \langle E^\sim \rangle \rightarrow \mathcal{M}\); the bijection sends each cylinder

\[ \begin{array}{c}
  E^\sim \times E \\
  K \downarrow \alpha \\
  X \xrightarrow{\sim} A
\end{array} \]

to the cell

\[ \begin{array}{c}
  E^\sim \times E \\
  K \downarrow \langle \alpha \rangle \\
  X \xrightarrow{\sim} A
\end{array} \]

which maps each \(E\)-arrow \(h : e \rightarrow e'\) to the \(M\)-arrow

\[ K(e', h) \circ \alpha_{e'} = K(h, e) \circ \alpha_e : K(e', e) \sim a \]

as indicated in

\[ \begin{array}{c}
  K(e', e) \xrightarrow{K(e', h)} K(e', e') \\
  K(h, e) \downarrow h : \langle \alpha \rangle \\
  K(e, e) \xrightarrow{\alpha_e} a
\end{array} \]

, and the inverse sends each cell

\[ \begin{array}{c}
  E^\sim \times E \\
  K \downarrow \Theta \\
  X \xrightarrow{\sim} A
\end{array} \]
to the cylinder \([\Theta] : K \sim a : E \sim M\) defined by

\[ [\Theta]_e = 1_e : \Theta \]

for \(e \in [E]\), where \(1_e : \Theta\) is the image of the identity \(e \rightarrow e\) under the function

\[ (e, e) (E^-) \xrightarrow{(e, e)(\Theta)} (e, e) (K(M) a) = (K(e, e)) (M) a \]

; moreover, \(\Psi_{E}^{E} \) is natural in \(M\) with \(M\) varying in \(\text{MOD}\); that is, the square

\[
\begin{array}{ccc}
(E, M) & \xrightarrow{\Psi_{E}^{E}} & ((E^-), M) \\
(E, \Phi) & \downarrow \downarrow & \downarrow \downarrow ((E^-), \Phi) \\
(E, N) & \xrightarrow{\Psi_{N}^{E}} & ((E^-), N)
\end{array}
\]

commutes for every cell \(\Phi : M \rightarrow N\).

**Proof.** Replacing \(M\) with \(x(M) K\) in Theorem 5.5.9, we have a bijection

\[ \Phi_{x(M)K}^{E} : \prod_{E} x(M) K \cong ((E)) (E^- \times E) (x(M) K) \]

, natural in \(x\) and \(K\). But

\[ \prod_{E} x(M) K = (x) (E, M)(K) \]

and

\[ ((E)) (E^- \times E) (x(M) K) = (x) ((E), M)(K) \]

by the definitions. Hence an isomorphism \(\Psi_{E}^{E} : (E, M) \cong ((E), M)(K)\) is given by

\[ (x) (\Psi_{E}^{E})(K) = \Phi_{x(M)K}^{E} \]

. The last assertion holds since for any cell

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi} & A \\
P & \downarrow \downarrow & \downarrow \downarrow \\
Y & \xrightarrow{\Phi} & B
\end{array}
\]

the square

\[
\begin{array}{ccc}
x(E, M) K & \xrightarrow{x(\Psi_{E}^{E})K} & x(E, M) K \\
x(E, \Phi)K & \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \\
(x : P) (E, N)(Q \circ K) & \xrightarrow{(x : P)(\Psi_{N}^{E})(Q \circ K)} & (x : P) ((E), N)(Q \circ K)
\end{array}
\]

, i.e.

\[
\begin{array}{ccc}
\prod_{E} x(M) K & \xrightarrow{\Phi_{x(M)K}^{E}} & ((E)) (E^- \times E) (x(M) K) \\
\prod_{E} x(\Phi)K & \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \\
\prod_{E} x(P(N) Q) K & \xrightarrow{\Phi_{x(P(N)Q)K}^{E}} & ((E)) (E^- \times E) (x(P(N) Q) K)
\end{array}
\]

commutes by the naturality of \(\Phi_{M}^{E}\). \(\square\)
6. Universals

6.1. Units of one-sided modules

Note. The Yoneda morphism for a right (resp. left) module (see Remark 5.2.6(2)) allows the following definition.

Definition 6.1.1.

\(\text{An arrow } u : r \to * \text{ of a right module } M : X \to * \text{ is called a unit}^{1} \text{ if the right module morphism } \)
\(X \uparrow u : (X) r \to M : X \to * \text{ is iso.} \)

\(\text{An arrow } u : * \to r \text{ of a left module } M : * \to A \text{ is called a unit if the left module morphism } \)
\(u \uparrow A : r (A) \to M : * \to A \text{ is iso.} \)

Remark 6.1.2.

(1) By the Yoneda lemma (Theorem 5.2.10), representations and units correspond one-to-one:

\(\text{if } u : r \to * \text{ is a unit of a right module } M : X \to *, \text{ then the object } r \text{ represents } M \text{ by the isomorphism } \)
\(X \uparrow u : (X) r \to M; \text{ conversely, if an object } r \in \|X\| \text{ and a right module isomorphism } \)
\(\Upsilon : (X) r \to M \text{ form a representation of } M, \text{ then the } M\text{-arrow } 1_r : \Upsilon : r \to * \text{ is a unit of } M. \)

\(\text{if } u : * \to r \text{ is a unit of a left module } M : * \to A, \text{ then the object } r \text{ represents } M \text{ by the isomorphism } u \uparrow A : r (A) \to M; \text{ conversely, if an object } r \in \|A\| \text{ and a left module isomorphism } \)
\(\Upsilon : r (A) \to M \text{ form a representation of } M, \text{ then the } M\text{-arrow } \Upsilon : 1_r : * \to r \text{ is a unit of } M. \)

(2) \(\text{Let } u : r \to * \text{ be a unit of a right module } M : X \to *. \text{ Given an } M\text{-arrow } m : x \to *, \text{ its inverse }\)
\(\text{image under } X \uparrow u \text{ is called the adjunct of } m \text{ along } u \text{ and written } m/u; \text{ that is, }\)
\(m/u := m : (X \uparrow u)^{-1} \)

; this is the unique \(X\)-arrow \(x \to r \) making the triangle

\[
\begin{array}{c}
\text{r} \\
\downarrow m \\
\text{x}
\end{array}
\begin{array}{c}
\uparrow u \\
\downarrow m
\end{array}
\begin{array}{c}
* \\
\text{m}
\end{array}
\]

\(\text{commute. An } M\text{-arrow } u : r \to * \text{ is a unit if and only if to every } M\text{-arrow } m : x \to * \text{ there is a unique } \)
\(X\text{-arrow } m/u : x \to r \text{ as above.} \)

\(\text{Let } u : * \to r \text{ be a unit of a left module } M : * \to A. \text{ Given an } M\text{-arrow } m : * \to a, \text{ its inverse }\)
\(\text{image under } u \uparrow A \text{ is called the adjunct of } m \text{ along } u \text{ and written } u\backslash m; \text{ that is, }\)
\(u\backslash m := (u \uparrow A)^{-1} : m\)

; this is the unique \(A\)-arrow \(r \to a \) making the triangle

\[
\begin{array}{c}
* \\
\downarrow m \\
\text{a}
\end{array}
\begin{array}{c}
\uparrow u \\
\downarrow m
\end{array}
\begin{array}{c}
\text{r} \\
\text{u\backslash m}
\end{array}
\]

\(\text{commute. An } M\text{-arrow } u : * \to r \text{ is a unit if and only if to every } M\text{-arrow } m : * \to a \text{ there is a unique } \)
\(A\text{-arrow } u\backslash m : r \to a \text{ as above.} \)

\(^{1}\text{A unit is called a universal element in the literature.} \)
Proposition 6.1.3.

- An arrow of a right module \( M \) is a unit if and only if it is a terminal object of the comma category \([M]\).
- An arrow of a left module \( M \) is a unit if and only if it is an initial object of the comma category \([M]\).

Proof. Immediate by the last sentence of Remark 6.1.2(2).

Proposition 6.1.4.

- A right module \( M \) is representable if and only if the comma category \([M]\) has a terminal object.
- A left module \( M \) is representable if and only if the comma category \([M]\) has an initial object.

Proof. Since \( M \) is representable iff it has a unit (see Remark 6.1.2(1)), this is immediate from Proposition 6.1.3.

Theorem 6.1.5.

- Suppose that a right module \( M : X \to * \) has a unit \( u : r \to * \). Then an \( X \)-arrow \( f : s \to r \) is iso if and only if the composite \( f \circ u : s \to * \) is a unit of \( M \); to put it the other way round, an \( M \)-arrow \( v : s \to * \) is a unit if and only if its adjunct \( v/u : s \to r \) along \( u \) is an iso \( X \)-arrow.
- Suppose that a left module \( M : * \to A \) has a unit \( u : * \to r \). Then an \( A \)-arrow \( f : r \to s \) is iso if and only if the composite \( u \circ f : * \to s \) is a unit of \( M \); to put it the other way round, an \( M \)-arrow \( v : * \to s \) is a unit if and only if its adjunct \( u/v : r \to s \) along \( u \) is an iso \( A \)-arrow.

Proof. By the naturality of the Yoneda morphism

\[
X \uparrow (f \circ u) = \langle X \rangle \circ f \circ X \uparrow u
\]

Since \( u \) is a unit, \( X \uparrow u \) is iso. Hence \( X \uparrow (f \circ u) \) is iso iff \( \langle X \rangle \circ f \) is iso. But \( X \uparrow (f \circ u) \) is iso iff \( f \circ u \) is a unit, and, since the Yoneda functor is fully faithful, \( \langle X \rangle \circ f \) is iso iff \( f \) is iso.

Theorem 6.1.6.

- Let \( M : X \to * \) be a right module. If \( u : r \to * \) and \( v : s \to * \) are two units, then the adjunct \( v/u : s \to r \) of \( v \) along \( u \) and the adjunct \( u/v : r \to s \) of \( u \) along \( v \), as shown in

\[
\begin{array}{c}
S \\
\downarrow v/u \\
\uparrow u/v \\
* \\
\downarrow \\_ \\
\_ \\
\end{array}
\]

are the inverse of each other.

- Let \( M : * \to A \) be a left module. If \( u : * \to r \) and \( v : * \to s \) are two units, then the adjunct \( u/v : r \to s \) of \( u \) along \( v \) and the adjunct \( v/u : s \to r \) of \( u \) along \( v \), as shown in

\[
\begin{array}{c}
* \\
\downarrow v/u \\
\uparrow u/v \\
S \\
\end{array}
\]

are the inverse of each other.

Proof. Since

\[
u/v \circ v/u \circ u = u/v \circ v = u
\]

and \( u \) is a unit, \( u/v \circ v/u = 1_r \) by the uniqueness of the factorization. Symmetrically, \( v/u \circ v/u = 1_s \).

Corollary 6.1.7. If a right (resp. left) module is representable, then a representing object is unique up to isomorphism.

Proof. Since a representing object has a unit associated with it (see Remark 6.1.2(1)), this is immediate from Theorem 6.1.6 (or Theorem 6.1.5).
Definition 6.1.8.

- A right module cell $\xymatrix{ X \ar[r]^-M & \ast }$ is said to $\xymatrix{ P \ar[r]^-\Phi & Y }$ to $\ast$.

1. preserve units if $\Phi$ sends each unit $u : r \leadsto \ast$ of $M$ to a unit $u : \Phi : r \leadsto \ast$ of $N$;
2. reflect units if an $M$-arrow $u : r \leadsto \ast$ is a unit whenever the $N$-arrow $u : \Phi : r \leadsto \ast$ is a unit;
3. create units if for every unit $v : s \leadsto \ast$ of $N$ there is exactly one $M$-arrow $u : r \leadsto \ast$ with $u : \Phi = v$.

- A left module cell $\xymatrix{ \ast \ar[r]^-M & A }$ is said to $\xymatrix{ P \ar[r]^-\Phi & Q }$ to $\ast$.

1. preserve units if $\Phi$ sends each unit $u : \ast \leadsto r$ of $M$ to a unit $u : \Phi : \ast \leadsto \ast$ of $N$;
2. reflect units if an $M$-arrow $u : \ast \leadsto r$ is a unit whenever the $N$-arrow $u : \Phi : \ast \leadsto \ast$ is a unit;
3. create units if for every unit $v : \ast \leadsto s$ of $N$ there is exactly one $M$-arrow $u : \ast \leadsto r$ with $u : \Phi = v$.

Proposition 6.1.9. If a right (resp. left) module cell creates units, then it reflects units.

Proof. Obvious by the definitions.

Proposition 6.1.10. If a right (resp. left) module cell preserves a unit, then it preserves every unit; that is,

- if a right module cell $\Phi : P \leadsto \ast : M \to N$ sends a unit $u : r \leadsto \ast$ of $M$ to a unit $u : \Phi : r \leadsto \ast$ of $N$, then $\Phi$ sends any other unit $v : s \leadsto \ast$ of $M$ to a unit $v : \Phi : s \leadsto \ast$ of $N$ as well.
- if a left module cell $\Phi : \ast \leadsto Q : M \to N$ sends a unit $u : \ast \leadsto r$ of $M$ to a unit $u : \Phi : \ast \leadsto \ast$ of $N$, then $\Phi$ sends any other unit $v : \ast \leadsto s$ of $M$ to a unit $v : \Phi : \ast \leadsto \ast$ of $N$ as well.

Proof. If $u$ is a unit of $M$, another unit $v$ of $M$ is written as $v = v/u \cdot u$, with $v/u$ an isomorphism by Theorem 6.1.5. $\Phi$ thus sends $v$ to the $N$-arrow $v : \Phi = (v/u : P) \circ (u : \Phi)$ (see Proposition 1.2.3), with $(v/u : P)$ an isomorphism (because any functor preserves isomorphisms). Hence if $u : \Phi$ is a unit of $N$, so is $v : \Phi$ again by Theorem 6.1.5.

Proposition 6.1.11. Consider a cell $\Phi$ as in Definition 6.1.8. If $N$ has a unit and $\Phi$ creates units, then $M$ has a unit as well and $\Phi$ preserves units.

Proof. Clearly $M$ has a unit. Since $\Phi$ preserves the units it has created, it preserves every unit by Proposition 6.1.10.

Proposition 6.1.12. If a right (resp. left) module cell is iso, then it preserves, reflects, and creates units.

Proof. Evident.

6.2. Universal arrows

Note. The right (resp. left) Yoneda morphism (see Definition 5.2.5) allows the following definition.

Definition 6.2.1. Let $\mathcal{M} : X \to A$ be a module.

- An $\mathcal{M}$-arrow $u : r \leadsto a$ is inverse universal if the right module morphism $X \downarrow u : (X) r \to (\mathcal{M}) a : X \to \ast$ is iso.
- An $\mathcal{M}$-arrow $u : x \leadsto r$ is direct universal if the left module morphism $u \downarrow A : r (A) \to x (\mathcal{M}) : \ast \to A$ is iso.
Remark 6.2.2.

(1) An \( \mathcal{M} \)-arrow \( u : x \to r \) is direct universal if and only if the \( \mathcal{M} \)-arrow \( u : r \to x \) is inverse universal.

(2) By Remark 5.2.6(2),
- an \( \mathcal{M} \)-arrow \( u : r \to a \) is inverse universal if and only if it is a unit of the right module \( \langle \mathcal{M} \rangle a : X \to \ast \), and a unit \( u : r \to \ast \) of a right module \( \mathcal{M} : X \to \ast \) is the same thing as an inverse universal arrow of \( \mathcal{M} \) regarded as a two-sided module from \( X \) to the terminal category.
- an \( \mathcal{M} \)-arrow \( u : x \to r \) is direct universal if and only if it is a unit of the left module \( x \langle \mathcal{M} \rangle : \ast \to A \), and a unit \( u : \ast \to r \) of a left module \( \mathcal{M} : \ast \to A \) is the same thing as a direct universal arrow of \( \mathcal{M} \) regarded as a two-sided module from the terminal category to \( A \).

(3) Remark 6.1.2(2) is repeated below in terms of universal arrows.
- Let \( u : r \to a \) be an inverse universal \( \mathcal{M} \)-arrow. Given an \( \mathcal{M} \)-arrow \( m : x \to a \), its inverse image under \( X \upharpoonright u \) is called the adjunct of \( m \) along \( u \) and written \( m \upharpoonright u \); that is,

\[
m \upharpoonright u := m \uparrow (X \upharpoonright u)^{-1}
\]

; this is the unique \( X \)-arrow \( x \to r \) making the triangle

\[
\begin{array}{c}
r \\
\downarrow^m \\
X \\
\end{array} \quad \begin{array}{c}
a \\
\downarrow^u \\
m \upharpoonright u
\end{array}
\]

commute. An \( \mathcal{M} \)-arrow \( u : r \to a \) is inverse universal if and only if to every \( \mathcal{M} \)-arrow \( m : x \to a \) there is a unique \( X \)-arrow \( m \upharpoonright u : x \to r \) as above.

- Let \( u : x \to r \) be a direct universal \( \mathcal{M} \)-arrow. Given an \( \mathcal{M} \)-arrow \( m : x \to a \), its inverse image under \( u \upharpoonright A \) is called the adjunct of \( m \) along \( u \) and written \( u \downarrow m \); that is,

\[
u \downarrow m := m \downarrow (u \upharpoonright A)^{-1}
\]

; this is the unique \( A \)-arrow \( r \to a \) making the triangle

\[
\begin{array}{c}
x \\
\downarrow^m \\
\end{array} \quad \begin{array}{c}
r \\
\downarrow^u \\
u \downarrow m
\end{array}
\]

commute. An \( \mathcal{M} \)-arrow \( u : x \to r \) is direct universal if and only if to every \( \mathcal{M} \)-arrow \( m : x \to a \) there is a unique \( A \)-arrow \( u \downarrow m : r \to a \) as above.

(4) An \( \mathcal{M} \)-arrow \( u : r \to s \) is called two-way universal if it is both inverse and direct universal.

Example 6.2.3. For any functor \( F : D \to E \), consider its representable module \( F(\mathcal{E}) : D \to E \) and corepresentable modules \( (\mathcal{E})F : E \to D \) (see Definition 2.3.5).

- An \( F(\mathcal{E}) \)-arrow \( u : r \to e \) (i.e. \( E \)-arrow \( u : r : F \to e \)) is inverse universal if the right module morphism \( D \upharpoonright u : (D) r \to F(\mathcal{E}) e : D \to \ast \) (see Example 5.2.9(2)) is iso; that is, if to every \( F(\mathcal{E}) \)-arrow \( f : d \to e \) (i.e. \( E \)-arrow \( f : d : F \to e \)), there is a unique \( D \)-arrow \( f \uparrow u : d \to r \), the adjunct of \( f \) along \( u \), such that the triangle

\[
\begin{array}{c}
r \\
\downarrow^f \\
f \uparrow u
\end{array} \quad \begin{array}{c}
e \\
\downarrow^e \\
\end{array}
\]

commutes. An \( E \)-arrow \( u : r : F \to e \) is said to be universal from \( F \) to \( e \) if the \( F(\mathcal{E}) \)-arrow \( u : r \to e \) is inverse universal.
6.2. Universal arrows

- An \((E)\) \(F\)-arrow \(u : e \sim r\) (i.e. \(E\)-arrow \(u : e \to F : r\)) is direct universal if the left module morphism \(u \| D : r(D) \to e(E) F : * \to D\) (see Example 5.2.9(2)) is iso; that is, if to every \((E)\) \(F\)-arrow \(f : e \sim d\) (i.e. \(E\)-arrow \(f : e \to F : d\)), there is a unique \(D\)-arrow \(u f : r \to d\), the adjunct of \(f\) along \(u\), such that the triangle

\[
\begin{array}{c}
e \\
\downarrow f \\
F : d \\
\downarrow u f \\
r \\
\end{array}
\]

commutes. An \(E\)-arrow \(u : e \to F : r\) is said to be universal from \(e\) to \(F\) if the \((E)\) \(F\)-arrow \(u : e \sim r\) is direct universal.

**Remark 6.2.4.** The module \(F(E) : D \to E\) and \((E) F : E \to D\) abstract the commutative diagrams above into

\[
\begin{array}{ccc}
r & u & e \\
\downarrow f u & \downarrow f & \downarrow u f \\
d & f & d \\
\end{array}
\]

and present a simpler and more conceptual view of universal arrows.

**Proposition 6.2.5.** For any arrow \(f : r \to s\) of a category \(C\), the following conditions are equivalent:

1. \(C\)-arrow \(f : r \to s\) is iso;
2. \((C)\)-arrow \(f : r \sim s\) is inverse universal;
3. \((C)\)-arrow \(f : r \sim s\) is direct universal;
4. \((C)\)-arrow \(f : r \sim s\) is two-way universal,

where \((C)\) is the hom of \(C\).

**Proof.** By definition, a \((C)\)-arrow \(f : r \sim s\) is inverse universal iff the right module morphism \(C f : (C)r \to (C)s\) is iso. By Proposition 5.2.8, \(C f = (C) f\), and by the fully faithfulness of the Yoneda functor, \((C) f\) is iso iff \(f\) is iso. The conditions (1) and (2) are thus equivalent. The equivalence of the conditions (1) and (3) is proved dually, and the equivalence of the conditions (1) and (4) follows.

**Note.** Theorem 6.2.6 and Theorem 6.2.7 are restatements of Theorem 6.1.5 and Theorem 6.1.6 in terms of universal arrows.

**Theorem 6.2.6.** Let \(M : X \to A\) be a module.

- Let \(u : r \sim a\) be an inverse universal \(M\)-arrow. Then an \(X\)-arrow \(f : s \to r\) is iso if and only if the composite \(f \circ u : s \sim a\) is an inverse universal \(M\)-arrow; to put it the other way round, an \(M\)-arrow \(v : s \sim a\) is universal if and only if its adjunct \(v u\) along \(u\) is an iso \(X\)-arrow.

- Let \(u : x \sim r\) be a direct universal \(M\)-arrow. Then an \(A\)-arrow \(f : r \to s\) is iso if and only if the composite \(u \circ f : x \sim s\) is a direct universal \(M\)-arrow; to put it the other way round, an \(M\)-arrow \(v : x \sim s\) is direct universal if and only if its adjunct \(u v\) along \(u\) is an iso \(A\)-arrow.

**Proof.** Since an inverse universal \(M\)-arrow \(u : r \sim a\) (resp. \(v : s \sim a\)) is the same thing as a unit of the right module \((M) a : X \to *\) (see Remark 6.2.2(2)), the assertion follows from Theorem 6.1.5.

**Theorem 6.2.7.** Let \(M : X \to A\) be a module.

- If \(u : r \sim a\) and \(v : s \sim a\) are two inverse universal \(M\)-arrows, then the adjunct \(v u : s \to r\) of \(v\) along \(u\) and the adjunct \(u v : r \to s\) of \(u\) along \(v\), as shown in

\[
\begin{array}{c}
s \\
\downarrow v \\
\downarrow u v \\
a \\
\end{array}
\]

are the inverse of each other.
Proof. Since an inverse universal $\mathcal{M}$-arrow $u : r \to a$ (resp. $v : s \to a$) is the same thing as a unit of the right module $\langle \mathcal{M} \rangle a : X \to \ast$ (see Remark 6.2.2(2)), the assertion follows from Theorem 6.1.6.

Corollary 6.2.8. Let $\mathcal{M} : X \to A$ be a module.

- For any object $a \in \|A\|$, inverse universal $\mathcal{M}$-arrows to $a$, if exist, are unique up to isomorphism; that is, if $u : r \to a$ and $v : s \to a$ are two inverse universal $\mathcal{M}$-arrows, then $r$ and $s$ are isomorphic.

- For any object $x \in \|X\|$, direct universal $\mathcal{M}$-arrows from $x$, if exist, are unique up to isomorphism; that is, if $u : x \to r$ and $v : x \to s$ are two direct universal $\mathcal{M}$-arrows, then $r$ and $s$ are isomorphic.

Proof. Immediate from Theorem 6.2.7 (or Theorem 6.2.6).

Theorem 6.2.9. Let $F : D \to E$ be a functor and $e$ be an object of $E$.

- If there is an iso $E$-arrow $u : r : F \to e$ universal from $F$ to $e$, then every $E$-arrow $v : s : F \to e$ universal from $F$ to $e$ is iso.

- If there is an iso $E$-arrow $u : e \to F : r$ universal from $e$ to $F$, then every $E$-arrow $v : s : e \to F : s$ universal from $e$ to $F$ is iso.

Proof. Consider the commutative diagram

```
\begin{align*}
\begin{array}{c}
\mathcal{F} \quad r \\
\downarrow v/u \\
\mathcal{S} \\
\end{array} & \quad \begin{array}{c}
\mathcal{F} \quad r : F \\
\downarrow (v/u) : F \\
\mathcal{S} : F \\
\end{array} \\
\end{align*}
```

We need to show that if $u$ is iso, so is $v$. For this, it suffices to show that $(v/u) : F$ is iso. But this holds because $v/u$ is iso by Theorem 6.2.6 and any functor preserves isomorphisms.

Theorem 6.2.10. Let $\mathcal{M} : X \to A$ be a module.

- Let $f : a \to b$ be an $A$-arrow such that the right module morphism $\langle \mathcal{M} \rangle f : \langle \mathcal{M} \rangle a \to \langle \mathcal{M} \rangle b : X \to \ast$ is iso. Then an $\mathcal{M}$-arrow $u : r \to a$ is inverse universal if and only if so is the composite $\mathcal{M}$-arrow $u \circ f : r \to b$.

- Let $f : y \to x$ be an $X$-arrow such that the left module morphism $f(\mathcal{M}) : x(\mathcal{M}) \to y(\mathcal{M}) : \ast \to A$ is iso. Then an $\mathcal{M}$-arrow $u : x \to r$ is direct universal if and only if so is the composite $\mathcal{M}$-arrow $f \circ u : y \to r$.

Proof. By the naturality of the Yoneda morphism (see Definition 5.2.5), we have

```
X(\mathcal{M}) f = X u \circ (\mathcal{M}) f
```

Since $\langle \mathcal{M} \rangle f$ is iso by assumption, $X(\mathcal{M}) f$ is iso (i.e. $u \circ f$ is inverse universal) iff $X u$ is iso (i.e. $u$ is inverse universal).

Remark 6.2.11. The pullback lemma follows from Theorem 6.2.10. Consider a commutative diagram

```
\begin{align*}
x & \xrightarrow{f} a' \\
\downarrow h & \downarrow h' \\
c & \xrightarrow{g} b' \\
\end{align*}
```
with the right square a pullback; the pullback lemma says that the left square is a pullback if and only if the outer rectangle is a pullback. To see this, first note (Definition 7.3.1) that a pullback in a category $C$ is the same thing as an inverse universal arrow of the module $(\ast E, C) : C \rightarrow [E, C]$ with $E$ be a category which looks like $0 \rightarrow 1 \leftarrow 2$. Now transform the commutative diagram into

$$
\begin{array}{ccc}
a & \xrightarrow{f} & a' \\
\downarrow{h} & & \downarrow{h'} \\
b & \xrightarrow{g} & b'
\end{array}
\begin{array}{ccc}
\downarrow{k} & & \downarrow{k \circ g} \\
c & \xrightarrow{e} & c
\end{array}
$$

, and observe that the triple $\tau := (f, g, c)$ forms a natural transformation from the cospan $S := (a \xrightarrow{h} b \xleftarrow{k} c)$ to the cospan $T := (a' \xrightarrow{h'} b' \xleftarrow{k \circ g} c)$, and that the composition $- \circ \tau$ maps the left square in the pullback lemma to the outer rectangle of it. This observation allows us to treat the pullback lemma as an instance of Theorem 6.2.10: it suffices to prove that the assignment $\alpha \mapsto \alpha \circ \tau$ is bijective. To see this bijectivity, let $\alpha : x \rightarrow T$ be a cone, i.e. a commutative square

$$
\begin{array}{ccc}
x & \xrightarrow{\alpha_0} & a' \\
\downarrow{\alpha_2} & & \downarrow{h'} \\
c & \xrightarrow{k} & b' \\
\end{array}
\begin{array}{ccc}
\end{array}
$$

. Then the unique arrow $\alpha'_0 : x \rightarrow a$ making the diagram

$$
\begin{array}{ccc}
x & \xrightarrow{\alpha_0} & a' \\
\downarrow{\alpha_2} & & \downarrow{h'} \\
c & \xrightarrow{k} & b' \\
\end{array}
\begin{array}{ccc}
\end{array}
$$

commute (the right square is a pullback by assumption) forms with $\alpha_2$ the unique cone $\alpha' : x \rightarrow S$ such that $\alpha = \alpha' \circ \tau$.

**Corollary 6.2.12.** Let $\mathcal{M} : X \rightarrow A$ be a module.
- Let $f : a \rightarrow b$ be an isomorphism in $A$. Then an $\mathcal{M}$-arrow $u : r \rightarrow a$ is inverse universal if and only if so is the composite $\mathcal{M}$-arrow $u \circ f : r \rightarrow b$.
- Let $f : y \rightarrow x$ be an isomorphism in $X$. Then an $\mathcal{M}$-arrow $u : x \rightarrow r$ is direct universal if and only if so is the composite $\mathcal{M}$-arrow $f \circ u : y \rightarrow r$.

**Proof.** Since (like any other functor) the functor $[\mathcal{M} \times] : A \rightarrow [X]$ preserves isomorphisms, if $f : a \rightarrow b$ is an isomorphism, so is the right module morphism $(\mathcal{M}) f : (\mathcal{M}) s \rightarrow (\mathcal{M}) t$. The assertion thus follows from Theorem 6.2.10.

**Theorem 6.2.13.** Let $\mathcal{M} : X \rightarrow A$ be a module and consider a commutative square

$$
\begin{array}{ccc}
x & \xrightarrow{\mu} & a \\
\downarrow{g} & & \downarrow{f} \\
y & \xrightarrow{\nu} & b
\end{array}
$$

consisting of $\mathcal{M}$-arrows $\mu$ and $\nu$, an iso $X$-arrow $g$, and an iso $A$-arrow $f$. In this square, if $\mu$ is inverse (resp. direct) universal, so is $\nu$.

**Proof.** Suppose that $\mu$ is inverse universal. By Corollary 6.2.12, $u \circ f$ is inverse inverse universal. Hence, by Theorem 6.2.6, $\nu = g^{-1} \circ u \circ f$ is inverse universal. 

\[\square\]
Theorem 6.2.14.  
- Consider a composable pair of a module and a functor

\[ X \xrightarrow{\mathcal{M}} A \xrightarrow{K} E \]

. An $\mathcal{M}$-arrow $u: r \rightsquigarrow K \cdot e$ is inverse universal if and only if so is the $(\mathcal{M}) K$-arrow $u: r \rightsquigarrow e$.

- Consider a composable pair of a module and a functor

\[ E \xrightarrow{K} X \xrightarrow{\mathcal{M}} A \]

. An $\mathcal{M}$-arrow $u: e \rightsquigarrow r$ is direct universal if and only if so is the $K(\mathcal{M})$-arrow $u: e \rightsquigarrow r$.

Proof. $u: r \rightsquigarrow e$ is an inverse universal $(\mathcal{M}) K$-arrow iff $u: r \rightsquigarrow \ast$ is a unit of $(\mathcal{M}) K \cdot e$, and $u: r \rightsquigarrow K \cdot e$ is an inverse universal $\mathcal{M}$-arrow iff $u: r \rightsquigarrow \ast$ is a unit of $(\mathcal{M}) (K \cdot e)$. But $(\mathcal{M}) K \cdot e = (\mathcal{M})(K \cdot e)$. 

Note. The following is a special case of Theorem 6.2.14 where $\mathcal{M}$ is given by the hom of a category.

Corollary 6.2.15. Let $F: D \to E$ be a functor.

- An $E$-arrow $u: r \to F \cdot d$ is iso if and only if the $(E)$ $F$-arrow $u: r \rightsquigarrow d$ is inverse universal.

- An $E$-arrow $u: d \cdot F \to r$ is iso if and only if the $F(E)$-arrow $u: d \rightsquigarrow r$ is direct universal.

Proof. By Proposition 6.2.5, $u: r \rightsquigarrow e$ is iso if it is an inverse universal $(E)$-arrow. The assertion thus follows from Theorem 6.2.14.

Theorem 6.2.16.

- Consider a composable pair of a functor and a module

\[ E \xrightarrow{K} X \xrightarrow{\mathcal{M}} A \]

with $K$ fully faithful. If an $\mathcal{M}$-arrow $u: r \rightsquigarrow a$ is inverse universal, so is the $K(\mathcal{M})$-arrow $u: r \rightsquigarrow a$.

- Consider a composable pair of a functor and a module

\[ X \xrightarrow{\mathcal{M}} A \xrightarrow{K} E \]

with $K$ fully faithful. If an $\mathcal{M}$-arrow $u: x \rightsquigarrow r$ is direct universal, so is the $(\mathcal{M})K$-arrow $u: x \rightsquigarrow r$.

Proof. By Example 5.2.9(1), $E|u$ is given by the composition $(K)r \circ K(X|u)$. Since $K$ is fully faithful, $(K)r$ is iso; since $u: r \cdot K \rightsquigarrow a$ is an inverse universal $\mathcal{M}$-arrow, $X|u$ is iso, and hence so is $K(X|u)$. $E|u$ is thus iso.

Corollary 6.2.17. As a special case of Theorem 6.2.16, for a fully faithful functor $F: D \to E$, consider its representable module $F(E): D \to E$ and corepresentable modules $(E)F: E \to D$.

- If an $E$-arrow $u: r \cdot F \to e$ is iso, then it is universal from $F$ to $e$, i.e. the $(E)F$-arrow $u: r \rightsquigarrow e$ is inverse universal.

- If an $E$-arrow $u: e \cdot F \to r$ is iso, then it is universal from $e$ to $F$, i.e. the $(E)F$-arrow $u: e \rightsquigarrow r$ is direct universal.

Proof. By Proposition 6.2.5, $u: r \cdot F \rightsquigarrow e$ is an inverse universal $(E)$-arrow. Hence, by Theorem 6.2.16, $u: r \rightsquigarrow e$ is an inverse universal $F(\mathcal{M})$-arrow.

Theorem 6.2.18. For a pair of fully faithful functors $E \xrightarrow{S} C \xrightarrow{T} D$, consider the composite module $S(C)T: E \to D$ (see Example 1.1.29(10)). If a $C$-arrow $u: r \cdot S \to T \cdot s$ is iso, then the $S(C)T$-arrow $u: r \rightsquigarrow s$ is two-way universal.
Proof. By Corollary 6.2.17, the \(S(C)\)-arrow \(u : r \sim s\) (resp. \(C\) \(T\)-arrow \(u : r : S \sim s\)) is inverse (resp. direct) universal; hence, by Theorem 6.2.14, the \(S(C)\) \(T\)-arrow \(u : r \sim s\) is inverse (resp. direct) universal.

Definition 6.2.19. A cell \(\begin{array}{c} X \\ P \end{array} \xrightarrow{M} \begin{array}{c} A \\ F \end{array} \begin{array}{c} Y \\ Q \end{array} \xrightarrow{N} \begin{array}{c} B \\ L \end{array} \) is said to

(1) preserve
- inverse universal arrows if \(\Phi\) sends each inverse universal \(M\)-arrow \(u : r \sim a\) to an inverse universal \(N\)-arrow \(u : \Phi : r : P \sim Q : a\).
- direct universal arrows if \(\Phi\) sends each direct universal \(M\)-arrow \(u : x \sim r\) to a direct universal \(N\)-arrow \(u : \Phi : x : P \sim Q : r\).

(2) reflect
- inverse universal arrows if an \(M\)-arrow \(u : r \sim a\) is inverse universal whenever the \(N\)-arrow \(u : \Phi : r : P \sim Q : a\) is inverse universal.
- direct universal arrows if an \(M\)-arrow \(u : x \sim r\) is direct universal whenever the \(N\)-arrow \(u : \Phi : x : P \sim Q : r\) is direct universal.

(3) create
- inverse universal arrows if for every object \(a \in \|A\|\) and for every inverse universal \(N\)-arrow \(v : s \sim Q : a\) there is exactly one \(M\)-arrow \(u : r \sim a\) with \(u : \Phi = v\), and if this \(u\) is inverse universal.
- direct universal arrows if for every object \(x \in \|X\|\) and for every direct universal \(N\)-arrow \(v : x : P \sim s\) there is exactly one \(M\)-arrow \(u : x \sim r\) with \(u : \Phi = v\), and if this \(u\) is direct universal.

Remark 6.2.20. By Remark 6.2.2(2),
- \(\Phi\) preserves (reflects, creates) inverse universal arrows if and only if each right slice (see Definition 2.1.8) preserves (reflects, creates) units in the sense of Definition 6.1.8.
- \(\Phi\) preserves (reflects, creates) direct universal arrows if and only if each left slice (see Definition 2.1.8) preserves (reflects, creates) units in the sense of Definition 6.1.8.

Proposition 6.2.21. If a cell creates inverse (resp. direct) universal arrows, then it reflects inverse (resp. direct) universal arrows.

Proof. Obvious by the definitions.

Remark 6.2.22. See also Theorem 6.4.12.

Proposition 6.2.23. Consider a cell \(\Phi\) as in Definition 6.2.19.
- If \(\Phi\) is fully faithful and \(P\) is iso, then \(\Phi\) preserves, reflects, and creates inverse universal arrows.
- If \(\Phi\) is fully faithful and \(Q\) is iso, then \(\Phi\) preserves, reflects, and creates direct universal arrows.

Proof. If \(\Phi\) is fully faithful and \(P\) is iso, then each right slice of \(\Phi\) is iso by Proposition 2.1.9. The assertion thus follows from Proposition 6.1.12 by noting Remark 6.2.20.

Proposition 6.2.24.
- Given a cell and a commutative square of functors as in

\[
\begin{array}{c|c|c|c}
X & A & D \\
\hline
P & F & \Phi & Q \\
\hline
Y & B & L & E \\
\end{array}
\]
6.3. Conjugation

, if the cell $\Phi$ preserves (reflects, creates) inverse universal arrows, so does the composite

\[
\begin{array}{ccc}
X & \xrightarrow{\{M\}K} & D \\
\downarrow P & & \downarrow F \\
Y & \xrightarrow{\{N\}L} & E
\end{array}
\]

Given a cell and a commutative square of functors as in

\[
\begin{array}{ccc}
D & \xrightarrow{K} & X & \xrightarrow{M} & A \\
\downarrow F & & \downarrow \Phi & & \downarrow Q \\
E & \xrightarrow{L} & Y & \xrightarrow{N} & B
\end{array}
\]

, if the cell $\Phi$ preserves (reflects, creates) direct universal arrows, so does the composite

\[
\begin{array}{ccc}
D & \xrightarrow{K\{M\}} & A \\
\downarrow F & & \downarrow Q \\
E & \xrightarrow{L\{N\}} & B
\end{array}
\]

Proof. Suppose that $\Phi$ preserves (resp. reflects) inverse universal arrows. By Theorem 6.2.14, we see that the cell $\langle \Phi \rangle K$ also preserves (resp. reflects) inverse universal arrows. Now suppose that $\Phi$ creates inverse universal arrows. To see that $\langle \Phi \rangle K$ also creates inverse universal arrows, let $d : e \Rightarrow d$ and suppose that an $\langle N \rangle$-arrow $v : s \Rightarrow F \cdot d$ is inverse universal. Then, by Theorem 6.2.14, $\langle N \rangle$-arrow $v : s \Rightarrow F \cdot d = Q \cdot K \cdot d$ is inverse universal. Hence there is exactly one $\langle M \rangle$-arrow $u : r \Rightarrow K \cdot d$ whose image under $\Phi$ is the $\langle N \rangle$-arrow $v : s \Rightarrow Q \cdot K \cdot d$; that is, there is exactly one $\langle M \rangle$-arrow $u : r \Rightarrow d$ whose image under $\langle \Phi \rangle K$ is the $\langle N \rangle$-arrow $v : s \Rightarrow F \cdot d$. Since the $\langle M \rangle$-arrow $u : r \Rightarrow K \cdot d$ is inverse universal, again by Theorem 6.2.14, so is the $\langle M \rangle$-arrow $u : r \Rightarrow d$. \qed

**Theorem 6.2.25.** A right (resp. left) Yoneda morphism preserves and reflects inverse (resp. direct) universal.

Proof. The right Yoneda morphism for a module $\mathcal{M} : X \to A$ (see Definition 5.2.5) sends each $\mathcal{M}$-arrow $u : r \Rightarrow a$ to the right module $X \upharpoonright u : \langle X \rangle r \to \langle \mathcal{M} \rangle a : X \to \ast$. But by definition, $u : r \Rightarrow a$ is inverse universal iff $X \upharpoonright u$ is iso, and by Proposition 6.2.5, $X \upharpoonright u$ is iso iff $X \upharpoonright u$ is inverse universal. \qed

6.3. Conjugation

**Definition 6.3.1.** Let $\mathcal{M} : X \to A$ be a module.

- Given a pair of inverse universal $\mathcal{M}$-arrows $u : r \Rightarrow a$ and $v : s \Rightarrow b$, the conjugate of an $A$-arrow $f : a \to b$ inverse along $(u, v)$ is the $X$-arrow given by $(u \circ f) / v$, the adjunct of the composite $u \circ f$ along $v$, i.e. the unique $X$-arrow $g : r \Rightarrow s$ making the square

\[
\begin{array}{ccc}
r & \xrightarrow{u} & a \\
g & \downarrow & \downarrow f \\
s & \xrightarrow{v} & b
\end{array}
\]

commute. This commutative square, or the assignment $f \mapsto g$, is called an inverse conjugation by $(u, v)$. 
Given a pair of direct universal $\mathcal{M}$-arrows $u : x \to r$ and $v : y \to s$, the conjugate of an $X$-arrow $g : x \to y$ direct along $(u, v)$ is the $A$-arrow given by $u \downarrow (g \cdot v)$, the adjunct of the composite $g \cdot v$ along $u$, i.e. the unique $A$-arrow $f : r \to s$ making the square

\[
\begin{array}{ccc}
x & \xrightarrow{u} & r \\
g & \downarrow & f \\
y & \xrightarrow{v} & s
\end{array}
\]

commute. This commutative square, or the assignment $g \mapsto f$, is called a direct conjugation by $(u, v)$.

**Remark 6.3.2.** Let $\mathcal{M} : X \to A$ be a module.

- The category $U_1(\mathcal{M})$ of inverse universal arrows of $\mathcal{M}$ is given by the full subcategory of the comma category $[\mathcal{M}]$ consisting of all inverse universal arrows; given inverse universal $\mathcal{M}$-arrows $u : r \to a$ and $v : s \to b$, an arrow $u \to v$ in $U_1(\mathcal{M})$ is a pair of an $X$-arrow $g : r \to s$ and an $A$-arrow $f : a \to b$ forming an inverse conjugation by $(u, v)$. The restriction of the comma fibration $\mathcal{M}_1 : [\mathcal{M}] \to A$ to $U_1(\mathcal{M})$ yields a fully faithful forgetful functor $\mathcal{M}_1 : U_1(\mathcal{M}) \to A$ sending each inverse conjugation

\[
\begin{array}{ccc}
r & \xrightarrow{u} & a \\
g & \downarrow & f \\
s & \xrightarrow{v} & b
\end{array}
\]

to the $A$-arrow $f : a \to b$.

- The category $U_0(\mathcal{M})$ of direct universal arrows of $\mathcal{M}$ is given by the full subcategory of the comma category $[\mathcal{M}]$ consisting of all direct universal arrows; given direct universal $\mathcal{M}$-arrows $u : x \to r$ and $v : y \to s$, an arrow $u \to v$ in $U_0(\mathcal{M})$ is a pair of an $X$-arrow $g : x \to y$ and an $A$-arrow $f : r \to s$ forming a direct conjugation by $(u, v)$. The restriction of the comma fibration $\mathcal{M}_0 : [\mathcal{M}] \to X$ to $U_0(\mathcal{M})$ yields a fully faithful forgetful functor $\mathcal{M}_0 : U_0(\mathcal{M}) \to X$ sending each direct conjugation

\[
\begin{array}{ccc}
x & \xrightarrow{u} & r \\
g & \downarrow & f \\
y & \xrightarrow{v} & s
\end{array}
\]

to the $X$-arrow $g : x \to y$.

**Proposition 6.3.3.** Let $\mathcal{M} : X \to A$ be a module.

1) Conjugation is functorial in the following sense:

a) for any inverse (resp. direct) universal $\mathcal{M}$-arrow $u$, the identities form an inverse (resp. direct) conjugation

\[
\begin{array}{ccc}
r & \xrightarrow{u} & s \\
1 & \downarrow & 1 \\
r & \xrightarrow{u} & s
\end{array}
\]

b) if, in the diagram

\[
\begin{array}{ccc}
r & \xrightarrow{u} & s \\
g & \downarrow & f \\
r' & \xrightarrow{u'} & s' \\
g' & \downarrow & f' \\
r'' & \xrightarrow{u''} & s''
\end{array}
\]

, each of the two inner squares is an inverse (resp. direct) conjugation, so is the outer rectangle.
(2) Conjugation preserve isomorphisms; that is, given an inverse (resp. direct) conjugation

\[
\begin{array}{ccc}
  r & \sim & u \\
  g \downarrow & & \downarrow f \\
  s & \sim & v \\
  \end{array}
\]

, if \( f \) (resp. \( g \)) is an isomorphism, so is \( g \) (resp. \( f \)).

(3) Conjugation is universal in the following sense:

\[
\begin{array}{ccc}
  r & \sim & u \\
  g \downarrow & & \downarrow f \\
  s & \sim & v \\
  \end{array}
\]

and any commutative square

\[
\begin{array}{ccc}
  x & \sim & a \\
  h \downarrow & & \downarrow f \\
  y & \sim & b \\
  \end{array}
\]

, the adjunct of \( m \) along \( u \) and the adjunct of \( n \) along \( v \) yield a unique pair of \( X \)-arrows making the diagram

\[
\begin{array}{ccc}
  r & \sim & u \\
  g \downarrow & & \downarrow f \\
  s & \sim & v \\
  m/\downarrow & & \downarrow f \\
  x & \sim & r \\
  h \downarrow & & \downarrow f \\
  y & \sim & s \\
  \end{array}
\]

commute.

\[
\begin{array}{ccc}
  x & \sim & a \\
  g \downarrow & & \downarrow h \\
  y & \sim & b \\
  \end{array}
\]

, the adjunct of \( m \) along \( u \) and the adjunct of \( n \) along \( v \) yield a unique pair of \( A \)-arrows making the diagram

\[
\begin{array}{ccc}
  x & \sim & r \\
  g \downarrow & & \downarrow f \\
  y & \sim & s \\
  m/\downarrow & & \downarrow f \\
  x & \sim & m \\
  h \downarrow & & \downarrow f \\
  y & \sim & n \\
  \end{array}
\]

commute.

Proof.

(1) Evident.

(2) Any functorial operation preserves isomorphisms.

(3) The uniqueness of such a pair follows from the uniqueness of the adjunct. The proof is thus complete if we show that the square

\[
\begin{array}{ccc}
  x & \sim & r \\
  h \downarrow & & \downarrow f \\
  y & \sim & s \\
  \end{array}
\]
commutes. But since
\[ h \circ n \circ v = h \circ n = m \circ f = m / u \circ u \circ f = m / u \circ g \circ v \]
and \( v \) is inverse universal, we have
\[ h \circ n / v = m \circ u \circ g \]
by the uniqueness of the factorization.

\[ \square \]

### 6.4. Units of two-sided modules

**Definition 6.4.1.** Let \( \mathcal{M} : X \to A \) be a module.
- A right cylinder \( X \xrightarrow{R} \mathcal{M} \xleftarrow{\mu} A \) or the pair \((R, \mu)\) is called a counit of \( \mathcal{M} \) if the module morphism \( X \uparrow \mu : (X) \to \mathcal{M} : X \to A \) is iso.
- A left cylinder \( X \xleftarrow{M} \mathcal{M} \xrightarrow{\mu} A \) or the pair \((M, \mu)\) is called a unit of \( \mathcal{M} \) if the module morphism \( \mu \uparrow A : R \langle A \rangle \to \mathcal{M} : X \to A \) is iso.

**Remark 6.4.2.** By the general Yoneda lemma (Corollary 5.3.18), representations and units correspond one-to-one:
- if a right cylinder \( \mu : R \to \mathcal{M} \) is a counit of a module \( \mathcal{M} : X \to A \), then the functor \( R \) corepresents \( \mathcal{M} \) by the isomorphism \( X \uparrow \mu : (X) \to \mathcal{M} \); conversely, if a functor \( R : A \to X \) and a module isomorphism \( \Upsilon : (X) \to \mathcal{M} \) form a corepresentation of \( \mathcal{M} \), then the right cylinder \( X \uparrow \Upsilon : R \to \mathcal{M} \) is a counit of \( \mathcal{M} \).
- if a left cylinder \( \mu : \mathcal{M} \to R \) is a unit of a module \( \mathcal{M} : X \to A \), then the functor \( \mathcal{M} \) represents \( \mathcal{M} \) by the isomorphism \( \mu \uparrow A : R \langle A \rangle \to \mathcal{M} \); conversely, if a functor \( \mathcal{M} : X \to A \) and a module isomorphism \( \Upsilon : R \langle A \rangle \to \mathcal{M} \) form a representation of \( \mathcal{M} \), then the left cylinder \( \Upsilon \uparrow A : \mathcal{M} \to R \) is a unit of \( \mathcal{M} \).

**Proposition 6.4.3.** Let \( \mathcal{M} : X \to A \) be a module.
- A right cylinder \( \mu : R \to \mathcal{M} \) is a counit of \( \mathcal{M} \) if and only if each component \( \mu_a : a : R \to a \) is an inverse universal \( \mathcal{M} \)-arrow.
- A left cylinder \( \mu : \mathcal{M} \to R \) is a unit of \( \mathcal{M} \) if and only if each component \( \mu_x : x \to R : x \) is a direct universal \( \mathcal{M} \)-arrow.

**Proof.** By Proposition 2.1.3, \( X \uparrow \mu \) is iso iff its each slice \( (X \uparrow \mu) a \) is iso. Since \( (X \uparrow \mu) a = X \uparrow \mu_a \) (see Corollary 5.3.12), \( (X \uparrow \mu) a \) is iso iff \( \mu_a \) is inverse universal. \[ \square \]

**Remark 6.4.4.**
(1) Proposition 6.4.3 gives an alternative definition of units of a module.
(2) By Proposition 6.4.3 and Remark 6.2.2(2), under the identification in Remark 4.3.2(2),
- a unit of a right module \( \mathcal{M} : X \to \ast \) is the same thing as a counit of \( \mathcal{M} \) regarded as the two-sided module from \( X \) to the terminal category.
- a unit of a left module \( \mathcal{M} : \ast \to A \) is the same thing as a unit of \( \mathcal{M} \) regarded as the two-sided module from the terminal category to \( A \).

**Proposition 6.4.5.** Let \( \mathcal{M} : X \to A \) be a module.
- A counit \( (R, \mu) \) of \( \mathcal{M} \) gives an inverse universal arrow \( \mu : R \to \mathcal{M} \) of the right general Yoneda module \( X \cdot \ast A \), i.e. a unit \( \mu : R \to \ast \) (in the sense of Definition 6.1.1) of the right module \( (X \cdot \ast A) (\mathcal{M}) \).
A unit \((R, \mu)\) of \(\mathcal{M}\) gives a direct universal arrow \(\mu: \mathcal{M} \rightarrow R\) of the left general Yoneda module \(\mathbf{X}^\ast \mathbf{A}\), i.e. a unit \(\mu: * \rightarrow R\) (in the sense of Definition 6.1.1) of the left module \((\mathcal{M}) (\mathbf{X}^\ast \mathbf{A})\).

**Proof.** For a right cylinder \(\alpha: G \rightarrow \mathcal{M}\), we need to show that there is a unique natural transformation \(\alpha/\mu: G \rightarrow R: A \rightarrow X\) making the triangle

\[
\begin{array}{c}
R \\
\downarrow \mu \downarrow \\
G
\end{array} \rightarrow
\begin{array}{c}
\alpha \\
\downarrow \\
\mathcal{M}
\end{array}
\]

commute. For an \(A\)-arrow \(f: a \rightarrow b\), consider the naturality squares

\[
\begin{array}{c}
a: R \overset{\mu_a}{\longrightarrow} a \\
f: R \downarrow \downarrow f \\
b: R \overset{\mu_b}{\longrightarrow} b
\end{array}
\]

\[
\begin{array}{c}
a: \mathbf{G} \overset{\alpha_a}{\longrightarrow} a \\
f: \mathbf{G} \downarrow \downarrow f \\
b: \mathbf{G} \overset{\alpha_b}{\longrightarrow} b
\end{array}
\]

given by \(\mu\) and \(\alpha\). Since \(\mu_a\) and \(\mu_b\) are inverse universal by Proposition 6.4.3, the left square forms an inverse conjugation. Hence, by Proposition 6.3.3(3), the adjunct of \(\alpha_a\) along \(\mu_a\) and the adjunct of \(\alpha_b\) along \(\mu_b\) yield a unique pair of \(X\)-arrows making the diagram

\[
\begin{array}{c}
a: R \overset{\mu_a}{\longrightarrow} a \\
f: R \downarrow \downarrow \alpha_a \\
b: R \overset{\mu_b}{\longrightarrow} b
\end{array}
\]

\[
\begin{array}{c}
a: \mathbf{G} \overset{\alpha_a}{\longrightarrow} a \\
f: \mathbf{G} \downarrow \downarrow \alpha_a \\
b: \mathbf{G} \overset{\alpha_b}{\longrightarrow} b
\end{array}
\]

commute. The family of \(X\)-arrows \(\alpha_a/\mu_a\), one for each \(a \in \| \mathbf{A} \|\), thus forms a unique natural transformation \(\alpha/\mu: G \rightarrow R\) such that \(\alpha/\mu \circ \mu = \alpha\).

**Remark 6.4.6.** The converse does not hold. See Example 6.4.7.

**Example 6.4.7.** Consider a module \(\mathcal{M}: 2 \rightarrow 2\) from the discrete category \(2 = \{0, 1\}\) to the interval category \(2\) which looks like

\[
\begin{array}{c}
0 \\
\downarrow \\
1
\end{array}
\]

Then \(\mathcal{M}\) admits only one right cylinder

\[
\begin{array}{c}
2 \\
\downarrow \mu \\
\mathcal{M}
\end{array} \rightarrow
\begin{array}{c}
2
\end{array}
\]

along it. This right cylinder \(\mu\) is an inverse universal arrow of the right general Yoneda module \(2^\ast 2\), being the sole \((2^\ast 2)\)-arrow to \(\mathcal{M}\); however, \(\mu\) is not a unit of \(\mathcal{M}\) since \(\mu_1: 0 \rightarrow 1\) is not an inverse universal \(\mathcal{M}\)-arrow.

**Theorem 6.4.8.** Let \(\mathcal{M}: \mathbf{X} \rightarrow \mathbf{A}\) be a module.

- If \((R, \mu)\) and \((S, \nu)\) are two counits of \(\mathcal{M}\), then \(R\) and \(S\) are isomorphic.
- If \((R, \mu)\) and \((S, \nu)\) are two units of \(\mathcal{M}\), then \(R\) and \(S\) are isomorphic.

**Proof.** By Proposition 6.4.5, \((R, \mu)\) and \((S, \nu)\) give inverse universal \((\mathbf{X}^\ast \mathbf{A})\)-arrows \(\mu: R \rightarrow \mathcal{M}\) and \(\nu: S \rightarrow \mathcal{M}\). Hence \(R\) and \(S\) are isomorphic by Corollary 6.2.8.

**Corollary 6.4.9.** A representing functor of a module, if exists, is unique up to isomorphism.
Proof. By Remark 6.4.2, this is just a restatement of Theorem 6.4.8.

Theorem 6.4.10. Let \( \mathcal{M} : X \rightarrow A \) be a module.

- If there is a family of inverse universal \( \mathcal{M} \)-arrows \( \mu_a : r_a \rightarrow a \), one for each object \( a \in \| A \| \), then there is a unique functor \( R : A \rightarrow X \) with \( R(a) = r_a \) such that \( \mu := (\mu_a)_{a \in \| A \|} \) forms a left cylinder \( R \rightsquigarrow \mathcal{M} \), and moreover \( \mu \) is a counit of \( \mathcal{M} \).

- If there is a family of direct universal \( \mathcal{M} \)-arrows \( \mu_x : x \rightarrow r_x \), one for each object \( x \in \| X \| \), then there is a unique functor \( R : X \rightarrow A \) with \( R(x) = r_x \) such that \( \mu := (\mu_x)_{x \in \| X \|} \) forms a left cylinder \( \mu : \mathcal{M} \rightarrow R \), and moreover \( \mu \) is a unit of \( \mathcal{M} \).

Proof. The arrow function of \( R \) is given by the inverse conjugation

\[
\begin{array}{ccc}
\mu_a & \sim & a \\
\mathcal{M} & \sim & f \\
\mu_b & \sim & b
\end{array}
\]

(see Definition 6.3.1) for each \( A \)-arrow \( f : a \rightarrow b \). \( R \) is functorial by Proposition 6.3.3(1) and the uniqueness \( R \) follows from the uniqueness of a conjugate. Since each \( \mu_a \) is inverse universal, \( \mu \) forms a counit of \( \mathcal{M} \) by Proposition 6.4.3.

Note. The axiom of choice is used in the proof of the following.

Corollary 6.4.11. Given a module \( \mathcal{M} : X \rightarrow A \),

- the following conditions are equivalent:
  1. \( \mathcal{M} \) is corepresentable;
  2. \( \mathcal{M} \) has a counit;
  3. for every object \( a \in \| A \| \), the right module \( (\mathcal{M}) a : X \rightarrow \ast \) is representable;
  4. for every object \( a \in \| A \| \), the right module \( (\mathcal{M}) a : X \rightarrow \ast \) has a unit; that is, for every \( a \in \| A \| \), there is an inverse universal \( \mathcal{M} \)-arrow to \( a \).

- the following conditions are equivalent:
  1. \( \mathcal{M} \) is representable;
  2. \( \mathcal{M} \) has a unit;
  3. for every object \( x \in \| X \| \), the left module \( x(\mathcal{M}) : \ast \rightarrow A \) is representable;
  4. for every object \( x \in \| X \| \), the left module \( x(\mathcal{M}) : \ast \rightarrow A \) has a unit; that is, for every \( x \in \| X \| \), there is a direct universal \( \mathcal{M} \)-arrow from \( x \).

Proof.

1. \( \Rightarrow \) (2) See Remark 6.4.2.
2. \( \Leftarrow \) (4) Immediate from Proposition 6.4.3.
3. \( \Rightarrow \) (4) Immediate from Proposition 6.4.10.
4. \( \Rightarrow \) (2) A family of inverse universal \( \mathcal{M} \)-arrows \( \mu_a : r_a \rightarrow a \), one chosen for each \( a \in \| A \| \), yields a counit of \( \mathcal{M} \) by Theorem 6.4.10.

Theorem 6.4.12. Consider a cell \( X \xrightarrow{\mathcal{M}} A \).

- If \( \mathcal{N} \) has a counit and \( \Phi \) creates inverse universal, then \( \mathcal{M} \) has a counit as well and \( \Phi \) preserves inverse universal arrows.

- If \( \mathcal{N} \) has a unit and \( \Phi \) creates direct universal, then \( \mathcal{M} \) has a unit as well and \( \Phi \) preserves direct universal arrows.

Proof. By the equivalence of (2) and (4) in Corollary 6.4.11, and by Remark 6.2.20, this is reduced to Proposition 6.1.11.
Theorem 6.4.13. Let \( M : X \to A \) be a module.

- For a counit \( \xymatrix{ X \ar@<1ex>[r]^-R & \ar@<1ex>[r]^-\mu_M A } \) of \( M \), the following conditions are equivalent:
  1. the functor \( R \) is fully faithful;
  2. each component of \( \mu \) is direct universal (hence two-way universal).
- For a unit \( \xymatrix{ X \ar@<1ex>[r]^-M \ar@<1ex>[l]^-R & \ar@<1ex>[r]^-\mu A } \) of \( M \), the following conditions are equivalent:
  1. the functor \( R \) is fully faithful;
  2. each component of \( \mu \) is inverse universal (hence two-way universal).

**Proof.** Let \( a \) and \( b \) be objects of \( A \). The commutativity of the naturality square

\[
\begin{array}{c}
a : R \xrightarrow{\mu_a} a \\
f : R \\
b : R \xrightarrow{\mu_b} b
\end{array}
\]

for every \( A \)-arrow \( f : a \to b \), translates into the commutativity of the triangle

\[
\begin{array}{c}
a(\mu)(A) \\
(a : R)(\mu)(X)(R)(b) \\
\end{array}
\]

Since \( \mu_b \) is inverse universal, the assignment \( h \mapsto h \circ \mu_b \) is bijective. Hence the assignment \( f \mapsto f : R \) is bijective (i.e. \( R \) is fully faithful) iff the assignment \( f \mapsto \mu_a \circ f \) is bijective (i.e. \( \mu_a \) is direct universal).

\[\square\]

### 6.5. Lifts

**Definition 6.5.1.**

- A cylinder \( \xymatrix{ E \ar@<1ex>[r]^-R & \ar@<1ex>[r]^-\mu_M A } \) is called
  1. inverse universal if it is an inverse universal \( (E, M) \)-arrow (see Definition 4.3.5);
  2. pointwise inverse universal if each component \( \mu_a : e : R \Rightarrow K \circ e \) is an inverse universal \( M \)-arrow.
  Given a functor \( K : E \to A \), an inverse universal (resp. pointwise inverse universal) cylinder \( \mu : R \Rightarrow K : E \Rightarrow M \) or the pair \( (R, \mu) \), or the functor \( R \) itself, is called a lift (resp. pointwise lift) of \( K \) inverse along \( M \).
- A cylinder \( \xymatrix{ E \ar@<1ex>[r]^-R & \ar@<1ex>[r]^-\mu_M A } \) is called
  1. direct universal if it is a direct universal \( (E, M) \)-arrow (see Definition 4.3.5);
  2. pointwise direct universal if each component \( \mu_a : e : K \Rightarrow R \circ e \) is a direct universal \( M \)-arrow.
  Given a functor \( K : E \to X \), a direct universal (resp. pointwise direct universal) cylinder \( \mu : K \Rightarrow R : E \Rightarrow M \) or the pair \( (R, \mu) \), or the functor \( R \) itself, is called a lift (resp. pointwise lift) of \( K \) direct along \( M \).

**Remark 6.5.2.**

1. A cylinder \( \mu : R \Rightarrow K : E \Rightarrow M \) is inverse universal if and only if to every cylinder \( \alpha : L \Rightarrow K : E \Rightarrow M \) there is a unique natural transformation \( \alpha / \mu : L \to R \) such that \( \alpha = \alpha / \mu \circ \mu \). Dually, a cylinder \( \mu : K \Rightarrow R : E \Rightarrow M \) is direct universal if and only if to every cylinder \( \alpha : K \Rightarrow L : E \Rightarrow M \), there is a unique natural transformation \( \mu / \alpha : R \to L \) such that \( \alpha = \mu \circ \mu / \alpha \).
(2) A cylinder \( \mu : R \to S : E \to M \) is called two-way universal (resp. pointwise two-way universal) if it is both inverse and direct universal (resp. pointwise inverse and direct universal).

**Proposition 6.5.3.**

- For a cylinder \( X \to M \to A \), the following conditions are equivalent:
  1. \( \mu \) is pointwise inverse universal;
  2. the module morphism \( (X,M) \mu : (X) R \to (M) K : X \to E \) is iso;
  3. the composite

\[
\begin{align*}
X & \to M \to A \\
\downarrow & \downarrow \\
[M] & \to [X]
\end{align*}
\]

of \( \mu \) and the right Yoneda morphism for \( M \) forms a natural isomorphism \( E \to [X] \).

- For a cylinder \( K \to M \to A \), the following conditions are equivalent:
  1. \( \mu \) is pointwise direct universal;
  2. the module morphism \( \mu (M) : R(\mathcal{A}) \to K(\mathcal{M}) : E \to A \) is iso;
  3. the composite

\[
\begin{align*}
K & \to M \to A \\
\downarrow & \downarrow \\
[M] & \to [A]
\end{align*}
\]

of \( \mu \) and the left Yoneda morphism for \( M \) forms a natural isomorphism \( E \to [A] \).

**Proof.**

1. \( \iff \) (3) The component of the natural transformation \( \langle (X,M) \rangle \delta \mu \) at \( e \in \|M\| \) is given by the right module morphism \( X \mu_e = (X,M) \mu_e \). Since a natural transformation is iso iff each component is an isomorphism, \( \langle (X,M) \rangle \delta \mu \) is iso iff \( X \mu_e \) is iso, i.e. \( \mu_e \) is inverse universal, for each \( e \in \|M\| \).

2. \( \iff \) (3) By Proposition 2.1.3, \( (X,M) \mu \) is iso iff \( \langle (X,M) \mu \rangle \rangle \) is iso. But \( \langle (X,M) \mu \rangle \rangle = \langle (X,M) \delta \mu \rangle \rangle \) (see Remark 5.3.11).

**Remark 6.5.4.** Proposition 6.5.3 gives alternative definitions of the pointwise universality of a cylinder.

**Proposition 6.5.5.**

- A counit

\[
\begin{align*}
X & \to R \to M \\
\downarrow & \downarrow \\
M & \to A
\end{align*}
\]

of a module \( M \) is the same thing as a pointwise lift

\[
\begin{align*}
R & \to A \\
\downarrow & \downarrow \\
M & \to A
\end{align*}
\]

of the identity \( A \to A \) inverse along \( M \).
- A unit
\[
\begin{array}{ccccc}
X & \longrightarrow & \mu & \longrightarrow & \mathcal{M} \\
\mu & \downarrow & & \downarrow & \mathcal{M} \\
R & & & \longrightarrow & \mu
\end{array}
\]
of a module \(\mathcal{M}\) is the same thing as a pointwise lift
\[
\begin{array}{ccccc}
X & \longrightarrow & \mu & \longrightarrow & \mathcal{M} \\
\mu & \downarrow & & \downarrow & \mathcal{M} \\
R & & & \longrightarrow & \mu
\end{array}
\]
of the identity \(X \rightarrow X\) direct along \(\mathcal{M}\).

**Proof.** By definition, \((R, \mu)\) is a unit of \(\mathcal{M}\) iff the module morphism \((X \uparrow \mathcal{M}) \mu : (X) R \rightarrow \mathcal{M}\) is iso. But by Proposition 6.5.3, this is the case iff \((R, \mu)\) is a pointwise lift of the identity \(A \rightarrow A\) inverse along \(\mathcal{M}\). \(\square\)

**Remark 6.5.6.** Proposition 6.4.3 now follows from Proposition 6.5.5.

**Proposition 6.5.7.**
- A pointwise lift
\[
\begin{array}{ccccc}
E & \downarrow & & \downarrow & \mathcal{M} \\
\mathcal{M} & \longrightarrow & \mu & \longrightarrow & \mathcal{M} \\
\mu & \downarrow & & \downarrow & \mathcal{M} \\
R & & & \longrightarrow & \mu
\end{array}
\]
of \(K\) inverse along \(\mathcal{M}\) is the same thing as a counit
\[
\begin{array}{ccccc}
X & \longrightarrow & \mu & \longrightarrow & \mathcal{M} \\
\mu & \downarrow & & \downarrow & \mathcal{M} \\
R & & & \longrightarrow & \mu
\end{array}
\]
of the composite module \((\mathcal{M}) K\).
- A pointwise lift
\[
\begin{array}{ccccc}
E & \downarrow & & \downarrow & \mathcal{M} \\
\mathcal{M} & \longrightarrow & \mu & \longrightarrow & \mathcal{M} \\
\mu & \downarrow & & \downarrow & \mathcal{M} \\
R & & & \longrightarrow & \mu
\end{array}
\]
of \(K\) direct along \(\mathcal{M}\) is the same thing as a unit
\[
\begin{array}{ccccc}
X & \longrightarrow & \mu & \longrightarrow & \mathcal{M} \\
\mu & \downarrow & & \downarrow & \mathcal{M} \\
R & & & \longrightarrow & \mu
\end{array}
\]
of the composite module \(K \uparrow \mathcal{M}\).

**Proof.** By definition, a right cylinder \(\begin{array}{ccccc}
X & \longrightarrow & \mu & \longrightarrow & E \\
\mu & \downarrow & & \downarrow & \mu \\
(\mathcal{M}) K & & & \longrightarrow & \mu
\end{array}\) is a unit iff the module morphism \((X \uparrow (\mathcal{M}) K) \mu : (X) R \rightarrow (\mathcal{M}) K\) is iso, and by Proposition 6.5.3, a two-sided cylinder \(\begin{array}{ccccc}
X & \longrightarrow & \mathcal{M} & \longrightarrow & \mu \\
\mu & \downarrow & & \downarrow & \mu \\
R & & & \longrightarrow & \mu
\end{array}\) is a pointwise lift iff the module morphism \((X \uparrow \mathcal{M}) \mu : (X) R \rightarrow (\mathcal{M}) K\) is iso. But these module isomorphisms coincide by Proposition 5.3.8. \(\square\)

**Proposition 6.5.8.** A pointwise inverse (resp. direct) universal cylinder is inverse (resp. direct) universal.

**Proof.** Let \(\mu : R \leadsto K : E \leadsto \mathcal{M}\) be a pointwise inverse universal cylinder. By Proposition 6.5.7, \((R, \mu)\) is a unit of \((\mathcal{M}) K\). Hence, by Proposition 6.4.5, \(\mu : R \leadsto (\mathcal{M}) K\) is an inverse universal \((E, \mathcal{M})\)-arrow, and by the identity in Theorem 5.1.8, this is the same thing as an inverse universal \((E, \mathcal{M})\)-arrow \(\mu : R \leadsto K\). \(\square\)
Proposition 6.5.9. For a natural transformation $\tau : R \to S : E \to C$, i.e. a cylinder $\tau : R \sim S : E \sim (C)$, the following conditions are equivalent:

1. $\tau$ is a natural isomorphism in $C$;
2. $\tau$ is an inverse universal cylinder along the hom $(C)$;
3. $\tau$ is a direct universal cylinder along the hom $(C)$;
4. $\tau$ is a two-way universal cylinder along the hom $(C)$;
5. $\tau$ is a pointwise inverse universal cylinder along the hom $(C)$;
6. $\tau$ is a pointwise direct universal cylinder along the hom $(C)$;
7. $\tau$ is a pointwise two-way universal cylinder along the hom $(C)$.

Proof. Since a natural transformation $\tau : R \to S : E \to C$ is the same thing as an arrow $\tau : R \to S$ in the category $\langle E, C \rangle$, and since a cylinder $\tau : R \sim S : E \sim (C)$ is the same thing as an arrow $\tau : R \sim S$ in the module $\langle E, (C) \rangle = \langle E, C \rangle$ (see Remark 4.3.6(3)), the equivalence of conditions (1), (2), (3), and (4) follows by applying Proposition 6.2.5 to the category $\langle E, C \rangle$. Since a natural transformation is iso iff each component is an isomorphism, the equivalence of conditions (1), (5), (6), and (7) follows again from Proposition 6.2.5.

Note. By Proposition 6.5.7, the following is a special case of Theorem 6.4.10 (and vice versa by Proposition 6.5.5).

Theorem 6.5.10. Let $M$ be a module and $K$ be a functor as in Definition 6.5.1.

- If there is a family of inverse universal $M$-arrows $\mu_e : r_e \sim K \cdot e$, one for each object $e \in |E|$, then there is a unique functor $R : E \to X$ with $e \cdot R = r_e$ such that $\mu := (\mu_e)_{e \in |E|}$ forms a cylinder $R \sim K : E \sim M$, and $\mu$ is pointwise inverse universal.

- If there is a family of direct universal $M$-arrows $\mu_e : e \cdot K \sim r_e$, one for each object $e \in |E|$, then there is a unique functor $R : E \to A$ with $r_e = R \cdot e$ such that $\mu := (\mu_e)_{e \in |E|}$ forms a cylinder $K \sim R : E \sim M$, and $\mu$ is pointwise direct universal.

Proof. Since an $M$-arrow $\mu_e : e \cdot K \sim r_e$ is inverse universal iff so is the $(M)K$-arrow $\mu_e : r_e \sim e$ (see Theorem 6.2.14), by Proposition 6.5.7, the assertion is reduced to an instance of Theorem 6.4.10 where $M$ is given by the composite module $(M)K$.

Theorem 6.5.11. Let $E$ be a category and $M : X \to A$ be a module.

- If a cylinder $\mu : R \sim K : E \sim M$ is inverse universal (resp. pointwise inverse universal), then a natural transformation $\tau : S \to R : E \to X$ is iso if and only if the cylinder $\tau \circ \mu : S \sim K : E \sim M$ is inverse universal (resp. pointwise inverse universal).

- If a cylinder $\mu : K \sim R : E \sim M$ is direct universal (resp. pointwise direct universal), then a natural transformation $\tau : R \to S : E \to A$ is iso if and only if the cylinder $\mu \circ \tau : K \sim S : E \sim M$ is direct universal (resp. pointwise direct universal).

Proof. Suppose that $\mu$ is inverse universal, i.e. an inverse universal $(E,M)$-arrow. Then the assertion is just an instance of Theorem 6.2.6 where $M$ is given by $(E,M)$. Now suppose that $\mu$ is pointwise inverse universal. Since a natural transformation is iso iff each component is an isomorphism, it suffices to show that, given $e \in |E|$, $[\tau \circ \mu]_e$ is an inverse universal $M$-arrow iff $\tau_e$ is an iso $X$-arrow. But since $[\tau \circ \mu]_e = \tau_e \circ \mu_e$, this follows again from Theorem 6.2.6.

Corollary 6.5.12. Let $E$ be a category and $M : X \to A$ be a module.

- If a functor $K : E \to A$ has a pointwise lift inverse along $M$, then every lift of $K$ inverse along $M$ is pointwise.

- If a functor $K : E \to X$ has a pointwise lift direct along $M$, then every lift of $K$ direct along $M$ is pointwise.
Proof. Let \( \mu : R \to K : E \to M \) and \( \nu : S \to K : E \to M \) be two lifts of \( K \) inverse along \( M \). Then, by Theorem 6.2.7, there are natural isomorphisms \( \nu/\mu : S \to R \) and \( \mu/\nu : R \to S \) inverse to each other making the diagram commute. Hence, by Theorem 6.5.11, if one of \( \mu \) and \( \nu \) is pointwise inverse universal, so is the other.

Theorem 6.5.13. Let \( H \) be a functor and \( \mu \) be a cylinder as in

\[
\begin{array}{ccc}
D & \overset{H}{\to} & E \\
\downarrow & & \downarrow \\
S & \overset{\mu}{\to} & X \\
\end{array}
\]

If \( \mu \) is pointwise inverse (resp. direct) universal, so is the composite \( H \circ \mu \) (see Definition 4.3.23).

Proof. Obvious since \( [H \circ \mu]_d = \mu((H \cdot d)) \) for each \( d \in \| D \| \).

Corollary 6.5.14. If a cylinder \( \mu \) is pointwise inverse (resp. direct) universal, so is the cylinder \( [D, \mu] \) (see Remark 4.3.34(2)) for any category \( D \), and moreover each component of \( [D, \mu] \) is a pointwise inverse (resp. direct) universal cylinder.

Proof. Since the component of \( [D, \mu] \) at a functor \( H \) is given by the cylinder \( H \circ \mu \), the assertion follows from Theorem 6.5.13 by noting that (see Proposition 6.5.8) a pointwise universal cylinder is universal.

Corollary 6.5.15. Given a functor \( H : D \to E \) and a module \( M : X \to A \), the precomposition cell \( (H, M) : (E, M) \to (D, M) \) (see Definition 4.3.27) preserves pointwise universality; that is, \( (H, M) \) sends each pointwise inverse (resp. direct) universal cylinder \( \mu : E \to M \) to a pointwise inverse (resp. direct) universal cylinder \( H \circ \mu : D \to M \).

Proof. Immediate from Theorem 6.5.13.

Note. By Proposition 6.5.5, Theorem 6.5.16 and Corollary 6.5.17 below are special cases of Theorem 6.5.13 and Corollary 6.5.14 (and vice versa by Proposition 6.5.7).

Theorem 6.5.16. Let \( \mu \) be a right (resp. left) cylinder and \( K \) be a functor as in

\[
\begin{array}{ccc}
X & \overset{R}{\leftarrow} & A \\
\downarrow & \overset{\mu}{\leftarrow} & \downarrow \\
E & \overset{K}{\leftarrow} & X \\
\end{array}
\]

If \( \mu \) is a counit (resp. unit) of \( M \), then the composite \( K \circ \mu \) (see Definition 4.3.25) is a pointwise inverse (resp. direct) universal cylinder.

Proof. Obvious since \( [K \circ \mu]_e = \mu((K \cdot e)) \) for each \( e \in \| E \| \).

Corollary 6.5.17. If a right (resp. left) cylinder \( \mu \) is a counit (resp. unit), so is the right (resp. left) cylinder \( [E, \mu] \) (see Remark 4.3.34(3)) for any category \( E \), and moreover each component of \( [E, \mu] \) is a pointwise inverse (resp. direct) universal cylinder.
Proof. Since the component of $[E, \mu]$ at a functor $K$ is given by the cylinder $K \circ \mu$, the assertion follows from Theorem 6.5.16 by noting that (see Proposition 6.5.8) a pointwise universal cylinder is universal.

Theorem 6.5.18. Let $\xymatrix{ S \ar[r]^-\mu \ar[d]_h \ar[r]_-\nu & T \ar[d]_e \ar[r]_-\lambda }$ be a cylinder and $D$ be an essentially wide subcategory of $E$. Then $\mu$ is pointwise inverse (resp. direct) universal if and only if so is its restriction to $D$ (see Remark 4.3.24(2)); that is, if and only if the component $\mu_d$ is inverse (resp. direct) universal for each $d \in |D|$.

Proof. The forward implication is immediate from Theorem 6.5.13. Assume now that $\mu$ is pointwise universal on $D$. We need to show that the component $\mu_e$ is universal for each $e \in |E|$. Since $D$ is essentially wide in $E$, there is an object $d \in |D|$ and an iso $E$-arrow $h : d \to e$, giving a naturality square

$$
\begin{array}{ccc}
 d \downarrow S \xrightarrow{\mu_d} T \downarrow d \\
 h \downarrow S \xrightarrow{T \downarrow h} E \downarrow e \xrightarrow{\mu_e} T \downarrow e \\
\end{array}
$$

with $h : S$ and $T : h$ iso (because any functor preserves isomorphisms). Now, since $\mu_d$ is universal by assumption, so is $\mu_e$ by Theorem 6.2.13.

Remark 6.5.19. Since a natural isomorphism is the same thing as a pointwise universal cylinder along the hom of a category (see Proposition 6.5.9), the lemma in Preliminary 19 is a special case of Theorem 6.5.18.

Theorem 6.5.20. Let $E$ be a category and $\mathcal{M} : X \to A$ be a module.

- Consider a pair of inverse universal cylinders $\xymatrix{ X \ar[r]^{\mu} \ar[d]_{\sigma} & A \ar@{=}[d] }$ and $\xymatrix{ X \ar[r]^{\mu'} \ar[d]_{\sigma'} & A \ar@{=}[d] }$. Then for any natural transformation $\sigma : K \to K'$, there exists a unique natural transformation $\tau : R \to R'$ making the square

$$
\begin{array}{ccc}
 R \xrightarrow{\mu} K \\
 \tau \downarrow \downarrow \sigma \\
 R' \xrightarrow{\mu'} K' \\
\end{array}
$$

commute in the module $(E, \mathcal{M}) : [E, X] \to [E, A]$.

- Consider a pair of direct universal cylinders $\xymatrix{ X \ar[r]^{\mu} \ar[d]_{\sigma} & A \ar@{=}[d] }$ and $\xymatrix{ X \ar[r]^{\mu'} \ar[d]_{\sigma'} & A \ar@{=}[d] }$. Then for any natural transformation $\sigma : K \to K'$, there exists a unique natural transformation $\tau : R \to R'$ making the square

$$
\begin{array}{ccc}
 K \xrightarrow{\mu} R \\
 \sigma \downarrow \downarrow \tau \\
 K' \xrightarrow{\mu'} R' \\
\end{array}
$$

commute in the module $(E, \mathcal{M}) : [E, X] \to [E, A]$.

Proof. The unique $\tau$ is given by the conjugate (see Definition 6.3.1) of $\sigma$ inverse along $(\mu, \mu')$.

Remark 6.5.21. Noting Remark 4.3.6(2), we see that the square

$$
\begin{array}{ccc}
 R \xrightarrow{\mu} K \\
 \tau \downarrow \downarrow \sigma \\
 R' \xrightarrow{\mu'} K' \\
\end{array} \text{ resp. } \\
\begin{array}{ccc}
 K \xrightarrow{\mu} R \\
 \sigma \downarrow \downarrow \tau \\
 K' \xrightarrow{\mu'} R' \\
\end{array}
$$
in Theorem 6.5.20 commutes if and only if the square
\[
\begin{array}{ccc}
e : R \xrightarrow{\mu_e} K & \xRightarrow{\tau_e} & e' : R' \xrightarrow{\mu_e'} K' \\
\downarrow{\sigma_e} & & \downarrow{\sigma_e'} \\
e : R' \xrightarrow{\mu_e} K' & \xRightarrow{\tau_e} & e' : R' \xrightarrow{\mu_e'} K'
\end{array}
\]
commutes in the module \( \mathcal{M} : X \to A \) for every object \( e \in |E| \). Hence if \( \mu \) and \( \mu' \) are pointwise inverse (resp. direct) universal, each \( \tau_e \) is given by the conjugate of \( \sigma_e \) along \( (\mu_e, \mu'_e) \); Theorem 6.5.20 says that if \( \sigma_e \) is natural in \( e \), so will be \( \tau_e \).

6.6. Kan lifts

Note. Remark 4.3.4(5) allows the following definition.

**Definition 6.6.1.** Given a pair of functors \( \text{D} \xrightarrow{F} \text{E} \leftarrow \text{C} \),

- a natural transformation

\[
\begin{array}{ccc}
\text{D} & \xrightarrow{\mu} & \text{E} \\
\downarrow{\text{R}} & & \downarrow{\text{K}} \\
\text{C} & \xleftarrow{\text{K}} & \text{E}
\end{array}
\]

from \( \text{R} \circ F \) to \( \text{K} \) or the pair \((\text{R}, \mu)\), or the functor \( \text{R} \) itself, is called a right Kan lift (resp. pointwise right Kan lift) of \( \text{K} \) along \( F \) if the cylinder

\[
\begin{array}{ccc}
\text{D} & \xrightarrow{\text{F(E)}} & \text{E}
\end{array}
\]

is inverse universal (resp. pointwise inverse universal).

- a natural transformation

\[
\begin{array}{ccc}
\text{E} & \xleftarrow{\mu} & \text{D} \\
\downarrow{\text{F}} & & \downarrow{\text{R}} \\
\text{C} & \xrightarrow{\text{K}} & \text{D}
\end{array}
\]

from \( \text{K} \) to \( \text{F} \circ \text{R} \) or the pair \((\text{R}, \mu)\), or the functor \( \text{R} \) itself, is called a left Kan lift (resp. pointwise left Kan lift) of \( \text{K} \) along \( F \) if the cylinder

\[
\begin{array}{ccc}
\text{E} & \xrightarrow{\text{(E)F}} & \text{D}
\end{array}
\]

is direct universal (resp. pointwise direct universal).

**Remark 6.6.2.**

1. A Kan lift (resp. pointwise Kan lift) is thus a special instance of a lift (resp. pointwise lift) defined in Definition 6.5.1 where \( \mathcal{M} \) is representable.

2. A natural transformation \( \mu : \text{R} \circ F \to \text{K} \) forms a right Kan lift if and only if to every natural transformation \( \alpha : \text{L} \circ F \to \text{K} \) there is a unique natural transformation \( \alpha/\mu : \text{L} \to \text{R} \) such that \( \alpha = [\alpha/\mu \circ F] \circ \mu \) (where \([\alpha/\mu \circ F] \circ \mu \) is the pasting composite of \( \alpha/\mu \) and \( \mu \)). Dually, a natural transformation \( \mu : \text{K} \to \text{F} \circ \text{R} \) forms a left Kan lift if and only if to every natural transformation \( \alpha : \text{K} \to \text{F} \circ \text{L} \) there is a unique natural transformation \( \mu \backslash \alpha : \text{R} \to \text{L} \) such that \( \alpha = \mu \circ [\mu \backslash \alpha \circ F] \).
(3) By Proposition 6.5.7,

- a pointwise right Kan lift of $K$ along $F$ is the same thing as a counit of the composite module $F(E)K : D \to C$.
- a pointwise left Kan lift of $K$ along $F$ is the same thing as a unit of the composite module $K(E)F : C \to D$.

(4) Using the terms introduced in Example 6.2.3, a pointwise Kan lift is described as follows:

- a natural transformation $\mu : R \circ F \to K$ forms a pointwise right Kan lift of $K$ along $F$ if and only if each component $\mu_c : c \circ R \circ F \to K \cdot c$ is universal from $F$ to $K \cdot c$.
- a natural transformation $\mu : K \to F \circ R$ forms a pointwise left Kan lift of $K$ along $F$ if and only if each component $\mu_c : K \cdot F \circ R \cdot c$ is universal from $c \cdot K$ to $F$.

(5) By Proposition 6.5.8, a pointwise Kan lift is a Kan lift. See Example 6.6.5(1) for an example of a non-pointwise Kan lift.

**Theorem 6.6.3.**

- Consider functors as in

\[
\begin{array}{ccc}
D & \overset{\mu}{\longrightarrow} & E \\
\downarrow & & \downarrow \\
F & \quad & C \\
\end{array}
\]

and assume that $\mu : R \circ F \to K$ is a natural isomorphism. Under this condition, if $F$ is fully faithful, then $\mu$ forms a pointwise right Kan lift of $K$ along $F$. The converse holds if we assume in addition that $R$ is essentially surjective.

- Consider functors as in

\[
\begin{array}{ccc}
E & \overset{\mu}{\longrightarrow} & D \\
\downarrow & & \downarrow \\
R & \quad & C \\
\end{array}
\]

and assume that $\mu : K \to F \circ R$ is a natural isomorphism. Under this condition, if $F$ is fully faithful, then $\mu$ forms a pointwise left Kan lift of $K$ along $F$. The converse holds if we assume in addition that $R$ is essentially surjective.

**Proof.** For each $c \in ||C||$, we have the commutative diagram

\[
\begin{array}{c}
\{D\} (R \cdot c) \\
\downarrow \text{D} \uparrow \mu_c \\
\{F(E)\} (F \cdot c) \\
\end{array}
\]

\[
\begin{array}{c}
\rightarrow \{F(E)\} (F \cdot R \cdot c) \\
\downarrow \text{F}(E) \uparrow \mu_{\cdot c} \\
\rightarrow \{F(E)\} (K \cdot c) \\
\end{array}
\]

by replacing $f : r \sim e$ in Example 5.2.9(2) with $\mu_c : R \cdot c \sim K \cdot c$. Since $\mu_c$ is an iso $E$-arrow by assumption, $\text{F}(E)\mu_c$ is iso. Hence $\text{D} \uparrow \mu_c$ is iso if $\text{F}(F) (R \cdot c)$ is iso. Suppose now that $F$ is fully faithful. Then $(F) (R \cdot c)$ and hence $\text{D} \uparrow \mu_c$ is iso for every $c \in ||C||$; $\mu$ is thus pointwise inverse universal. Suppose conversely that $\mu$ is pointwise inverse universal. Then $\text{D} \uparrow \mu_c$ and hence $(F) (R \cdot c)$ is iso for every $c \in ||C||$; $F$ is thus fully faithful by Proposition 2.1.5 under the condition that $R$ is essentially surjective.

**Remark 6.6.4.** If $F$ is not fully faithful in Theorem 6.6.3, then a natural isomorphism $\mu$, even an identity, need not form a Kan lift. See Example 6.6.5(2) for an example.

**Example 6.6.5.**

(1) Example 6.4.7 is repeated below in terms of Kan lifts. Let 2 and 2 be as in Example 6.4.7 and let $F : 2 \to 2$ be the inclusion functor. Then the representable module of $F$ looks like

\[
\begin{array}{c}
0 \sim 0 \\
\downarrow \\
1 \sim 1 \\
\end{array}
\]
The identity $2 \to 2$ has only one natural transformation along $F$. The constant functor $\Delta 0$ and $\mu$ form a right Kan lift of the identity $2 \to 2$ along $F$; however, the lift is not pointwise since $\mu_1 : 0 : F \to 1$ is not universal from $F$ to $1$.

(2) Consider functors as in $\begin{array}{c} \ast \downarrow \\ \downarrow 1 \\ \ast \end{array}$, where $\ast$ is the terminal category and $E$ is any category. Given $e \in [E]$, the functor $e : \ast \to E$, and the identity natural transformation $\begin{array}{c} E \downarrow \\ \downarrow 1 \\ \ast \end{array}$ form a right Kan lift of the identity $\ast \to \ast$ along the unique functor $E \to \ast$ only when $e$ is a terminal object of $E$. 
7. Limits

7.1. Limits

**Definition 7.1.1.** Let $E$ be a category and $M : X \to A$ be a module.

- A cone $\begin{array}{c} * \downarrow \pi \\ X \xrightarrow{M} A \end{array}$ is called universal if it is an inverse universal $\langle *E, M \rangle$-arrow (see Definition 4.6.5). Given a functor $K : E \to A$, a universal cone $\begin{array}{c} r \downarrow \pi \\ X \xrightarrow{M} A \end{array}$, or the object $r$ itself, is called a limit of $K$ along $M$, with the object $r$ denoted by $\prod E K$ (or just by $\prod K$).

- A cone $\begin{array}{c} E \downarrow \pi \\ K \xrightarrow{M} A \end{array}$ is called universal if it is a direct universal $\langle E *, M \rangle$-arrow (see Definition 4.6.5). Given a functor $K : E \to X$, a universal cone $\begin{array}{c} K \downarrow r \\ E \xrightarrow{M} A \end{array}$, or the object $r$ itself, is called a colimit of $K$ along $M$, with the object $r$ denoted by $\coprod E K$ (or just by $\coprod K$).

**Remark 7.1.2.**

1. A cone $\begin{array}{c} r \downarrow \pi \\ X \xrightarrow{M} A \end{array}$ is universal if and only if to every cone $\begin{array}{c} x \downarrow \alpha \\ X \xrightarrow{M} A \end{array}$ there is a unique $X$-arrow $\alpha \rangle \pi : x \to r$ such that $\alpha \rangle \pi = \alpha \circ \pi$. Dually, a cone $\begin{array}{c} r \downarrow \pi \\ X \xrightarrow{M} A \end{array}$ is universal if and only if to every cone $\begin{array}{c} r \downarrow \alpha \rangle \pi \\ X \xrightarrow{M} A \end{array}$ there is a unique $A$-arrow $\pi \rangle \alpha : r \to a$ such that $\alpha = \pi \circ \alpha \rangle \pi$.

2. A limit $\prod E K$ (resp. colimit $\coprod E K$) is called a product (resp. coproduct) when $E$ is discrete.

**Theorem 7.1.3.**

- Given a module and functors as in

\[
\begin{array}{c} X \xrightarrow{M} A \xleftarrow{K} E \xleftarrow{F} D \\
\end{array}
\]

a cone

\[
\begin{array}{c} * \Downarrow \pi \\
X \xrightarrow{M} E \xrightarrow{K} A \xrightarrow{F} D \\
\end{array}
\]

forms a limit of $F$ along the composite module $\langle M \rangle K$ if and only if $\pi$ depicted as in

\[
\begin{array}{c} * \Downarrow \pi \\
X \xrightarrow{M} E \xrightarrow{K \circ F} A \\
\end{array}
\]

forms a limit of the composite functor $K \circ F$ along $M$.

- Given a module and functors as in

\[
\begin{array}{c} D \xrightarrow{F} E \xrightarrow{K} X \xrightarrow{M} A \\
\end{array}
\]

a cone

\[
\begin{array}{c} D \Downarrow \pi \\
E \xrightarrow{M} A \xrightarrow{K} X \xrightarrow{F} D \\
\end{array}
\]

forms a limit of the composite functor $K \circ F$ along $M$. 

203
forms a colimit of $F$ along the composite module $K(M)$ if and only if $\pi$ depicted as in
\[
\begin{array}{ccc}
D & \xrightarrow{\pi} & * \\
\downarrow F \circ K & & \downarrow r \\
X & \xrightarrow{\_} & A
\end{array}
\]
forms a colimit of the composite functor $F \circ K$ along $M$.

Proof. By Proposition 6.2.23, the cell
\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & [D,E] \\
\downarrow 1 & & \downarrow 1 \\
X & \xrightarrow{\_} & [D,K]
\end{array}
\]
(see Proposition 4.6.7) preserves and reflects inverse universal arrows; that is, an $(*D,(M)K)$-arrow $\pi : r \leadsto F$ is inverse universal iff so is the $(*D,M)$-arrow $\pi : r \leadsto [D,K] \leadsto F; that is, a cone $\pi : r \leadsto F \leadsto (M)K$ is universal iff so is the cone $\pi : r \leadsto K \circ F \leadsto *D \leadsto M$. □

Definition 7.1.4. A module $M : X \rightarrow A$ is said to
- have limits over a category $E$ if every functor $E \rightarrow A$ has a limit along $M$.
- have colimits over a category $E$ if every functor $E \rightarrow X$ has a colimit along $M$.

Proposition 7.1.5.
- A module $M : X \rightarrow A$ has limits over a category $E$ if and only if the module $(*E,M)$ has a counit
\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & [E,A] \\
\downarrow (\_,*E,M) & & \downarrow \Pi
\end{array}
\]
- A module $M : X \rightarrow A$ has colimits over a category $E$ if and only if the module $(E*,M)$ has a unit
\[
\begin{array}{ccc}
[E,X] & \xrightarrow{\pi} & A \\
\downarrow (E*,M) & & \downarrow \Pi
\end{array}
\]

Proof. Since a limit $\pi : \prod K \leadsto K$ of $*E \leadsto M$ of a functor $K : E \rightarrow A$ is defined as an inverse universal $(*E,M)$-arrow to $K$, the assertion follows from the equivalence of (2) and (4) in Corollary 6.4.11. □

Definition 7.1.6. A module $M : X \rightarrow A$ is called complete (resp. cocomplete) if it has limits (resp. colimits) over any small category $E$.

Proposition 7.1.7.
- A module $M$ is complete if and only if for any small category $E$ the module $(*E,M)$ has a counit.
- A module $M$ is cocomplete if and only if for any small category $E$ the module $(E*,M)$ has a unit.

Proof. Immediate from Proposition 7.1.5. □

Definition 7.1.8. A cell $\begin{array}{ccc}
X & \xrightarrow{\_} & A \\
\downarrow p & & \downarrow \phi \\
Y & \xrightarrow{\_} & B
\end{array}$ is said to
- preserve (reflect, create) limits over a category $E$ if the postcomposition cell $(*E,\phi)$ (see Definition 4.6.15) preserves (reflects, creates) inverse universal arrows.
- preserve (reflect, create) colimits over a category $E$ if the postcomposition cell $(E*,\phi)$ (see Definition 4.6.15) preserves (reflects, creates) direct universal arrows.
Remark 7.1.9.

(1) Recalling the definition of limits (resp. colimits) and the definition of the postcomposition cell, Definition 7.1.8 can be stated in elementary terms as follows: \( \Phi \) is said to

a) preserve

- limits over \( E \) if each universal cone \( \pi : r \rightarrow K : \ast E \rightarrow M \) yields by composition with \( \Phi \) a universal cone \( \pi \circ \Phi : r : P \rightarrow Q \circ K : \ast E \rightarrow N \).
- colimits over \( E \) if each universal cone \( \pi : K \rightarrow r : E \ast \rightarrow M \) yields by composition with \( \Phi \) a universal cone \( \pi \circ \Phi : K \circ P \rightarrow Q \circ r : E \ast \rightarrow N \).

b) reflect

- limits over \( E \) if a cone \( \pi : r \rightarrow K : \ast E \rightarrow M \) is universal whenever the cone \( \pi \circ \Phi : r : P \rightarrow Q \circ K : \ast E \rightarrow N \) is universal.
- colimits over \( E \) if a cone \( \pi : K \rightarrow r : E \ast \rightarrow M \) is universal whenever the cone \( \pi \circ \Phi : K \circ P \rightarrow Q \circ r : E \ast \rightarrow N \) is universal.

c) create

- limits over \( E \) if for every functor \( K : E \rightarrow A \) and for every universal cone \( \kappa : s \rightarrow Q \circ K : \ast E \rightarrow N \) there is exactly one cone \( \pi : r \rightarrow K : \ast E \rightarrow M \) with \( \pi \circ \Phi = \kappa \), and if this \( \pi \) is universal.
- colimits over \( E \) if for every functor \( K : E \rightarrow X \) and for every universal cone \( \kappa : K \circ P \rightarrow s : E \ast \rightarrow N \) there is exactly one cone \( \pi : K \rightarrow r : E \ast \rightarrow M \) with \( \pi \circ \Phi = \kappa \), and if this \( \pi \) is universal.

(2) A cell \( \Phi \) is said to

- preserve (reflect, create) limits (resp. small limits) if it preserves (reflects, creates) limits over any category (resp. small category).
- preserve (reflect, create) colimits (resp. small colimits) if it preserves (reflects, creates) colimits over any category (resp. small category).

(3) We also say that a cell \( \Phi : M \rightarrow N \) is

- continuous if \( M \) is compete and \( \Phi \) preserves small limits.
- cocontinuous if \( M \) is cocomplete and \( \Phi \) preserves small colimits.

Proposition 7.1.10. Consider a cell \( \Phi \) as in Definition 7.1.8.

- If \( \Phi \) is fully faithful and \( P \) is iso, then \( \Phi \) preserves, reflects, and creates limits.
- If \( \Phi \) is fully faithful and \( Q \) is iso, then \( \Phi \) preserves, reflects, and creates colimits.

Proof. The assertion is equivalent to saying that if \( \Phi \) is fully faithful and \( P \) is iso, then for any category \( E \), the postcomposition cell \( (\ast E, \Phi) \) preserves, reflects, and creates inverse universal arrows. But this follows from Proposition 6.2.23 because if \( \Phi \) is fully faithful, so is \( (\ast E, \Phi) \) by Proposition 4.6.17.

Proposition 7.1.11. Let \( E \) be a category and let \( \Phi \) be a cell as in Definition 7.1.8.

- If \( N \) has limits over \( E \) and \( \Phi \) creates limits from them, then \( M \) has limits over \( E \) as well and \( \Phi \) preserves them.
- If \( N \) has colimits over \( E \) and \( \Phi \) creates colimits from them, then \( M \) has colimits over \( E \) as well and \( \Phi \) preserves them.

Proof. Noting Proposition 7.1.5, we see that this is an instance of Theorem 6.4.12 where \( \Phi \) is given by the postcomposition cell \( (\ast E, \Phi) \).

Proposition 7.1.12. Consider a cell \( \Phi \) as in Definition 7.1.8.

- If \( N \) is complete and \( \Phi \) creates small limits, then \( M \) is complete as well and \( \Phi \) is continuous.
- If \( N \) is cocomplete and \( \Phi \) creates small colimits, then \( M \) is cocomplete as well and \( \Phi \) is cocontinuous.

Proof. Immediate from Proposition 7.1.11.
7.2. Limits with parameters

Definition 7.2.1. Let $E$ and $D$ be categories and $M : X \to A$ be a module.

A cone

\[
\begin{array}{ccc}
* & \leftarrow & ! \\
\downarrow R & \omega & \downarrow K \\
[E, X] & \to & [E, A]
\end{array}
\]

along the module $(E, M)$ (see Definition 4.3.5) is called

(1) universal if it is an inverse universal $(\ast D, (E, M))$-arrow (see Definition 4.6.5);

(2) pointwise universal if each slice

\[
\begin{array}{ccc}
* & \leftarrow & ! \\
\downarrow R & \omega & \downarrow K \\
[E, X] & \to & [E, A]
\end{array}
\]

(see Remark 4.8.8(2)) is a universal cone, i.e. an inverse universal $(\ast D, M)$-arrow.

Given a functor $K : D \to [E, A]$, a pointwise universal cone $\omega : R \Rightarrow K : \ast D \Rightarrow (E, M)$ or the pair $(R, \omega)$, or the functor $R$ itself, is called a pointwise limit of $K$ along $(E, M)$, with the functor $R$ denoted by $\prod^E D K$ (or just by $\prod^E K$).

A cone

\[
\begin{array}{ccc}
D & \longrightarrow & * \\
\downarrow K & \omega & \downarrow R \\
[E, X] & \to & [E, A]
\end{array}
\]

along the module $(E, M)$ (see Definition 4.3.5) is called

(1) universal if it is a direct universal $(D \ast, (E, M))$-arrow (see Definition 4.6.5);

(2) pointwise universal if each slice

\[
\begin{array}{ccc}
D & \longrightarrow & * \\
\downarrow K & \omega & \downarrow R \\
[E, X] & \to & [E, A]
\end{array}
\]

(see Remark 4.8.8(2)) is a universal cone, i.e. a direct universal $(D \ast, M)$-arrow.

Given a functor $K : D \to [E, X]$, a pointwise universal cone $\omega : K \Rightarrow R : D \ast \Rightarrow (E, M)$ or the pair $(R, \omega)$, or the functor $R$ itself, is called a pointwise colimit of $K$ along $(E, M)$, with the functor $R$ denoted by $\prod^B D K$ (or just by $\prod^E K$).

Proposition 7.2.2.

The following conditions are equivalent:

(1) a cone

\[
\begin{array}{ccc}
* & \leftarrow & ! \\
\downarrow R & \omega & \downarrow K \\
[E, X] & \to & [E, A]
\end{array}
\]

is universal (resp. pointwise universal);
(2) its transpose

\[
\begin{array}{ccc}
E & \rightarrow & E \\
\downarrow R & & \downarrow K^\dagger \\
X & \rightarrow & \omega^\dagger \rightarrow [D, A]
\end{array}
\]

(see Definition 4.8.7) is an inverse universal (resp. pointwise inverse universal) cylinder.

- The following conditions are equivalent:
  (1) a cone

\[
D \overset{!}{\rightarrow} * \\
\downarrow K \downarrow \omega \downarrow R \\
[E, X] \rightarrow \omega^\dagger \rightarrow [E, A]
\]

is universal (resp. pointwise universal);
  (2) its transpose

\[
\begin{array}{ccc}
E & \rightarrow & E \\
\downarrow K^\dagger & & \downarrow R \\
[D, X] & \rightarrow & \omega^\dagger \rightarrow A
\end{array}
\]

(see Definition 4.8.7) is a direct universal (resp. pointwise direct universal) cylinder.

Proof. By the isomorphism in Remark 4.8.8(1), \(\omega\) is universal iff \(\omega^\dagger\) is inverse universal. Since the slice of \(\omega\) at each \(e \in \|E\|\) is given by the component of \(\omega^\dagger\) at \(e\), \(\omega\) is pointwise universal iff \(\omega^\dagger\) is pointwise inverse universal.

Remark 7.2.3. By Proposition 7.2.2 and by the bijectiveness of transposition, we see that given a module \(M : X \rightarrow A\),
  - a limit (resp. pointwise limit) of a functor \(K : D \rightarrow [E, A]\) along the module \((E, M)\) is the same thing as a lift (resp. pointwise lift) of \(K^\dagger : E \rightarrow [D, A]\) inverse along the module \((\ast D, M)\).
  - a colimit (resp. pointwise colimit) of a functor \(K : E \rightarrow [D, X]\) along the module \((E, M)\) is the same thing as a lift (resp. pointwise lift) of \(K^\dagger : E \rightarrow [D, X]\) direct along the module \((D^\ast, M)\).

Proposition 7.2.4. A pointwise universal cone is universal.

Proof. This is reduced to Proposition 6.5.8 by the equivalence of (1) and (2) in Proposition 7.2.2.

Theorem 7.2.5. Let \(E\) and \(D\) be categories and \(M : X \rightarrow A\) be a module.
  - If a functor \(K : D \rightarrow [E, A]\) has a pointwise limit along \((E, M)\), then every limit of \(K\) along \((E, M)\) is pointwise.
  - If a functor \(K : D \rightarrow [E, X]\) has a pointwise colimit along \((E, M)\), then every colimit of \(K\) along \((E, M)\) is pointwise.

Proof. By the equivalence of (1) and (2) in Proposition 7.2.2, this is reduced to an instance of Corollary 6.5.12 where \(M\) is given by the module \((\ast D, M)\).

Theorem 7.2.6. Given a module \(M\) and a functor \(K\) as in Definition 7.2.1, the family of evaluations \((e, M)\), one for each object \(e \in \|E\|\), collectively creates a pointwise limit (resp. colimit) for \(K\) in the following sense:
  - if there is a family of universal cones \(\omega_e : \ast D \rightarrow [e, A] \rightarrow (e, M)\), one for each \(e \in \|E\|\), then there is a unique cone \(\omega : \ast D \rightarrow (e, M)\) such that \(\omega \circ (e, M) = \omega_e\), and \(\omega\) is pointwise universal.
  - if there is a family of universal cones \(\omega_e : K \circ [e, X] \rightarrow \ast D \rightarrow (e, M)\), one for each \(e \in \|E\|\), then there is a unique cone \(\omega : K \circ [e, X] \rightarrow (e, M)\) such that \(\omega \circ (e, M) = \omega_e\), and \(\omega\) is pointwise universal.
Proof. By the equivalence of (1) and (2) in Proposition 7.2.2, this is reduced to an instance of Theorem 6.5.10 where \( \mathcal{M} \) is given by the module \( (\ast \mathcal{D}, \mathcal{M}) \).

Remark 7.2.7. It follows that if \( \mathcal{M} \) has limits (resp. colimits) over \( \mathcal{D} \), so does the module \( (E, \mathcal{M}) \). This fact is described later in Theorem 7.2.13 in terms of units (cf. Proposition 7.1.5).

Definition 7.2.8. Let \( E \) and \( D \) be categories and \( \mathcal{M} : X \to A \) be a module.

- A wedge \( E \xleftarrow{\mathcal{E}^!} E \times D \) is called
  \[
  \begin{array}{c}
  E \xleftarrow{\mathcal{E}^!} E \times D \\
  R \downarrow \omega \downarrow K \\
  X \xrightarrow{\mathcal{M}} \ast \mathcal{A}
  \end{array}
  \]
  (1) universal if it is an inverse universal \( (E \times \ast \mathcal{D}, \mathcal{M}) \)-arrow (see Definition 4.8.3);
  (2) pointwise universal if each left slice
  \[
  \begin{array}{c}
  \ast \xleftarrow{!} D \\
  R(e) \downarrow \omega_e \downarrow K^*(e) \\
  X \xrightarrow{\mathcal{M}} \ast \mathcal{A}
  \end{array}
  \]
  (see Definition 4.8.5) is a universal cone, i.e. an inverse universal \( (\ast \mathcal{D}, \mathcal{M}) \)-arrow.

Given a functor \( K : E \times D \to A \), a pointwise universal wedge \( \omega : R \Rightarrow K : E \times \ast \mathcal{D} \to \mathcal{M} \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called an \( E \)-parameterized limit of \( K \) along \( \mathcal{M} \), with the functor \( R \) denoted by \( \prod^E_D K \) (or just by \( \prod^E K \)).

- A wedge \( E \times D \xrightarrow{\mathcal{E}^!} E \) is called
  \[
  \begin{array}{c}
  E \times D \xrightarrow{\mathcal{E}^!} E \\
  K \downarrow \omega \downarrow R \\
  X \xrightarrow{\mathcal{M}} \ast \mathcal{A}
  \end{array}
  \]
  (1) universal if it is a direct universal \( (E \times \ast \mathcal{D}, \mathcal{M}) \)-arrow (see Definition 4.8.3);
  (2) pointwise universal if each left slice
  \[
  \begin{array}{c}
  D \xrightarrow{!} \ast \\
  K^*(e) \downarrow \omega_e \downarrow R(e) \\
  X \xrightarrow{\mathcal{M}} \ast \mathcal{A}
  \end{array}
  \]
  (see Definition 4.8.5) is a universal cone, i.e. a direct universal \( (\ast \mathcal{D}, \mathcal{M}) \)-arrow.

Given a functor \( K : E \times D \to X \), a pointwise universal wedge \( \omega : K \Rightarrow R : E \times \mathcal{D} \to \mathcal{M} \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called an \( E \)-parameterized colimit of \( K \) along \( \mathcal{M} \), with the functor \( R \) denoted by \( \prod^E_D K \) (or just by \( \prod^E K \)).

Proposition 7.2.9.

- The following conditions are equivalent:
  (1) a wedge
  \[
  \begin{array}{c}
  E \xleftarrow{\mathcal{E}^!} E \times D \\
  R \downarrow \omega \downarrow K \\
  X \xrightarrow{\mathcal{M}} \ast \mathcal{A}
  \end{array}
  \]
  is universal (resp. pointwise universal);

(2) its right exponential transpose
  \[
  \begin{array}{c}
  \ast \xleftarrow{!} D \\
  R \downarrow \omega^* \downarrow K^* \\
  [E, X] \xrightarrow{\mathcal{M}} [E, A]
  \end{array}
  \]
  (see Definition 4.8.5) is a universal (resp. pointwise universal) cone;
(3) its left exponential transpose

\[
\begin{array}{c}
E \\ \downarrow \omega^- \\
\cdashline{1-2}
X \\
\end{array} \\
\begin{array}{c}
\llap{[D, A]} \\
\end{array}
\]

(see Definition 4.8.5) is an inverse universal (resp. pointwise inverse universal) cylinder.

The following conditions are equivalent:

(1) a wedge

\[
\begin{array}{c}
E \times D \\ \llap{[D, A]} \\ \llap{[E, X]}
\end{array} \\
\begin{array}{c}
\llap{[E, A]} \\
\end{array}
\]

is universal (resp. pointwise universal);

(2) its right exponential transpose

\[
\begin{array}{c}
D \\
\llap{[E, A]}
\end{array} \\
\begin{array}{c}
\llap{[E, X]} \\
\end{array}
\]

(see Definition 4.8.5) is a universal (resp. pointwise universal) cone;

(3) its left exponential transpose

\[
\begin{array}{c}
E \\ \downarrow \omega^- \\
\cdashline{1-2}
X \\
\end{array} \\
\begin{array}{c}
\llap{[D, A]} \\
\end{array}
\]

(see Definition 4.8.5) is a direct universal (resp. pointwise direct universal) cylinder.

Proof. By the isomorphisms in Remark 4.8.6, \(\omega\) is universal iff \(\omega^-\) is universal iff \(\omega^+\) is inverse universal. Since the left slice of \(\omega\) at each \(e \in [E]\) is given by the component of \(\omega^-\) at \(e\), \(\omega\) is pointwise universal iff \(\omega^-\) is pointwise inverse universal. By Proposition 7.2.2, \(\omega^-\) is pointwise universal iff \([\omega^-]^+ = \omega^-\) by the commutative diagram in Remark 4.8.8(1). \(\Box\)

Remark 7.2.10. By Proposition 7.2.9 and by the bijectiveness of exponential transposition, we see that given a module \(\mathcal{M} : X \rightarrow A\),

- the following are the same thing:
  1. an \(E\)-parameterized limit of a functor \(K : E \times D \rightarrow A\) along \(\mathcal{M}\);
  2. a pointwise limit of \(K^- : D \rightarrow [E, A]\) along the module \(\langle E, \mathcal{M}\rangle\);
  3. a pointwise lift of \(K^- : E \rightarrow [D, A]\) inverse along the module \(\langle *D, \mathcal{M}\rangle\).

- the following are the same thing:
  1. an \(E\)-parameterized colimit of a functor \(K : E \times D \rightarrow X\) along \(\mathcal{M}\);
  2. a pointwise colimit of \(K^- : D \rightarrow [E, X]\) along the module \(\langle E, \mathcal{M}\rangle\);
  3. a pointwise lift of \(K^- : E \rightarrow [D, X]\) direct along the module \(\langle D*, \mathcal{M}\rangle\).

Proposition 7.2.11. A pointwise universal wedge is universal.

Proof. This is reduced to Proposition 6.5.8 by the equivalence of (1) and (3) in Proposition 7.2.9. \(\Box\)

Theorem 7.2.12. Let \(\mathcal{M} : X \rightarrow A\) be a module.

- If a wedge \(\omega : R \rightarrow K : E \times *D \rightarrow \mathcal{M}\) is universal (resp. pointwise inverse universal), then a natural transformation \(\tau : S \rightarrow R\) is iso if and only if the wedge \(\tau \circ \omega : S \rightarrow K : E \times *D \rightarrow \mathcal{M}\) is universal (resp. pointwise inverse universal).
If a wedge \( \omega : K \Rightarrow R : E \times D \Rightarrow M \) is universal (resp. pointwise direct universal), then a natural transformation \( \tau : R \Rightarrow S : E \times D \Rightarrow M \) is iso if and only if the wedge \( \omega \circ \tau : K \Rightarrow S : E \times D \Rightarrow M \) is universal (resp. pointwise direct universal).

**Proof.** This is reduced to Theorem 6.5.11 by the equivalence of (1) and (3) in Proposition 7.2.9: the statement faithfully translates, along the iso cell \( \dashv \) in Remark 4.8.6, into (hence reflects) an instance of Theorem 6.5.11 where \( M \) is given by the module \( \ast D, M \).

**Theorem 7.2.13.** Let \( E \) and \( D \) be categories and let \( M : X \Rightarrow A \) be a module.

- Suppose that \( M \) has limits over \( D \); that is (see Proposition 7.1.5), suppose that the module \( \ast D, M \) has a counit \( X \Rightarrow \sum_{D} M \).

\[
\begin{array}{c}
X \xrightarrow{\prod} [D, A]
\end{array}
\]

Then this counit yields

1. a counit

\[
\begin{array}{c}
[E, X] \xrightarrow{\prod_{D} \varphi_{E}} [E \times D, A]
\end{array}
\]

of the module of wedges \( E \times \ast D \Rightarrow M \), giving, for each functor \( K : E \times D \Rightarrow A \), a universal wedge

\[
\varphi^{E}_{K} : \prod_{D} K \Rightarrow K : E \times \ast D \Rightarrow M
\]

, in fact a pointwise universal wedge, with each left slice

\[
\prod_{D} K \Rightarrow \prod_{D} [K^{e} (e)] = \varphi_{K^{e} (e)}
\]

giving a limit of the functor \( K^{e} (e) : D \Rightarrow A \).

2. a counit

\[
\begin{array}{c}
[E, X] \xrightarrow{\prod_{D} \varphi_{E}} [D, [E, A]]
\end{array}
\]

of the module of cones \( \ast D \Rightarrow [E, M] \), giving, for each functor \( K : D \Rightarrow [E, A] \), a universal cone

\[
\varphi^{E}_{K} : \prod_{D} K \Rightarrow \ast D \Rightarrow [E, M]
\]

, in fact a pointwise universal cone, with each slice

\[
\prod_{D} K \Rightarrow \prod_{D} [K^{e} (e)] = \varphi_{K^{e} (e)}
\]

giving a limit of the functor \( K^{e} (e) : D \Rightarrow A \).

- Suppose that \( M \) has colimits over \( D \); that is (see Proposition 7.1.5), suppose that the module \( \ast D, M \) has a unit

\[
\begin{array}{c}
[D, X] \xrightarrow{\prod_{D} \varphi_{E}} A
\end{array}
\]

. Then this unit yields
(1) a unit

\[ [E \times D, X] \xrightarrow{\varphi^E} \Pi^E [E, A] \]

of the module of wedges \( E \times D \Rightarrow M \), giving, for each functor \( K : E \times D \to X \), a universal wedge

\[ [\varphi^E]^E_K : \prod^E K \Rightarrow K : E \times D \Rightarrow M \]

, in fact pointwise universal wedge, with each slice

\[ \left[ \prod^E K \right] (e) = \prod E [K^e (e)] \quad \left[ \left[ [\varphi^E]^E \right] \right] (e) = \varphi^E (e) \]

giving a colimit of the functor \( K^e : D \to X \).

(2) a unit

\[ [D, [E, X]] \xrightarrow{\varphi^E} \Pi^E [E, A] \]

of the module of cones \( D \Rightarrow (E, M) \), giving, for each functor \( K : D \to [E, X] \), a universal cone

\[ [\varphi^E]^E_K : \prod^E K \Rightarrow K : D \Rightarrow (E, M) \]

, in fact a pointwise universal cone, with each slice

\[ \left[ \prod^E K \right] (e) = \prod E [K^e (e)] \quad \left[ \left[ [\varphi^E]^E \right] \right] (e) = \varphi^E (e) \]

giving a colimit of the functor \( K^e : D \to X \).

**Proof.** By Corollary 6.5.17, a counit \( \varphi \) of the module \( (*D, M) \) yields a counit

\[ [E, X] \xrightarrow{\varphi^E} \Pi^E [E, A] \]

of the module \( (E, (*D, M)) \), and, from this counit, the iso cells in Remark 4.8.6 and Remark 4.8.8(1) create a counit \( \varphi^E \) of the module \( (E \times *D, M) \) and a counit \( \varphi^E \) of the module \( (*D, (E, M)) \).

**Corollary 7.2.14.** If a module \( M : X \Rightarrow A \) has limits (resp. colimits) over a category \( D \), so is the module \( (E, M) \) for any category \( E \), and every limit (resp. colimit) over \( D \) is given pointwise.

**Proof.** By Theorem 7.2.13, every functor \( K : D \to [E, A] \) has a pointwise limit \( \prod^E K \) along \( (E, M) \), and every limit of \( K \) is pointwise by Theorem 7.2.5.

**Theorem 7.2.15.** If a module \( M : X \Rightarrow A \) is complete (resp. cocomplete), so is the module \( (E, M) \) for any category \( E \), and all small limits (resp. colimits) are given pointwise.

**Proof.** Immediate from Corollary 7.2.14.

**Theorem 7.2.16.** If a module \( M : X \Rightarrow A \) is complete (resp. cocomplete), then for any functor \( H : E' \Rightarrow E \) the precomposition cell

\[ [E, X] \xrightarrow{(E,M)} [E, A] \]
\[ [H \times X] \xrightarrow{(H,M)} [H, A] \]
\[ [E', X] \xrightarrow{(E',M)} [E', A] \]

(see Definition 4.3.27) is continuous (resp. cocomplete).
7.3. Limits in a category

Note. A limit of a functor in a category is defined as below as a special case of Definition 7.1.1 where \( \mathcal{M} \) is given by the hom of a category and coincides with the usual definition of a limit in the literature.

Definition 7.3.1.

- A cone \( \pi : \mathbf{r} \to \mathbf{K} : \mathbf{E} \to \mathbf{C} \) is called universal if it is an inverse universal \( \langle \mathbf{E}, \mathbf{C} \rangle \)-arrow (see Definition 4.9.3). Given a functor \( \mathbf{K} : \mathbf{E} \to \mathbf{C} \), a universal cone \( \pi : \mathbf{r} \to \mathbf{K} : \mathbf{E} \to \mathbf{C} \) or the pair \( (\mathbf{r}, \pi) \), or the object \( \mathbf{r} \) itself, is called a limit of \( \mathbf{K} \) in \( \mathbf{C} \).

- A cone \( \pi : \mathbf{K} \to \mathbf{r} : \mathbf{E}^* \to \mathbf{C} \) is called universal if it is a direct universal \( \langle \mathbf{E}^*, \mathbf{C} \rangle \)-arrow (see Definition 4.9.3). Given a functor \( \mathbf{K} : \mathbf{E} \to \mathbf{C} \), a universal cone \( \pi : \mathbf{K} \to \mathbf{r} : \mathbf{E}^* \to \mathbf{C} \) or the pair \( (\mathbf{r}, \pi) \), or the object \( \mathbf{r} \) itself, is called a colimit of \( \mathbf{K} \) in \( \mathbf{C} \).

Remark 7.3.2. A cone \( \pi : \mathbf{r} \to \mathbf{K} : \mathbf{E} \to \mathbf{C} \) is universal if and only if to every cone \( \alpha : \mathbf{s} \to \mathbf{K} : \mathbf{E} \to \mathbf{C} \) there is a unique \( \mathbf{C} \)-arrow \( \alpha / \pi : \mathbf{s} \to \mathbf{r} \) such that \( \alpha = \alpha / \pi \circ \pi \). Dually, a cone \( \pi : \mathbf{K} \to \mathbf{r} : \mathbf{E}^* \to \mathbf{C} \) is universal if and only if to every cone \( \alpha : \mathbf{K} \to \mathbf{t} : \mathbf{E}^* \to \mathbf{C} \) there is a unique \( \mathbf{C} \)-arrow \( \pi \backslash \alpha : \mathbf{r} \to \mathbf{t} \) such that \( \alpha = \pi \circ \pi \backslash \alpha \).

Definition 7.3.3. A category \( \mathbf{C} \) is said to have limits (resp. colimits) over a category \( \mathbf{E} \) if every functor \( \mathbf{E} \to \mathbf{C} \) has a limit (resp. colimit) in \( \mathbf{C} \).

Note. The following is a special case of Proposition 7.1.5 where \( \mathcal{M} \) is given by the hom of a category.

Proposition 7.3.4.

- A category \( \mathbf{C} \) has limits over a category \( \mathbf{E} \) if and only if the module \( \langle \mathbf{E}, \mathbf{C} \rangle \) has a counit

\[
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\Pi} & \mathbf{[E, C]} \\
\langle \mathbf{E}, \mathbf{C} \rangle & \xrightarrow{\pi} & \\
\end{array}
\]

- A category \( \mathbf{C} \) has colimits over a category \( \mathbf{E} \) if and only if the module \( \langle \mathbf{E}^*, \mathbf{C} \rangle \) has a unit

\[
\begin{array}{ccc}
\mathbf{[E, C]} & \xleftarrow{\Pi} & \mathbf{C} \\
\langle \mathbf{E}^*, \mathbf{C} \rangle & \xleftarrow{\pi} & \\
\end{array}
\]

Proof. Since a limit of a functor \( \mathbf{K} : \mathbf{E} \to \mathbf{C} \) is defined as an inverse universal \( \langle \mathbf{E}, \mathbf{C} \rangle \)-arrow to \( \mathbf{K} \), the assertion follows from the equivalence of (2) and (4) in Corollary 6.4.11. □
Remark 7.3.5. Since the module \( \text{hom}(\ast E, C) \) (resp. \( \text{hom}(E\ast, C) \)) is represented (resp. corepresented) by the diagonal functor \([!_E, C] \) (see Remark 4.9.4(2)), Proposition 7.3.4 is alternatively stated as below in terms of adjunctions (see Section 8.3).

- A category \( C \) has limits over a category \( E \) if and only if the diagonal functor \([!_E, C] \) has a right adjoint.
- A category \( C \) has colimits over a category \( E \) if and only if the diagonal functor \([!_E, C] \) has a left adjoint.

Definition 7.3.6. A category \( C \) is called complete (resp. cocomplete) if it has limits (resp. colimits) over any small category \( E \).

Remark 7.3.7. A category \( C \) is complete (resp. cocomplete) if and only if the hom \( (C) \) is complete (resp. cocomplete) in the sense of Definition 7.1.6.

Note. The following is a special case of Proposition 7.1.7 where \( M \) is given by the hom of a category.

Proposition 7.3.8.

- A category \( C \) is complete if and only if for any small category \( E \) the module \( \text{hom}(\ast E, C) \) has a counit.
- A category \( C \) is cocomplete if and only if for any small category \( E \) the module \( \text{hom}(E\ast, C) \) has a unit.

Proof. Immediate from Proposition 7.3.4.

Note. The preservation (reflection, creation) of limits by a functor is defined as below as a special case of Definition 7.1.8 where \( \Phi \) is given by the hom of a functor and coincides with the usual notion of preservation (reflection, creation) of limits defined in the literature.

Definition 7.3.9. A functor \( H : C \to B \) is said to

- preserve (reflect, create) limits over a category \( E \) if the postcomposition cell \( (\ast E, H) \) (see Definition 4.9.5) preserves (reflects, creates) inverse universal arrows.
- preserve (reflect, create) colimits over a category \( E \) if the postcomposition cell \( (E\ast, H) \) (see Definition 4.9.5) preserves (reflects, creates) direct universal arrows.

Remark 7.3.10.

1. Since \( (\ast E, H) = (\ast E, (H)) \) (resp. \( (E\ast, H) = (E\ast, (H)) \)), a functor \( H \) preserves (reflects, creates) limits (resp. colimits) over a category \( E \) if and only if the hom cell \( (H) \) does the same in the sense of Definition 7.1.8.

2. A functor \( H \) is said to

- preserve (reflect, create) limits (resp. small limits) if it preserves (reflects, creates) limits over any category (resp. small category).
- preserve (reflect, create) colimits (resp. small colimits) if it preserves (reflects, creates) colimits over any category (resp. small category).

3. We also say that functor \( H : C \to B \) is

- continuous if \( C \) is complete and \( H \) preserves small limits.
- cocontinuous if \( C \) is cocomplete and \( H \) preserves small colimits.

Note. Proposition 7.3.11 and Proposition 7.3.12 below are special cases of Proposition 7.1.11 and Proposition 7.1.12 where \( \Phi \) is given by the hom of a functor.

Proposition 7.3.11. Let \( H : C \to B \) be a functor.

- If \( B \) has limits over \( E \) and \( H \) creates limits from them, then \( C \) has limits over \( E \) as well and \( H \) preserves them.
- If \( B \) has colimits over \( E \) and \( H \) creates colimits from them, then \( C \) has colimits over \( E \) as well and \( H \) preserves them.
 Proof. Noting Proposition 7.3.4, we see that this is an instance of Theorem 6.4.12 where \( \Phi \) is given by the postcomposition cell \( \ast \mathcal{E}, \mathcal{H} \).

\[ \square \]

**Proposition 7.3.12.** Let \( \mathcal{H} : \mathcal{C} \to \mathcal{B} \) be a functor.

- If \( \mathcal{B} \) is complete and \( \mathcal{H} \) creates small limits, then \( \mathcal{C} \) is complete as well and \( \mathcal{H} \) is continuous.
- If \( \mathcal{B} \) is cocomplete and \( \mathcal{H} \) creates small colimits, then \( \mathcal{C} \) is cocomplete as well and \( \mathcal{H} \) is cocomplete.

**Proof.** Immediate from 7.3.11.

\[ \square \]

**Theorem 7.3.13.** Let \( \mathcal{M} : \mathcal{X} \to \mathcal{A} \) be a module.

- Any corepresentation \( X \xrightarrow{\mathcal{M}} \mathcal{A} \) of \( \mathcal{M} \) preserves, reflects, and creates limits.

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{\mathcal{M}} \mathcal{A} \\
\downarrow \mathcal{Y} \quad \downarrow \mathcal{R} \\
\mathcal{X} \xrightarrow{(\mathcal{Y})} \mathcal{X}
\end{array}
\]

- Any representation \( X \xrightarrow{\mathcal{M}} \mathcal{A} \) of \( \mathcal{M} \) preserves, reflects, and creates colimits.

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{\mathcal{M}} \mathcal{A} \\
\downarrow \mathcal{R} \quad \downarrow \mathcal{Y} \\
\mathcal{A} \xrightarrow{(\mathcal{Y})} \mathcal{A}
\end{array}
\]

**Proof.** Immediate from Proposition 7.1.10.

\[ \square \]

**Corollary 7.3.14.** Let \( \mathcal{M} : \mathcal{X} \to \mathcal{A} \) be a module.

- If \( \mathcal{M} \) is corepresentable and \( \mathcal{X} \) is complete, then \( \mathcal{M} \) is complete as well.
- If \( \mathcal{M} \) is representable and \( \mathcal{A} \) is cocomplete, then \( \mathcal{M} \) is cocomplete as well.

**Proof.** Since a corepresentation of \( \mathcal{M} \) creates limits by Theorem 7.3.13, the assertion follows from Proposition 7.1.12.

\[ \square \]

**Note.** The following is a special case of Theorem 7.2.6 where \( \mathcal{M} \) is given by the hom of a category.

**Theorem 7.3.15.** Given a functor \( \mathcal{K} : \mathcal{D} \to [\mathcal{E}, \mathcal{C}] \), the family of evaluations \( [\mathcal{e}, \mathcal{C}] : [\mathcal{E}, \mathcal{C}] \to \mathcal{C} \), one for each object \( \mathcal{e} \in \mathcal{E} \), collectively creates a pointwise limit (resp. colimit) for \( \mathcal{K} \) in the following sense:

- if there is a family of universal cones \( \omega_{\mathcal{e}} : \mathcal{R}_{\mathcal{e}} \sim [\mathcal{e}, \mathcal{C}] \circ \mathcal{K} : \ast \mathcal{D} \to \mathcal{C} \), one for each \( \mathcal{e} \in \mathcal{E} \), then there is a unique cone \( \omega : \mathcal{R} \sim \mathcal{K} : \ast \mathcal{D} \to [\mathcal{E}, \mathcal{C}] \) such that \( \omega \circ [\mathcal{e}, \mathcal{C}] = \omega_{\mathcal{e}} \), and \( \omega \) is pointwise universal.
- if there is a family of universal cones \( \omega_{\mathcal{e}} : \mathcal{K} \circ [\mathcal{e}, \mathcal{C}] \sim \mathcal{R}_{\mathcal{e}} : \mathcal{D}^{\ast} \to \mathcal{C} \), one for each \( \mathcal{e} \in \mathcal{E} \), then there is a unique cone \( \omega : \mathcal{K} \sim \mathcal{R} : \mathcal{D}^{\ast} \to [\mathcal{E}, \mathcal{C}] \) such that \( \omega \circ [\mathcal{e}, \mathcal{C}] = \omega_{\mathcal{e}} \), and \( \omega \) is pointwise universal.

**Proof.** Since a cone in \( \mathcal{C} \) (resp. \( [\mathcal{E}, \mathcal{C}] \)) is the same thing as a cone along the hom \( \langle \mathcal{C} \rangle \) (resp. \( [\mathcal{E}, \mathcal{C}] \)), the assertion follows from Theorem 7.2.6 by replacing \( \mathcal{M} \) with the hom of \( \mathcal{C} \) and recalling that \( \langle \mathcal{E}, \langle \mathcal{C} \rangle \rangle = \langle \mathcal{E}, \mathcal{C} \rangle \) (see Remark 4.3.6(3)) and \( \omega \circ \langle \mathcal{e}, \langle \mathcal{C} \rangle \rangle = \omega \circ [\mathcal{e}, \mathcal{C}] \) (see Remark 4.3.14(2) and Remark 4.3.31).

\[ \square \]

**Note.** Theorem 7.3.16 and Theorem 7.3.17 below are special cases of Theorem 7.2.15 and Theorem 7.2.16 where \( \mathcal{M} \) is given by the hom of a category.

**Theorem 7.3.16.** If a category \( \mathcal{C} \) is complete (resp. cocomplete), so is the functor category \( [\mathcal{E}, \mathcal{C}] \) for any category \( \mathcal{E} \), and all small limits (resp. colimits) are given pointwise.

**Proof.** Since \( \mathcal{C} \) (resp. \( [\mathcal{E}, \mathcal{C}] \)) is complete iff its hom \( \langle \mathcal{C} \rangle \) (resp. \( [\mathcal{E}, \mathcal{C}] \)) is complete in the sense of Definition 7.1.6, the assertion follows from Theorem 7.2.15 by replacing \( \mathcal{M} \) with the hom of \( \mathcal{C} \) and recalling that \( \langle \mathcal{E}, \langle \mathcal{C} \rangle \rangle = \langle \mathcal{E}, \mathcal{C} \rangle \) (see Remark 4.3.6(3)).

\[ \square \]
Theorem 7.3.17. If a category $\mathcal{C}$ is complete (resp. cocomplete), then for any functor $\mathbf{H}: \mathbf{E}' \to \mathbf{E}$ the precomposition functor $[\mathbf{H}, \mathcal{C}]: [\mathbf{E}, \mathcal{C}] \to [\mathbf{E}', \mathcal{C}]$ is continuous (resp. cocontinuous).

Proof. Since $\mathcal{C}$ is complete iff its hom $(\mathcal{C})$ is complete in the sense of Definition 7.1.6, and $[\mathbf{H}, \mathcal{C}]$ is continuous iff its hom $(\mathcal{H})$ is continuous in the sense of Definition 7.1.8, the assertion follows from Theorem 7.2.16 by replacing $\mathcal{M}$ with the hom of $\mathcal{C}$ and recalling that $(\mathbf{H}, (\mathcal{C})) = (\mathbf{H}, (\mathcal{C}))$ (see Remark 4.3.28(3)).

7.4. Limits of modules

Theorem 7.4.1. Given a small left module $\mathcal{M}: \star \to \mathbf{E}$, i.e. a functor $\mathcal{M}: \mathbf{E} \to \mathbf{Set}$ with $\mathbf{E}$ small,

- a limit of $\mathcal{M}$ is given by an equalizer $\pi$ as in

$$\begin{array}{ccc}
\prod_{\mathbf{E}} \mathcal{M} & \xrightarrow{\pi} & \prod_{e \in \mathbf{E}} \langle \mathcal{M} \rangle e
\\
\downarrow & & \downarrow \pi
\\
\prod_{e' \in \mathbf{E}} \langle \mathcal{M} \rangle e' & \xrightarrow{\prod_{e' \in \mathbf{E}} (\Delta_{e,e'})_{e \in \mathbf{E}}} & \prod_{e \in \mathbf{E}} \prod_{e' \in \mathbf{E}} \langle e \mathbf{E} \rangle e', \langle \mathcal{M} \rangle e'
\end{array}$$

, where

$$\chi_{e,e'}: \langle \mathcal{M} \rangle e \to \langle e \mathbf{E} \rangle e', \langle \mathcal{M} \rangle e'; m \mapsto [h \mapsto m \circ h]$$

is the function given by the exponential transpose of the composition between $\mathcal{M}$-arrows and $\mathbf{E}$-arrows, and

$$\Delta_{e,e'}: \langle \mathcal{M} \rangle e' \to \langle e \mathbf{E} \rangle e', \langle \mathcal{M} \rangle e'; m \mapsto [h \mapsto m]$$

is the diagonal function.

- a colimit of $\mathcal{M}$ is given by a coequalizer $\pi$ as in

$$\begin{array}{ccc}
\coprod_{e \in \mathbf{E}} \mathcal{M} & \xrightarrow{\coprod_{e \in \mathbf{E}} (\chi_{e,e'})_{e \in \mathbf{E}}} & \coprod_{e' \in \mathbf{E}} \langle \mathcal{M} \rangle e'
\\
\downarrow & & \downarrow \pi
\\
\coprod_{e' \in \mathbf{E}} \langle \mathcal{M} \rangle e' & \xrightarrow{\coprod_{e' \in \mathbf{E}} (\Delta_{e,e'})_{e \in \mathbf{E}}} & \coprod_{e \in \mathbf{E}} \prod_{e' \in \mathbf{E}} \langle e \mathbf{E} \rangle e', \langle \mathcal{M} \rangle e'
\end{array}$$

, where

$$\chi_{e,e'}: \langle \mathcal{M} \rangle e \times \langle e \mathbf{E} \rangle e' \to \langle \mathcal{M} \rangle e'; (m, h) \mapsto m \circ h$$

is the function given by the composition between $\mathcal{M}$-arrows and $\mathbf{E}$-arrows, and

$$\Delta_{e,e'}: \langle \mathcal{M} \rangle e \times \langle e \mathbf{E} \rangle e' \to \langle \mathcal{M} \rangle e; (m, h) \mapsto m$$

is the projection.

Proof. By the claim below, a limit of $\mathcal{M}$ is given by a universal fork on $\prod_{e \in \mathbf{E}} (\chi_{e,e'})_{e \in \mathbf{E}}$ and $\prod_{e' \in \mathbf{E}} (\Delta_{e,e'})_{e \in \mathbf{E}}$, i.e. by an equalizer of them. 

Claim. Given a small set $S$, a family of functions $\alpha_e: S \to \langle \mathcal{M} \rangle e$, one for each $e \in \mathbf{E}$, forms a cone $S \to \mathcal{M}$ if and only if the diagram

$$\begin{array}{ccc}
S & \xrightarrow{(\alpha_e)_{e \in \mathbf{E}}} & \prod_{e \in \mathbf{E}} \langle \mathcal{M} \rangle e
\\
\downarrow (\alpha_{e'})_{e' \in \mathbf{E}} & & \downarrow \prod_{e' \in \mathbf{E}} (\chi_{e,e'})_{e' \in \mathbf{E}}
\\
\prod_{e' \in \mathbf{E}} \langle \mathcal{M} \rangle e' & \xrightarrow{\prod_{e' \in \mathbf{E}} (\Delta_{e,e'})_{e \in \mathbf{E}}} & \prod_{e \in \mathbf{E}} \prod_{e' \in \mathbf{E}} \langle e \mathbf{E} \rangle e', \langle \mathcal{M} \rangle e'
\end{array}$$

commutes, i.e. if and only if the function $\alpha_S: S \to \prod_{e \in \mathbf{E}} \langle \mathcal{M} \rangle e$ forms a fork on the functions $\prod_{e \in \mathbf{E}} (\chi_{e,e'})_{e \in \mathbf{E}}$ and $\prod_{e' \in \mathbf{E}} (\Delta_{e,e'})_{e \in \mathbf{E}}$. 

Proof. The diagram in the claim commutes iff the diagram

\[
\begin{array}{c}
S \\
\downarrow^{(\alpha_e')_e\epsilon|E|} \\
\prod_{e'\in|E|}(\mathcal{M})e' \downarrow^{\chi_{e,e'}} \prod_{e'\in|E|}[e\mathcal{E}e',(\mathcal{M})e'] \\
\end{array}
\]

commutes for each \( e \in |E| \), and this diagram commutes iff the diagram

\[
\begin{array}{c}
S \\
\downarrow^{\alpha_e'} \\
(\mathcal{M})e' \downarrow^{\Delta_{e,e'}} [e\mathcal{E}e',(\mathcal{M})e']
\end{array}
\]

commutes for each \( e' \in |E| \), and this diagram commutes iff the triangle

\[
\begin{array}{c}
S \\
\downarrow^{\alpha_e} \\
\downarrow^{\alpha_e'} (\mathcal{M})e
\end{array}
\]

commutes for each \( E \)-arrow \( h : e \to e' \) since for any \( s \in S \),

\[
s : [\alpha_e \circ (\mathcal{M})h] = (s : \alpha_e) \circ h = h : [(s : \alpha_e) \circ \chi_{e,e'}] = h : [s : [\alpha_e \circ \chi_{e,e'}]]
\]

and

\[
s : \alpha_{e'} = h : [(s : \alpha_{e'}) \circ \Delta_{e,e'}] = h : [s : [\alpha_{e'} \circ \Delta_{e,e'}]]
\]

by the definitions of \( \chi_{e,e'} \) and \( \Delta_{e,e'} \).

Corollary 7.4.2. Set is complete and cocomplete.

Proof. Since Set has equalizers and coequalizers, the assertion follows from Theorem 7.4.1.

Corollary 7.4.3. For any pair of categories \( X \) and \( A \), the module category \( [X : A] \) is complete and cocomplete, and all small limits and colimits are given pointwise.

Proof. Since Set is complete and cocomplete, so is the category \( [X : A] := [X^{-} \times A, Set] \) by Theorem 7.3.16.

Corollary 7.4.4. For any pair of functors \( S : E \to X \) and \( T : D \to A \), the precomposition functor \( [S : T] : [X : A] \to [E : D] \) is continuous and cocontinuous.

Proof. Since Set is complete and cocomplete, the precomposition functor \( [S : T] := [S^{-} \times T, Set] \) is continuous and cocontinuous by Theorem 7.3.17.

Theorem 7.4.5.
- For a small left module \( \mathcal{M} : * \to E \), i.e. a functor \( \mathcal{M} : E \to Set \) with \( E \) small, the set of frames of \( \mathcal{M} \) gives a limit of \( \mathcal{M} \) with the universal cone

\[
\varphi_{\mathcal{M}} : \prod_{\mathcal{M} \to \mathcal{M} : * \to *E}
\]

defined by

\[
\alpha : (\varphi_{\mathcal{M}})e = \alpha_e
\]

such that the component of \( \varphi_{\mathcal{M}} \) at \( e \in |E| \) sends each frame \( \alpha \) of \( \mathcal{M} \) to its component at \( e \).
For a small right module $M : E \to \ast$, i.e. a functor $M : E^\to \to \text{Set}$ with $E$ small, the set of frames of $M$ gives a limit of $M$ with the universal cone

$$\varphi_M : \prod_{E^\ast} M \sim M : E^\ast \to \ast$$

defined by

$$\alpha : e(\varphi_M) = \alpha_e$$

such that the component of $\varphi_M$ at $e \in \|E\|$ sends each frame $\alpha$ of $M$ to its component at $e$.

**Proof.** This follows from the observation that an element $\alpha$ of the product $\prod_{e \in \|E\|} (M)_e$ forms a frame of $M$ iff it lies in the equalizer in Theorem 7.4.1. \qed

**Remark 7.4.6.** We thus have

$$\prod_{E^\ast} M \cong \prod_{E} M \quad \prod_{E^\ast} M \cong \prod_{E^\ast} M .$$

**Corollary 7.4.7.** Let $E$ be a small category.

- The family of universal cones $\varphi_M : \prod_{E^\ast} M \sim M : E^\ast \to \ast$, one for each left module $M : E \to \ast$, forms a counit

$$\text{Set} \xrightarrow{\prod_{E^\ast}} [E :]$$

of the module $\langle E^\ast \rangle$.

- The family of universal cones $\varphi_M : \prod_{E^\ast} M \sim M : E \to \ast$, one for each right module $M : E \to \ast$, forms a counit

$$\text{Set} \xrightarrow{\prod_{E^\ast}} [E :]$$

of the module $\langle E^\ast \rangle$.

**Proof.** By Theorem 7.4.5, each component of $\varphi$ is universal. Hence it remains to verify that $\varphi$ satisfies the naturality condition, i.e. verify that the square

$$\begin{array}{ccc}
\prod_{E^\ast} M & \sim & M \\
\Phi : & & \downarrow \Phi \\
\prod_{E^\ast} N & \sim & N
\end{array}$$

commutes for any left module morphism $\Phi : M \to N$. But for any frame $\alpha \in \prod_{E^\ast} M$ and any object $e \in \|E\|$, 

$$\alpha : \langle \varphi_M \circ \Phi \rangle e = \alpha : \langle \varphi_M \rangle e : \langle \Phi \rangle e = \alpha_e : \langle \Phi \rangle e$$

and

$$\alpha : \left( \prod_{E^\ast} \Phi \circ \varphi_N \right) e = \alpha : \prod_{E^\ast} \Phi : \langle \varphi_N \rangle e = [\alpha \circ \Phi] : \langle \varphi_N \rangle e = [\alpha \circ \Phi]_e = \alpha_e : \langle \Phi \rangle e$$

\qed

**Corollary 7.4.8.** Let $E$ be a small category.
Given a category $X$, the module $(X: *E)$ has a counit

$$\left[\pi^X_{\e}\right] [X:] \xrightarrow[e]{\epsilon^X_{\e}} [X:E]$$

, giving, for each module $\mathcal{M} : X \rightarrow E$, a universal wadge

$$\left[\varphi^X\right]_{\mathcal{M}} : \prod_{\e} X \Rightarrow M : X \rightarrow *E$$

, in fact a pointwise universal wadge, with each slice

$$\prod_{\e} \left(x\left(\prod_{\e} M\right)\right) = \prod_{\e} x(M) \quad \left(\left[\varphi^X\right]_{\mathcal{M}}\right) \cdot x = \varphi_{x(M)}$$

giving a limit of the left module $x(M) : * \rightarrow E$.

Given a category $A$, the module $(E*: A)$ has a counit

$$\left[\pi^A_{\e}\right] [:A] \xrightarrow[e]{\epsilon^A_{\e}} [E:A]$$

, giving, for each module $\mathcal{M} : E \rightarrow A$, a universal wadge

$$\left[\varphi^A\right]_{\mathcal{M}} : \prod_{E*} A \Rightarrow M : E* \rightarrow A$$

, in fact a pointwise universal wadge, with each slice

$$\prod_{E*} \left(\prod_{E*} M\right) = \prod_{E*} (M) \cdot a \quad \left(\left[\varphi^A\right]_{\mathcal{M}}\right) \cdot a = \varphi_{(M)a}$$

giving a limit of the right module $(M) a : E \rightarrow *$.

Proof. Apply Theorem 7.2.13 to the counit in Corollary 7.4.7.

Remark 7.4.9.
(1) By Remark 4.3.2(4), the counit

$$\left[\pi^A_{\e}\right] [:A] \xrightarrow[e]{\epsilon^A_{\e}} [E:A]$$

is the same thing as the unit

$$[E:A] \xrightarrow[\pi^A_{\e}] \left[\varphi^X\right]_{\mathcal{M}} : \prod_{\e} X \xrightarrow[e]{\epsilon^X_{\e}} [X:E]$$

(2) Given a small category $E$ and a module $\mathcal{M} : X \rightarrow A$, the triangle

$$\left[\pi^X_{\e}\right] [X:] \xrightarrow[e]{\epsilon^X_{\e}} [X:E]$$

$$\left[\pi^A_{\e}\right] [:A] \xrightarrow[e]{\epsilon^A_{\e}} [E:A]$$

$$\left[\varphi^X\right]_{\mathcal{M}} : \prod_{\e} X \Rightarrow M : X \rightarrow *E$$

$$\left[\varphi^A\right]_{\mathcal{M}} : \prod_{E*} A \Rightarrow M : E* \rightarrow A$$
commutes, where \((\ast E, M) \cdot \triangleright\) is the right exponential transpose of the module \((\ast E, M) : X \to [E, A]\) and \(M \cdot \triangleright E\) is the right action of \(M\) on \([E, A]\). Indeed, for any \(K \in [E, A]\) and any \(x \in X\),
\[
(x)(\ast E, M)(K) = \prod_{E} x(M)K = x\left(\prod_{E} (M)K\right)
\]
. Dually, the triangle
\[
\begin{array}{ccc}
[E, X] & \xrightarrow{\phi X} & [E : A] \\
\downarrow \cong & & \downarrow \cong \\
\prod_{E}^A & \xrightarrow{\cdot A} & [ : A]
\end{array}
\]
commutes.

### 7.5. Limits and Yoneda morphisms

**Note.** Remark 7.4.9 allows the following definition.

**Definition 7.5.1.** Given a small category \(E\) and a module \(M : X \to A\),

- the cylinder

\[
\begin{array}{ccc}
[X] & \xleftarrow{\prod_{E}^X} & [X : E] \\
\downarrow \cong & & \downarrow \cong \\
\prod_{E}^X & \xleftarrow{\cdot E} & [ : A]
\end{array}
\]

is defined by the composition

\[
[X] \xleftarrow{\prod_{E}^X} [X : E] \xrightarrow{M \cdot \triangleright E} [E, A]
\]

(see Definition 4.3.25).

- the cylinder

\[
\begin{array}{ccc}
[E, X] & \xrightarrow{E \cdot \triangleright M} & [E : A] \\
\downarrow \cong & & \downarrow \cong \\
\prod_{E}^E & \xrightarrow{\cdot E} & [ : A]
\end{array}
\]

is defined by the composition

\[
[E, X] \xrightarrow{E \cdot \triangleright M} [E : A] \xrightarrow{(E \cdot \triangleright A)} [ : A]
\]

(see Definition 4.3.25).

**Remark 7.5.2.** By Theorem 6.5.16, the cylinder \([M \ast E]\) (resp. \([E \ast M]\)) is pointwise inverse (resp. direct) universal;

- the component of \([M \ast E]\) at a functor \(K : E \to A\) is the pointwise universal wadge

\[
[M \ast E]_K : (\ast E, M)(K) \triangleright (M)K : X \to \ast E
\]
given by \([M \ast E]_K = [\varphi X]_{(M)K}\) (the component of the counit \(\varphi X\) at \((M)K\)), and the slice of \([M \ast E]_K\) at \(x \in [X]\) is the universal cone

\[
x([M \ast E]_K) : (x)(\ast E, M)(K) \triangleright x(M)K : \ast \to \ast E
\]
given by \(x([M \ast E]_K) = x(\varphi X)_{(M)K} = \varphi_{x(M)K}\) (see Corollary 7.4.8).
7.5. Limits and Yoneda morphisms

- the component of \([E \ast M]\) at a functor \(K : E \to X\) is the pointwise universal wadge

\[
[E \ast M]_K : (K \langle E \ast, M \rangle) \sim K \langle M \rangle : E \ast \to A
\]

given by \([E \ast M]_K = \varphi^A\) (the component of the unit \(\varphi^A\) at \(K \langle M \rangle\)), and the slice of \([E \ast M]_K\) at \(a \in \|A\|\) is the universal cone

\[
\langle [E \ast M]_K \rangle a : (K \langle E \ast, M \rangle) a \sim K \langle M \rangle a : E \ast \to *
\]

given by \(\langle [E \ast M]_K \rangle a = \langle [\varphi^A]_{K \langle M \rangle} a \rangle\) (see Corollary 7.4.8).

**Note.** Given a category \(E\) and a module \(M : X \to A\), we have two Yoneda morphisms for \(\langle E \ast, M \rangle\),

\[
\begin{align*}
X &\sim \langle E \ast \rangle (\langle E \ast, M \rangle) \sim [E, A] \\
X &\sim (\langle X \ast, E \rangle) (\langle E \ast, M \rangle) \sim X \sim X \ast \sim [X] \ast \sim [X : A]
\end{align*}
\]

, one defined in Definition 5.2.5 and the other in Definition 5.4.1. The following reconciles these two Yoneda morphisms.

**Theorem 7.5.3.** Let \(E\) be a small category and \(M : X \to A\) be a module.

- The right general Yoneda morphism for \(\langle E \ast, M \rangle\) is obtained by “pasting” the cylinder \([M \ast E]\) defined in Definition 7.5.1 to the right Yoneda morphism for \(\langle E \ast, M \rangle\) as shown in

\[
\begin{array}{ccc}
X &\sim \langle E \ast \rangle (\langle X \ast, E \rangle) (\langle E \ast, M \rangle) &\sim [E, A] \\
\xrightarrow{\langle X \rangle} & \xrightarrow{(\langle X \ast, E \rangle) (\langle E \ast, M \rangle)} &\sim X \sim X \ast \sim [X] \ast \sim [X : A]
\end{array}
\]

; that is, given a cone \(\alpha : r \sim K : E \ast \sim M\), the wedge

\[
\langle X \rangle (\langle X \ast, E \rangle) (\langle E \ast, M \rangle) \alpha : (X) r \sim (M) K : X \to E
\]

is obtained by the composition of the right module morphism

\[
\langle X \rangle (\langle E \ast, M \rangle) \alpha : (X) r \to (E \ast, M) \langle K \rangle : X \to *
\]

and the wedge

\[
[M \ast E]_K : (E \ast, M) \langle K \rangle \sim (M) K : X \to E
\]

- The left general Yoneda morphism for \(\langle E \ast, M \rangle\) is obtained by “pasting” the cylinder \([E \ast M]\) defined in Definition 7.5.1 to the left Yoneda morphism for \(\langle E \ast, M \rangle\) as shown in

\[
\begin{array}{ccc}
E \ast \sim \langle E \ast \rangle (\langle E \ast, M \rangle) (\langle E \ast, A \rangle) &\sim [E, A] \\
\xleftarrow{(\langle E \ast, M \rangle)} & \xleftarrow{(\langle E \ast, A \rangle)} &\sim [E : A] \sim [A]
\end{array}
\]

; that is, given a cone \(\alpha : K \sim r : E \ast \sim M\), the wedge

\[
\alpha (\langle M \rangle A) : r (A) \sim K (M) : E \ast \to A
\]
is obtained by the composition of the left module morphism
\[ \alpha((E*,M)|A) : r(A) \rightarrow (K)(E*,M) : * \rightarrow A \]
and the wedge
\[ [E,M]_K : (K)(E*,M) \sim K(M) : E* \rightarrow A \]

**Proof.** We need to show that
\[ x((X|M)|x)e = x([M*E]_x)e \]
for \( x \in [X] \) and \( e \in [E] \). But for an \( X \)-arrow \( h : x \rightarrow r \),
\[ h : x((X|M)|x)e = h \circ \alpha_e \]
and by Remark 7.5.2 and Theorem 7.4.5,
\[ h : x((X|M)|x)e = (h \circ \alpha)(\varphi_{x(M)}_K)e = (h \circ \alpha)_e = h \circ \alpha_e \]

**Theorem 7.5.4.** Let \( E \) be a category and \( M : X \rightarrow A \) be a module.

- The right general Yoneda morphism for \((E*,M)\) preserves and reflects inverse universal arrows in the following sense: a cone \( \pi : r \sim K : *E \sim M \) is universal if and only if the wedge \((X|M)\pi : (X)r \sim (M)K : X \rightarrow *E \) is pointwise universal; that is, if and only if the cone \( x((X|M)|x) \pi : x(X)r \sim x(M)K : * \rightarrow *E \) is universal for every \( x \in [X] \).
- The left general Yoneda morphism for \((E*,M)\) preserves and reflects direct universal arrows in the following sense: a cone \( \pi : K \sim r : *E \sim M \) is universal if and only if the wedge \( \pi(M|A) : r(A) \sim K(M) : E* \rightarrow A \) is pointwise universal; that is, if and only if the cone \( \pi(M|A) \alpha : r(A) \sim K(M) \alpha : E* \rightarrow * \) is universal for every \( \alpha \in [A] \).

**Proof.** First enlarge the universe if necessary so that \( E \) become small. For a cone \( \pi : r \sim K : *E \sim M \), by Theorem 7.5.3, \((X|M)\pi\) is given by the composition of \((X|(*E,M))\pi\) and \([M*E]_K\). Since \([M*E]_K\) is a pointwise universal wedge (see Remark 7.5.2), by Theorem 7.2.12, \((X|M)\pi\) is pointwise universal iff \((X|(*E,M))\pi\) is iso, i.e. iff \( \pi \) is universal.

**Corollary 7.5.5.** Let \( E \) be a category and \( M : X \rightarrow A \) be a module.

- The right Yoneda morphism for \( M \) preserves and reflects limits in the following sense: a cone \( \pi : r \sim K : *E \sim M \) is universal if and only if its composite

\[
\begin{array}{ccc}
        & E & \sim \downarrow \pi \\
X & \sim \downarrow \gamma & \sim \downarrow A \\
\end{array}
\]

with the right Yoneda morphism for \( M \) is pointwise universal (see Definition 7.2.1).

- The left Yoneda morphism for \( M \) preserves and reflects colimits in the following sense: a cone \( \pi : K \sim r : E* \sim M \) is universal if and only if its composite

\[
\begin{array}{ccc}
        & E & \sim \downarrow \pi \\
K & \sim \downarrow r & \sim \downarrow A \\
\end{array}
\]

with the left Yoneda morphism for \( M \) is pointwise universal (see Definition 7.2.1).
Proof. Since the right exponential transpose of the wedge $\langle X \uparrow M \rangle \pi$ is given by the cone $\langle (X \uparrow M) \rightharpoonup \rangle \delta \pi$ (see Remark 5.4.5), by the equivalence of (1) and (2) in Proposition 7.2.9, the assertion is reduced to Theorem 7.5.4 (and vice versa).

Note. Theorem 7.5.6 and Corollary 7.5.7 below are special cases of Theorem 7.5.4 and Corollary 7.5.5 where $M$ is given by the hom of a category.

Theorem 7.5.6. Let $C$ and $E$ be categories.

- The right Yoneda morphism for $(\ast E, C)$ (see Definition 5.4.6) preserves and reflects inverse universal arrows in the following sense: a cone $\pi : r \rightharpoonup K : \ast E \rightarrow C$ is universal if and only if the wedge $\langle C \rangle \pi : \langle C \rangle \rightharpoonup \langle C \rangle K : C \rightarrow \ast E$ is pointwise universal; that is, if and only if the cone $c \langle C \rangle \pi : c \langle C \rangle K : \ast \rightarrow c \ast \ast E$ is universal for every $c \in \parallel C \parallel$.

- The left general Yoneda morphism for $(E \ast, C)$ (see Definition 5.4.6) preserves and reflects direct universal arrows in the following sense: a cone $\pi : K \rightharpoonup r : E \ast \rightarrow C$ is universal if and only if the wedge $\langle r \rangle \pi : \langle r \rangle K : \langle r \rangle C \rightarrow \ast E$ is pointwise universal; that is, if and only if the cone $\langle r \rangle \langle r \rangle c \rightharpoonup \langle r \rangle \langle r \rangle C : \ast \rightarrow \ast \ast E$ is universal for every $c \in \parallel C \parallel$.

Proof. By Remark 5.4.7(2), this is a special case of Theorem 7.5.4 where $M$ is given by the hom of $C$.

Corollary 7.5.7. Let $C$ and $E$ be categories.

- The right Yoneda functor for $C$ preserves and reflects limits in the following sense: a cone $\pi : r \rightharpoonup K : \ast E \rightarrow C$ is universal if and only if its composite

\[ C \xrightarrow{\pi} \ast \xrightarrow{K} E \]

with the right Yoneda functor for $C$ is pointwise universal (see Definition 7.2.1).

- The left Yoneda functor for $C$ preserves and reflects colimits in the following sense: a cone $\pi : K \rightharpoonup r : E \ast \rightarrow C$ is universal if and only if its composite

\[ C \xrightarrow{\pi} \ast \xrightarrow{r} E \]

with the left Yoneda functor for $C$ is pointwise universal (see Definition 7.2.1).

Proof. Since the right exponential transpose of the wedge $\langle C \rangle \pi$ is given by the cone $\langle C \rangle \pi \delta \pi$ (see Remark 5.4.10), by the equivalence of (1) and (2) in Proposition 7.2.9, the assertion is reduced to Theorem 7.5.6 (and vice versa).

Corollary 7.5.8.

- A functor $H : C \rightarrow B$ preserves limits over a category $E$ if and only if for every $b \in \parallel B \parallel$ the left module $b (B) H : \ast \rightarrow C$ does the same.

- A functor $H : C \rightarrow B$ preserves colimits over a category $E$ if and only if for every $b \in \parallel B \parallel$ the right module $H (B) b : C \rightarrow \ast$ does the same.

Proof. By Theorem 7.5.6, for any cone $\pi : \ast E \rightarrow C$, $H \delta \pi$ is universal iff $b (B) [H \delta \pi] = (b (B) H) \pi$ is universal for every $b \in \parallel B \parallel$. 

\[ \square \]
Corollary 7.5.9.

- A representable left module $M : * \to A$ preserves limits; that is, if a cone $\pi : r \to K : *E \to A$ is universal, so is the cone $\langle M \rangle \pi : r \langle M \rangle K : * \to *E$.
- A representable right module $M : X \to *$ preserves colimits; that is, if a cone $\pi : K \to r : E* \to X$ is universal, so is the cone $\pi \langle M \rangle : r \langle M \rangle K : E* \to *$.

Proof. Since $M$ is representable, $M \cong a(A)$ for some $a \in |A|$. But by Theorem 7.5.6, if a cone $\pi : r \to K : *E \to A$ is universal, so is the cone $a(A) \pi : a(A) r \to a(A) K : * \to *E$. □

Note. In the following, a module and the corresponding collage are given the same name and identified with each other.

Corollary 7.5.10.

- A left module $M : * \to A$ preserves limits over a category $E$ if and only if the inclusion $M : A \to [M]$ does the same.
- A right module $M : X \to *$ preserves colimits over a category $E$ if and only if the inclusion $M : X \to [M]$ does the same.

Proof. Since the objects of the collage category $[M]$ consists of all objects of $A$ and the object $*$, by Corollary 7.5.8, the inclusion $M : A \to [M]$ preserves limits over $E$ iff the following conditions hold:

1. for every $a \in |A|$, $a(|M|) M = a(A)$ preserves limits over $E$;

Since the first condition always holds by Corollary 7.5.9, the assertion follows. □

Remark 7.5.11.

- Suppose that a left module $M : * \to A$ preserves limits over a category $E$. Under this condition, Corollary 7.5.10 says that, if a cone $\pi : r \to K : *E \to A$ is universal in $A$, then it is universal in the collage category $[M]$ as well; that is, to every cone $\alpha : * \to K : *E \to M$ there is a unique $M$-arrow $\alpha / \pi : * \to r$, the adjunct of $\alpha$ along $\pi$ in $M$, such that $\alpha = \alpha / \pi \circ \pi$.
- Suppose that a right module $M : X \to *$ preserves colimits over a category $E$. Under this condition, Corollary 7.5.10 says that, if a cone $\pi : K \to r : E* \to X$ is universal in $X$, then it is universal in the collage category $[M]$ as well; that is, to every cone $\alpha : K \to * : E* \to M$ there is a unique $M$-arrow $\pi \\backslash \alpha : r \to *$, the adjunct of $\alpha$ along $\pi$ in $M$, such that $\alpha = \pi \\backslash \pi \alpha$.

7.6. Limits in comma categories

Theorem 7.6.1. Consider the comma and collage

![Diagram](image)

of a module $M : X \to A$.

- If the inclusion $M_1$ preserves limits over a category $E$, then the pair of comma fibrations $M_0^1$ and $M_1^1$ creates limits over $E$ in the following sense: given a functor $K : E \to [M]$, if $M_0^1 \circ K$ and $M_1^1 \circ K$ have limits $\rho : r \to M_0^1 \circ K$ and $\sigma : s \to M_1^1 \circ K$, then there is a unique cone $\pi : m \to K$ in $[M]$ such that $\pi \circ M_0^1 = \rho$ and $\pi \circ M_1^1 = \sigma$, and moreover $\pi$ is universal.
- If the inclusion $M_0$ preserves colimits over a category $E$, then the pair of comma fibrations $M_0^1$ and $M_1^1$ creates colimits over $E$ in the following sense: given a functor $K : E \to [M]$, if $K \circ M_0^1$ and $K \circ M_1^1$ have colimits $\rho : K \circ M_0^1 \to r$ and $\sigma : K \circ M_1^1 \to s$, then there is a unique cone $\pi : K \to m$ in $[M]$ such that $\pi \circ M_0^1 = \rho$ and $\pi \circ M_1^1 = \sigma$, and moreover $\pi$ is universal.
Proof. We use the commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\mathcal{M}_0} & [\mathcal{M}] & \xrightarrow{\mathcal{M}_1} & A \\
\downarrow & & \downarrow & & \downarrow \\
[\mathcal{M}] & \xrightarrow{([0],[\mathcal{M}])} & [2, [\mathcal{M}]] & \xrightarrow{([1],[\mathcal{M}])} & [\mathcal{M}]
\end{array}
\]

in Definition 3.2.17(1). We first observe that, by the construction of a collage, \(\mathcal{M}_0\) preserves limits. Now suppose that \(\mathcal{M}_0 \circ K\) and \(\mathcal{M}_1 \circ K\) have limits \(\rho : r \sim [0, [\mathcal{M}]] \circ K\) and \(\sigma : s \sim [1, [\mathcal{M}]] \circ K\). Since \(\mathcal{M}_0\) and \(\mathcal{M}_1\) preserve limits, they yield universal cones \(\rho : r \sim [0, [\mathcal{M}]] \circ K\) and \(\sigma : s \sim [1, [\mathcal{M}]] \circ K\). Hence, by Theorem 7.3.15, there is a unique cone \(\pi : m \sim K\) in \([2, [\mathcal{M}]\)] such that \(\pi \circ [0, [\mathcal{M}]] = \rho\) and \(\pi \circ [1, [\mathcal{M}]] = \sigma\), and \(\pi\) is universal. Clearly, \(\pi\) is in \([\mathcal{M}]\), and the assertion follows.

Corollary 7.6.2.
- If a left module \(\mathcal{M} : \ast \to A\) preserves limits over a category \(E\), then the left comma fibration \(\mathcal{M}^1 : [\mathcal{M}] \to A\) creates limits over \(E\).
- If a right module \(\mathcal{M} : X \to \ast\) preserves colimits over a category \(E\), then the right comma fibration \(\mathcal{M}^1 : [\mathcal{M}] \to X\) creates colimits over \(E\).

Proof. Since the inclusion \(\mathcal{M} : A \to [\mathcal{M}]\) preserves limits over \(E\) by Corollary 7.5.10, the assertion follows as a special case of Theorem 7.6.1 where \(X\) is the terminal category.

Corollary 7.6.3.
- Given a left module \(\mathcal{M} : \ast \to A\), if \(A\) is complete and \(\mathcal{M}\) is continuous, then the comma category \([\mathcal{M}]\) is complete and the left comma fibration \(\mathcal{M}^1 : [\mathcal{M}] \to A\) is continuous.
- Given a right module \(\mathcal{M} : X \to \ast\), if \(X\) is cocomplete and \(\mathcal{M}\) is cocontinuous, then the comma category \([\mathcal{M}]\) is cocomplete and the right comma fibration \(\mathcal{M}^1 : [\mathcal{M}] \to X\) is cocontinuous.

Proof. Since \(\mathcal{M}\) preserves small limits, the left comma fibration \(\mathcal{M}^1 : [\mathcal{M}] \to A\) creates small limits by Corollary 7.6.2. The assertion thus follows from Proposition 7.3.12.

Corollary 7.6.4. Let \(C\) be a category and \(c\) be an object of \(C\).
- The left comma fibration \(c \downarrow C : [c(C)] \to C\) (see Remark 3.2.30) creates limits. Hence if \(C\) is complete, so is the comma category \([c(C)]\) and the left comma fibration \(c \downarrow C : [c(C)] \to C\) is continuous.
- The right comma fibration \(C \downarrow c : [(C)c] \to C\) (see Remark 3.2.30) creates colimits. Hence if \(C\) is cocomplete, so is the comma category \([(C)c]\) and the right comma fibration \(C \downarrow c : [(C)c] \to C\) is cocontinuous.

Proof. By Corollary 7.5.9, the representable left module \(c(C) : \ast \to C\) is continuous. The first assertion is thus an instance of Corollary 7.6.2 where \(\mathcal{M} : \ast \to A\) is given by the representable left module \(c(C) : \ast \to C\). The second assertion now follows from Proposition 7.3.12.

Corollary 7.6.5.
- For any right module \(\mathcal{M} : X \to \ast\), the right comma fibration \(\mathcal{M}^1 : [\mathcal{M}] \to X\) creates limits over a connected category.
- For any left module \(\mathcal{M} : \ast \to A\), the left comma fibration \(\mathcal{M}^1 : [\mathcal{M}] \to A\) creates colimits over a connected category.

Proof. The assertion follows as a special case of Theorem 7.6.1 where \(A\) is the terminal category, observing that the inclusion \(\mathcal{M}_1 : \ast \to [\mathcal{M}]\) preserves limits over a connected category (an immediate consequence of [ML98] p90 Exercise 8).
Remark 7.6.6. Pullbacks and equalizers are examples of limits over a connected category. The right comma fibration \( \mathcal{M}^\downarrow : [\mathcal{M}] \to \mathbf{X} \) thus creates pullbacks and equalizers. Hence, by Proposition 7.3.11, if \( \mathbf{X} \) has pullbacks (resp. equalizers), so does the comma category \( [\mathcal{M}] \) and the right comma fibration \( \mathcal{M}^\downarrow : [\mathcal{M}] \to \mathbf{X} \) preserves them.

Corollary 7.6.7. Let \( \mathbf{C} \) be a category and \( c \) be an object of \( \mathbf{C} \).

- The right comma fibration \( \mathbf{C} \downarrow c : [\langle c \rangle] \to \mathbf{C} \) (see Remark 3.2.30) creates limits over a connected category.
- The left comma fibration \( c \downarrow \mathbf{C} : [\{c \rangle] \to \mathbf{C} \) (see Remark 3.2.30) creates colimits over a connected category.

Proof. This is an instance of Corollary 7.6.5 where \( \mathcal{M} : \mathbf{X} \to * \) is given by the representable right module \( \langle c \rangle \mathbf{C} : \mathbf{C} \to * \).

Remark 7.6.8. For example, the right comma fibration \( \mathbf{C} \downarrow c : [\langle c \rangle] \to \mathbf{C} \) creates pullbacks and equalizers (cf. Remark 7.6.6); hence, if \( \mathbf{C} \) has pullbacks (resp. equalizers), so does the comma category \( [\langle c \rangle] \) and the right comma fibration \( \mathbf{C} \downarrow c : [\langle c \rangle] \to \mathbf{C} \) preserves them.

Corollary 7.6.9. Let \( \mathbf{C} \) be a category and \( c \) be an object of \( \mathbf{C} \).

- If \( \mathbf{C} \) is complete, so is the comma category \( [\langle c \rangle] \).
- If \( \mathbf{C} \) is cocomplete, so is the comma category \( [\{c \rangle] \).

Proof. Suppose that \( \mathbf{C} \) is complete. Then \( [\langle c \rangle] \) has equalizers as we saw in Remark 7.6.8. Now since all limits can be constructed from products and equalizers (see [ML98] p113 Theorem 1), we are done if we show that \( [\langle c \rangle] \) has products. But this is immediate because products in \( [\langle c \rangle] \) are “the same thing as” (multiple) pullbacks in \( \mathbf{C} \).

Remark 7.6.10. We saw in 7.6.8 that the right comma fibration \( \mathbf{C} \downarrow c : [\langle c \rangle] \to \mathbf{C} \) preserves pullbacks and equalizers, but this is not the case for products (limits over a discrete category); because products in the comma category \( [\langle c \rangle] \) are given by pullbacks in \( \mathbf{C} \), we cannot expect the right comma fibration \( \mathbf{C} \downarrow c : [\langle c \rangle] \to \mathbf{C} \) to preserve products.
8. Adjeuctions

8.1. Symmetric cells

**Definition 8.1.1.** Given a pair of modules $\mathcal{M} : X \rightarrow A$ and $\mathcal{N} : Y \rightarrow B$, and given a pair of functors $P : X \rightarrow Y$ and $Q : B \rightarrow A$,

- a right symmetric cell $\Phi : Q \sim P : \mathcal{M} \rightarrow \mathcal{N}$, written diagrammatically variously as

\[
\begin{array}{cccc}
X & \xrightarrow{M} & A \\
\downarrow{p} & \phi & \uparrow{Q} \\
Y & \xrightarrow{\mathcal{N}} & B
\end{array}
\quad
\begin{array}{cccc}
Y & \xrightarrow{\mathcal{N}} & B \\
\downarrow{p} & \phi & \uparrow{Q} \\
X & \xrightarrow{M} & A
\end{array}
\quad
\begin{array}{cccc}
A & \xrightarrow{Q} & B \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xrightarrow{p} & Y
\end{array}
\quad
\begin{array}{cccc}
B & \xrightarrow{Q} & A \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xrightarrow{p} & Y
\end{array}
\]

is defined by a module morphism $\Phi : \langle \mathcal{M} \rangle Q \rightarrow P \langle \mathcal{N} \rangle : X \rightarrow B$.

- a left symmetric cell $\Phi : Q \sim P : \mathcal{M} \rightarrow \mathcal{N}$, written diagrammatically as

\[
\begin{array}{cccc}
X & \xrightarrow{M} & A \\
\downarrow{p} & \phi & \uparrow{Q} \\
Y & \xrightarrow{\mathcal{N}} & B
\end{array}
\quad
\begin{array}{cccc}
Y & \xrightarrow{\mathcal{N}} & B \\
\downarrow{p} & \phi & \uparrow{Q} \\
X & \xrightarrow{M} & A
\end{array}
\quad
\begin{array}{cccc}
A & \xrightarrow{Q} & B \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xrightarrow{p} & Y
\end{array}
\quad
\begin{array}{cccc}
B & \xrightarrow{Q} & A \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xrightarrow{p} & Y
\end{array}
\]

is defined by a module morphism $\Phi : P \langle \mathcal{N} \rangle \rightarrow \langle \mathcal{M} \rangle Q : X \rightarrow B$.

**Remark 8.1.2.**

1. For a pair of objects $x \in \|X\|$ and $b \in \|B\|$,

- the component of a right symmetric cell

\[
\begin{array}{cc}
A & \xrightarrow{Q} B \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xrightarrow{p} Y
\end{array}
\]

at $(x, b)$ is the function

\[
x \langle \mathcal{M} \rangle (Q \cdot b) = x \langle \langle \mathcal{M} \rangle Q \rangle b \xrightarrow{x(\Phi)b} x \langle P \langle \mathcal{N} \rangle \rangle b = (x \cdot P) \langle \mathcal{N} \rangle b
\]

which sends each $\mathcal{M}$-arrow $m : x \sim Q \cdot b$ to the $\mathcal{N}$-arrow $m \cdot \Phi : x \cdot P \sim b$.

- the component of a left symmetric cell

\[
\begin{array}{cc}
A & \xrightarrow{Q} B \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xrightarrow{p} Y
\end{array}
\]

at $(x, b)$ is the function

\[
x \langle \mathcal{M} \rangle (Q \cdot b) = x \langle \langle \mathcal{M} \rangle Q \rangle b \xleftarrow{x(\Phi)b} x \langle P \langle \mathcal{N} \rangle \rangle b = (x \cdot P) \langle \mathcal{N} \rangle b
\]

which sends each $\mathcal{N}$-arrow $n : x \cdot P \sim b$ to the $\mathcal{M}$-arrow $\Phi \cdot n : x \sim Q \cdot b$.

2. The identity module morphism $\mathcal{M} \rightarrow \mathcal{M}$ yields the identity right and left symmetric cells

\[
\begin{array}{cc}
A & \xleftarrow{1} A \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xleftarrow{1} X
\end{array}
\quad
\begin{array}{cc}
A & \xleftarrow{1} A \\
\downarrow{M} & \phi & \uparrow{\mathcal{N}} \\
X & \xleftarrow{1} X
\end{array}
\]

. 226
Definition 8.1.3.
- Given a pair of right symmetric cells as in
  \[
  \begin{array}{c}
  A \xleftarrow{Q} A' \xleftarrow{Q'} A'' \\
  M \xleftarrow{\Phi} M' \xleftarrow{\Phi'} M'' \\
  X \xleftarrow{p} X' \xleftarrow{p'} X''
  \end{array}
  \]
, their composite \(\Phi \circ \Phi' = \Phi' \circ \Phi\) is the right symmetric cell
  \[
  \begin{array}{c}
  A \xleftarrow{Q \circ Q'} A'' \\
  M \xleftarrow{\Phi \circ \Phi'} M'' \\
  X \xleftarrow{p \circ p'} X''
  \end{array}
  \]
defined by the module morphism \(\Phi \circ \Phi' : \langle M \rangle [Q \circ Q'] \to \langle P \circ P' \rangle \langle M'' \rangle : X \to A''\) given by the composition
  \[
  \langle M \rangle [Q \circ Q'] = \langle \langle M \rangle Q \rangle (P \langle M' \rangle) Q' = P \langle \langle M' \rangle Q' \rangle \xrightarrow{P(\Phi')} P \langle P' \langle M'' \rangle \rangle = [P \circ P'] \langle M'' \rangle
  \]
- Given a pair of left symmetric cells as in
  \[
  \begin{array}{c}
  A \xleftarrow{Q} A' \xleftarrow{Q'} A'' \\
  M \xleftarrow{\Phi} M' \xleftarrow{\Phi'} M'' \\
  X \xleftarrow{p} X' \xleftarrow{p'} X''
  \end{array}
  \]
, their composite \(\Phi \circ \Phi' = \Phi' \circ \Phi\) is the left symmetric cell
  \[
  \begin{array}{c}
  A \xleftarrow{Q \circ Q'} A'' \\
  M \xleftarrow{\Phi \circ \Phi'} M'' \\
  X \xleftarrow{p \circ p'} X''
  \end{array}
  \]
defined by the module morphism \(\Phi \circ \Phi' : [P \circ P'] \langle M'' \rangle \to \langle M \rangle [Q \circ Q'] : X \to A''\) given by the composition
  \[
  \langle M \rangle [Q \circ Q'] = \langle \langle M \rangle Q \rangle (P \langle M' \rangle) Q' = P \langle \langle M' \rangle Q' \rangle \xrightarrow{P(\Phi')} P \langle P' \langle M'' \rangle \rangle = [P \circ P'] \langle M'' \rangle
  \]

Proposition 8.1.4. Modules and right (resp. left) symmetric cells among them form a category with the composition given in Definition 8.1.3 and the identity symmetric cells in Remark 8.1.2(2).

Proof. The only non-trivial part is the verification of the associativity of the composition. Consider cells as in
  \[
  \begin{array}{c}
  A \xleftarrow{Q} A' \xleftarrow{Q'} A'' \xleftarrow{Q''} A''' \\
  M \xleftarrow{\Phi} M' \xleftarrow{\Phi'} M'' \xleftarrow{\Phi''} M'''
  \end{array}
  \]
. The two cell compositions \(\langle \Phi \circ \Phi' \rangle \circ \Phi''\) and \(\Phi \circ \langle \Phi' \circ \Phi'' \rangle\) are given by the module morphisms \(\langle \langle \Phi \rangle Q' \circ P \langle \Phi' \rangle \rangle \circ \Phi'' \circ [P \circ P'] \langle \Phi'' \rangle\) and \(\Phi \circ \langle \langle \Phi' \rangle Q'' \circ P \langle \Phi'' \rangle \rangle\) respectively. But by the functoriality and associativity of the composition, we have
  \[
  \langle \langle \Phi \rangle Q' \circ P \langle \Phi' \rangle \rangle \circ \Phi'' = \langle \langle \Phi \rangle Q' \rangle Q'' \circ (P \langle \Phi' \rangle) \circ \Phi'' = \Phi' \circ (P \langle \Phi' \rangle)
  \]
desc
Definition 8.1.5. Given a pair of modules $\mathcal{M} : X \to A$ and $\mathcal{N} : Y \to B$, 

- the module of right symmetric cells $\mathcal{M} \to \mathcal{N}$, 
  
  \[
  \langle \mathcal{M} \mathcal{N} \rangle : [B, A] \to [X, Y]^-
  \]

, is defined by the composition

\[
[B, A] \xrightarrow{M \cdot B} [X : B] \xrightarrow{(X B)^\circ} [X : B] \xrightarrow{X \cdot N} [X, Y]^-
\]

(cf. Example 1.1.29(10)), where $M \cdot B$ is the right action of $\mathcal{M}$ on the functor category $[B, A]$ and $X \cdot N$ is the left action of $\mathcal{N}$ on the functor category $[X, Y]$.

- the module of left symmetric cells $\mathcal{M} \to \mathcal{N}$, 
  
  \[
  \langle \mathcal{M} \mathcal{N} \rangle : [B, A]^\circ \to [X, Y]^-
  \]

, is defined by the composition

\[
[B, A]^\circ \xrightarrow{M \cdot B} [X : B]^\circ \xrightarrow{(X B)^\circ} [X : B]^\circ \xrightarrow{X \cdot N} [X, Y]^-
\]

(cf. Example 1.1.29(10)), where $M \cdot B$ is the right action of $\mathcal{M}$ on the functor category $[B, A]$ and $X \cdot N$ is the left action of $\mathcal{N}$ on the functor category $[X, Y]$.

Remark 8.1.6.

1. For each pair of functors $Q : B \to A$ and $P : X \to Y$, 
   - the set $(Q) \mathcal{M} \mathcal{N} (P)$ consists of all module morphisms $\langle \mathcal{M} \rangle Q \to P \langle \mathcal{N} \rangle : X \to B$, i.e. all right symmetric cells $Q \sim P : \mathcal{M} \to \mathcal{N}$.
   - the set $(Q) \mathcal{M} \mathcal{N} (P)$ consists of all module morphisms $P \langle \mathcal{N} \rangle \to \langle \mathcal{M} \rangle Q : X \to B$, i.e. all left symmetric cells $Q \sim P : \mathcal{M} \to \mathcal{N}$.

2. Given a right symmetric cell $\Phi : Q \sim P$ and natural transformations $\tau : Q' \to Q$ and $\sigma : P' \to P$ as in

\[
\begin{array}{ccc}
A & \xrightarrow{Q'} & B \\
| & \Phi \downarrow & | \\
M \downarrow & \downarrow \sigma & \downarrow \sigma' \\
X & \xrightarrow{\tau} & Y
\end{array}
\]

, their composite

\[
\begin{array}{ccc}
Q \xrightarrow{\Phi} & P \\
\tau \downarrow & \downarrow \sigma' \\
Q' & \xrightarrow{\tau \circ \Phi \circ \sigma'} & P'
\end{array}
\]

(where $\sigma' : P \to P'$ denotes the opposite of $\sigma : P' \to P$ (see Preliminary 6)) in the module $\langle \mathcal{M} \mathcal{N} \rangle$ is the right symmetric cell

\[
\begin{array}{ccc}
A & \xrightarrow{Q'} & B \\
| & \Phi \downarrow & | \\
M \downarrow & \downarrow \sigma' & \downarrow \sigma' \\
X & \xrightarrow{\tau} & Y
\end{array}
\]

defined by the module morphism $\tau \circ \Phi \circ \sigma' : \langle \mathcal{M} \rangle Q' \to P' \langle \mathcal{N} \rangle$ given by the composition

\[
\begin{array}{ccc}
\langle \mathcal{M} \rangle Q' & \xrightarrow{(M \tau)} & \langle \mathcal{M} \rangle Q \\
| & \Phi \downarrow & | \\
P \langle \mathcal{N} \rangle & \xrightarrow{\sigma} & P' \langle \mathcal{N} \rangle
\end{array}
\]
Given a left symmetric cell \( \Phi : Q \Rightarrow P \) and natural transformations \( \tau : Q \to Q' \) and \( \sigma : P \to P' \) as in

\[
\begin{array}{c}
A \xleftarrow{Q} \xrightarrow{Q'} B \\
\downarrow M \quad \downarrow M' \\
X \xrightarrow{\phi} Y
\end{array}
\]

, their composite

\[
\begin{array}{c}
Q \xrightarrow{\Phi} P \\
\tau \downarrow \quad \downarrow \sigma \\
Q' \xrightarrow{\tau \circ \Phi \circ \sigma} P'
\end{array}
\]

(where \( \tau^- : Q' \to Q \) denotes the opposite of \( \tau : Q \to Q' \) (see Preliminary 6)) in the module \( (\mathcal{M} \downarrow \mathcal{N}) \) is the left symmetric cell

\[
\begin{array}{c}
A \xleftarrow{Q'} \xrightarrow{Q} B \\
\downarrow M \quad \downarrow M' \\
X \xrightarrow{\tau \circ \Phi \circ \sigma} Y
\end{array}
\]

defined by the module morphism \( \tau^- \circ \Phi \circ \sigma : P' (\mathcal{N}) \to (\mathcal{M}) Q' \) given by the composition

\[
(\mathcal{M}) Q' \xrightarrow{(\mathcal{M}) \tau} (\mathcal{M}) Q \xrightarrow{\Phi} P (\mathcal{N}) \xrightarrow{\sigma (\mathcal{N})} P' (\mathcal{N})
\]

(3) For any modules \( \mathcal{M} \) and \( \mathcal{N} \),

\[
(\mathcal{M} \downarrow \mathcal{N})^{-} \cong (\mathcal{N}^{-} \downarrow \mathcal{M}^{-})
\]

Note. The postcomposition in Definition 1.2.14 and the identity in Proposition 1.2.10 allow the following definition (cf. Definition 1.2.23).

**Definition 8.1.7.** Let \( \mathcal{J} : \mathcal{E} \to \mathcal{D} \) be a module.
- Given a right symmetric cell

\[
\begin{array}{c}
A \xleftarrow{Q} \xrightarrow{Q'} B \\
\downarrow M \quad \downarrow M' \\
X \xrightarrow{\phi} Y
\end{array}
\]

, the right symmetric cell

\[
\begin{array}{c}
[D, A] \xleftarrow{(D, Q)} [D, B] \\
(\mathcal{J}, \mathcal{M}) \xleftarrow{(\mathcal{J}, \phi)} (\mathcal{J}, \mathcal{N}) \\
[E, X] \xrightarrow{(E, P)} [E, Y]
\end{array}
\]

, “postcomposition with \( \Phi \)”, is defined by the module morphism

\[
(\mathcal{J}, \mathcal{M}) [D, Q] = (\mathcal{J}, (\mathcal{M}) Q) \xrightarrow{(\mathcal{J}, \phi)} (\mathcal{J}, P (\mathcal{N})) = [E, P] (\mathcal{J}, \mathcal{N})
\]

, postcomposition with \( \Phi : (\mathcal{M}) Q \to P (\mathcal{N}) \).
Given a left symmetric cell

\[
\begin{array}{c}
\begin{array}{ccc}
\mathbf{A} & \xleftarrow{\Phi} & \mathbf{B} \\
\mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\
\mathbf{X} & \xrightarrow{\rho} & \mathbf{Y}
\end{array}
\end{array}
\]

, the left symmetric cell

\[
\begin{array}{c}
\begin{array}{ccc}
[D, A] & \xleftarrow{[D, \Phi]} & [D, B] \\
(\mathcal{J}, \mathcal{M}) & \xrightarrow{\phi} & (\mathcal{J}, \mathcal{N}) \\
[E, X] & \xrightarrow{\rho} & [E, Y]
\end{array}
\end{array}
\]

, “postcomposition with \( \Phi \)”, is defined by the module morphism

\[
\langle \mathcal{J}, \mathcal{M} \rangle [D, Q] = \langle \mathcal{J}, (\mathcal{M} \circ \mathcal{Q}) \rangle \xrightarrow{\langle \mathcal{J}, \mathcal{M} \rangle \circ \Phi} \langle \mathcal{J}, \mathcal{N} \rangle = [E, P] \langle \mathcal{J}, \mathcal{N} \rangle
\]

Remark 8.1.8. Given a pair of functors \( S : E \to X \) and \( T : D \to B \),

- the right symmetric cell \( \langle \mathcal{J}, \Phi \rangle \) sends each cell \( \Theta : S \Rightarrow Q \delta T : \mathcal{J} \to \mathcal{M} \) to the cell \( \Theta \circ \Phi : S \Rightarrow P \Rightarrow T : \mathcal{J} \to \mathcal{N} \) defined by the module morphism \( \Theta \circ \Phi : \mathcal{J} \to [S \Rightarrow P] \langle \mathcal{N} \rangle \) given by the composition

\[
\mathcal{J} \xrightarrow{\Theta} S \langle \mathcal{M} \rangle \xrightarrow{\langle \mathcal{M}, Q \rangle \Phi} T \xrightarrow{S(\Phi)T} S \langle \mathcal{P} \rangle \langle \mathcal{N} \rangle \xrightarrow{T} [S \Rightarrow P] \langle \mathcal{N} \rangle \xrightarrow{\Theta} \mathcal{J}
\]

- the left symmetric cell \( \langle \mathcal{J}, \Phi \rangle \) sends each cell \( \Theta : S \Rightarrow Q \delta T : \mathcal{J} \to \mathcal{M} \) to the cell \( \Theta \circ \Phi : S \Rightarrow Q \delta T : \mathcal{J} \to \mathcal{M} \) defined by the module morphism \( \Theta \circ \Phi : \mathcal{J} \to [S \Rightarrow P] \langle \mathcal{N} \rangle \) given by the composition

\[
S \langle \mathcal{M} \rangle [Q \delta T] = \langle \langle \mathcal{M}, Q \rangle \rangle \xrightarrow{S(\Phi)T} S \langle \mathcal{P} \rangle \langle \mathcal{N} \rangle \xrightarrow{T} [S \Rightarrow P] \langle \mathcal{N} \rangle \xrightarrow{\Theta} \mathcal{J}
\]

Note. The postcomposition in Definition 4.3.11 and the identity in Proposition 4.3.7 allow the following definition (cf. Definition 4.3.15).

Definition 8.1.9. Let \( E \) be a category.

- Given a right symmetric cell

\[
\begin{array}{c}
\begin{array}{ccc}
\mathbf{A} & \xleftarrow{\Phi} & \mathbf{B} \\
\mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\
\mathbf{X} & \xrightarrow{\rho} & \mathbf{Y}
\end{array}
\end{array}
\]

, the right symmetric cell

\[
\begin{array}{c}
\begin{array}{ccc}
[E, A] & \xleftarrow{[E, \Phi]} & [E, B] \\
(\mathcal{E}, \mathcal{M}) & \xrightarrow{\phi} & (\mathcal{E}, \mathcal{N}) \\
[E, X] & \xrightarrow{\rho} & [E, Y]
\end{array}
\end{array}
\]

, “postcomposition with \( \Phi \)”, is defined by the module morphism

\[
\langle \mathcal{E}, \mathcal{M} \rangle [E, Q] = \langle \mathcal{E}, (\mathcal{M} \circ \mathcal{Q}) \rangle \xrightarrow{\langle \mathcal{E}, \mathcal{M} \rangle \circ \Phi} \langle \mathcal{E}, \mathcal{N} \rangle = [E, P] \langle \mathcal{E}, \mathcal{N} \rangle
\]

, postcomposition with \( \Phi : \langle \mathcal{M} \rangle Q \to P \langle \mathcal{N} \rangle \).
8.2. Adjoining pairs of modules

- Given a left symmetric cell

\[
\begin{array}{c}
\text{A} \xleftarrow{Q} \text{B} \\
\text{M} \xleftarrow{\Phi} \text{N} \\
\text{X} \xleftarrow{P} \text{Y}
\end{array}
\]

, the left symmetric cell

\[
\begin{array}{c}
\text{[E, A]} \xleftarrow{[E, Q]} \text{[E, B]} \\
\text{[E, M]} \xleftarrow{[E, \Phi]} \text{[E, N]} \\
\text{[E, X]} \xleftarrow{[E, P]} \text{[E, Y]}
\end{array}
\]

“postcomposition with \( \Phi \)”, is defined by the module morphism

\[
\langle E, M \rangle [E, Q] = \langle E, (M) Q \rangle \xleftarrow{(E, \Phi)} \langle E, P (N) \rangle = [E, P] \langle E,N \rangle
\]

postcomposition with \( \Phi : P (N) \to (M) Q \).

Remark 8.1.10. Given a pair of functors \( S : E \to X \) and \( T : E \to B \),

- the right symmetric cell \( \langle E, \Phi \rangle \) sends each cylinder \( \alpha : S \to Q \alpha T : E \to M \) to the cylinder \( \alpha \circ \Phi : S \circ P \to T : E \to N \) defined by

\[
\alpha \circ \Phi = \alpha \circ S(\Phi) T = \alpha : \prod_E S(\Phi) T
\]

, the image of a frame \( \alpha \in \prod_E S \langle M \rangle [Q \circ T] \) under the function

\[
\prod_E S \langle M \rangle [Q \circ T] = \prod_E S \langle (M) Q \rangle T \xleftarrow{\prod_E S(\Phi) T} \prod_E S \langle P (N) \rangle T = \prod_E [S \circ P] \langle N \rangle T
\]

- the left symmetric cell \( \langle E, \Phi \rangle \) sends each cylinder \( \alpha : S \circ P \to T : E \to N \) to the cylinder \( \alpha \circ \Phi : S \to Q \circ T : E \to M \) defined by

\[
\alpha \circ \Phi = \alpha \circ S(\Phi) T = \alpha : \prod_E S(\Phi) T
\]

, the image of a frame \( \alpha \in \prod_E [S \circ P] \langle N \rangle T \) under the function

\[
\prod_E S \langle M \rangle [Q \circ T] = \prod_E S \langle (M) Q \rangle T \xleftarrow{\prod_E S(\Phi) T} \prod_E S \langle P (N) \rangle T = \prod_E [S \circ P] \langle N \rangle T
\]

8.2. Adjoining pairs of modules

Definition 8.2.1.

- A right symmetric cell \( \begin{array}{c} A \xleftarrow{Q} \text{B} \\
\text{M} \xleftarrow{\Phi} \text{N} \\
\text{X} \xleftarrow{P} \text{Y} \end{array} \)

is called adjunctive (or an adjunction) if the module morphism \( \Phi : \langle M \rangle Q \to \langle N \rangle : X \to B \) is iso. If \( \Phi : Q \to P : M \to N \) is an adjunctive right symmetric cell, then the pair \( (Q, \Phi) \), or the functor \( Q \) itself, is called a right adjoint of \( P \) along \( M \) and \( N \).

- A left symmetric cell \( \begin{array}{c} A \xleftarrow{Q} \text{B} \\
\text{M} \xleftarrow{\Phi} \text{N} \\
\text{X} \xleftarrow{P} \text{Y} \end{array} \)

is called adjunctive (or an adjunction) if the module morphism \( \Phi : \langle N \rangle \to \langle M \rangle Q : X \to B \) is iso. If \( \Phi : Q \to P : M \to N \) is an adjunctive left symmetric cell, then the pair \( (P, \Phi) \), or the functor \( P \) itself, is called a left adjoint of \( Q \) along \( N \) and \( M \).
Recalling Definition 8.1.5 and noting Remark 8.1.6(1), we have the following description of an

\[
\text{The naturalities of the bijections } \Phi \text{ of Proposition 8.2.4.}
\]

and often write an adjunctive symmetric cell, left or right, just as

\[
A \xleftarrow{Q} B.
\]

Remark 8.2.2.

1. If \( \Phi \) is an adjunctive right symmetric cell, then the inverse \( \Phi^{-1} \) of the module

isomorphism \( \Phi : (M) Q \to P(N) \) gives the corresponding adjunctive left symmetric cell (and vice versa). Because of this, we often do not care about the direction of the isomorphism \( \Phi \), and often write an adjunctive symmetric cell, left or right, just as

\[A \xleftarrow{Q} B\]

and

\[X \xrightarrow{P} Y\]

By Proposition 1.1.16, a right symmetric cell \( \Phi : Q \to P : M \to N \) is adjunctive if and only if all its components

\[
x(M)(Q \cdot b) \xrightarrow{x(\Phi)b} (x : P)(N) b
\]

(see Remark 8.1.2(1)) are bijective; and a left symmetric cell \( \Phi : Q \to P : M \to N \) is adjunctive if and only if all its components

\[
x(M)(Q \cdot b) \xrightarrow{x(\Phi)b} (x : P)(N) b
\]

are bijective. For each \( M \)-arrow \( m : x \to Q \cdot b \), the corresponding \( N \)-arrow \( m : x : P \to b \) is called the left adjunct of \( m \); and for each each \( N \)-arrow \( n : x : P \to b \), the corresponding \( M \)-arrow \( n : x \to Q \cdot b \) is called the right adjunct of \( n \).

(3) The naturalities of the bijections \( x \langle \Phi \rangle b \) in (2) above are expressed by the commutativities of

\[
\begin{array}{c}
\xymatrix{
\mathfrak{X}'(M)(Q \cdot b) \ar[r]^{x(\Phi)b} & (x : P)(N) b \\
\mathfrak{X}(M)(Q \cdot b) \ar[r]_{x(\Phi)b} & (x : P)(N) b
}
\end{array}
\]

for any \( X \)-arrow \( k : x' \to x \) and any \( B \)-arrow \( h : b \to b' \), and these commutativities are in turn expressed by the identities

\[
(\kappa \circ m) \cdot \Phi = (k : P) \circ (m : \Phi) \quad \Phi \cdot (h \circ n) = (Q : h) \circ (\Phi : n)
\]

for any \( M \)-arrow \( m : x \to Q \cdot b \) and \( N \)-arrow \( n : x : P \to b \).

(4) Recalling Definition 8.1.5 and noting Remark 8.1.6(1), we have the following description of an adjunctive right (resp. left) symmetric cell as an arrow of the module \( \langle M \downarrow N \rangle \) (resp. \( \langle M \downarrow N \rangle \)):

\[\begin{array}{l}
\begin{array}{l}
\text{• an adjunctive right symmetric cell } \Phi : Q \to P : M \to N \text{ is an } \langle M \downarrow N \rangle \text{-arrow } \Phi : Q \to P \text{ given by an isomorphism } \Phi : (M) Q \to P(N) \text{ in the category } [X : B].
\end{array}
\end{array}\]

\[\begin{array}{l}
\begin{array}{l}
\text{• an adjunctive left symmetric cell } \Phi : Q \to P : M \to N \text{ is an } \langle M \downarrow N \rangle \text{-arrow } \Phi : Q \to P \text{ given by an isomorphism } \Phi : P(N) \to (M) Q \text{ in the category } [X : B].
\end{array}
\end{array}\]

Notation 8.2.3. We write

\[
\begin{array}{c}
Q : b \xleftarrow{m} b \\
x \xleftarrow{n} x : P
\end{array}
\]

to express that \( m \) and \( n \) are the adjunct of each other (somewhat unfortunately, \( m \) is the left adjunct of \( n \) and \( n \) is the right adjunct of \( m \)), and call the diagram an adjunct diagram.

Proposition 8.2.4. Given an adjunctive symmetric cell

\[
\begin{array}{c}
A \xleftarrow{Q} B \\
M \xrightarrow{\Phi} N \\
X \xrightarrow{P} Y
\end{array}
\]
8.2. Adjunctions for modules

, if the middle square in

\[
\begin{array}{ccc}
Q \cdot b' & \leftarrow & b' \\
\downarrow h & & \downarrow h \\
Q \cdot b & \leftarrow & b
\end{array}
\]

is an adjunct diagram, so are all three rectangles in the diagram for any X-arrow \( k : x' \to x \) and B-arrow \( h : b \to b' \).

Proof. This is a restatement of Remark 8.2.2(3) in terms of an adjunct diagram, stating the naturality of the bijection \( x(\Phi)b \).

**Proposition 8.2.5.**

1. The identity right (resp. left) symmetric cell is adjunctive.
2. If two composable right (resp. left) symmetric cells are adjunctive, so is their composite.

**Proof.**

(1) Evident.

(2) Consider a pair of right symmetric cells as in Definition 8.1.3. Then the cell \( \Phi \circ \Phi' : M \to M'' \) is defined by the composite module morphism \( (\Phi)Q' \circ P(\Phi') \). Since the module morphisms \( \Phi \) and \( \Phi' \) are iso by the definition of adjunctiveness, so are the module morphisms \( (\Phi)Q' \) and \( P(\Phi') \) by Proposition 1.1.31. Hence \( (\Phi)Q' \circ P(\Phi') \) is an isomorphism, and the cell \( \Phi \circ \Phi' \) is adjunctive.

**Remark 8.2.6.** Modules and adjunctive symmetric cells among them thus constitute a subcategory of the category of modules and symmetric cells (see Proposition 8.1.4).

**Proposition 8.2.7.**

- If a right symmetric cell \( \Phi : Q \to P \) is adjunctive, then for any natural isomorphisms \( \tau : Q' \to Q \) and \( \sigma : P' \to P \), the composite right symmetric cell \( \tau \circ \Phi \circ \sigma : Q' \to P' \) (see Remark 8.1.6(2)) is adjunctive.
- If a left symmetric cell \( \Phi : Q \to P \) is adjunctive, then for any natural isomorphisms \( \tau : Q \to Q' \) and \( \sigma : P \to P' \), the composite left symmetric cell \( \tau \circ \Phi \circ \sigma : Q' \to P' \) (see Remark 8.1.6(2)) is adjunctive.

**Proof.** By the description of an adjunctive symmetric cell in Remark 8.2.2(4), we see that this is an instance of Proposition 1.1.33.

**Proposition 8.2.8.** In Definition 8.1.7, if a cell \( \Phi \) is adjunctive, so is the postcomposition cell \( (J, \Phi) \).

**Proof.** Like all functors, the functor \( (J, \cdot) : [X : B] \to [[E, X] : [D, B]] \) (see Remark 1.2.15(2)) preserves isomorphisms.

**Proposition 8.2.9.** In Definition 8.1.9, if a cell \( \Phi \) is adjunctive, so is the postcomposition cell \( (E, \Phi) \).

**Proof.** Like all functors, the functor \( (E, \cdot) : [X : B] \to [[E, X] : [E, B]] \) (see Remark 4.3.12(2)) preserves isomorphisms.
8.3. Adjunctions for categories

**Definition 8.3.1.** Given a pair of functors \( \mathbf{X} \xrightarrow{G} \mathbf{F} \xrightarrow{\mathbf{A}} \mathbf{A} \),

- an adjunction \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) is defined by a module isomorphism
  \[ \Upsilon : \langle \mathbf{X} \rangle \mathbf{G} \to \mathbf{F} (\mathbf{A}) : \mathbf{X} \to \mathbf{A} \]

  - If \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) is an adjunction, then the pair \((\mathbf{G}, \Upsilon)\), or the functor \(\mathbf{G}\) itself, is called a right adjoint of \(\mathbf{F}\).
- an adjunction \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) is defined by a module isomorphism
  \[ \Upsilon : \mathbf{F} (\mathbf{A}) \to \langle \mathbf{X} \rangle \mathbf{G} : \mathbf{X} \to \mathbf{A} \]

  - If \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) is an adjunction, then the pair \((\mathbf{F}, \Upsilon)\), or the functor \(\mathbf{F}\) itself, is called a left adjoint of \(\mathbf{G}\).

**Remark 8.3.2.**

(1) The two forms of adjunctions, \( \mathbf{X} \to \mathbf{A} \) and \( \mathbf{X} \to \mathbf{A} \), are referred to as the right and left forms. If \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) is the right form of an adjunction, then the inverse \( \Upsilon^{-1} \) of the module isomorphism \( \Upsilon : \langle \mathbf{X} \rangle \mathbf{G} \to \mathbf{F} (\mathbf{A}) \) gives the corresponding left form of the adjunction (and vice versa). Because of this, we often do not care about the direction of the isomorphism \( \Upsilon \), and often regard \( \Upsilon : \mathbf{X} \to \mathbf{A} \) and \( \Upsilon^{-1} : \mathbf{X} \to \mathbf{A} \) as the same thing.

(2) Recalling the definition of representation (Definition 2.3.10), we see that
- a right adjoint \( \mathbf{G} : \mathbf{A} \to \mathbf{X} \) of a functor \( \mathbf{F} : \mathbf{X} \to \mathbf{A} \) gives a corepresentation of the representable module \( \mathbf{F} (\mathbf{A}) : \mathbf{X} \to \mathbf{A} \).
- a left adjoint \( \mathbf{F} : \mathbf{X} \to \mathbf{A} \) of a functor \( \mathbf{G} : \mathbf{A} \to \mathbf{X} \) gives a representation of the corepresentable module \( \langle \mathbf{X} \rangle \mathbf{G} : \mathbf{X} \to \mathbf{A} \).

(3) By (2) above and by the uniqueness of representation (Corollary 6.4.9), an adjoint, if exists, is unique up to isomorphism.

(4) An adjunction for categories \( \mathbf{X} \) and \( \mathbf{A} \) is a special case of an adjunction defined in Definition 8.2.1 where \( \mathcal{M} \) and \( \mathcal{N} \) are given by the hom of \( \mathbf{X} \) and the hom of \( \mathbf{A} \): an adjunction \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) (resp. \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \)) for categories \( \mathbf{X} \) and \( \mathbf{A} \) is the same thing as an adjunctive symmetric cell

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\mathbf{G}} & \mathbf{A} \\
\langle \mathbf{X} \rangle & \xrightarrow{\Upsilon} & \langle \mathbf{A} \rangle \\
\mathbf{X} & \xrightarrow{\mathbf{F}} & \mathbf{A}
\end{array}
\]

(5) Because of (4) above, adjunctions for categories inherit the notion of adjunct from adjunctions for modules (see Remark 8.2.2(2)). For a pair of objects \( x \in \| \mathbf{X} \| \) and \( a \in \| \mathbf{A} \| \),

- the component of an adjunction \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) at \((x, a)\) is the bijection
  \[ x \langle \mathbf{X} \rangle (\mathbf{G} \cdot a) = x \langle \langle \mathbf{X} \rangle \mathbf{G} \rangle a \xrightarrow{x(\Upsilon)a} x \langle \mathbf{F} (\mathbf{A}) \rangle a = (x \cdot \mathbf{F}) \langle \mathbf{A} \rangle a \]

  , which sends each \( \mathbf{X} \)-arrow \( g : x \to \mathbf{G} \cdot a \) to the \( \mathbf{A} \)-arrow \( g \cdot \Upsilon : x \cdot \mathbf{F} \to a \).
- the component of an adjunction \( \Upsilon : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A} \) at \((x, a)\) is the bijection
  \[ x \langle \mathbf{X} \rangle (\mathbf{G} \cdot a) = x \langle \langle \mathbf{X} \rangle \mathbf{G} \rangle a \xleftarrow{x(\Upsilon)a} x \langle \mathbf{F} (\mathbf{A}) \rangle a = (x \cdot \mathbf{F}) \langle \mathbf{A} \rangle a \]

  , which sends each \( \mathbf{A} \)-arrow \( f : x \cdot \mathbf{F} \to a \) to the \( \mathbf{X} \)-arrow \( \Upsilon \cdot f : x \to \mathbf{G} \cdot a \).

For each \( \mathbf{X} \)-arrow \( g : x \to \mathbf{G} \cdot a \), the corresponding \( \mathbf{A} \)-arrow \( \Upsilon \cdot g : x \cdot \mathbf{F} \to a \) is called the left adjunct of \( g \); and for each \( \mathbf{A} \)-arrow \( f : x \cdot \mathbf{F} \to a \), the corresponding \( \mathbf{X} \)-arrow \( \Upsilon \cdot f : x \to \mathbf{G} \cdot a \) is called the right adjunct of \( f \). The adjunct diagram

\[
\begin{array}{cccc}
\mathbf{G} \cdot a & \xleftarrow{\mathbf{g}} & a & \xrightarrow{f} \\
\mathbf{X} & \xleftarrow{\mathbf{x}} & x \cdot \mathbf{F}
\end{array}
\]
expresses that \( f \) and \( g \) are the adjunct of each other (cf. Notation 8.2.3).

(6) An adjunction \( \mathcal{Y} : G \dashv F : X \to A \) is expressed diagrammatically variously as

\[
\begin{array}{cccc}
X & \xrightarrow{\gamma(X)} & A & \xrightarrow{\delta} & X
\end{array}
\]

with \( \mathcal{Y} \) often omitted if it is understood or unimportant.

**Proposition 8.3.3.** Given an adjunction \( \mathcal{Y} : G \dashv F : X \to A \), if the middle square in

\[
\begin{array}{ccc}
G \cdot a' & \rightarrow & a' \\
G \cdot h & \xrightarrow{\eta} & h \\
G \cdot a & \rightarrow & a
\end{array}
\]

is an adjunct diagram, so are all three rectangles in the diagram for any \( X \)-arrow \( k : x' \to x \) and \( A \)-arrow \( h : a \to a' \).

**Proof.** This is a special instance of Proposition 8.2.4, stating the naturality of the bijection \( x(\mathcal{Y})a \) in Remark 8.3.2(5).

\( \Box \)

**Note.** Theorem 5.5.1 justifies the following definition.

**Definition 8.3.4.**
- If \( \mathcal{Y} : G \dashv F : X \to A \) is an adjunction, then
  - the counit of \( \mathcal{Y} \) is the natural transformation \( \epsilon : G \circ F \to 1_A \) such that \( X \cdot \epsilon = \mathcal{Y} \);
  - the unit of \( \mathcal{Y} \) is the natural transformation \( \eta : 1_X \to G \circ F \) such that \( \eta \uplus A = \mathcal{Y}^{-1} \).
- If \( \mathcal{Y} : G \dashv F : X \to A \) is an adjunction, then
  - the unit of \( \mathcal{Y} \) is the natural transformation \( \eta : 1_X \to G \circ F \) such that \( \eta \uplus A = \mathcal{Y} \);
  - the counit of \( \mathcal{Y} \) is the natural transformation \( \epsilon : G \circ F \to 1_A \) such that \( X \cdot \epsilon = \mathcal{Y}^{-1} \).

**Remark 8.3.5.**

(1) Theorem 5.5.1 shows how the counit (resp. unit) of an adjunction is obtained and how an adjunction is recovered from its counit (resp. unit).

a) The component \( \epsilon_a : a : G : F \to a \) of the counit \( \epsilon : G \circ F \to 1_A \) at \( a \in \{ A \} \) is given by the left adjunct \( 1_{\{ G : a \}} : \mathcal{Y} \) of the identity \( X \)-arrow \( G \cdot a \to G \cdot a \), and dually the component \( \eta_x : x \to G \cdot F : x \) of the unit \( \eta : 1_X \to G \circ F \) at \( x \in \{ X \} \) is given by the right adjunct \( \mathcal{Y} \cdot 1_{\{ x : F \}} \) of the identity \( A \)-arrow \( x : F \to x : F \), as shown in the adjunct diagrams

\[
\begin{array}{ccc}
G : a & \xrightarrow{\epsilon_a} & a \\
\uparrow & \uparrow \eta_a & \uparrow 1 \\
G : a & \xrightarrow{\eta} & G : F
\end{array}
\]

b) Conversely, given an \( X \)-arrow \( g : x \to G \cdot a \), its left adjunct \( g : \mathcal{Y} : X : F \to a \) is given by the composite \( (g : F) \circ \epsilon_a \), and dually, given an \( A \)-arrow \( f : x : F \to a \), its right adjunct \( \mathcal{Y} \cdot f : x : G : a \) is given by the composite \( \eta_x \circ (\mathcal{Y} \cdot f) \), as shown in

\[
\begin{array}{ccc}
G : a & \xrightarrow{\epsilon_a} & a \\
\uparrow & \uparrow \eta_x & \uparrow f \\
x & \xrightarrow{\mathcal{Y} \cdot f} & G : F : x
\end{array}
\]

\[
\begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
\downarrow & \downarrow & \downarrow \\
g : F & \xrightarrow{\gamma} & G : a
\end{array}
\]
(2) The adjunct diagram in Remark 8.3.2(5) is depicted more elaborately as

\[ \begin{array}{ccc}
G : a & \xrightarrow{\epsilon_a} & a \\
g & \downarrow & \downarrow \\
x & \xrightarrow{\eta_x} & x : F
\end{array} \]

incorporating the commutative diagrams

\[ \begin{array}{ccc}
G : a & \xrightarrow{\epsilon_a} & a \\
g \downarrow & & \downarrow \\
x & \xrightarrow{\eta_x} & x : F
\end{array} \]

\[ \begin{array}{ccc}
G : a & \xrightarrow{\epsilon_a} & a \\
f \downarrow & & \downarrow \\
x & \xrightarrow{\eta_x} & x : F
\end{array} \]

in (1b) above to show how the left (resp. right) adjunct is obtained via the counit (resp. unit). From this adjunct diagram we see that the component \( \epsilon_a \) of the counit \( \epsilon \) at \( a \in \mathbb{A} \) is universal from \( F \) to \( a \), and dually that the component \( \eta_x \) of the unit \( \eta \) at \( x \in \mathbb{X} \) is universal from \( x \) to \( G \) (a formal proof is given in Proposition 8.3.8); the right adjunct of \( f : x : F \to a \) is the same thing as the adjunct of \( f \) along \( \epsilon_a \), and dually the left adjunct of \( g : x \to G : a \) is the same thing as the adjunct of \( g \) along \( \eta_x \) (cf. Example 6.2.3).

(3) An adjunction is also denoted using its unit or counit (or both). For example, if \( \eta \) is the unit and \( \epsilon \) is the counit, we write the right (resp. left) form of an adjunction as \((\eta, \epsilon) : G \vdash F : \mathbb{X} \to \mathbb{A}\) (resp. \((\eta, \epsilon) : G : F : \mathbb{X} \to \mathbb{A}\)), or diagrammatically as

\[ \begin{array}{ccc}
\mathbb{X} & \xrightarrow{\gamma(\eta, \epsilon)} & \mathbb{A} \\
F \downarrow & \downarrow \gamma(\eta, \epsilon) \\
G \mathbb{A} & \xrightarrow{\gamma(\eta, \epsilon)} & \mathbb{X} \\
A \downarrow & \downarrow \gamma(\eta, \epsilon) \\
G \mathbb{A} & \xrightarrow{\gamma(\eta, \epsilon)} & \mathbb{X}
\end{array} \]

**Proposition 8.3.6.**

- The counit \( \epsilon \) of an adjunction \( \gamma : G \vdash F : \mathbb{X} \to \mathbb{A} \) makes the diagram

\[ \begin{array}{ccc}
G \mathbb{A} & \xrightarrow{\epsilon \mathbb{A}} & \mathbb{A} \\
G\mathbb{X} & \downarrow G\mathbb{Y} & \downarrow \epsilon \mathbb{A} \\
G \mathbb{F} \mathbb{A} & \xrightarrow{[G \circ \mathbb{F} \mathbb{A}]} & \mathbb{A}
\end{array} \]

commute, yielding the adjunct diagram

\[ \begin{array}{ccc}
G : a' & \xrightarrow{\epsilon_{a'}} & a' \\
G : h & \uparrow \hline & \uparrow \hline \\
G : a & \xrightarrow{\epsilon_a} & a : G : F
\end{array} \]

for each \( A \)-arrow \( h : a \to a' \).

- The unit \( \eta \) of an adjunction \( \gamma : G \vdash F : \mathbb{X} \to \mathbb{A} \) makes the diagram

\[ \begin{array}{ccc}
\mathbb{X} & \xrightarrow{\eta \mathbb{X}} & \mathbb{A} \\
F \mathbb{X} & \downarrow (\eta \mathbb{X}) & \downarrow (\eta \mathbb{X})F \\
\mathbb{X} \mathbb{A} & \xrightarrow{\eta \mathbb{X} \mathbb{A}} & \mathbb{X} \mathbb{A} \\
G \mathbb{X} \mathbb{A} & \xrightarrow{[G \circ \mathbb{X} \mathbb{A}]} & \mathbb{X} \mathbb{A}
\end{array} \]

commute, yielding the adjunct diagram

\[ \begin{array}{ccc}
G : F : x & \xrightarrow{\eta_x} & x : F \\
G : F : x & \xrightarrow{\eta_x} & x : F \\
G : F : x & \xrightarrow{\eta_x} & x : F \\
G : F : x & \xrightarrow{\eta_x} & x : F
\end{array} \]

for each \( X \)-arrow \( k : x' \to x \).
Proof. The commutativity of the diagram follows from Proposition 5.3.7, and implies that
\[ G(\Upsilon) \circ (G \circ h) = (G(\Upsilon) \circ G) \circ h = \epsilon \langle A \rangle \circ h = h \circ \epsilon_a \]
for any \( A \)-arrow \( h : a \to a' \).

\[ \square \]

Remark 8.3.7. Proposition 8.3.3 and the adjunct diagrams in Remark 8.3.5(1a) also yield the adjunct diagrams in Proposition 8.3.6 by
\[
\begin{array}{ccc}
G' : a' & \leftarrow & a' \\
\downarrow & & \downarrow h \\
G : a & \leftarrow & a \\
\downarrow & & \downarrow \epsilon_a \\
G' : a & \leftarrow & a : G \\
\downarrow & & \downarrow k \\
x : x' & \leftarrow & x : F \\
\end{array}
\]

Proposition 8.3.8. Consider a pair of functors \( X \xleftarrow{G} F \xrightarrow{\eta} A \).

- For a natural transformation \( \epsilon : G \circ F \to 1_A \), the following conditions are equivalent:
  1. the module morphism \( X \uparrow \epsilon : (X) G \to F(\langle A \rangle) : X \to A \) is iso;
  2. \( \epsilon \) is the counit of an adjunction \( G \dashv F : X \to A \);
  3. \( \epsilon \) regarded as a right cylinder \( X \leftarrow \frac{G}{\eta} \frac{F(\langle A \rangle)}{A} \) is a counit of the representable module \( F(\langle A \rangle) \);
  4. \( \epsilon \) regarded as a two-sided cylinder \( X \xleftarrow{\epsilon} A \) is a pointwise lift of the identity \( 1_A \) inverse along the representable module \( F(\langle A \rangle) \).
  5. \( \epsilon \) is a pointwise right Kan lift of the identity \( 1_A \) along \( F \);
  6. each component \( \epsilon_a : a : G : F \to a \) is universal from \( F \) to \( a \).

- For a natural transformation \( \eta : 1_X \to G \circ F \), the following conditions are equivalent:
  1. the module morphism \( \eta \uparrow A : F(\langle A \rangle) \to (X) G : X \to A \) is iso;
  2. \( \eta \) is the unit of an adjunction \( G \dashv F : X \to A \);
  3. \( \eta \) regarded as a left cylinder \( X \leftarrow \frac{\pi}{\eta} \frac{F(\langle A \rangle)}{A} \) is a unit of the corepresentable module \( (X) G \);
  4. \( \eta \) regarded as a two-sided cylinder \( X \xleftarrow{\eta} A \) is a pointwise lift of the identity \( 1_X \) direct along the corepresentable module \( (X) G \).
  5. \( \eta \) is a pointwise left Kan lift of the identity \( 1_X \) along \( G \);
  6. each component \( \eta_x : x \to G : F : x \) is universal from \( x \) to \( G \).

Proof. By definition, \( (1) \Leftrightarrow (2) \Leftrightarrow (3) \) and \( (4) \Leftrightarrow (5) \Leftrightarrow (6) \). The equivalence \( (3) \Leftrightarrow (4) \) is stated in Proposition 6.5.5. \[ \square \]

Note. By Remark 8.3.2(2), Theorem 8.3.9 and Corollary 8.3.10 are special cases of Theorem 6.4.10 and Corollary 6.4.11 where \( M \) is given by a representable (resp. corepresentable) module.

Theorem 8.3.9.

- Given a functor \( F : X \to A \), suppose that there is a family of \( A \)-arrows \( \epsilon_a : r_a : F \to a \), one for each object \( a \in |A| \), universal from \( F \) to \( a \). Then there is a unique functor \( G : A \to X \) with \( G : a = r_a \) such that \( \epsilon := (\epsilon_a)_{a \in |A|} \) forms a natural transformation \( \epsilon : G \circ F \to 1_A \), and \( G \) is a right adjoint of \( F \) with \( \epsilon \) the counit of the adjunction.
• Given a functor $G : A \to X$, suppose that there is a family of $X$-arrow $\eta_x : x \to G \circ r_x$, one for each object $x \in |X|$, universal from $x$ to $G$. Then there is a unique functor $F : X \to A$ with $r_x = x \dashv F$ such that $\eta := (\eta_x)_{x \in |X|}$ forms a natural transformation $\eta : 1_X \to G \circ F$, and $F$ is a left adjoint of $G$ with $\eta$ the unit of the adjunction.

Proof. As noted above, this follows as a special case of Theorem 6.4.10 with $M$ given by the representable module $F(A)$ (resp. the corepresentable module $(X) G$). □

Corollary 8.3.10.

The following conditions are equivalent for a functor $F : X \to A$:

1. $F$ has a right adjoint;
2. for every object $a \in |A|$, the right module $F(A) a : X \to *$ is representable;
3. for every object $a \in |A|$, there is an object $r_a \in |X|$ and an $A$-arrow $\epsilon_a : r_a \dashv F \to a$ universal from $F$ to $a$.

The following conditions are equivalent for a functor $G : A \to X$:

1. $G$ has a left adjoint;
2. for every object $x \in |X|$, the left module $x(X) G : * \to A$ is representable;
3. for every object $x \in |X|$, there is an object $r_x \in |A|$ and an $X$-arrow $\eta_x : x \to G \circ r_x$ universal from $x$ to $G$.

Proof. Since $F$ has a right adjoint iff $F(A)$ is corepresentable iff $F(A)$ has a counit, the assertion is a special case of Corollary 6.4.11 where $M$ is given by the representable module of $F$. □

Theorem 8.3.11. Consider a pair of functors $X \xrightarrow{G} A$. A pair of natural transformations $\eta : 1_X \to G \circ F$ and $\epsilon : G \circ F \to 1_A$ form an adjunction $(\eta, \epsilon) : G \dashv F : X \to A$ if and only if the pasting compositions

$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & F \\
\downarrow & & \downarrow \\
G & \xrightarrow{\epsilon} & A
\end{array}
$$

yield the identity natural transformations $F \to F$ and $G \to G$; that is, if and only if the triangles

$$
\begin{array}{ccc}
F & \xrightarrow{\eta \circ F \circ G} & F \\
\downarrow & & \downarrow \\
F \circ F & \xrightarrow{\epsilon \circ F} & F
\end{array}
\quad
\begin{array}{ccc}
G & \xrightarrow{\epsilon \circ G} & G \\
\downarrow & & \downarrow \\
G \circ F & \xrightarrow{\eta \circ G} & F
\end{array}
$$

commute.

Proof. By Corollary 5.3.22, the triangles

$$
\begin{array}{ccc}
F(A) & \xrightarrow{[\eta \circ F] \circ [F \circ \epsilon]} & F(A) \\
\downarrow & & \downarrow \\
(X) G & \xrightarrow{X \circ \epsilon} & X \circ \epsilon
\end{array}
\quad
\begin{array}{ccc}
(X) G & \xrightarrow{[G \circ \eta] \circ [\epsilon \circ G]} & (X) G \\
\downarrow & & \downarrow \\
(\eta \circ A) & \xrightarrow{X \circ \epsilon} & (\eta \circ A)
\end{array}
$$

commute. Since the general Yoneda functor is fully faithful (see Corollary 5.3.15), $[[\eta \circ F] \circ [F \circ \epsilon]](A)$ (resp. $[[G \circ \eta] \circ [\epsilon \circ G]](X)$) is the identity iff so is $[\eta \circ F] \circ [F \circ \epsilon]$ (resp. $[G \circ \eta] \circ [\epsilon \circ G]$). Hence the identities $[\eta \circ F] \circ [F \circ \epsilon] = 1_F$ and $[G \circ \eta] \circ [\epsilon \circ G] = 1_G$ hold iff $\eta \circ A$ and $X \circ \epsilon$ are the inverse of each other; but this is the case iff $\eta$ and $\epsilon$ are the unit and counit of an adjunction $G \dashv F : X \to A$. □

Theorem 8.3.12. For an adjunction $\Upsilon : G \dashv F : X \to A$,

1. the functor $G : A \to X$ is fully faithful if and only if the counit $\epsilon : G \circ F \to 1_A$ is a natural isomorphism.
the functor \( F : X \rightarrow A \) is fully faithful if and only if the unit \( \eta : 1_X \rightarrow G \circ F \) is a natural isomorphism.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
G(X) & \xrightarrow{(G)} & A \\
G(T) & \downarrow \cong & (A) \\
G(F(A)) & \longrightarrow & [G \circ F](A)
\end{array}
\]

in Proposition 8.3.6. Since \( \Upsilon : (X) \rightarrow F(A) \) is an isomorphism, so is \( G(\Upsilon) \) by Proposition 1.1.31. Hence \( (G) \) is an isomorphism iff \( \epsilon(A) \) is an isomorphism. But \( (G) \) is an isomorphism iff \( G \) is fully faithful (Proposition 1.2.29), and, since the general Yoneda functor is fully faithful (Corollary 5.3.15), \( \epsilon(A) \) is an isomorphism iff \( \epsilon \) is an isomorphism. \( \square \)

**Theorem 8.3.13.** Suppose that a module \( M : X \rightarrow A \) has a counit \( X \xleftarrow{\rho} \xrightarrow{\lambda} M \) and a unit \( X \xrightarrow{\lambda} A \). Then there is an adjunction \( \Upsilon : G \dashv F : X \rightarrow A \) with the module isomorphism \( \Upsilon : X(G) \rightarrow F(A) : X \rightarrow A \) defined by the composition

\[
X(G) \xrightarrow{X(\rho)} M \xrightarrow{(\lambda M)^{-1}} F(A)
\]

, and the counit \( \epsilon : G \circ F \rightarrow 1_A \) and the unit \( \eta : 1_X \rightarrow G \circ F \) of the adjunction are given by the compositions

\[
\begin{array}{cc}
\xymatrix{
X \ar[r]^{\lambda} & A \\
A \ar[r]^{\rho} \ar[u]_{\lambda} & X \ar[u]_{1}
}
\end{array}
\quad \begin{array}{cc}
\xymatrix{
X \ar[r]^{\rho} & M \\
A \ar[r]^{\lambda} \ar[u]_{X(\rho)} & X \ar[u]_{(X\rho)^{-1}}
}
\end{array}
\]

; that is, the component of \( \epsilon \) at \( a \in [A] \) is given by the adjunct of \( \rho_a \) along \( \lambda_{(a : G)} \) and the component of \( \eta \) at \( x \in [X] \) is given by the adjunct of \( \lambda_X \) along \( \rho_{(F : X)} \) as indicated in

\[
\begin{array}{cc}
\xymatrix{
a : G \ar[r]^{\lambda_{(a : G)}} & F \ar[r]^{\beta} & \ar[l]_{\rho_a} X \\
a \ar[u]_{\lambda} & X \ar[u]_{\rho_{(F : X)}} & \ar[l]_{\lambda_X} X
}
\end{array}
\]

. \( \square \)

**Proof.** The first assertion is obvious since \( X(\rho) \) and \( \lambda(A) \) are isomorphisms by the definition of units. The counit and unit are given by

\[
\begin{align*}
\epsilon &= X(\rho) \circ (\lambda(A)^{-1}) \\
&= [X(\rho)] \circ (\lambda(A)^{-1}) \\
&= \rho \circ (\lambda(A)^{-1}) \\
\end{align*}
\]

\[
\begin{align*}
\eta &= (\lambda(1_A)) \circ (X(\rho)^{-1}) \\
&= (\lambda(1_A)) \circ (X(\rho)^{-1}) \\
&= \lambda \circ (X(\rho)^{-1})
\end{align*}
\]

(*\(^1\) by Corollary 5.3.20). \( \square \)
8.4. Adjunctions as universal arrows

Definition 8.4.1. Given a pair of categories \(X\) and \(A\),
- the module
  \[
  \langle X \downarrow A \rangle : [A, X] \to \mathcal{M}\]
  is defined by the composition
  \[
  [A, X] \xrightarrow{X \circ A} [X : A] \xrightarrow{(X \circ A)} [X : A] \xleftarrow{X \circ A} [X, A]^\sim
  \]
  (cf. Example 1.1.29(10)), where \(X \circ A\) is the right general Yoneda functor for the functor category \([A, X]\) and \(X \circ A\) is the left general Yoneda functor for the functor category \([X, A]\).
- the module
  \[
  \langle X \uparrow A \rangle : [A, X]^\sim \to [X, A]
  \]
  is defined by the composition
  \[
  [A, X]^\sim \xrightarrow{X \circ A} [X : A]^\sim \xrightarrow{(X \circ A)} [X : A]^\sim \xleftarrow{X \circ A} [X, A]
  \]
  (cf. Example 1.1.29(10)), where \(X \circ A\) is the right general Yoneda functor for the functor category \([A, X]\) and \(X \circ A\) is the left general Yoneda functor for the functor category \([X, A]\).

Remark 8.4.2.
(1) The module \(\langle X \downarrow A \rangle\) (resp. \(\langle X \uparrow A \rangle\)) is a special case of the module defined in Definition 8.1.5 where \(\mathcal{M}\) and \(\mathcal{N}\) are given by the hom of \(X\) and the hom of \(A\); that is,
  \[
  \langle X \downarrow A \rangle = \langle (X) \downarrow (A) \rangle \quad \langle X \uparrow A \rangle = \langle (X) \uparrow (A) \rangle
  \]
  For any categories \(X\) and \(A\),
  \[
  \langle X \downarrow A \rangle^\sim \cong \langle A^\sim \downarrow X^\sim \rangle
  \]
  (cf. Remark 8.1.6(3)).

(2) For each pair of functors \(X \xrightarrow{F} A\),
- the set \((G) \langle X \uparrow A \rangle (F)\) consists of all module morphisms \((G) \circ (X) G : X \to A\); that is,
  \[
  (G) \langle X \uparrow A \rangle (F) = ((X) G) \langle X : A \rangle (F(A))
  \]
- the set \((G) \langle X \downarrow A \rangle (F)\) consists of all module morphisms \(F(A) \to (X) G : X \to A\); that is,
  \[
  (G) \langle X \downarrow A \rangle (F) = (F(A)) \langle X : A \rangle ((X) G)
  \]

(3) Noting (2) above, we have the following description of an adjunction as an arrow of the module \(\langle X \uparrow A \rangle\) (resp. \(\langle X \downarrow A \rangle\)):
- an adjunction \(\mathcal{T} : G \dashv F : X \to A\) is an \(\langle X \uparrow A \rangle\)-arrow \(\mathcal{T} : G \sim F\) given by an isomorphism \(\mathcal{T} : (X) G \to F(A)\) in the category \([X : A]\).
- an adjunction \(\mathcal{T} : G \dashv F : X \to A\) is an \(\langle X \downarrow A \rangle\)-arrow \(\mathcal{T} : G \sim F\) given by an isomorphism \(\mathcal{T} : F(A) \to (X) G\) in the category \([X : A]\).
(4) Given a module morphisms $\Phi : \langle X \rangle G \to F \langle A \rangle : X \to A$ (i.e. an $\langle X \mid A \rangle$-arrow $\Phi : G \to F$) and natural transformations $\tau : G' \to G : A \to X$ and $\sigma : F' \to F : X \to A$, their composite

$$\begin{array}{ccc}
G & \xrightarrow{\Phi} & F \\
\uparrow{\tau} & & \uparrow{\sigma} \\
G' & \xrightarrow{\phi \circ \sigma} & F'
\end{array}$$

(where $\sigma^- : F \to F'$ denotes the opposite of $\sigma : F \to F$ (see Preliminary 6)) in the module $\langle X \mid A \rangle$ is the module morphism $\tau \circ \phi \circ \sigma^- : \langle X \rangle G' \to F' \langle A \rangle : X \to A$ given by the composition

$$\langle X \rangle G' \xrightarrow{(X)\tau} \langle X \rangle G \xrightarrow{\phi} F \langle A \rangle \xrightarrow{\sigma(A)} F' \langle A \rangle$$

(cf. Remark 8.1.6(2)).

(4) Given a module morphisms $\Phi : F \langle A \rangle \to \langle X \rangle G : X \to A$ (i.e. an $\langle X \mid A \rangle$-arrow $\Phi : G \to F$) and natural transformations $\tau : G \to G' : A \to X$ and $\sigma : F \to F' : X \to A$, their composite

$$\begin{array}{ccc}
G & \xrightarrow{\phi} & F \\
\uparrow{\tau} & & \uparrow{\sigma} \\
G' & \xrightarrow{\phi \circ \sigma} & F'
\end{array}$$

(where $\tau^- : G' \to G$ denotes the opposite of $\tau : G \to G'$ (see Preliminary 6)) in the module $\langle X \mid A \rangle$ is the module morphism $\tau^- \circ \phi \circ \sigma : F' \langle A \rangle \to \langle X \rangle G' : X \to A$ given by the composition

$$\langle X \rangle G' \xleftarrow{(X)\tau^-} \langle X \rangle G \xleftarrow{\phi} F \langle A \rangle \xleftarrow{\sigma(A)} F' \langle A \rangle$$

(cf. Remark 8.1.6(2)).

(5) The bijections in Theorem 5.5.1 are now written respectively as

$$(G \circ F) \langle A, A \rangle (1_A) = (G) \langle X \cdot A \rangle (F \langle A \rangle) \cong (G) \langle X \mid A \rangle (F)$$

$$(1_X) \langle X, X \rangle (G \circ F) = ((X) G) \langle X \cdot A \rangle (F) \cong (G) \langle X \mid A \rangle (F)$$

, and the module isomorphisms in Remark 5.5.2 are written respectively as

$$\langle X \mid A \rangle \langle X \cdot A \rangle : \langle X \cdot A \rangle \langle X \cdot A \rangle \to \langle X \mid A \rangle : [A, X] \to [X, A]^\sim$$

$$\langle X \cdot A \rangle \langle X \mid A \rangle : \langle X \cdot A \rangle \langle X \cdot A \rangle \to \langle X \mid A \rangle : [A, X]^\sim \to [X, A]$$

. The inverse of the module isomorphism $\langle X \mid A \rangle \langle X \cdot A \rangle$ (resp. $\langle X \cdot A \rangle \langle X \mid A \rangle$) is presented in the form of a fully faithful cell

$$\begin{array}{ccc}
[A, X] & \xrightarrow{\cdot} & [X, A]^\sim \\
\downarrow{1} & & \downarrow{1} \\
\langle X \cdot A \rangle \langle X \cdot A \rangle & \xrightarrow{\cdot} & X \cdot A \\
X \cdot A & \xrightarrow{\cdot} & [X, A]
\end{array}$$

, where $\langle X \mid A \rangle$ (resp. $\langle X \cdot A \rangle$) denotes the inverse of the general Yoneda representation (resp. corepresentation). By (3) above and recalling Definition 8.3.4, we see that

- the cell $\langle X \mid A \rangle \langle X \cdot A \rangle$ sends each adjunction $\Upsilon : G \dashv F : X \to A$ to its counit.
- the cell $\langle X \cdot A \rangle \langle X \mid A \rangle$ sends each adjunction $\Upsilon : G \dashv F : X \to A$ to its unit.

Note. By Remark 8.4.2(1), the following is a special case of Proposition 8.2.7.
**Proposition 8.4.3.**
- If \( \Upsilon: G \dashv F: X \to A \) is an adjunction and \( \tau: G' \to G: A \to X \) and \( \sigma: F \to F': X \to A \) are natural isomorphisms, then their composite (see Remark 8.4.2(4)) is an adjunction \( \tau \circ \Upsilon \circ \sigma^{-1}: G' \dashv F' : X \to A \).
- If \( \Upsilon: G \dashv F: X \to A \) is an adjunction and \( \tau: G \to G': A \to X \) and \( \sigma: F \to F': X \to A \) are natural isomorphisms, then their composite (see Remark 8.4.2(4)) is an adjunction \( \tau^{-1} \circ \Upsilon \circ \sigma: G' \dashv F': X \to A \).

**Proof.** By the description of an adjunction in Remark 8.4.2(3), we see that this is an instance of Proposition 1.1.33. \( \square \)

**Proposition 8.4.4.** Let \( X \) and \( A \) be categories.
- The right form \( \Upsilon: G \dashv F: X \to A \) of an adjunction is a two-way universal \( (X \vdash A) \)-arrow.
- The left form \( \Upsilon: G \dashv F: X \to A \) of an adjunction is a two-way universal \( (X \downarrow A) \)-arrow.

**Proof.** Since the general Yoneda functors \( X \Rightarrow A \) and \( X \Leftarrow A \) are fully faithful (see Corollary 5.3.15) and \( \Upsilon \) is an isomorphism in \([X: A]\), the assertion follows from Theorem 6.2.18. \( \square \)

**Remark 8.4.5.** The converse does not hold in general: a two-way universal \( (X \vdash A) \)-arrow need not be an adjunction \( X \to A \). To see this, consider the module

\[
(2 \vdash 2): [2, 2] \to [2, 2]^{-}
\]

made from the discrete category \( 2 = \{0, 1\} \) and the interval category \( 2 \). The functor category \( [2, 2] \cong 2 \) consists of the two constant functors \( \Delta 0 \) and \( \Delta 1 \), and their corepresentable modules look like

\[
\begin{array}{cc}
0 & 0 \\
1 & 1 \\
\end{array}
\]

respectively. On the other hand, the opposite of the functor category \( [2, 2] \) is depicted as

\[
\begin{array}{c}
\{01\} \\
\{00\} \\
\{11\} \\
\{10\}
\end{array}
\]

, where each \( [ij] \) denotes the functor \( 2 \to 2 \) sending \( 0 \) to \( i \) and \( 1 \) to \( j \); the representable modules of the functors \( [00], [01], [10], \) and \( [11] \) look like

\[
\begin{array}{cc}
0 & 0 \\
1 & 1 \\
\end{array}
\]

respectively. We now see that the module \( (2 \vdash 2) \) consists of the following arrows

\[
\begin{array}{c}
\Delta 0 \searrow [01] \\
\{00\} \downarrow \Upsilon
\end{array}
\]

\[
\begin{array}{c}
\Delta 1 \leftarrow [10] \\
\{00\} \downarrow \Upsilon
\end{array}
\]

. As we can read off from the diagram above, \( \Upsilon: (2)(\Delta 0) \to [01](2) \) is a two-way universal \( (2 \vdash 2) \)-arrow, but it is not an adjunction because the function \( 1(\Upsilon): 1(\{2\}) 0 \to 1(\{2\}) 1 \), the component of \( \Upsilon \) at \((1, 1)\), is not bijective (note that \( \Upsilon \) corresponds to the non-pointwise Kan lift in Example 6.6.5(1) under the module isomorphism \( (2 \vdash 2)[2 \Leftarrow 2]: (2 \vdash 2) [2 \Leftarrow 2] \to (2 \vdash 2) \) (see Remark 8.4.2(5)).)
8.5. Conjugation for adjunctions

Definition 8.5.1.

Given two adjunctions \( \Upsilon : G \dashv F : X \to A \) and \( \Upsilon' : G' \dashv F' : X \to A \), a pair of natural transformations \( \tau : G \to G' : A \to X \) and \( \sigma : F \to F' : X \to A \) are called conjugate if the square

\[
\begin{array}{ccc}
G & \xrightarrow{\langle X \rangle} & F \langle A \rangle \\
\downarrow \tau & & \downarrow \sigma \\
G' & \xrightarrow{\langle X \rangle} & F' \langle A \rangle
\end{array}
\]

commutes; that is, if the square

\[
\begin{array}{ccc}
G & \xrightarrow{\Upsilon} & F \\
\downarrow \tau & & \downarrow \sigma \\
G' & \xrightarrow{\Upsilon'} & F'
\end{array}
\]

(where \( \sigma^\sim : F \to F' \) denotes the opposite of \( \sigma : F' \to F \)) is a two-way conjugation along the module \( \langle X \mid A \rangle \) in the sense of Definition 6.3.1.

Given two adjunctions \( \Upsilon : G \dashv F : X \to A \) and \( \Upsilon' : G' \dashv F' : X \to A \), a pair of natural transformations \( \tau : G \to G' : A \to X \) and \( \sigma : F \to F' : X \to A \) are called conjugate if the square

\[
\begin{array}{ccc}
G & \xleftarrow{\langle X \rangle} & F \langle A \rangle \\
\downarrow \tau & & \downarrow \sigma \\
G' & \xleftarrow{\langle X \rangle} & F' \langle A \rangle
\end{array}
\]

commutes; that is, if the square

\[
\begin{array}{ccc}
G & \xleftarrow{\Upsilon} & F \\
\downarrow \tau & & \downarrow \sigma \\
G' & \xleftarrow{\Upsilon'} & F'
\end{array}
\]

(where \( \tau^\sim : G' \to G \) denotes the opposite of \( \tau : G \to G' \)) is a two-way conjugation along the module \( \langle X \mid A \rangle \) in the sense of Definition 6.3.1.

Remark 8.5.2.

1. The commutativities of the conjugation squares in the above definition are expressed component-wise by the commutativities of

\[
\begin{array}{ll}
G : a & \xrightarrow{x} (X : G : a) \xrightarrow{x(Y)a} \langle x : F \rangle \langle A \rangle : a \\
\tau_a & \xrightarrow{x(x)\tau_a} \xrightarrow{x_{\sigma x}(A)a} (X : F) \langle A \rangle : a & \xrightarrow{x_{\sigma x}} (X : F : a) \xrightarrow{x(Y)a} \langle x : F \rangle \langle A \rangle : a & \xrightarrow{x : F} x : F
\end{array}
\]

\[
\begin{array}{ll}
G' : a & \xrightarrow{x} (X : G' : a) \xrightarrow{x(Y)a} \langle x : F' \rangle \langle A \rangle : a \\
\tau_a & \xrightarrow{x(x)\tau_a} \xrightarrow{x_{\sigma x}(A)a} (X : F') \langle A \rangle : a & \xrightarrow{x_{\sigma x}} (X : F' : a) \xrightarrow{x(Y)a} \langle x : F' \rangle \langle A \rangle : a & \xrightarrow{x : F'} x : F'
\end{array}
\]

for every pair of objects \( x \in [X] \) and \( a \in [A] \), and these commutativities are in turn expressed by the identities

\[
(g \circ \tau_a) : \Upsilon' = \sigma_x \circ (g : \Upsilon) \quad \Upsilon' : (f \circ \sigma_x) = \tau_a \circ (\Upsilon : f)
\]

for any \( X \)-arrow \( g : x \to G : a \) and any \( A \)-arrow \( x : F \to a \).

2. Given categories \( X \) and \( A \), the category of adjunctions \( X \to A \), denoted by \( \text{ADJ}[X, A] \), is given by the full subcategory of the comma category \( [X] \mid A \) consisting of all adjunctions \( X \to A \); given two adjunctions \( \Upsilon : G \dashv F : X \to A \) and \( \Upsilon' : G' \dashv F' : X \to A \), an arrow \( \Upsilon \to \Upsilon' \) is a conjugate pair of natural transformations \( \tau : G \to G' : A \to X \) and \( \sigma : F \to F' : X \to A \). Two fully faithful forgetful functors \( \text{ADJ}[X, A] \to [A, X] \) and \( \text{ADJ}[X, A] \to [X, A] \) are given by
restricting the comma fibrations \((X | A)_{\delta} : [X | A] \to [A, X]\) and \((X | A)_{\delta} : [X | A] \to [X, A]^\vee\) to \(\text{ADJ}[X, A]\); they send each conjugation
\[
\begin{array}{ccc}
G & \xrightarrow{\gamma} & F \\
\tau \downarrow & & \downarrow \sigma \\
G' & \xrightarrow{\gamma'} & F'
\end{array}
\]
to the natural transformations \(\tau : G \to G'\) and \(\sigma : F' \to F\) respectively. Note that the category of adjunctions \(X \to A\) is a full subcategory of the category of universal arrows (see Remark 6.3.2) of the module \((X | A)\).

(3) We say that two adjunctions \(\Upsilon : G \vdash F : X \to A\) and \(\Upsilon' : G' \vdash F' : X \to A\) are isomorphic if they are so in the category \(\text{ADJ}[X, A]\); that is, when there is a conjugate pair \((\tau, \sigma)\) of natural isomorphisms for \(\Upsilon\) and \(\Upsilon'\). The composite \(\tau \circ \Upsilon \circ \sigma : G' \vdash F' : X \to A\) in Proposition 8.4.3 is an adjunction isomorphic to \(\Upsilon : G \vdash F : X \to A\), and \(\tau^{-1} : G \to G'\) is conjugate to \(\sigma : F' \to F\).

Conversely, if adjunctions \(\Upsilon : G \vdash F : X \to A\) and \(\Upsilon' : G' \vdash F' : X \to A\) are isomorphic with a conjugate pair \((\tau, \sigma)\) of natural isomorphisms, then \(\Upsilon'\) is given by the composite \(\tau^{-1} \circ \Upsilon \circ \sigma\).

**Proposition 8.5.3.** Consider a conjugate pair \((\tau, \sigma)\) of natural transformations as in Definition 8.5.1. If one of \((\tau, \sigma)\) is an isomorphism, so is the other (and thus adjunctions \(\Upsilon\) and \(\Upsilon'\) are isomorphic).

**Proof.** This is an instance of Proposition 6.3.3(2). \(\square\)

**Proposition 8.5.4.** Consider a conjugate pair \((\tau, \sigma)\) of natural transformations as in Definition 8.5.1. Then if the inner parallelogram in
\[
\begin{array}{ccc}
G' \vdash a & \xleftarrow{\eta} & a \\
\tau \downarrow & & \downarrow f \\
G \vdash a & \xrightarrow{x} & F \\
g \uparrow & & \uparrow x : \sigma \\
x \xleftarrow{x : \sigma} & F'
\end{array}
\]
is an adjunct diagram of \(\Upsilon\), the outer rectangle is an adjunct diagram of \(\Upsilon'\).

**Proof.** This is a restatement of Remark 8.5.2(1) in terms of an adjunct diagram, stating the commutativity of the conjugation square component-wise. \(\square\)

**Proposition 8.5.5.** Consider adjunctions and natural transformations as in Definition 8.5.1, and let \(\epsilon\) and \(\eta\) be the counit and unit of \(\Upsilon\), and \(\epsilon'\) and \(\eta'\) be the counit and unit of \(\Upsilon'\).

1. \(\tau\) and \(\sigma\) are conjugate if and only if the square
\[
\begin{array}{ccc}
G \circ F' & \xrightarrow{\tau \circ F'} & G' \circ F' \\
\downarrow G \circ \sigma & & \downarrow \epsilon' \\
G \circ F & \xrightarrow{\epsilon} & 1_A
\end{array}
\]

commutes.

2. \(\tau\) and \(\sigma\) are conjugate if and only if the square
\[
\begin{array}{ccc}
1_X & \xrightarrow{\eta} & G \circ F \\
\eta' \downarrow & & \downarrow \tau \circ F \\
G' \circ F' & \xrightarrow{G' \circ \sigma} & G' \circ F
\end{array}
\]

commutes.
(2) Given \( \tau \), its conjugate \( \sigma \) is obtained by the composition
\[
\begin{array}{c}
F' \xrightarrow{\sigma} F \\
\downarrow \Downarrow \\
F \circ G \circ F' \xrightarrow{\Downarrow} G \circ F' \circ F'
\end{array}
\]
that is, by the pasting composition
\[
\begin{array}{c}
X \xrightarrow{\eta} F \\
\downarrow \Downarrow \\
X \xrightarrow{\tau} A
\end{array}
\begin{array}{c}
F' \xrightarrow{\sigma} A \\
\downarrow \Downarrow \\
A \xrightarrow{1}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
G \\
\downarrow \Downarrow \\
G' \circ G \circ F' \circ G \\
\downarrow \Downarrow \\
G' \circ F' \circ G \circ F
\end{array}
\end{array}
\]

Given \( \sigma \), its conjugate \( \tau \) is obtained by the composition
\[
\begin{array}{c}
\begin{array}{c}
G' \circ G' \circ F' \circ G' \\
\downarrow \Downarrow \\
G' \circ F' \circ G' \circ F
\end{array}
\end{array}
\]
that is, by the pasting composition
\[
\begin{array}{c}
\begin{array}{c}
A \\
\downarrow \Downarrow \\
A \\
\downarrow \Downarrow \\
X
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
\downarrow \Downarrow \\
1
\end{array}
\end{array}
\]

(3) If \( \tau \) and \( \sigma \) are conjugate, then
- the component \( x \cdot \sigma : x \cdot F' \rightarrow x \cdot F \) of \( \sigma \) at \( x \in \parallel X \parallel \) is given by the left adjunct of the composite
\[
\begin{array}{c}
x \xrightarrow{\eta \cdot x} G' \circ F' \circ x \\
\downarrow \Downarrow \\
x \xrightarrow{\tau \cdot F' \cdot x} G' \circ F' \circ x
\end{array}
\]
as shown in the adjunct diagram
\[
\begin{array}{c}
G' \circ F' \cdot x \xleftarrow{x \cdot \sigma} x \cdot F \\
\downarrow \Downarrow \\
G' \circ F' \cdot x \\
\downarrow \Downarrow \\
x \xrightarrow{\eta \cdot x} x \cdot F'
\end{array}
\]
- the component \( \tau' \cdot a : G' \circ a \rightarrow G' \circ F' \) of \( \tau \) at \( a \in \parallel A \parallel \) is given by the right adjunct of the composite
\[
\begin{array}{c}
a \circ G' \circ F' \xrightarrow{a \circ G' \circ \sigma} a \circ G \circ F \\
\downarrow \Downarrow \\
a \circ G' \circ a \xrightarrow{a \circ G' \circ \sigma} a \circ G \circ F'
\end{array}
\]

\[
\begin{array}{c}
G' \circ a \xrightarrow{a \circ \epsilon} a \\
\downarrow \Downarrow \\
\tau' \cdot a \xrightarrow{a \circ \sigma} a \circ G \circ F \\
\downarrow \Downarrow \\
G \circ a \rightarrow a \circ G \circ F'
\end{array}
\]
Proof.

(1) Since the bijection \((G \circ F) (A, A) \cong (G) (X \uparrow A) (F)\) in Remark 8.4.2(5) is natural in \(G\) and \(F\), the square

\[
\begin{array}{ccc}
G & \xrightarrow{\eta} & F \\
\downarrow{\tau} & & \downarrow{\sigma} \\
G' & \xrightarrow{\eta'} & F'
\end{array}
\]

commutes iff the square

\[
\begin{array}{ccc}
G \circ F' & \xrightarrow{\tau \circ F'} & G' \circ F' \\
\downarrow{G \circ \sigma} & & \downarrow{G' \circ \sigma} \\
G \circ F & \xrightarrow{\sigma} & 1_A
\end{array}
\]

commutes.

(2) The commutativity of the square in (1) says that the two pasting compositions

\[
\begin{array}{cc}
X & \xleftarrow{\eta} & A \\
\downarrow{F'} & & \downarrow{1} \\
\end{array}
\]

yield the same natural transformation \(G \circ F' \rightarrow 1_A\). Hence the two pasting compositions

\[
\begin{array}{cc}
X & \xleftarrow{\eta} & A \\
\downarrow{F'} & & \downarrow{1} \\
\end{array}
\]

yield the same natural transformation, which is \(\sigma\) by Theorem 8.3.11.

(3) Replacing \(f: x: F \rightarrow a\) with the identity \(x: F \rightarrow x: F\) in the adjunct diagram in Proposition 8.5.4, and recalling from Remark 8.3.5(1a) that the component of the unit \(\eta\) at \(x \in \|X\|\) is given by the right adjunct of the identity \(x: F \rightarrow x: F\), we have the required adjunct diagram

\[
\begin{array}{ccc}
\xrightarrow{\tau: F: x} & \xleftarrow{x: F} & \xleftarrow{\eta: x} \\
\xrightarrow{\eta: F: x} & \xrightarrow{x: F} & \xrightarrow{x: \sigma} \\
\xrightarrow{\tau: F: x} & \xrightarrow{x: F} & \xrightarrow{x: \sigma}
\end{array}
\]

Remark 8.5.6. Note that (3) in Proposition 8.5.5 is in fact the component-wise description of (2). Indeed, by Remark 8.3.5(1b), (3) amounts to saying that the component of \(\sigma\) at \(x \in \|X\|\) is given by the composition

\[
\begin{array}{ccc}
G' \circ F' \circ x & \xleftarrow{x: F} & x: F \\
\downarrow{\tau \circ F'} & & \downarrow{\tau \circ F'} \\
G \circ F \circ x & \xrightarrow{x: \sigma} & x: F'
\end{array}
\]

, which is exactly the component at \(x\) of the commutative square in (2).

Note. We saw the uniqueness of adjoint in Remark 8.3.2(3). The following gives a more direct and detailed view of this fact.
Proposition 8.5.7.

- If \( \Upsilon : G \mapsto F : X \rightarrow A \) and \( \Upsilon' : G' \mapsto F' : X \rightarrow A \) are adjunctions, then there is a unique natural transformation \( \tau : G \rightarrow G' : A \rightarrow X \) such that \( \Upsilon = \tau \circ \Upsilon' \). Moreover, \( \tau \) is an isomorphism (any two right adjoints \( G \) and \( G' \) of a functor \( F : X \rightarrow A \) are thus isomorphic) and commutes with the units and counit of the adjunctions as shown in

\[
\begin{array}{c}
G \circ F \xrightarrow{\tau \circ F} G' \circ F \\
\epsilon \downarrow \quad \downarrow \epsilon' \\
1_A \\
\end{array}
\quad
\begin{array}{c}
1_X \xrightarrow{\eta} G \circ F \\
\eta' \downarrow \quad \downarrow \tau \circ F \\
G' \circ F \\
\end{array}
\]

, where \( \epsilon \) and \( \eta \) are the counit and unit of \( \Upsilon \), and \( \epsilon' \) and \( \eta' \) are the counit and unit of \( \Upsilon' \).

- If \( \Upsilon : G \mapsto F : X \rightarrow A \) and \( \Upsilon' : G \mapsto F' : X \rightarrow A \) are adjunctions, then there is a unique natural transformation \( \sigma : F' \mapsto F : X \rightarrow A \) such that \( \Upsilon = \Upsilon' \circ \sigma \). Moreover, \( \sigma \) is an isomorphism (any two left adjoints \( F \) and \( F' \) of a functor \( G : A \rightarrow X \) are thus isomorphic) and commutes with the units and counit of the adjunctions as shown in

\[
\begin{array}{c}
G \circ F' \xrightarrow{\sigma} F' \circ F \\
\epsilon \downarrow \quad \downarrow \epsilon' \\
1_A \\
\end{array}
\quad
\begin{array}{c}
1_X \xrightarrow{\eta} G \circ F' \\
\eta' \downarrow \quad \downarrow \sigma \\
G \circ F' \\
\end{array}
\]

, where \( \epsilon \) and \( \eta \) are the counit and unit of \( \Upsilon \), and \( \epsilon' \) and \( \eta' \) are the counit and unit of \( \Upsilon' \).

Proof. A unique natural transformation \( \tau : G \rightarrow G' \) is given by the adjunct of \( \Upsilon \) along \( \Upsilon' \) as shown in

\[
\begin{array}{c}
G' \xrightarrow{\Upsilon'} F \\
\tau \uparrow \quad \uparrow \Upsilon \\
G \\
\end{array}
\]

; that is; by the conjugate of the identity \( F \rightarrow F \) as shown in

\[
\begin{array}{c}
G' \xrightarrow{\Upsilon'} F \\
\tau \uparrow \quad \uparrow 1 \\
G \xrightarrow{\Upsilon} F \\
\end{array}
\]

. This natural transformation \( \tau \) is an isomorphism by Proposition 8.5.3. The commutative squares in Proposition 8.5.5(1) for the conjugation above shrink to the commutative triangles in the assertion. \( \square \)

Theorem 8.5.8. Let \( X \) and \( A \) be categories.

- Given a family of functors \( (F_e : X \rightarrow A)_{e \in \mathcal{E}} \), let \( E \) denote the full subcategory of \([X,A]\) generated by \( \{F_e : e \in \mathcal{E}\} \). If each \( F_e \) has a right adjoint \( G_e : A \rightarrow X \) via an adjunction \( \Upsilon_e : G_e \mapsto F_e : X \rightarrow A \), then there is a fully faithful contravariant functor \( R : E^\sim \rightarrow [A,X] \) which sends each \( F_e \) to its right adjoint \( G_e \) and sends each natural transformation \( \sigma : F_e \mapsto F_{e'} \) to its conjugate \( \tau : G_e \rightarrow G_{e'} \).

- Given a family of functors \( (G_e : A \rightarrow X)_{e \in \mathcal{E}} \), let \( E \) denote the full subcategory of \([A,X]\) generated by \( \{G_e : e \in \mathcal{E}\} \). If each \( G_e \) has a left adjoint \( F_e : X \rightarrow A \) via an adjunction \( \Upsilon_e : G_e \mapsto F_e : X \rightarrow A \), then there is a fully faithful contravariant functor \( R : E^\sim \rightarrow [X,A] \) which sends each \( G_e \) to its left adjoint \( F_e \) and sends each natural transformation \( \sigma : G_e \mapsto G_{e'} \) to its conjugate \( \tau : F_e \rightarrow F_{e'} \).

Proof. Since \( \Upsilon_e : G_e \mapsto F_e \) is a two-way universal \((X|A)-arrow \) (see Proposition 8.4.4), by Theorem 6.4.10, there is a unique functor \( R : E^\sim \rightarrow [A,X] \) which sends each \( F_e \) to \( G_e \) and forms with \( (\Upsilon_e)_{e \in \mathcal{E}} \) a unit of \((X|A)E^\sim \) (the restriction of \((X|A)E^\sim \) to \([A,X] \times E^\sim \)). As the proof of Theorem 6.4.10 shows, the functor \( R \) sends each natural transformation \( \sigma : F_e \mapsto F_{e'} \) to its conjugate \( \tau : G_e \mapsto G_{e'} \). Finally, \( R \) is fully faithful by Theorem 6.4.13. \( \square \)
8.6. Adjunctions with parameters

Definition 8.6.1.

1. Given a covariant functor $F : E \to [X, A]$ and a contravariant functor $G : E^\sim \to [A, X]$, an $E$-parameterized adjunction $\Upsilon : G \vdash F : X \to A$ is defined by a cylinder

$$
\begin{array}{c}
\xymatrix{
E 
\ar@/^{10pt}/[rr]^G & & [A, X] \\
\ar@/_{10pt}/[rr]_F & & X \times A \\
\end{array}
$$

such that each component is an adjunction $\Upsilon_e : G(e) \dashv F(e) : X \to A$.

2. Given a covariant functor $G : E \to [A, X]$ and a contravariant functor $F : E^\sim \to [X, A]$, an $E$-parameterized adjunction $\Upsilon : G \vdash F : X \to A$ is defined by a cylinder

$$
\begin{array}{c}
\xymatrix{
E^\sim 
\ar@/^{10pt}/[rr]^G & & [A, X] \\
\ar@/_{10pt}/[rr]_F & & X \times A \\
\end{array}
$$

such that each component is an adjunction $\Upsilon_e : G(e) \dashv F(e) : X \to A$.

Remark 8.6.2.

1. By Proposition 8.4.4, if $\Upsilon : G \vdash F : X \to A$ (resp. $\Upsilon : G \vdash F : X \to A$) is an $E$-parameterized adjunction, then the cylinder $\Upsilon : G \vdash F : E^\sim \to (X \uparrow A)$ (resp. $\Upsilon : G \vdash F : E^\sim \to (X \downarrow A)$) is pointwise two-way universal.

2. Given a pair of functors $F : E \to [X, A]$ and $G : E^\sim \to [A, X]$, a family of adjunctions $\Upsilon_e : G(e) \dashv F(e) : X \to A$, one for each object $e \in \|E\|$, forms an $E$-parameterized adjunction $\Upsilon : G \vdash F : X \to A$ if and only if the square

$$
\begin{array}{c}
\xymatrix{
G(e) 
\ar[r]^-{\langle X \rangle [G(e)]} & [F(e)] \langle A \rangle \\
G(h) \ar@{|->}[u] \ar[r]_-{\langle X \rangle [G(h)]} & [F(h)] \langle A \rangle \\
G(e') \ar[r]^-{\langle X \rangle [G(e')]} & [F(e')] \langle A \rangle \\
G(h) \ar@{|->}[u] \ar[r]_-{\langle X \rangle [G(h)]} & [F(h)] \langle A \rangle \\
}
\end{array}
$$

commutes (i.e. $G(h)$ and $F(h)$ are conjugate) for every $E$-arrow $h : e' \to e$.

3. Given a pair of functors $G : E \to [A, X]$ and $F : E^\sim \to [X, A]$, a family of adjunctions $\Upsilon_e : G(e) \dashv F(e) : X \to A$, one for each object $e \in \|E\|$, forms an $E$-parameterized adjunction $\Upsilon : G \vdash F : X \to A$ if and only if the square

$$
\begin{array}{c}
\xymatrix{
G(e) 
\ar[r]^-{\langle X \rangle [G(e)]} & [F(e)] \langle A \rangle \\
G(h) \ar@{|->}[u] \ar[r]_-{\langle X \rangle [G(h)]} & [F(h)] \langle A \rangle \\
G(e') \ar[r]^-{\langle X \rangle [G(e')]} & [F(e')] \langle A \rangle \\
G(h) \ar@{|->}[u] \ar[r]_-{\langle X \rangle [G(h)]} & [F(h)] \langle A \rangle \\
}
\end{array}
$$

commutes (i.e. $G(h)$ and $F(h)$ are conjugate) for every $E$-arrow $h : e' \to e$.

(3) For each object $e \in \|E\|$, the counit $\epsilon_e : G(e) \circ F(e) \to 1_A$ of the adjunction $\Upsilon_e : G(e) \dashv F(e) : X \to A$ is given by the component of the composite cylinder

$$
\begin{array}{c}
\xymatrix{
E 
\ar@/^{10pt}/[rr]^G & & [A, X] \\
\ar@/_{10pt}/[rr]_F & & X \times A \\
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{
1_A 
\ar@{|->}[d] & & [X \times A] \\
\langle X \rangle [X \times A] \\
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{
A 
\ar@/^{10pt}/[rr]^G & & [X, A] \\
\ar@/_{10pt}/[rr]_F & & X \times A \\
\end{array}
$$
at \( e \in \|E\| \) (see Remark 8.4.2(5) for the cell \( \langle X | A \rangle [X \times A] \)). The family of counits \( \epsilon_e \) is thus natural in \( e \in \|E\| \); that is, the square

\[
\begin{array}{ccc}
G(e) \circ F(e') & \xrightarrow{G(e) \circ F(h)} & G(e) \circ F(e') \\
\downarrow \epsilon_e & & \downarrow \epsilon_e' \\
G(e) \circ F(e) & \xrightarrow{\epsilon_e} & 1_A
\end{array}
\]

commutes for every \( E \)-arrow \( h : e' \to e \).

For each object \( e \in \|E\| \), the unit \( \eta_e : 1_X \to G(e) \circ F(e) \) of the adjunction \( \Upsilon_e : G(e) \vdash F(e) : X \to A \) is given by the component of the composite cylinder

\[
\begin{array}{ccc}
G & \xrightarrow{\Upsilon} & E^- \\
\downarrow & & \downarrow F \\
[A, X]^\rightarrow & \xrightarrow{-X|A} & [X, A] \\
\downarrow & & \downarrow 1 \\
[X : A]^\rightarrow & \xrightarrow{-X|A} & [X, A]
\end{array}
\]

at \( e \in \|E\| \) (see Remark 8.4.2(5) for the cell \( [X \times A] (X | A) \)). The family of units \( \eta_e \) is thus natural in \( e \in \|E\| \); that is, the square

\[
\begin{array}{ccc}
1_X & \xrightarrow{\eta_e} & G(e) \circ F(e) \\
\downarrow & & \downarrow G(e) \circ F(h) \\
G(e') \circ F(e') & \xrightarrow{G(h) \circ F(e') \circ \eta_e} & G(e) \circ F(e')
\end{array}
\]

commutes for every \( E \)-arrow \( h : e' \to e \).

(4) The functors \( F : E \to [X, A] \) and \( G : E^\rightarrow \to [A, X] \) in an \( E \)-parameterized adjunction \( \Upsilon : G \vdash F : X \to A \) are often presented as bifunctors \( F : X \times E \to A \) and \( G : E^\rightarrow \times A \to X \) and identified with their exponential transposes. We say “adjunction \( \Upsilon_e : G(e, -) \vdash F(-, e) \) is parametrized by \( e' \) when a family of adjunctions \( \Upsilon_e : G(e, -) \vdash F(-, e) \), one for each object \( e \in \|E\| \), forms an \( E \)-parameterized adjunction \( \Upsilon : G \vdash F \).

**Proposition 8.6.3.** Let \( F : X \times E \to A \) and \( G : E^\rightarrow \times A \to X \) be bifunctors and suppose that there is an adjunction \( \Upsilon_e : G(e, -) \vdash F(-, e) : X \to A \) parameterized by \( e \) (see Remark 8.6.2(4)). Then for any \( E \)-arrow \( h : e' \to e \), if the inner parallelogram in

\[
\begin{array}{ccc}
G(e', a) & \xleftarrow{a} & a \\
\downarrow G(h,a) & & \downarrow f \\
G(e,a) & \xleftarrow{f} & F(x,e) \\
\downarrow g & & \downarrow F(x,h) \\
x & \xleftarrow{f} & F(x,e')
\end{array}
\]

is an adjunct diagram of \( \Upsilon_e \), the outer rectangle is an adjunct diagram of \( \Upsilon_{e'} \).

**Proof.** Apply Proposition 8.5.4 to the conjugate pair \((G(h), F(h))\) in Remark 8.6.2(2).

**Remark 8.6.4.** Proposition 8.6.3 states the naturality of the bijection

\[
x(X)(G(e, a)) \xrightarrow{x(\Upsilon_e)a} (F(x,e))(A)a
\]

in \( e \), i.e. the commutativity of the diagram

\[
\begin{array}{ccc}
e & \xrightarrow{h} & e' \\
x(X)(G(e, a)) & \xleftarrow{x(\Upsilon_e)a} & (F(x,e))(A)a & \xrightarrow{h} & e'
\end{array}
\]

for an \( E \)-arrow \( h : e' \to e \).
Theorem 8.6.5.

> Given a functor $F : E \to [X, A]$, suppose that there is a family of adjunctions $\Upsilon_e : G_e \dashv F(e) : X \to A$, one for each object $e \in \|E\|$. Then there is a unique functor $G : E^\ast \to [A, X]$ with $G(e) = G_e$ such that $\Upsilon := (\Upsilon_e)_{e \in \|E\|}$ forms a cylinder $G \sim F : E^\ast \sim (X \dashv A)$, and $\Upsilon$ is an $E$-parameterized adjunction.

> Given a functor $G : E \to [A, X]$, suppose that there is a family of adjunctions $\Upsilon_e : G(e) \dashv F_e : X \to A$, one for each object $e \in \|E\|$. Then there is a unique functor $F : E^\ast \to [X, A]$ with $F(e) = F_e$ such that $\Upsilon := (\Upsilon_e)_{e \in \|E\|}$ forms a cylinder $G \sim F : E^\ast \sim (X \dashv A)$, and $\Upsilon$ is an $E$-parameterized adjunction.

Proof. Since an adjunction $X \to A$ is a two-way universal $(X \dashv A)$-arrow (see Proposition 8.4.4), this is an instance of Theorem 6.5.10 where $M$ is given by the module $(X \dashv A)$. \qed

Remark 8.6.6. Theorem 8.5.8 may be seen as a special case of the above result where $F : E \to [X, A]$ (resp. $G : E \to [A, X]$) is an inclusion (cf. Proposition 6.5.5).

Theorem 8.6.7. Let $\Upsilon : G \dashv F : X \to A$ and $\Upsilon' : G' \dashv F' : X \to A$ be $E$-parameterized adjunctions as in Definition 8.6.1. If $\sigma : F' \to F$ (resp. $\tau : G \to G'$) is a natural transformation, then there exists a unique natural transformation $\tau : G \to G'$ (resp. $\sigma : F' \to F$) making the square

\[
\begin{array}{ccc}
G & \sim & F \\
\Upsilon \downarrow & & \downarrow \sigma^- \\
G' & \sim & F'
\end{array}
\]

(where $\sigma^- : F \to F'$ denotes the opposite of $\sigma : F' \to F$) commute in the module $(E^\ast, X \dashv A) : [E^\ast, [A, X]] \to [E^\ast, [X, A]]$.

Proof. Recalling from Remark 8.6.2(1) that $\Upsilon$ and $\Upsilon'$ are (pointwise) two-way universal cylinders $E^\ast \sim (X \dashv A)$, we see that the assertion is an instance of Theorem 6.5.20 where $M$ is given by the module $(X \dashv A)$. \qed

Remark 8.6.8. The square in Theorem 8.6.7 commutes if and only if the square

\[
\begin{array}{ccc}
G(e) & \sim & F(e) \\
\Upsilon_e \downarrow & & \downarrow \sigma^-_e \\
G'(e) & \sim & F'(e)
\end{array}
\]

commutes in the module $(X \dashv A) : [A, X] \to [X, A]$ for every object $e \in \|E\|$ (cf. Remark 6.5.21). Hence the components of $\tau$ and $\sigma$ at each $e \in \|E\|$ are conjugate in the sense of Definition 8.5.1; Theorem 8.6.7 says that if $\sigma_e$ (resp. $\tau_e$) is natural in $e$, so will be $\tau_e$ (resp. $\sigma_e$).

8.7. Composition of adjunctions

Note. In Remark 8.3.2(4) we noted that an adjunction between two categories is a special instance of a symmetric cell. We may thus define the composition of two adjunctions as in Definition 8.1.3.

Definition 8.7.1. Given two adjunctions as in

\[
\begin{array}{ccc}
X \xrightarrow{G_0} C & \xleftarrow{\tau(T_0)} & C \xrightarrow{G_1} A \\
\sim_{F_0} & & \sim_{F_1} \tau(T_1)
\end{array}
\]

Note.
The composite
\[
\begin{array}{ccc}
\text{X} & \xrightarrow{G_0 \circ G_1} & \text{A} \\
\end{array}
\]
defined by the module isomorphism \( Y_0 \circ Y_1 : (X) [G_0 \circ G_1] \rightarrow [F_0 \circ F_1] (A) : X \rightarrow A \) given by the composition
\[
\begin{array}{c}
\langle X \rangle [G_0 \circ G_1] = \langle \langle X \rangle G_0 \rangle G_1 \xrightarrow{(Y_0)G_1 \circ \gamma(Y_0 \circ Y_1)} \langle F_0 (C) \rangle G_1 = F_0 \langle \langle C \rangle G_1 \rangle \xrightarrow{F_0 (Y_1) \circ \gamma(Y_0 \circ Y_1)} F_0 (F_1 (A)) = [F_0 \circ F_1] (A)
\end{array}
\]

Remark 8.7.2.
(1) The composite \( Y_0 \circ Y_1 \) is indeed an adjunction by Proposition 8.2.5(2).
(2) By this composition, categories and adjunctions among them form a category, fully embeddable into the category of modules and adjunctions among them (see Remark 8.2.6). For any category \( C \), the identity adjunction \( 1 : 1_C \dashv 1_C : C \rightarrow C \) is given by the identity module morphism \( 1 : \{ C \} \rightarrow \{ C \} \).
(3) Given a pair of objects \( x \in \{ X \} \) and \( a \in \{ A \} \), the components of the composite adjunction \( Y_0 \circ Y_1 \) at \( (x, a) \) is given by the composite function
\[
\begin{array}{c}
\langle X \rangle (G_0 \cdot G_1 : a) \xrightarrow{x \cdot F_0 (G_1 : a)} \langle x \cdot (G_0 : a) \circ G_1 : a \rangle \xrightarrow{(x \cdot F_0 (Y_1) : a)} \langle F_0 (F_1 : a) \rangle = \langle A \rangle a
\end{array}
\]
; it first sends an \( X \)-arrow \( f : x \rightarrow G_0 \cdot G_1 : a \) to the \( C \)-arrow \( f : Y_0 \circ Y_1 : x \cdot F_0 \rightarrow G_1 : a \), the left adjunct of \( f \) under \( Y_0 \), and then sends this \( C \)-arrow to the \( A \)-arrow \( f : Y_0 \cdot Y_1 : x \cdot F_0 \cdot F_1 \rightarrow a \), the left adjunct of \( f : Y_0 \) under \( Y_1 \), all as shown in the adjunct diagram

\[
\begin{array}{cccc}
G_0 \cdot G_1 : a & \xleftarrow{f} & G_1 : a & \xrightarrow{\gamma(Y_0 \circ Y_1)} a \\
\uparrow & & \uparrow & \\
x & \xrightarrow{\gamma(Y_0)} & x \cdot F_0 & \xrightarrow{\gamma(Y_1)} x \cdot F_0 : F_1
\end{array}
\]

Proposition 8.7.3. Consider
\[
\begin{array}{ccc}
G_0 \cdot G_1 : a & \xleftarrow{f} & G_1 : a & \xrightarrow{\gamma(Y_0 \circ Y_1)} a \\
\uparrow & & \uparrow & \\
x & \xrightarrow{\gamma(Y_0)} & x \cdot F_0 & \xrightarrow{\gamma(Y_1)} x \cdot F_0 : F_1
\end{array}
\]
in the situation of Definition 8.7.1. If the left square is an adjunct diagram of \( Y_0 \) and the right square is an adjunct diagram of \( Y_1 \), then the outer rectangle is an adjunct diagram of \( Y_0 \circ Y_1 \). Conversely, every adjunct diagram of \( Y_0 \circ Y_1 \) is given by a composite of adjunct diagrams of \( Y_0 \) and \( Y_1 \) as above.

Proof. Obvious by Remark 8.7.2(3).

Proposition 8.7.4. Consider
\[
\begin{array}{ccc}
G_0 \cdot G_1 : a & \xleftarrow{f} & G_1 : a & \xrightarrow{\gamma(Y_0 \circ Y_1)} a \\
G_0 : k & \xrightarrow{k} & k & \xrightarrow{g} \\\nG_0 \cdot c & \xleftarrow{h} & c & \xrightarrow{\gamma(Y_0 \circ Y_1)} c : F_1 \\
\uparrow & & \uparrow & \\
x & \xrightarrow{h} & x \cdot F_0 & \xrightarrow{h \cdot F_1} x \cdot F_0 : F_1
\end{array}
\]
in the situation of Definition 8.7.1. If the bottom left square is an adjunct diagram of \( Y_0 \) and the top right square is an adjunct diagram of \( Y_1 \), then the outer square is an adjunct diagram of \( Y_0 \circ Y_1 \).
Proposition 8.3.3. Hence the outer square is an adjunct diagram of \( \Upsilon_0 \) (resp. \( \Upsilon_1 \)) by Proposition 8.3.3. Hence the outer square is an adjunct diagram of \( \Upsilon_0 \circ \uparrow \Upsilon_1 \) by Proposition 8.7.3.

Proposition 8.7.5. Consider two adjunctions as in Definition 8.7.1.

- If \( \varepsilon_0 : G_0 \circ F_0 \to 1_C \) and \( \varepsilon_1 : G_1 \circ F_1 \to 1_A \) are the counits of the adjunctions \( \Upsilon_0 \) and \( \Upsilon_1 \), then the counit \( \varepsilon : G_1 \circ G_0 \circ F_0 \circ F_1 \to 1_A \) of the composite adjunction \( \Upsilon_0 \circ \Upsilon_1 \) is given by the composite

\[
G_1 \circ G_0 \circ F_0 \circ F_1 \xrightarrow{G_1 \circ \varepsilon_0 \circ F_1} G_1 \circ 1_C \circ F_1 = G_1 \circ F_1 \xrightarrow{\varepsilon_1} 1_A
\]

- If \( \eta_0 : 1_X \to G_0 \circ F_0 \) and \( \eta_1 : 1_C \to G_1 \circ F_1 \) are the units of the adjunctions \( \Upsilon_0 \) and \( \Upsilon_1 \), then the unit \( \eta : 1_X \to G_0 \circ G_1 \circ F_0 \circ F_1 \) of the composite adjunction \( \Upsilon_0 \circ \Upsilon_1 \) is given by the composite

\[
1_X \xrightarrow{\eta_0} G_0 \circ F_0 = G_0 \circ G_1 \circ F_0 \xrightarrow{G_0 \circ \eta_1 \circ F_0} G_0 \circ G_1 \circ F_1 \circ F_0
\]

Proof. By Remark 8.3.5(1a), the component of \( \varepsilon \) at \( a \in [A] \) is given by

\[
a \cdot \varepsilon = 1_{(a : G_1 : G_0)} : (\Upsilon_0 \circ \Upsilon_1)
\]

But

\[
1_{(a : G_1 : G_0)} : (\Upsilon_0 \circ \Upsilon_1) = \left( 1_{(a : G_1 : G_0)} : \Upsilon_0 \right) : \Upsilon_1
\]

\[
= (a : G_1 : \varepsilon_0) : \Upsilon_1
\]

\[
= (a : G_1 : \varepsilon_0 : F_1) \circ (a : \varepsilon_1)
\]

\( (\ast^1 \text{ by Remark } 8.7.2(3); \ast^2 \text{ by Remark } 8.3.5(1a); \ast^3 \text{ by Remark } 8.3.5(1b)) \) as shown in the adjunct diagram

\[
\begin{array}{ccc}
\text{a} : G_1 : G_0 & \leftarrow & \text{a} : G_1 : \varepsilon_1 \\
\downarrow^{\text{\(a : G_1 : \varepsilon_0\)}} & & \downarrow^{\text{\(a : \varepsilon\)}} \\
\text{a} : G_1 : F_0 & \rightarrow & \text{a} : G_1 : F_1
\end{array}
\]

Hence we have

\[
a \cdot \varepsilon = (a : G_1 : \varepsilon_0 : F_1) \circ (a : \varepsilon_1) = a : ((G_1 \circ \varepsilon_0 \circ F_1) \circ \varepsilon_1)
\]

as required.

\[\Box\]

Theorem 8.7.6. Consider adjunctions

\[
\begin{array}{ccc}
X & \xrightarrow{G_0 \circ F_0} & C & \xrightarrow{\varepsilon_0} & A \\
\uparrow^{\tau_0} & & \uparrow^{\tau_1} & & \uparrow^{\tau_1} \\
F_0 & \xrightarrow{\tau_0} & \varepsilon_0 & \xrightarrow{\tau_1} & F_1
\end{array}
\]

and conjugate pairs of natural transformations

\[
\left( G_0 \xrightarrow{\sigma_0} G_0' \right) \quad \left( G_1 \xrightarrow{\sigma_1} G_1' \right)
\]

making the squares

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\tau_0} & F_0 \\
\downarrow^{\sigma_0} & & \downarrow^{\tau_1} \\
G_0' & \xrightarrow{\tau_1} & F_0'
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\tau_0} & F_1 \\
\downarrow^{\sigma_0} & & \downarrow^{\tau_1} \\
G_1' & \xrightarrow{\tau_1} & F_1'
\end{array}
\]
8.8. Exponentials in a category

In this section, we look at some basic properties of exponentials in a category (in particular, those in a cartesian closed category) as an example of a (parameterized) adjunction.

**Definition 8.8.1.** Let $\mathbf{C}$ be a category with finite products. An object $p \in \| \mathbf{C} \|$ is said to be exponentiable if the functor $[- \times p] : \mathbf{C} \to \mathbf{C}$ has a right adjoint $[p \triangleright -] : \mathbf{C} \to \mathbf{C}$, and when this is the case, for any $a \in \| \mathbf{C} \|$, the object $p \triangleright a \in \| \mathbf{C} \|$ is called the exponential of $p$ and $a$. 

**Remark 8.7.7.** The result above says that a bifunctor

\[ \text{ADJ}[\mathbf{X}, \mathbf{C}] \times \text{ADJ}[\mathbf{C}, \mathbf{A}] \to \text{ADJ}[\mathbf{X}, \mathbf{A}] \]

on the adjunction categories (see Remark 5.5.2(2)) is given so that the diagram

\[
\begin{array}{ccc}
\text{ADJ}[\mathbf{X}, \mathbf{C}] & \times & \text{ADJ}[\mathbf{C}, \mathbf{A}] \\
\downarrow & & \downarrow \\
\mathbf{X} \downarrow \times \mathbf{C} \downarrow & & \mathbf{X} \downarrow \times \mathbf{A} \\
\end{array}
\]

commutes, where the top and bottom horizontal arrows are the bifunctors on the functor categories and all vertical arrows are the forgetful functors.

### Proof

From the commutative squares

\[
\begin{array}{c}
G_0 \circ G_1 \xrightarrow{\tau_0 \circ \tau_1} F_0 \circ F_1 \\
\downarrow \text{commute} \downarrow \text{commute} \\
G_0' \circ G_1' \xrightarrow{\tau_0' \circ \tau_1'} F_0' \circ F_1'
\end{array}
\]

of the given conjugate pairs, we can construct the commutative diagram

\[
\begin{array}{c}
\langle X \rangle [G_0 \circ G_1] = \langle (X) G_0 \circ G_1 \rangle = \langle F_0 \circ (C) \rangle G_1 = F_0 \circ (\langle C \rangle G_1) = F_0 \circ (\langle A \rangle F_1) = [F_0 \circ F_1]\langle A \rangle \\
\langle X \rangle [G_0' \circ G_1'] = \langle (X) G_0' \circ G_1' \rangle = \langle F_0' \circ (C) \rangle G_1' = F_0' \circ (\langle C \rangle G_1') = F_0' \circ (\langle A \rangle F_1') = [F_0' \circ F_1']\langle A \rangle
\end{array}
\]

of conjugation. 

Remark 8.8.2.

(1) The adjunction \([\mathbf{p} \vdash -] \vdash [- \times \mathbf{p}]\) is called the exponential adjunction at \(\mathbf{p}\), and written diagrammatically as

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{\epsilon_{\mathbf{p}}} & \text{C} \\
\text{p} \times \mathbf{p} & \xrightarrow{f} & \mathbf{p} \\
\end{array}
\]

using its counit \(\epsilon_{\mathbf{p}}\). Given a \(\text{C}\)-arrow \(f : \mathbf{x} \times \mathbf{p} \to \mathbf{a}\) (resp. \(g : \mathbf{x} \to \mathbf{p} \vdash \mathbf{a}\)), its right (resp. left) adjunct under the exponential adjunction is denoted by \(f^\ast : \mathbf{x} \to \mathbf{p} \vdash \mathbf{a}\) (resp. \(g^\ast : \mathbf{x} \times \mathbf{p} \to \mathbf{a}\)) as in the adjunct diagram

\[
\begin{array}{ccc}
\mathbf{p} \vdash \mathbf{a} & \xleftarrow{f^\ast} & \mathbf{a} \\
\mathbf{x} \times \mathbf{p} & \xleftarrow{f} & \mathbf{x} \\
\end{array}
\]

, and called the left exponential transpose of \(f\) (resp. \(g\)). The counit \(\epsilon_{\mathbf{p}}\) of the exponential adjunction \([\mathbf{p} \vdash -] \vdash [- \times \mathbf{p}]\) is called the evaluation; its component at \(\mathbf{a} \in \|\text{C}\\|\) is a \(\text{C}\)-arrow \(\epsilon_{\mathbf{p},\mathbf{a}} : (\mathbf{p} \vdash \mathbf{a}) \times \mathbf{p} \to \mathbf{a}\), whose universality from \([- \times \mathbf{p}]\) to \(\mathbf{a}\) is expressed by the commutative diagram

\[
\begin{array}{ccc}
\mathbf{p} \vdash \mathbf{a} & \xleftarrow{(\mathbf{p} \vdash \mathbf{a}) \times \mathbf{p}} & \mathbf{a} \\
\mathbf{x} \times \mathbf{p} & \xleftarrow{f} & \mathbf{x} \\
\end{array}
\]

(cf. Remark 8.3.5(2)).

(2) If we use the functor \([\mathbf{p} \times -] : \text{C} \to \text{C}\) instead of \([- \times \mathbf{p}] : \text{C} \to \text{C}\), the exponential adjunction \(\epsilon_{\mathbf{p}} : [\mathbf{p} \vdash -] \vdash [\mathbf{p} \times -]\) maps a \(\text{C}\)-arrow \(f : \mathbf{p} \times \mathbf{x} \to \mathbf{a}\) (resp. \(g : \mathbf{x} \to \mathbf{p} \vdash \mathbf{a}\)) to its right exponential transpose \(f^\ast : \mathbf{x} \to \mathbf{p} \vdash \mathbf{a}\) (resp. \(g^\ast : \mathbf{x} \times \mathbf{p} \to \mathbf{a}\)) as shown in the adjunct diagram

\[
\begin{array}{ccc}
\mathbf{p} \vdash \mathbf{a} & \xleftarrow{f^\ast} & \mathbf{a} \\
\mathbf{p} \times \mathbf{x} & \xleftarrow{f} & \mathbf{x} \\
\end{array}
\]

; in this case, the evaluation has the form \(\epsilon_{\mathbf{p},\mathbf{a}} : \mathbf{p} \times (\mathbf{p} \vdash \mathbf{a}) \to \mathbf{a}\), and its universality is expressed by the commutative diagram

\[
\begin{array}{ccc}
\mathbf{p} \vdash \mathbf{a} & \xleftarrow{\mathbf{p} \times (\mathbf{p} \vdash \mathbf{a})} & \mathbf{a} \\
\mathbf{x} & \xleftarrow{\mathbf{p} \times f^\ast} & \mathbf{p} \times \mathbf{x} \\
\end{array}
\]

(3) If objects \(\mathbf{p}\) and \(\mathbf{q}\) in \(\text{C}\) are exponentiable, the composition

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{\psi_{\mathbf{q}}} & \text{C} \\
\text{C} & \xrightarrow{\psi_{\mathbf{p}}} & \text{C} \\
\end{array}
\]

of the exponential adjunctions yields the adjunction

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{\psi_{\mathbf{p} \times \mathbf{q}}} & \text{C} \\
\text{C} & \xrightarrow{\psi_{\mathbf{p} \vdash \mathbf{q}}} & \text{C} \\
\end{array}
\]

. By Proposition 8.7.5, the component of the counit \(\epsilon_{\psi_{\mathbf{p} \times \mathbf{q}}}\) at \(\mathbf{a} \in \|\text{C}\\|\) is given by the composite

\[
((\mathbf{p} \vdash (\mathbf{q} \vdash \mathbf{a})) \times \mathbf{p}) \xrightarrow{\epsilon_{\psi_{\mathbf{p} \times \mathbf{q}}}} (\mathbf{q} \vdash \mathbf{a}) \times \mathbf{q} \xrightarrow{\epsilon_{\psi_{\mathbf{q},\mathbf{a}}}} \mathbf{a}
\]
as shown in the adjunct diagram

\[
\begin{array}{ccccccc}
p \triangleright (q \triangleright a) & \xrightarrow{\epsilon_{p,q,a}} & q \triangleright a & \xleftarrow{\epsilon_{q,a}} & a \\
1 & \uparrow{\epsilon_{p,q,a}} & (q \triangleright a) \times q & \downarrow{\tau_{(p,q),a}} & (p \triangleright (q \triangleright a)) \times p \xrightarrow{\epsilon_{p,q,a \times q}} (p \triangleright (q \triangleright a)) \times p \times q
\end{array}
\]

\[\text{Proposition 8.8.3.} \text{ For any } C\text{-arrow } h : a \to a', \text{ the exponential transpose of a composite } C\text{-arrow } x \times p \xrightarrow{f} a \xrightarrow{h} a' \text{ is given by the composite } x \xrightarrow{f} p \triangleright a \xrightarrow{p \triangleright h} p \triangleright a', \text{ and for any } C\text{-arrow } k : x' \to x, \text{ the exponential transpose of a composite } C\text{-arrow } x' \times p \xrightarrow{k \times p} x \times p \xrightarrow{f} a \text{ is given by the composite } x' \xrightarrow{k} x \xrightarrow{f} p \triangleright a, \text{ as shown in the adjunct diagrams}
\]

\[
\begin{array}{ccccccc}
p \triangleright a' & \xleftarrow{p \triangleright h} & a' \\
p \triangleright a & \xleftarrow{f} & a \\
x & \xleftarrow{k \times p} & x \times p
\end{array}
\]

\[\text{Proof.} \text{ This is an instance of Proposition 8.3.3, stating the naturality of the bijection}
\]

\[x(C)(p \triangleright a) \cong (x \times p)(C)a
\]

\[
in x \text{ and } a. \]

\[\text{□}
\]

\[\text{Proposition 8.8.4. Let } C \text{ be a category with finite products. Then the terminal object } 1 \text{ of } C \text{ is exponentiable. Moreover, there is an isomorphism } a \cong 1 \triangleright a \text{ natural in } a, \text{ given by the exponential transpose of the canonical isomorphism } a \times 1 \cong a \text{ as shown in the adjunct diagram}
\]

\[
\begin{array}{ccc}
1 \triangleright a & \xleftarrow{z} & a \\
z & \downarrow{z} & a \times 1
\end{array}
\]

\[\text{Proof.} \text{ By Proposition 8.4.3 and Proposition 8.5.7, the canonical natural isomorphism } \sigma : 1_C \to [- \times 1] : C \to C \text{ yields an adjunction } \epsilon_1 : [1 \triangleright -] \dashv [- \times 1] \text{ isomorphic to the identity adjunction } 1 : 1_C \dashv 1_C \text{ (see Remark 8.7.2(2)) as shown in the conjugation diagram}
\]

\[
\begin{array}{ccc}
[1 \triangleright -] & \xrightarrow{\epsilon_1} & [- \times 1] \\
\uparrow{\tau} & \uparrow{\sigma^\circ} & \uparrow{\sigma^\circ} \\
1_C & \xrightarrow{1} & 1_C
\end{array}
\]

\[. \text{ By Proposition 8.5.4, the conjugate pair } (\tau, \sigma) \text{ yields the adjunct diagram}
\]

\[
\begin{array}{ccc}
a & \xleftarrow{\tau_a} & a \\
\uparrow{\sigma_a^{-1}} & & \uparrow{\sigma_a^{-1}} \\
[1, a] & \xleftarrow{\sigma_a} & a \times 1 \\
\uparrow{\sigma_a} & & \uparrow{\sigma_a}
\end{array}
\]

\[\text{for each } a \in [C]. \text{ Since the right adjunct of the identity } a \to a \text{ under the identity adjunction } 1 : 1_C \dashv 1_C \text{ is the identity } a \to a, \text{ we have } [\sigma_a^{-1}]^* = \tau_a^{-1}. \]

\[\text{□}
\]
Proposition 8.8.5. Let $C$ be a category with finite products. If objects $p$ and $q$ in $C$ are exponentiable, so is the product $p \times q$. Moreover, there is a canonical isomorphism $p \to (q \to a) \cong (p \times q) \to a$ natural in $a$, given by the exponential transpose of the composite

$$(p \to (q \to a)) \times (p \times q) \cong ((p \to (q \to a)) \times p) \times q \xrightarrow{\epsilon_{pq,a\times q}} (q \to a) \times q \xrightarrow{\epsilon_{q,a}} a$$

as shown in the adjoint diagram

\[
\begin{array}{ccc}
(p \times q) \to a & \cong & a \\
\downarrow & & \downarrow \\
(q \to a) \times q & \cong & (p \to (q \to a)) \times (p \times q)
\end{array}
\]

Proof. By Proposition 8.4.3 and Proposition 8.5.7, the canonical natural isomorphism $[- \times (p \times q)] \cong [(- \times p) \times q]$ (see [ML98] p73 Proposition 1) yields an adjunction $\epsilon_{pq} : [(p \times q) \to -] \dashv [- \times (p \times q)]$ isomorphic to the adjunction $\epsilon_{(p,q)} : [p \to (q \to -)] \dashv [(- \times p) \times q]$ in Remark 8.8.2(3) as shown in the conjugation diagram

\[
\begin{array}{ccc}
[p \to (q \to -)] & \cong & [(- \times p) \times q] \\
\downarrow & & \downarrow \\
[(p \times q) \to -] & \cong & [- \times (p \times q)]
\end{array}
\]

. Recalling from Remark 8.8.2(3) that the component of the counit $\epsilon_{(p,q)}$ at $a \in |C|$ is given by the composite $(\epsilon_{pq,a} \times q) \circ \epsilon_{q,a}$, we have the second assertion as an instance of Proposition 8.5.5(3). □

Definition 8.8.6. A category $C$ is cartesian closed if it has finite products and all objects of $C$ are exponentiable.

Remark 8.8.7.

1. If $C$ is cartesian closed, by applying Theorem 8.6.5 to the bifunctor $(x,p) \mapsto x \times p : C \times C \to C$ and the exponential adjunction $\epsilon_p : [p \to -] \dashv [- \times p]$ at each $p \in |C|$, we have a bifunctor $(p,a) \mapsto p \to a : C^\to \times C \to C$, “exponentiation bifunctor”, and a $C$-parameterized adjunction

\[
\begin{array}{ccc}
[C,-] & \cong & [- \times p] \\
\downarrow & & \downarrow \\
C \to [C,-] & \cong & [p,-]
\end{array}
\]

, whose component at $p \in |C|$ is the exponential adjunction $\epsilon_p : [p \to -] \dashv [- \times p]$. By Remark 8.8.2(3), the evaluation $\epsilon_{p,a} : (p \to a) \times p \to a$ (i.e. the counit of the exponential adjunction) is natural in $p$ as well as in $a$.

2. In a cartesian closed category $C$, the adjunction $\epsilon_{(p,q)} : [p \to (q \to -)] \dashv [(- \times p) \times q]$ in Remark 8.8.2(3) is also parameterized (by $p$ and $q$) with the bifunctors $(x,(p,q)) \mapsto (x \times p) \times q : C \times (C \times C) \to C$ and $(p,q,a) \mapsto p \to (q \to a) : (C \times C)^\to \times C \to C$ (apply Theorem 8.6.5 to the bifunctor $(x,(p,q)) \mapsto (x \times p) \times q$ and the adjunction $\epsilon_{(p,q)} : [p \to (q \to -)] \dashv [(- \times p) \times q]$).

Proposition 8.8.8. If $C$ is cartesian closed, the assignment $(p,a) \mapsto p \to a$ defines a bifunctor $C^\to \times C \to C$, and the evaluation $\epsilon_{p,a} : (p \to a) \times p \to a$ is natural in $p$ and $a$.

Proof. See Remark 8.8.7(1) for the whole story.
Proposition 8.8.9. Let $C$ be a cartesian closed category. Then for any $C$-arrow $k : p' \to p$, the exponential transpose of a composite $C$-arrow $x \times p' \xrightarrow{p} f \times x \xrightarrow{\text{}} x \xrightarrow{\text{}} a$ is given by the composite $x \xrightarrow{\text{}} p' \triangleright a \xrightarrow{k \circ a} p' \triangleright a$ as shown in the adjunct diagram.

\[
\begin{array}{c}
p' \triangleright a & \xleftarrow{k \circ a} & a \\
\downarrow f & & \downarrow f \\
p \triangleright a & \xleftarrow{x \times p} & x \times p \\
\downarrow f & & \downarrow x \times k \\
x & \xleftarrow{x \times f} & x \times p'
\end{array}
\]

Proof. This is an instance of Proposition 8.6.3, stating the naturality of the bijection

\[
x \langle C \rangle (p \triangleright a) \cong (x \times p) \langle C \rangle a
\]

in $p$, i.e. the commutativity of the diagram

\[
\begin{array}{c}
p & x \langle C \rangle (p \triangleright a) & \xleftarrow{\sim} (x \times p) \langle C \rangle a & p' \\
\downarrow k & & \downarrow \langle x \times k \rangle (C \langle a \rangle) & \downarrow k \\
p' & x \langle C \rangle (p' \triangleright a) & \xleftarrow{\sim} (x \times p') \langle C \rangle a & p'
\end{array}
\]

for a $C$-arrow $k : p' \to p$ (cf. Remark 8.6.4).

Proposition 8.8.10. In a cartesian closed category $C$, the canonical isomorphism $p \triangleright (q \triangleright a) \cong (p \times q) \triangleright a$ in Proposition 8.8.5 is natural in $p$ and $q$ as well.

Proof. First note that the composition of the bifunctor $(p, q) \mapsto p \times q : C \times C \to C$ and the parameterized adjunction in Remark 8.8.7(1) yields the adjunction $\epsilon_{pq} : (\langle p \times q \rangle \triangleright -) \dashv [\times (p \times q)]$ parameterized by $p$ and $q$ with the bifunctors $(x, (p, q)) \mapsto x \times (p \times q) : C \times (C \times C) \to C$ and $((p, q), a) \mapsto (p \times q) \triangleright a : (C \times C)^{-} \times C \to C$. Consider now the conjugation

\[
\begin{array}{c}
[p \triangleright (q \triangleright -)] & \xleftarrow{\epsilon_{pq}} & [- \times (p \times q)] \\
\xleftarrow{\sim} & & \xleftarrow{\sim} \\
[(p \times q) \triangleright -] & \xleftarrow{\epsilon_{pq}} & - \times (p \times q)
\end{array}
\]

in the proof of Proposition 8.8.5. The adjunction $\epsilon_{pq} : (\langle p \times q \rangle \triangleright -) \dashv [\times (p \times q)]$ is parameterized by $p$ and $q$ as we have just noted, and so is the adjunction $\epsilon_{pq} : [p \triangleright (q \triangleright -)] \dashv [\times (p \times q)]$ (Remark 8.8.7(2)). Now since the natural isomorphism $[\times (p \times q)] \cong [\times (p \times q)]$ is natural in $p$ and $q$ (see [ML98] p73 Proposition 1), so will be the natural isomorphism $[p \triangleright (q \triangleright -)] \cong [(p \times q) \triangleright -]$ by Theorem 8.6.7 (see Remark 8.6.8).

8.9. Adjoint of a cell

Note. We define adjoints of a cell using the notion of adjunction between modules (see Definition 8.2.1).

Definition 8.9.1. Given a cell $A \xrightarrow{\Phi} \phi \xleftarrow{\Phi'} B$ (as defined in Definition 1.2.1),

\[
\begin{array}{c}
X \xrightarrow{\phi} Y \\
\xrightarrow{\Phi} \phi \xleftarrow{\Phi'} B
\end{array}
\]
8.9. Adjoints of a cell

* a right adjoint \((R, \Upsilon)\) of the collage functor \([\Phi]\) (see Definition 3.1.9) along the left hom (see Definition 5.2.1) of \(\mathcal{M}\) and the left hom of \(\mathcal{N}\) as depicted in

\[
\begin{align*}
\mathcal{A} & \xleftarrow{R} \mathcal{B} \\
[M] & \xrightarrow{\Phi} [N] \\
\end{align*}
\]

is called a right adjoint of \(\Phi\).

* a left adjoint \((R, \Upsilon)\) of the collage functor \([\Phi]\) (see Definition 3.1.9) along the right hom (see Definition 5.2.1) of \(\mathcal{M}\) and the right hom of \(\mathcal{N}\) as depicted in

\[
\begin{align*}
[X] & \xrightarrow{\Upsilon} [Y] \\
\mathcal{A} & \xleftarrow{R} \mathcal{B} \\
\end{align*}
\]

is called a left adjoint of \(\Phi\).

Remark 8.9.2.

(1) If \((R, \Upsilon)\) is a right adjoint of \(\Phi\), then the restrictions of \(\Upsilon\) to \(X\) and \(A\) yield right adjoints

\[
\begin{align*}
\mathcal{A} & \xrightarrow{R} \mathcal{B} \\
\mathcal{M} & \xrightarrow{\Upsilon_{Y}} \mathcal{N} \\
X & \xrightarrow{p} Y \\
\end{align*}
\]

of \(P\) and \(Q\). Conversely, a pair of right adjoints

\[
\begin{align*}
\mathcal{A} & \xrightarrow{R} \mathcal{B} \\
\mathcal{M} & \xrightarrow{\Upsilon_{Y}} \mathcal{N} \\
X & \xrightarrow{p} Y \\
\end{align*}
\]

form a right adjoint of \(\Phi\) if they satisfy the following naturality condition for each object \(b \in [B]\) and each \(\mathcal{M}\)-arrow \(k : x \sim a\): for any \(\mathcal{A}\)-arrow \(m : a \rightarrow R \cdot b\) as in

\[
\begin{align*}
\text{natural diagram}
\end{align*}
\]

the triangle

\[
\begin{align*}
\text{natural diagram}
\end{align*}
\]

commutes (cf. Remark 8.2.2(3)).

* If \((R, \Upsilon)\) is a left adjoint of \(\Phi\), then the restrictions of \(\Upsilon\) to \(X\) and \(A\) yield left adjoints

\[
\begin{align*}
\mathcal{X} & \xleftarrow{P} \mathcal{Y} \\
[X] & \xleftarrow{\Upsilon_{Y}} [Y] \\
\mathcal{A} & \xrightarrow{Q} \mathcal{B} \\
\end{align*}
\]

of \(P\) and \(Q\). Conversely, a pair of left adjoints

\[
\begin{align*}
\mathcal{X} & \xleftarrow{P} \mathcal{Y} \\
[X] & \xleftarrow{\Upsilon_{Y}} [Y] \\
\mathcal{A} & \xrightarrow{Q} \mathcal{B} \\
\end{align*}
\]
form a left adjoint of $\Phi$ if they satisfy the following naturality condition for each object $y \in \|Y\|$ and each $\mathcal{M}$-arrow $k : x \rightarrow a$: for any $\mathcal{X}$-arrow $n : y \rightarrow x$ as in

![Diagram](image)

, the triangle

![Diagram](image)

commutes (cf. Remark 8.2.2(3)).

(2) A right adjoint of a functor $Q : A \rightarrow B$ is the same thing as a right adjoint of the hom cell $\langle Q \rangle : \langle A \rangle \rightarrow \langle B \rangle$ (to see this, replace $A \xleftarrow{\Phi} B$ with $A \xleftarrow{Q} B$ in the argument of (1) above). Dually, a left adjoint of a functor $P : X \rightarrow Y$ is the same thing as a left adjoint of the hom cell $\langle P \rangle : \langle X \rangle \rightarrow \langle Y \rangle$.

**Theorem 8.9.3.** If a functor $R : B \rightarrow A$ (resp. $R : Y \rightarrow X$) is a right (resp. left) adjoint of a cell

![Diagram](image)

, then the postcomposition functor $[D, R] : [D, B] \rightarrow [D, A]$ (resp. $[E, R] : [E, Y] \rightarrow [E, X]$) gives a right (resp. left) adjoint of the postcomposition cell

![Diagram](image)

(see Definition 1.2.23) for any module $J : E \rightarrow D$.

**Proof.** Suppose that a cell $\Phi$ has a right adjoint $(R, \Upsilon)$. The postcompositions with the adjunctive symmetric cells $X(T)$ and $A(T)$ in Remark 8.9.2(1) yield the adjunctive symmetric cells

![Diagram](image)

by Proposition 8.2.8 and Proposition 8.2.9. The proof will be complete if we show that this pair or right adjoints satisfy the naturality condition in Remark 8.9.2(1) for each functor $K : D \rightarrow B$ and each cell $\Theta : S \rightarrow T : J \rightarrow M$, i.e. if we show that for any natural transformation $\tau : T \rightarrow R \circ K : D \rightarrow A$ as in

![Diagram](image)
8.9. Adjoints of a cell

, the triangle

\[
\begin{array}{c}
T \circ Q \xrightarrow{\tau \circ \gamma} K \\
\theta \circ \phi \downarrow \\
S \circ P
\end{array}
\]

commutes. But this is reduced to the commutativity of the triangle

\[
\begin{array}{c}
d \cdot [T \circ Q] \xrightarrow{\tau \circ \gamma} K \cdot d \\
j \cdot \theta \circ \phi \downarrow \\
e \cdot [S \circ P]
\end{array}
\]

, i.e.

\[
\begin{array}{c}
d \cdot T \circ Q \xrightarrow{\tau \circ \gamma} K \cdot d \\
j \cdot \theta \circ \phi \downarrow \\
e \cdot S \circ P
\end{array}
\]

(see Remark 1.2.9(2) and Remark 4.3.14(1)) for each \( \mathcal{J} \)-arrow \( j : e \sim d \), i.e. to the naturality of \( \gamma \).

\[\square\]

**Corollary 8.9.4.** If a cell \( \Phi \) has a right (resp. left) adjoint, each of the following postcomposition cells (see Definition 4.5.7, Definition 4.3.15, and Definition 4.6.15) has a right (resp. left) adjoint as well.

- \( \langle F \circ \Phi \rangle \) and \( \langle F \ast \Phi \rangle \),
- \( \langle E, \Phi \rangle \),
- \( \langle E^*, \Phi \rangle \) and \( \langle E^*, \Phi \rangle \).

**Proof.** By the isomorphism in Theorem 5.5.3, the cell \( \langle F \circ \Phi \rangle \) is identified with the cell \( \langle (E) F, \Phi \rangle \). Hence \( \langle F \circ \Phi \rangle \) has an adjoint by Theorem 8.9.3. Since \( \langle E, \Phi \rangle = \langle 1_E \circ \Phi \rangle \) (see Remark 4.5.8(2)) and \( \langle E^*, \Phi \rangle = \langle !_E \circ \Phi \rangle \) (see Corollary 4.6.21), \( \langle E, \Phi \rangle \) and \( \langle E^*, \Phi \rangle \) have an adjoint as well. The same holds for \( \langle F \ast \Phi \rangle \) and \( \langle E^*, \Phi \rangle \) by duality. \[\square\]

**Theorem 8.9.5.**

- If a cell has a left adjoint, then it preserves inverse universal arrows.
- If a cell has a right adjoint, then it preserves direct universal arrows.

**Proof.** A left adjoint \( (R, \gamma) \) of \( \Phi \) in Definition 8.9.1 is depicted more elaborately as

\[\square\]

, and the right exponential transposition yields a natural isomorphism \( \gamma^\ast \) as in
8.10. Equivalence of categories

Definition 8.10.1. A functor \( H : C \to B \) is called an equivalence provided that it is fully faithful and essentially surjective.

Proposition 8.10.2. If a functor \( H : C \to B \) is an equivalence, so is any functor \( C \to B \) isomorphic to \( H \).

Proof. Evident.

Definition 8.10.3. A module \( M : X \to A \) is called an equivalence provided that
- for each object \( a \in \| A \| \) there exist an object \( x \in \| X \| \) and a two-way universal \( M \)-arrow \( u : x \sim a \), and
- for each object \( x \in \| X \| \) there exist an object \( a \in \| A \| \) and a two-way universal \( M \)-arrow \( u : x \sim a \).

Proposition 8.10.4. If a module \( M : X \to A \) is an equivalence, so is any module \( X \to A \) isomorphic to \( M \).

Proof. Evident since any module isomorphism preserves universal arrows.
Proposition 8.10.5.

- If a functor $G : A \to X$ is an equivalence, then its corepresentable module $(X)G : X \to A$ is an equivalence.
- If a functor $F : X \to A$ is an equivalence, then its representable module $F(A) : X \to A$ is an equivalence.

Proof. By Theorem 6.2.18, for each $a \in \|A\|$, the identity $X$-arrow $a : G \to G \cdot a$ gives a two-way universal $(X)G$-arrow $a : G \cong a$, and for each $x \in \|X\|$, an iso $X$-arrow $u : x \to G \cdot a$ gives a two-way universal $(X)G$-arrow $u : x \to a$.

Remark 8.10.6. We will see that the converse also holds in Corollary 8.10.9.

Note. The axiom of choice is used in the proof of the following.

Theorem 8.10.7. Any equivalence module $\mathcal{M} : X \to A$ has

- a counit $\rho : G \cong \mathcal{M}$ with $G : A \to X$ an equivalence functor.
- a unit $\lambda : \mathcal{M} \cong F$ with $F : X \to A$ an equivalence functor.

Proof. Choose a two-way universal $\mathcal{M}$-arrow $\rho_a : r_a \cong a$ for each $a \in \|A\|$. By Theorem 6.4.10, there is a functor $G : A \to X$ such that $\rho := (\rho_a)_{a \in \|A\|}$ forms a counit $\rho : G \cong \mathcal{M}$. We claim that $G$ is an equivalence. By Theorem 6.4.13, $G$ is fully faithful. It remains to show that $G$ is essentially surjective. Let $x \in \|X\|$. Since $\mathcal{M}$ is an equivalence, there exist $a \in \|A\|$ and a two-way universal $\mathcal{M}$-arrow $u : x \cong a$, and because $\rho_a : a : G \cong a$ is an inverse universal $\mathcal{M}$-arrow by Proposition 6.4.3, we have $x \cong a : G$ by Corollary 6.2.8.

Corollary 8.10.8.

- Suppose that a module $\mathcal{M} : X \to A$ has a counit $X \xrightarrow{G} A$ such that $\mathcal{M}$ is an equivalence if and only if $G$ is an equivalence. Moreover, if these equivalent conditions hold, then each component of $\rho$ is two-way universal.
- Suppose that a module $\mathcal{M} : X \to A$ has a unit $X \xleftarrow{\lambda} A$ such that $\mathcal{M}$ is an equivalence if and only if $F$ is an equivalence. Moreover, if these equivalent conditions hold, then each component of $\rho$ is two-way universal.

Proof. Suppose that $\mathcal{M}$ is an equivalence. Then, by Theorem 8.10.7, there is a counit $\rho' : G' \cong \mathcal{M}$ with $G'$ an equivalence. Since $G \cong G'$ by Theorem 6.4.8, $G$ is an equivalence by Proposition 8.10.2. Conversely, suppose that $G$ is an equivalence. Then the corepresentable module $(X)G$ is an equivalence by Proposition 8.10.5. Since $\rho$ yields $\mathcal{M} \cong (X)G$ (see Remark 6.4.2), $\mathcal{M}$ is an equivalence by Proposition 8.10.4. The second assertion follows from Theorem 6.4.13.

Corollary 8.10.9.

- Suppose that a module $\mathcal{M} : X \to A$ is corepresented by a functor $G : A \to X$. Then $\mathcal{M}$ is an equivalence if and only if $G$ is an equivalence.
- Suppose that a module $\mathcal{M} : X \to A$ is represented by a functor $F : X \to A$. Then $\mathcal{M}$ is an equivalence if and only if $F$ is an equivalence.

Proof. Since a corepresenting functor has a counit associated with it (see Remark 6.4.2), this is immediate from Corollary 8.10.8.

Definition 8.10.10. An equivalence of categories $(G, F, \eta, \epsilon) : X \cong A$ consists of a pair of functors

$$X \xrightarrow{G} A \xleftarrow{F} A$$
and a pair of natural isomorphisms

\[ \eta : 1_X \to G \circ F : X \to X \quad \epsilon : G \circ F \to 1_A : A \to A \]

In this situation, the functors \(G\) and \(F\) are said to be quasi-inverse to each other. Two categories \(X\) and \(A\) are called equivalent, written \(X \simeq A\), when there is an equivalence of categories between them.

**Proposition 8.10.11.** If \((G, F, \eta, \epsilon) : X \simeq A\) is an equivalence of categories, then \(G\) and \(F\) are equivalences.

**Proof.** Since \(\eta_X : x \to G \circ F \cdot x\) (resp. \(\epsilon_a : a \cdot G \circ F \to a\)) is an isomorphism for each \(x \in \|X\|\) (resp. \(a \in \|A\|\)), \(G\) (resp. \(F\)) is essentially surjective. Hence the proof is complete if we show that \(G\) and \(F\) are fully faithful. For any arbitrary \(x, y \in \|X\|\) and \(a, b \in \|A\|\), consider the commutative diagrams

\[
\begin{array}{ccc}
  x \cdot (X) & \xrightarrow{g \cdot g^{-1} : F} & (x \cdot F) \cdot (A) \cdot (F \cdot y) \\
  \downarrow{g^{-1} : F \cdot G} & & \downarrow{q^{-1} : G} \\
  (x \cdot F) \cdot (X) & \xrightarrow{a \cdot (G \cdot X) \cdot (G \cdot b)} & (a \cdot G) \cdot (X) \cdot (G \cdot b)
\end{array}
\]

For any \(X\)-arrow \(g : x \to y\) and any \(A\)-arrow \(f : a \to b\), the squares

\[
\begin{array}{ccc}
  x & \xrightarrow{\eta_X} & x \cdot F : G \\
  \downarrow{g} & & \downarrow{g \cdot g^{-1} : F} \\
  y & \xrightarrow{\eta_Y} & y \cdot F : G
\end{array}
\]

\[
\begin{array}{ccc}
  a \cdot G : F & \xrightarrow{\epsilon_a} & a \\
  \downarrow{f \cdot g : F} & & \downarrow{f} \\
  b \cdot G : F & \xrightarrow{\epsilon_b} & b
\end{array}
\]

commutes by the naturality of \(\eta\) and \(\epsilon\). Since \(\eta_X\) and \(\eta_Y\) are isomorphisms, \(g \mapsto g \cdot F : G\) is bijective, and hence \(g \mapsto g \cdot F\) is injective and \(q \mapsto q \cdot G\) is surjective. Symmetrically, since \(\epsilon_a\) and \(\epsilon_b\) are isomorphisms, \(f \mapsto f \cdot G : F\) is bijective, and hence \(f \mapsto f \cdot G\) is injective and \(p \mapsto p \cdot F\) is surjective. Since \(x, y, a, b\) are arbitrary, the injectivity of \(g \mapsto g \cdot F\) (resp. \(f \mapsto f \cdot G\)) implies the injectivity of \(p \mapsto p \cdot F\) (resp. \(q \mapsto q \cdot G\)). Hence \(q \mapsto q \cdot G\) and \(p \mapsto p \cdot F\) are both injective and surjective, i.e. bijective. The bijectivity of \(f \mapsto f \cdot G\) and \(g \mapsto g \cdot F\) now follows from the bijectivities of the other edges in the diagram, proving the fully faithfulness of \(G\) and \(F\). \(\square\)

**Theorem 8.10.12.** If \((G, F, \eta, \epsilon) : X \simeq A\) is an equivalence of categories, then for any category \(E\)

- the pair of postcomposition functors

\[
\begin{array}{ccc}
  [E, X] & \xrightarrow{[E, G]} & [E, A] \\
  \downarrow{[E, F]} & & \downarrow{[E, F]} \\
  [E, A] & \xrightarrow{[E, F]} & [E, A]
\end{array}
\]

and the pair of postcomposition natural transformations

\[ [E, \eta] : 1_{[E, X]} \to [E, G] \circ [E, F] \]

\[ [E, \epsilon] : [E, G] \circ [E, F] \to 1_{[E, A]} \]

(see Preliminary 9) form an equivalence of categories \([E, X] \simeq [E, A]\).

- the pair of precomposition functors

\[
\begin{array}{ccc}
  \downarrow{[G, E]} & & \downarrow{[G, E]} \\
\end{array}
\]

and the pair of precomposition natural transformations


\[ [\epsilon, E] : [F, E] \circ [G, E] \to 1_{[A, E]} \]

(see Preliminary 9) form an equivalence of categories \([X, E] \simeq [A, E]\).
Proof. First note that the postcomposition natural transformations
\[
[E, \eta] : [E, 1_X] \to [E, G \circ F] \quad [E, \epsilon] : [E, G \circ F] \to [E, 1_A]
\]
yield
\[
[E, \eta] : 1_{[E, X]} \to [E, G] \circ [E, F] \quad [E, \epsilon] : [E, G] \circ [E, F] \to 1_{[E, A]}
\]
by the functoriality of the postcomposition operation \([E, -]\). Since (like any other functor) the functors \([E, -] : [X, X] \to [[E, X], [E, X]]\) and \([E, -] : [A, A] \to [[E, A], [E, A]]\) preserve isomorphisms, \([E, \eta]\) and \([E, \epsilon]\) are natural isomorphisms. \(\square\)

Remark 8.10.13. Theorem 8.10.12 says that the functor \([E, -]\) preserves equivalences of categories. In fact, this is immediate if we use the notion of 2-categories and define \([E, -]\) as a 2-functor.

Theorem 8.10.14. For any adjunction \(X \overset{G}{\underset{F}{\leftrightarrow}} A\), the following conditions are equivalent:

1. \(\eta\) and \(\epsilon\) are natural isomorphisms, i.e. \((G, F, \eta, \epsilon)\) forms an equivalence of categories \(X \simeq A\);
2. \(G\) and \(F\) are fully faithful;
3. \(G\) and \(F\) are equivalences;
4. \(G\) (resp. \(F\)) is an equivalence.

Proof.

1. \(\Leftrightarrow\) (2) Immediate from Theorem 8.3.12.
2. \(\Rightarrow\) (3) This is Proposition 8.10.11.
3. \(\Rightarrow\) (2) Immediate by definition.
4. \(\Rightarrow\) (3) A well known tautology.
5. \(\Rightarrow\) (3) By Corollary 8.10.9, showing this is equivalent to showing that if the corepresentable module \((X)G\) (resp. the representable module \(F(A)\)) is an equivalence, then \((X)G\) and \(F(A)\) are equivalences. But since \((X)G \cong F(A)\) by the definition of an adjunction, this follows from Proposition 8.10.4. \(\square\)

Definition 8.10.15. An adjoint equivalence of categories \(X\) and \(A\) is an adjunction \((\eta, \epsilon) : G \dashv F : X \to A\) satisfying the equivalent conditions in Theorem 8.10.14.

Theorem 8.10.16. Given a functor \(G : A \to X\) (resp. \(F : X \to A\)), the following conditions are equivalent:

1. \(G\) (resp. \(F\)) is an equivalence;
2. \(G\) (resp. \(F\)) is a part of an adjoint equivalence \((\eta, \epsilon) : G \dashv F : X \to A\);
3. \(G\) (resp. \(F\)) is a part of an equivalence of categories \((G, F, \eta, \epsilon) : X \simeq A\).

Proof.

1. \(\Rightarrow\) (2) Since \(G\) is an equivalence, its corepresentable module \((X)G\) is an equivalence by Proposition 8.10.5, and thus has a unit \(\eta : (X)G \to F\) by Theorem 8.10.7. Hence, by the equivalence of (2) and (3) in Proposition 8.3.8, \(G\) is a part of an adjunction \((\eta, \epsilon) : G \dashv F : X \to A\), in fact an adjoint equivalence because \(G\) is an equivalence.
2. \(\Rightarrow\) (3) Immediate by definition.
3. \(\Rightarrow\) (1) This is Proposition 8.10.11. \(\square\)
8.11. Equivalence of modules

**Note.** Recall from Remark 1.2.2(1) that \( \Phi_0 : M_0 \to N_0 \) and \( \Phi_1 : M_1 \to N_1 \) denote the left and right components of a cell \( \Phi : M \to N \).

**Definition 8.11.1.** A cell \( \Phi : M \to N \) is called an equivalence if \( \Phi \) is fully faithful and the functors \( \Phi_0 : M_0 \to N_0 \) and \( \Phi_1 : M_1 \to N_1 \) (the left and right components of \( \Phi \)) are equivalences.

**Remark 8.11.2.** As we will see below in Proposition 8.11.3 and Proposition 8.11.4, the notion of equivalence cells and the notion of equivalence functors subsume each other.

**Proposition 8.11.3.** A functor \( H : C \to B \) is an equivalence if and only if the hom cell \( \langle H \rangle : \langle C \rangle \to \langle B \rangle \) is an equivalence.

**Proof.** Immediate from the definitions. \( \square \)

**Proposition 8.11.4.** A cell \( \Phi : M \to N \) is an equivalence if and only if the collage functor \( \lfloor \Phi \rfloor : \lfloor M \rfloor \to \lfloor N \rfloor \) (see Definition 3.1.9) is an equivalence.

**Proof.** By the construction of \( \lfloor \Phi \rfloor \), we can easily see that \( \lfloor \Phi \rfloor \) is fully faithful iff so are \( \Phi_0, \Phi_1, \) and \( \Phi \), and that \( \lfloor \Phi \rfloor \) is essentially surjective iff so are \( \Phi_0 \) and \( \Phi_1 \). \( \square \)

**Theorem 8.11.5.** Any equivalence cell preserves and reflects inverse (resp. direct) universal arrows.

**Proof.** Suppose that a cell \( \begin{array}{ccc} X & \xrightarrow{\lambda} & A \\ \downarrow P & \Phi & \downarrow Q \\ Y & \xrightarrow{\lambda} & B \end{array} \) is an equivalence and let \( u : r \sim a \) be an \( M \)-arrow. We need to show that \( u \) is inverse universal iff so is \( \Phi \cdot u \); that is, \( X \uparrow u \) is isomorphism iff \( Y \uparrow (\Phi \cdot u) \) is isomorphism. By Example 5.2.9(1), the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{r} & Y \\
\downarrow X \uparrow (\Phi \cdot u) & & \downarrow Y \uparrow (\Phi \cdot u) \\
\langle P \rangle Q & \xrightarrow{a} & P \langle Q \rangle A
\end{array}
\]

commutes. Since \( P \) is fully faithful, \( \langle P \rangle r \) is isomorphism. Hence, by the commutativity of the diagram above, \( X \uparrow (\Phi \cdot u) \) is isomorphism iff \( P \langle Y \rangle (\Phi \cdot u) \) is isomorphism. Since \( \Phi \) is fully faithful (i.e. \( \Phi : M \to P \langle N \rangle Q \) is isomorphism), \( X \uparrow u \) is isomorphism iff \( X \uparrow (\Phi \cdot u) \) is isomorphism. Since \( P \) is essentially surjective, by Proposition 1.1.31, \( Y \uparrow (\Phi \cdot u) \) is isomorphism iff \( P \langle Y \rangle (\Phi \cdot u) \) is isomorphism. Hence \( X \uparrow u \) is isomorphism iff \( Y \uparrow (\Phi \cdot u) \) is isomorphism as required. \( \square \)

**Definition 8.11.6.** An equivalence of modules \( \langle \Psi, \Phi, \eta, \epsilon \rangle : M \simeq N \) consists of a pair of cells

\[
\begin{array}{ccc}
M & \xrightarrow{\Psi} & N \\
\Phi & \downarrow & \\
M & \xrightarrow{\psi} & N
\end{array}
\]

and a pair of cell isomorphisms

\[
\eta : 1_M \to \Psi \circ \Phi : M \to M \hspace{1cm} \epsilon : \Psi \circ \Phi \to 1_N : N \to N
\]

In this situation, the cells \( \Psi \) and \( \Phi \) are said to be quasi-inverse to each other. Two modules \( M \) and \( N \) are called equivalent, written \( M \simeq N \), if there is an equivalence of modules between them.

**Remark 8.11.7.** As we will see below in Proposition 8.11.8 and Proposition 8.11.9, the notion of equivalence of modules and the notion of equivalence of categories subsume each other.
Proposition 8.11.8. A pair of natural transformations
\[ \eta : 1_X \to G \circ F : X \to X \quad \epsilon : G \circ F \to 1_A : A \to A \]
as in Definition 8.10.10 form an equivalence of categories \((G, F, \eta, \epsilon) : X \simeq A\) if and only if their hom cell morphisms
\[ \langle \eta \rangle : 1_{\langle X \rangle} \to \langle G \circ (F) \rangle : \langle X \rangle \to \langle X \rangle \quad \langle \epsilon \rangle : \langle G \circ (F) \rangle \to 1_{\langle A \rangle} : \langle A \rangle \to \langle A \rangle \]
(see Definition 1.3.5) form an equivalence of modules \(((G) \circ (F), \langle \eta \rangle, \langle \epsilon \rangle) : \langle X \rangle \simeq \langle A \rangle \).

Proof. Immediate from Theorem 1.3.7. \[ \square \]

Proposition 8.11.9. For a pair of cell morphisms
\[ \eta : 1_M \to \Psi \circ \Phi : M \to M \quad \epsilon : \Psi \circ \Phi \to 1_N : N \to N \]
as in Definition 8.11.6, the following conditions are equivalent:
1. \(\eta\) and \(\epsilon\) form an equivalence of modules \((\Psi, \Phi, \eta, \epsilon) : M \simeq N\);
2. the collage natural transformations
\[ [\eta] : 1_{[M]} \to [\Psi] \circ [\Phi] : [M] \to [M] \quad [\epsilon] : [\Psi] \circ [\Phi] \to 1_{[N]} : [N] \to [N] \]
of \(\eta\) and \(\epsilon\) (see Definition 3.1.7) form an equivalence of categories \(([\Psi], [\Phi], [\eta], [\epsilon]) : [M] \simeq [N] \);
3. the left and right components
\[ \eta_b : 1_{M_i} \to \Psi_i \circ \Phi_i : M_i \to M_i \quad \epsilon_i : \Psi_i \circ \Phi_i \to 1_{N_i} : N_i \to N_i \] (i = 0, 1)
of \(\eta\) and \(\epsilon\) (see Definition 1.3.1) form an equivalence of categories \((\Psi_i, \Phi_i, \eta_b, \epsilon_i) : M_i \simeq N_i \) (i = 0, 1).

Proof. (1) \(\iff\) (2) First note that the collage natural transformations
\[ [\eta] : 1_{[M]} \to [\Psi] \circ [\Phi] \quad [\epsilon] : [\Psi] \circ [\Phi] \to 1_{[N]} \]
yield
\[ [\eta] : 1_{[M]} \to [\Psi] \circ [\Phi] \quad [\epsilon] : [\Psi] \circ [\Phi] \to 1_{[N]} \]
by the functoriality of the collage operation \([-]\). Since (like any other full embedding) the full embeddings \([-] : [M] : M \to [[M], [M]]\) and \([-] : [N] : N \to [[N], [N]]\) (see Remark 3.1.15(4)) preserve and reflect isomorphisms, the two conditions are equivalent.

(1) \(\iff\) (3) Immediate from Proposition 1.3.4. \[ \square \]

Theorem 8.11.10. If \((\Psi, \Phi, \eta, \epsilon) : M \simeq N\) is an equivalence of modules, then
1. for any module \(J\), the pair of postcomposition cells
\[ \langle J, M \rangle \xrightarrow{(\langle J, \Psi \rangle)} \langle J, N \rangle \]
(see Definition 1.2.23) and the pair of postcomposition cell morphisms
\[ \langle J, \eta \rangle : 1_{\langle J, M \rangle} \to \langle J, \Psi \rangle \circ \langle J, \Phi \rangle \quad \langle J, \epsilon \rangle : \langle J, \Psi \rangle \circ \langle J, \Phi \rangle \to 1_{\langle J, N \rangle} \]
(see Definition 1.3.8) form an equivalence of modules \((\langle J, M \rangle) \simeq (\langle J, N \rangle)\).
Given a cell

\[ \Psi : N \to M \] (resp. \( \Phi : M \to N \)), the following conditions are equivalent:

1. \( \Psi \) (resp. \( \Phi \)) is an equivalence;
2. \( \Psi \) (resp. \( \Phi \)) is a part of an equivalence of modules \((\Psi, \Phi, \eta, \epsilon) : M \simeq N\).
Proof.
(1) ⇒ (2) We depict \( \Psi : \mathcal{N} \to \mathcal{M} \) as
\[
\xymatrix{ \mathcal{N}_0 \ar[d]_{\Psi_0} \ar[r]^\sim & \mathcal{N}_1 \ar[d]_{\Psi_1} \\
\mathcal{M}_0 \ar[r]^\sim & \mathcal{M}_1 }
\]
First note that if \( \Psi \) is an equivalence, then the functors \( \Psi_0 \) and \( \Psi_1 \) and the collage functor \( [\Psi] \) are equivalences by Definition 8.11.1 and Proposition 8.11.4. Now since \( \Psi_0 \) and \( \Psi_1 \) are equivalences and thus essentially surjective, we may choose, for each object \( x \in \mathcal{M}_0 \), an object \( r_x \in \mathcal{N}_0 \) and an iso \( \mathcal{M}_0 \text{-arrow} \( \epsilon_x : x \to \Psi_0 \circ r_x \), and for each object \( x' \in \mathcal{M}_1 \), an object \( r_{x'} \in \mathcal{N}_1 \) and an iso \( \mathcal{M}_1 \text{-arrow} \( \epsilon_{x'} : x' \to \Psi_1 \circ r_{x'} \). Since \( [\Psi] \) is fully faithful, by Corollary 6.2.17, each \( \epsilon_x \) is universal from \( x \) to \( [\Psi] \) and each \( \epsilon_{x'} \) is universal from \( x' \) to \( [\Psi] \). Hence, by Theorem 8.3.9, \( [\Psi] \) has a left adjoint \( F : \mathcal{M} \to \mathcal{N} \) with \( r_x = F \circ x \) and \( r_{x'} = F \circ x' \); clearly, \( F \) defines a cell \( \Phi : \mathcal{M} \to \mathcal{N} \) such that \( F = [\Phi] \). Let \( \{ \eta \} : 1_{\mathcal{M}} \to [\Psi] \circ [\Phi] \) and \( \{ \epsilon \} : [\Psi] \circ [\Phi] \to 1_{\mathcal{N}} \) be the collage natural transformations forming the unit and counit of the adjunction. Since \( [\Psi] \) is an equivalence, \( (\{ \Psi \}, \{ \Phi \}, \{ \eta \}, \{ \epsilon \}) \) forms an equivalence of categories \( \mathcal{M} \simeq \mathcal{N} \) by the equivalence of (1) and (4) in Theorem 8.10.14. Hence, by the equivalence of (1) and (2) in Proposition 8.11.9, \( (\Psi, \Phi, \eta, \epsilon) \) forms an equivalence of modules \( \mathcal{M} \simeq \mathcal{N} \).

(2) ⇒ (1) By the equivalence of (1) and (2) in Proposition 8.11.9, \( (\{ \Psi \}, \{ \Phi \}, \{ \eta \}, \{ \epsilon \}) \) is an equivalence of categories \( \mathcal{M} \simeq \mathcal{N} \). Hence, by Theorem 8.10.16, \( [\Psi] \) is an equivalence. \( \Psi \) is thus an equivalence by Proposition 8.11.4.

\[ \square \]

**Theorem 8.11.13.** If a cell \( \Phi \) is an equivalence, so are the postcomposition cells \( \langle J, \Phi \rangle, \langle E, \Phi \rangle, \langle *E, \Phi \rangle \), and \( (E*, \Phi) \).

**Proof.** By the equivalence of (1) and (2) in Theorem 8.11.12, this follows from Theorem 8.11.10. \[ \square \]

**Corollary 8.11.14.** Any equivalence cell preserves and reflects limits (resp. colimits).

**Proof.** By definition (see Definition 7.1.8), the assertion is equivalent to saying that if a cell \( \Phi \) is an equivalence, then for any category \( E \) the postcomposition cell \( \langle *E, \Phi \rangle \) preserves and reflects universal arrows. But since \( \langle *E, \Phi \rangle \) is an equivalence as well by Theorem 8.11.13, this follows from Theorem 8.11.5. \[ \square \]

**Note.** The following is a special case of Corollary 8.11.14 where a cell is given by the hom of a functor.

**Corollary 8.11.15.** Any equivalence functor preserves and reflects limits (resp. colimits).

**Proof.** Since a functor preserves and reflects limits (resp. colimits) iff its hom cell preserves and reflects limits (resp. colimits) (see 7.3.10(1)), and since a functor is an equivalence iff its hom cell is an equivalence (see Proposition 8.11.3), the assertion is reduced to Corollary 8.11.14. \[ \square \]

**Theorem 8.11.16.** In the situation of Theorem 8.3.13, suppose that the following equivalent conditions (see Corollary 8.10.8) hold:
1. \( \mathcal{M} \) is an equivalence;
2. \( \mathcal{G} \) is an equivalence;
3. \( \mathcal{F} \) is an equivalence.

Then \( \mathcal{M} \) is equivalent to the hom of \( X \) (resp. \( A \)). Specifically,
The pair of isomorphisms. But since Proof. Once we verify the claim below, it remains to show that Claim. The inverse of the module morphism \(X \vdash \rho : (X) G \to \mathcal{M} : X \to A\), a corepresentation of \(\mathcal{M}\) by \(G\), and the module morphism \(X \vdash \lambda : (X) \to (\mathcal{M}) F : X \to X\) (see Example 5.3.5(1)) yield the equivalence cells
\[
\begin{align*}
X & \xrightarrow{\mathcal{M}} A \\
\xrightarrow{1} (X, \rho)^{-1} \xrightarrow{G} \\
X & \xrightarrow{(X)} X \\
\xrightarrow{X, \lambda} \xrightarrow{F} \\
X & \xrightarrow{\mathcal{M}^{-1}} A
\end{align*}
\]
which is quasi-inverse to each other, forming an equivalence
\[
\begin{array}{c}
\xrightarrow{(X, \rho)^{-1}} \\
X, \lambda
\end{array}
\xrightarrow{\mathcal{M}}
\]
of \((X)\) and \(\mathcal{M}\) with the cell isomorphisms
\[
(1_X, \eta) : 1_{(X)} \to (X, \rho)^{-1} \circ (X, \lambda) \quad (1_X, \epsilon) : (X, \rho)^{-1} \circ (X, \lambda) \to 1_{\mathcal{M}}
\]
where \(\eta\) and \(\epsilon\) are the unit and counit of the adjunction in Theorem 8.3.13.

The inverse of the module morphism \(\lambda \vdash A : F(A) \to \mathcal{M} : X \to A\), a representation of \(\mathcal{M}\) by \(F\), and the module morphism \(\rho \vdash A : (A) \to G(\mathcal{M}) : A \to A\) (see Example 5.3.5(1)) yield the equivalence cells
\[
\begin{align*}
X & \xrightarrow{\mathcal{M}} A \\
\xrightarrow{F} (\lambda, A)^{-1} \xrightarrow{1} \\
A & \xrightarrow{\rho(\lambda)} A \\
\xrightarrow{G} (\rho(\lambda))^{-1} \xrightarrow{1} \\
A & \xrightarrow{\mathcal{M}^{-1}} A
\end{align*}
\]
which is quasi-inverse to each other, forming an equivalence
\[
\begin{array}{c}
\mathcal{M} \xrightarrow{\rho(A)} \\
(\lambda, A)^{-1}
\end{array}
\xrightarrow{\rho(\lambda)}
\]
of \(\mathcal{M}\) and \((A)\) with the cell isomorphisms
\[
(\eta, 1_A) : 1_\mathcal{M} \to (\rho(\lambda))^{-1} \circ (\lambda, A) \quad (\epsilon, 1_A) : (\rho(\lambda))^{-1} \circ (\lambda, A) \to 1_A
\]
where \(\eta\) and \(\epsilon\) are the unit and counit of the adjunction in Theorem 8.3.13.

Proof. Once we verify the claim below, it remains to show that \((1_X, \eta)\) and \((1_X, \epsilon)\) are cell isomorphisms. For this, we just need to show that \(\eta : 1_X \to G \circ F\) and \(\epsilon : G \circ F \to 1_A\) are natural isomorphisms. But since \(G\) and \(F\) are equivalences, this follows from Theorem 8.10.14. \(\square\)

Claim.

(1) The pair \((1_X, \eta)\) forms a cell morphism \(1_X \to (X, \rho)^{-1} \circ (X, \lambda)\); that is, the diagram
\[
\begin{array}{c}
y \\
\xrightarrow{f}
\end{array}
\xrightarrow{1_y} \\
\begin{array}{c}
y \\
\xrightarrow{f : (X, \lambda)^{-1} \circ (X, \rho)}
\end{array}
\xrightarrow{G \circ F} \xrightarrow{\eta_x} \\
\begin{array}{c}
x \\
\xrightarrow{f : y \to x}
\end{array}
\]
commutes for every \(X\)-arrow \(f : y \to x\).

(2) The pair \((1_X, \epsilon)\) forms a cell morphism \((X, \rho)^{-1} \circ (X, \lambda) \to 1_\mathcal{M}\); that is, the diagram
\[
\begin{array}{c}
x \\
\xrightarrow{m : (X, \rho)^{-1} \circ (X, \lambda)} \\
\xrightarrow{1_x}
\end{array}
\xrightarrow{F : G \circ F} \\
\begin{array}{c}
a \\
\xrightarrow{m : x \to a}
\end{array}
\]
commutes for every \(\mathcal{M}\)-arrow \(m : x \to a\).
Proof.

(1) By Theorem 8.3.13, the diagram

\[ \begin{array}{ccc}
    y & \xrightarrow{f \circ \lambda_X} & x \\
    f \downarrow & & \downarrow \eta_X \\
    x & \xrightarrow{\lambda_X} & F \cdot x \\
    \eta \downarrow & & \downarrow \rho(F \cdot x) \\
    x \cdot F : G
\end{array} \]

commutes; that is,

\[ f \circ \lambda_X = f \circ \eta_X \circ \rho(F \cdot x) \]

But

\[ f \circ \lambda_X = f \cdot (X \upharpoonright \lambda) \quad \text{and} \quad f \circ \eta_X \circ \rho(F \cdot x) = (f \circ \eta_X) \cdot (X \upharpoonright \rho) \]

Hence

\[ f \cdot (X \upharpoonright \lambda) = (f \circ \eta_X) \cdot (X \upharpoonright \rho) \]

i.e.

\[ f \cdot (X \upharpoonright \lambda) \cdot (X \upharpoonright \rho)^{-1} = f \circ \eta_X \]

as required.

(2) By Theorem 8.3.13, the diagram

\[ \begin{array}{ccc}
    a : G & \xrightarrow{\lambda(a : G)} & F : G \cdot a \\
    m \upharpoonright \rho_a \uparrow & & \downarrow \epsilon_a \\
    x & \xrightarrow{\rho_a} & a
\end{array} \]

commutes, where \( m / \rho_a \) is the adjunct of \( m \) along \( \rho_a \). Since

\[ m / \rho_a \circ \lambda(a : G) = (m \cdot (X \upharpoonright \rho)^{-1}) \circ \lambda(a : G) = m \cdot (X \upharpoonright \rho)^{-1} \cdot (X \upharpoonright \lambda) \]

we have

\[ m = m / \rho_a \circ \lambda(a : G) \circ \epsilon_a = (m \cdot (X \upharpoonright \rho)^{-1} \cdot (X \upharpoonright \lambda)) \circ \epsilon_a \]

as required.
9. Adjoint Functor Theorem

9.1. Cofinality

Definition 9.1.1. A right (resp. left) module $\mathcal{M}$ is said to be connected if the comma category $[\mathcal{M}]$ is non-empty and connected.

Remark 9.1.2. A right module $\mathcal{M} : \mathbf{E} \to \ast$ is connected if and only if there exists at least one $\mathcal{M}$-arrow and any two $\mathcal{M}$-arrows $m : e \to \ast$ and $m' : e' \to \ast$ are connected by a finite sequence of commutative diagrams as in

\[
e \xrightarrow{f_0} e_1 \xleftarrow{f_1} e_2 \longrightarrow \cdots \xrightarrow{f_n} e' \quad \text{(the direction of each E-arrow $f_i$ is arbitrary)}.
\]

Proposition 9.1.3. A representable left (resp. right) module is connected.

Proof. If a left module $\mathcal{M}$ is representable, then the comma category $[\mathcal{M}]$ has an initial object (see Proposition 6.1.4). Clearly, a category with an initial object is connected. \qed

Definition 9.1.4. (1) A functor $H : \mathbf{D} \to \mathbf{E}$ is called

- left (or downward) cofinal\(^1\) if for every object $e \in [\mathbf{E}]$ the right module $H(e) : \mathbf{D} \to \ast$ is connected,
- right (or upward) cofinal if for every object $e \in [\mathbf{E}]$ the left module $e(H) : \ast \to \mathbf{D}$ is connected.

(2) A subcategory $\mathbf{D}$ of a category $\mathbf{E}$ is called

- left cofinal in $\mathbf{E}$ if the inclusion $\mathbf{D} \to \mathbf{E}$ is left cofinal,
- right cofinal in $\mathbf{E}$ if the inclusion $\mathbf{D} \to \mathbf{E}$ is right cofinal.

(3) A set $\mathcal{D}$ of objects in a category $\mathbf{E}$ is called

- left cofinal in $\mathbf{E}$ if the full subcategory of $\mathbf{E}$ generated by $\mathcal{D}$ is left cofinal,
- right cofinal in $\mathbf{E}$ if the full subcategory of $\mathbf{E}$ generated by $\mathcal{D}$ is right cofinal.

Remark 9.1.5. A functor $H : \mathbf{D} \to \mathbf{E}$ is cofinal if and only if its image is cofinal in $\mathbf{E}$.

Proposition 9.1.6. If $d$ is an initial object of a category $\mathbf{E}$, then the set $\{d\}$ is left cofinal in $\mathbf{E}$.

Proof. Evident. \qed

Proposition 9.1.7. If a functor $H : \mathbf{D} \to \mathbf{E}$ has a right (resp. left) adjoint, then $H$ is left (resp. right) cofinal.

Proof. Since $H$ has a right adjoint iff for every object $e \in [\mathbf{E}]$ the right module $H(e) : \mathbf{D} \to \ast$ is representable (see Corollary 8.3.10), the assertion follows from Proposition 9.1.3. \qed

Remark 9.1.8. As an immediate consequence, if $\mathbf{D}$ is a coreflective (resp. reflective) subcategory of a category $\mathbf{E}$, then $\mathbf{D}$ is left (resp. right) cofinal in $\mathbf{E}$.

\(^1\)Some authors use the term “final” instead of “cofinal”.

271
Proposition 9.1.9. An equivalence functor is both left and right cofinal.

Proof. Since an equivalence functor has both left and right adjoints (see Theorem 8.10.16), the assertion follows from Proposition 9.1.7.

Theorem 9.1.10.

- Let \( \mathcal{M} : \ast \to E \) be a left module. If \( H : D \to E \) is a left cofinal functor, then for any frame \( \alpha \) of the composite left module \( \langle \mathcal{M} \rangle H : \ast \to D \) there exists a unique frame \( \alpha' \) of \( \mathcal{M} \) such that \( \alpha = \alpha' \circ H \).

- Let \( \mathcal{M} : E \to \ast \) be a right module. If \( H : D \to E \) is a right cofinal functor, then for any frame \( \alpha \) of the composite right module \( H(\mathcal{M}) : D \to \ast \) there exists a unique frame \( \alpha' \) of \( \mathcal{M} \) such that \( \alpha = H \circ \alpha' \).

Proof. Let \( \alpha \) be a frame of \( \langle \mathcal{M} \rangle H \). Given \( e \in |E| \), choose \( d \in |D| \) and an \( E \)-arrow \( f : H \cdot d \to e \) (the left cofinality of \( H \) allows such choices), and define an \( \mathcal{M} \)-arrow \( \alpha'_e : \ast \to e \) by \( \alpha'_e = \alpha_d \circ f \). We claim that

1. the definition of \( \alpha'_e \) is independent of the choices of \( d \) and \( f \);
2. \( \alpha' := (\alpha'_e)_{e \in |E|} \) forms a frame of \( \mathcal{M} \);
3. \( \alpha' \) is the only frame of \( \mathcal{M} \) such that \( \alpha = \alpha' \circ H \).

Let \( d' \) and \( f' \) be alternative choices for defining \( \alpha'_e \). Then the diagram

\[
\begin{array}{ccc}
\ast & \rightarrow & \ast \\
\alpha_d & \rightarrow & \alpha_{d'} \\
H \cdot d & \rightarrow & H \cdot d' \\
f & \rightarrow & f'
\end{array}
\]

commutes because, by the left cofinality of \( H \), the interior of the diagram is divided into a finite number of commutative diagrams as in

\[
\begin{array}{ccccccccc}
\ast & \rightarrow & \ast & \rightarrow & \ast & \rightarrow & \ast & \rightarrow & \ast \\
\alpha_d & \rightarrow & \alpha_{d'} & \rightarrow & \alpha_{d_1} & \rightarrow & \alpha_{d_2} & \rightarrow & \alpha_{d_3} \\
H \cdot d & \rightarrow & H \cdot d_1 & \rightarrow & H \cdot d_2 & \rightarrow & H \cdot d_3 & \rightarrow & H \cdot d_4 \\
f & \rightarrow & f_1 & \rightarrow & f_2 & \rightarrow & f_3 & \rightarrow & f_4
\end{array}
\]

Hence the definition of \( \alpha'_e \) is independent of the choices of \( d \) and \( f \). Let \( \alpha'_e = \alpha_d \circ f \) and \( \alpha'_{e'} = \alpha_{d'} \circ f' \) be two \( \mathcal{M} \)-arrows defined as above and let \( h : e \to e' \) be an \( E \)-arrow. Then the diagram

\[
\begin{array}{ccc}
e & \searrow & e' \\
h & \downarrow & \downarrow
\end{array}
\]

i.e.

\[
\begin{array}{ccccccccc}
\ast & \rightarrow & \ast & \rightarrow & \ast & \rightarrow & \ast & \rightarrow & \ast \\
\alpha_d & \rightarrow & \alpha_{d'} & \rightarrow & \alpha_{d_1} & \rightarrow & \alpha_{d_2} & \rightarrow & \alpha_{d_3} \\
H \cdot d & \rightarrow & H \cdot d_1 & \rightarrow & H \cdot d_2 & \rightarrow & H \cdot d_3 & \rightarrow & H \cdot d_4 \\
f & \rightarrow & f_1 & \rightarrow & f_2 & \rightarrow & f_3 & \rightarrow & f_4
\end{array}
\]

commutes by the first claim. \( \alpha' := (\alpha'_e)_{e \in |E|} \) thus forms a frame of \( \mathcal{M} \). By setting \( e = H \cdot d \) and \( f = 1_{H \cdot d} \), we have \( \alpha_d = \alpha_H \cdot d \) for every \( d \in |D| \); hence \( \alpha = \alpha' \circ H \). It is clear that this is the only possible construction of \( \alpha' \) with \( \alpha = \alpha' \circ H \) if \( \alpha' \) is to satisfy the naturality condition.

Corollary 9.1.11. Let \( H : D \to E \) be a functor and \( \mathcal{M} : X \to A \) be a module.
9.1. Cofinality

- If \(H\) is left cofinal, then the precomposition cell

\[
\begin{array}{c}
X \xrightarrow{(\mathcal{E}, \mathcal{M})} [E, A] \\
\downarrow \quad (\mathcal{H}, \mathcal{M}) \\
X \xrightarrow{(\mathcal{D}, \mathcal{M})} [D, A]
\end{array}
\]

(see Definition 4.6.27) is fully faithful.

- If \(H\) is right cofinal, then the precomposition cell

\[
\begin{array}{c}
[E, X] \xrightarrow{(\mathcal{E}, \mathcal{M})} A \\
[H, A] \quad (\mathcal{H}, \mathcal{M}) \\
[D, X] \xrightarrow{(\mathcal{D}, \mathcal{M})} A
\end{array}
\]

(see Definition 4.6.27) is fully faithful.

Proof. Let \(x\) be an object in \(X\) and \(K\) be a functor \(E \to A\). By Theorem 9.1.10, for a cone \(\alpha : x \leadsto K \delta H : * \mathcal{D} \to \mathcal{M}\), i.e., a frame \(\alpha\) of the left module \(x(\mathcal{M})[K \delta H] = (x(\mathcal{M}) K) H : * \to \mathcal{D}\), there is a unique frame \(\alpha'\) of the left module \(x(\mathcal{M}) K : * \to \mathcal{E}\), i.e., a unique cone \(\alpha' : x \leadsto K : * \mathcal{E} \to \mathcal{M}\), such that \(\alpha = \alpha' \delta H\).

\(\square\)

Remark 9.1.12.

1. As a consequence, by Proposition 6.2.23,
   - the precomposition cell \((\mathcal{H}, \mathcal{M})\) preserves, reflects, and creates inverse universal arrows:
     a) a cone \(\pi : r \leadsto K : * \mathcal{E} \to \mathcal{M}\) is universal if and only if the composite cone \(\pi \delta H : r \leadsto K \delta H : * \mathcal{D} \to \mathcal{M}\) is universal;
     b) given a functor \(K : E \to A\), if the composite functor \(K \delta H : D \to A\) has a limit \(\pi : r \leadsto K \delta H : * \mathcal{D} \to \mathcal{M}\), then the unique cone \(\pi' : r \leadsto K : * \mathcal{E} \to \mathcal{M}\) with \(\pi = \pi' \delta H\) is a limit of \(K\).
   - the precomposition cell \((\mathcal{H}, \mathcal{M})\) preserves, reflects, and creates direct universal arrows:
     a) a cone \(\pi : K \leadsto r : E \to \mathcal{M}\) is universal if and only if the composite cone \(H \delta \pi : H \delta K \leadsto r : D \to \mathcal{M}\) is universal;
     b) given a functor \(K : E \to X\), if the composite functor \(H \delta K : D \to X\) has a colimit \(\pi : H \delta K \leadsto r : D \to \mathcal{M}\) with \(\pi = H \delta \pi'\) is a colimit of \(K\).
2. If \(H\) is an equivalence, so are the precomposition cells \((\mathcal{H}, \mathcal{M})\) and \((\mathcal{H}, \mathcal{M})\) (because \([H, A]\) is an equivalence by Theorem 8.10.12, and \((\mathcal{H}, \mathcal{M})\) and \((\mathcal{H}, \mathcal{M})\) are fully faithful by Proposition 9.1.9 and 9.1.11).

Theorem 9.1.13.

- Let \(\mathcal{M} : * \to E\) be a left module. If \(D\) is a left cofinal subcategory of \(E\), then a frame of the left module \((\mathcal{M}) D : * \to D\) (the restriction of \(\mathcal{M}\) to \(D\)) uniquely extends to a frame of \(\mathcal{M}\).
- Let \(\mathcal{M} : E \to *\) be a right module. If \(D\) is a right cofinal subcategory of \(E\), then a frame of the right module \(\mathcal{M} D : D \to *\) (the restriction of \(\mathcal{M}\) to \(D\)) uniquely extends to a frame of \(\mathcal{M}\).

Proof. This is a special case of Theorem 9.1.10 where \(H : D \to E\) is the inclusion. \(\square\)

Corollary 9.1.14. Let \(\mathcal{M} : X \to A\) be a module and let \(D\) be a subcategory of a category \(E\).

- If \(D\) is left cofinal in \(E\), then the cell

\[
\begin{array}{c}
X \xrightarrow{(\mathcal{E}, \mathcal{M})} [E, A] \\
\downarrow \quad (\mathcal{D}, \mathcal{M}) \\
X \xrightarrow{(\mathcal{D}, \mathcal{M})} [D, A]
\end{array}
\]

, “restriction to \(D\)” (see Example 4.6.29(1)), is fully faithful.
- If $D$ is right cofinal in $E$, then the cell

$$
\begin{array}{ccc}
[E,X] & \xrightarrow{(E\times M)} & A \\
[D,X] & \xrightarrow{(D\times M)} & A
\end{array}
$$

, “restriction to $D$” (see Example 4.6.29(1)), is fully faithful.

Proof. Let $x$ be an object in $X$ and $K$ be a functor $E \to A$. By Theorem 9.1.13, a cone $\alpha : x \rightharpoonup K \circ D : *E \rightharpoonup M$, i.e. a frame $\alpha$ of the left module $x(M) [K \circ D] = \langle x(M) K \rangle D : * \rightharpoonup D$, uniquely extends to a frame $\alpha'$ of the left module $x(M) K : * \rightharpoonup E$, i.e. a cone $\alpha' : x \rightharpoonup K : *E \rightharpoonup M$. □

Remark 9.1.15.

(1) As a consequence, by Proposition 6.2.23,

- the cell $(D, M)$ preserves, reflects, and creates inverse universal arrows:
  a) a cone $\pi : r \rightharpoonup K : *E \rightharpoonup M$ is universal if and only if its restriction $\pi \circ D : r \rightharpoonup K \circ D : *D \rightharpoonup M$ to $D$ is universal;
  b) if the restriction of a functor $K : E \rightharpoonup C$ to $D$ has a limit $\pi : r \rightharpoonup K \circ D : *D \rightharpoonup M$, then its unique extension $\pi' : r \rightharpoonup K : *E \rightharpoonup M$ is a limit of $K$.

- the cell $(D, M)$ preserves, reflects, and creates direct universal arrows:
  a) a cone $\pi : K \rightharpoonup r : E* \rightharpoonup M$ is universal if and only if its restriction $D \circ \pi : D \circ K \rightharpoonup r : D* \rightharpoonup M$ to $D$ is universal;
  b) if the restriction of a functor $K : E \rightharpoonup C$ to $D$ has a colimit $\pi : D \circ K \rightharpoonup r : D* \rightharpoonup M$, then its unique extension $\pi' : K \rightharpoonup r : E* \rightharpoonup M$ is a colimit of $K$.

(2) As a special case where $M$ is given by the hom of a category of $C$,

- if $D$ is left cofinal in $E$, the cell

$$
\begin{array}{ccc}
C & \xrightarrow{(E\times C)} & [E, C] \\
1 & \xrightarrow{(D, C)} & [D, C]
\end{array}
$$

(see Example 4.9.9(1)) is fully faithful, and thus preserves, reflects, and creates inverse universal arrows:

a) a cone $\pi : r \rightharpoonup K : *E \rightharpoonup C$ is universal if and only if its restriction $\pi \circ D : r \rightharpoonup K \circ D : *D \rightharpoonup C$ to $D$ is universal;

b) if the restriction of a functor $K : E \rightharpoonup C$ to $D$ has a limit $\pi : r \rightharpoonup K \circ D : *D \rightharpoonup C$, then its unique extension $\pi' : r \rightharpoonup K : *E \rightharpoonup C$ is a limit of $K$.

- if $D$ is right cofinal in $E$, the cell

$$
\begin{array}{ccc}
[E, C] & \xrightarrow{(E\times C)} & C \\
[D, C] & \xrightarrow{(D, C)} & [D, C]
\end{array}
$$

(see Example 4.9.9(1)) is fully faithful, and thus preserves, reflects, and creates direct universal arrows:

a) a cone $\pi : K \rightharpoonup r : E* \rightharpoonup C$ is universal if and only if its restriction $D \circ \pi : D \circ K \rightharpoonup r : D* \rightharpoonup C$ to $D$ is universal;

b) if the restriction of a functor $K : E \rightharpoonup C$ to $D$ has a colimit $\pi : D \circ K \rightharpoonup r : D* \rightharpoonup C$, then its unique extension $\pi' : K \rightharpoonup r : E* \rightharpoonup C$ is a colimit of $K$. 
Theorem 9.1.16.  

- Let $\mathcal{M} : \ast \to \mathbf{E}$ be a left module. If $d$ is an initial object of $\mathbf{E}$, then for any $\mathcal{M}$-arrow $m : \ast \to d$ there is a unique frame $\alpha$ of $\mathcal{M}$ with $\alpha_d = m$. Specifically, $\alpha$ is given by the family of $\mathcal{M}$-arrows $\alpha_e = m \circ v_e$, one for each $e \in \| \mathbf{E} \|$, with $v_e$ the unique $\mathbf{E}$-arrow $d \to e$.

- Let $\mathcal{M} : \mathbf{E} \to \ast$ be a right module. If $d$ is a terminal object of $\mathbf{E}$, then for any $\mathcal{M}$-arrow $m : d \to \ast$ there is a unique frame $\alpha$ of $\mathcal{M}$ with $\alpha_d = m$. Specifically, $\alpha$ is given by the family of $\mathcal{M}$-arrows $\alpha_e = v_e \circ m$, one for each $e \in \| \mathbf{E} \|$, with $v_e$ the unique $\mathbf{E}$-arrow $e \to d$.

**Proof.** Evident. \qed

**Remark 9.1.17.** If $d$ is an initial object of $\mathbf{E}$, the functor $d : \ast \to \mathbf{E}$ is left cofinal by Proposition 9.1.6. Dually, if $d$ is a terminal object of $\mathbf{E}$, the functor $d : \ast \to \mathbf{E}$ is right cofinal. Hence Theorem 9.1.16 is seen a special case of Theorem 9.1.10 where $\mathbf{D}$ is the terminal category.

**Corollary 9.1.18.** Let $\mathcal{M} : \mathbf{X} \to \mathbf{A}$ be a module and $\mathbf{E}$ be a category.

- If $d$ is an initial object of $\mathbf{E}$, then the cell

$$
\begin{array}{c}
\mathbf{X} \xrightarrow{(\ast, \mathcal{M})} [\mathbf{E}, \mathbf{A}] \\
\downarrow \downarrow \downarrow \downarrow \\
\mathbf{X} \xrightarrow{(d, \mathcal{M})} [\mathbf{E}, \mathbf{A}]
\end{array}
$$

, “evaluation at $d$” (see Example 4.6.29(2)), is fully faithful. Specifically, for an $\mathcal{M}$-arrow $m : \mathbf{X} \to K : \mathbf{E} \to \mathbf{A}$, there is a unique cone $\alpha : \mathbf{X} \to K : \mathbf{E} \to \mathcal{M}$ with $\alpha_d = m$, given by the family of $\mathcal{M}$-arrows $\alpha_e = m \circ (K \cdot v_e)$, one for each $e \in \| \mathbf{E} \|$, with $v_e$ the unique $\mathbf{E}$-arrow $d \to e$.

- If $d$ is a terminal object of $\mathbf{E}$, then the cell

$$
\begin{array}{c}
[\mathbf{E}, \mathbf{X}] \xrightarrow{(\ast, \mathcal{M})} \mathbf{A} \\
\downarrow \downarrow \downarrow \downarrow \\
[\mathbf{E}, \mathbf{X}] \xrightarrow{(d, \mathcal{M})} \mathbf{A}
\end{array}
$$

, “evaluation at $d$” (see Example 4.6.29(2)), is fully faithful. Specifically, for an $\mathcal{M}$-arrow $m : \mathbf{X} \to K : \mathbf{E} \to \mathbf{A}$, there is a unique cone $\alpha : \mathbf{X} \to K : \mathbf{E} \to \mathcal{M}$ with $\alpha_d = m$, given by the family of $\mathcal{M}$-arrows $\alpha_e = (v_e \cdot K) \circ m$, one for each $e \in \| \mathbf{E} \|$, with $v_e$ the unique $\mathbf{E}$-arrow $e \to d$.

**Proof.** Let $x$ be an object in $\mathbf{X}$ and $K$ be a functor $\mathbf{E} \to \mathbf{A}$. By Theorem 9.1.16, for an $\mathcal{M}$-arrow $m : x \to K : d$, i.e. an arrow $m : \ast \to d$ in the left module $x(\mathcal{M})$, there is a unique cone $\alpha$ of $x(\mathcal{M}) \cdot K$, i.e. a unique cone $\alpha : x \to K : \mathbf{E} \to \mathcal{M}$, with $\alpha_d = m$, and the component of $\alpha$ at $e \in \| \mathbf{E} \|$ is given by the $x(\mathcal{M})$-arrow $\alpha_e = m \circ v_e$, i.e. the $\mathcal{M}$-arrow $\alpha_e = m \circ (K \cdot v_e)$. \qed

**Remark 9.1.19.**

1. As a consequence, by Proposition 6.2.23,

- the cell $(d, \mathcal{M})$ preserves, reflects, and creates inverse universal arrows:

a) a cone $\pi : r \to K : \mathbf{E} \to \mathcal{M}$ is universal if and only if the $\mathcal{M}$-arrow $\pi_d : r \to K : d$ is inverse universal;

b) given a functor $K : \mathbf{E} \to \mathbf{A}$, if its value at $d$ has an inverse universal $\mathcal{M}$-arrow $u : r \to K : d$, then the unique cone $\pi : r \to K : \mathbf{E} \to \mathcal{M}$ with $\pi_d = u$ is a limit of $K$.

- the cell $(d \cdot, \mathcal{M})$ preserves, reflects, and creates direct universal arrows:

a) a cone $\pi : K \to r : \mathbf{E} \to \mathcal{M}$ is universal if and only if the $\mathcal{M}$-arrow $\pi_d : d \cdot K \to r$ is direct universal;
b) given a functor $K : E \to X$, if its value at $d$ has a direct universal $M$-arrow $u : d : K \to r$, then the unique cone $\pi : K \to r \cdot E \to M$ with $\pi_d = u$ is a colimit of $K$.
(2) As a special case where $M$ is given by the hom of a category of $C$,
\begin{itemize}
  \item if $d$ is an initial object of $E$, the cell
\[
\begin{array}{c}
C - \overset{(*E,C)}{\to} [E,C] \\
| \downarrow (d,C) \quad | \downarrow \quad (d,C) \\
C - \overset{(C)}{\to} C
\end{array}
\]
(see Example 4.9.9(2)) is fully faithful, and thus preserves, reflects, and creates inverse universal arrows:
\begin{itemize}
  \item a cone $\pi : r \to K : *E \to C$ is universal if and only if the $C$-arrow $\pi_d : r \to K \cdot d$ is iso (see Proposition 6.2.5);
  \item for a functor $K : E \to C$ and an iso $C$-arrow $u : r \to K \cdot d$, there is a unique cone $\pi : r \to K : *E \to C$ with $\pi_d = u$, given by the family of $C$-arrows $\pi_e = u \circ (K \cdot v_e)$, one for each $e \in |E|$, with $v_e$ the unique $E$-arrow $d \to e$, and this unique cone is a limit of $K$.
\end{itemize}
\begin{itemize}
  \item if $d$ is a terminal object of $E$, the cell
\[
\begin{array}{c}
[E,C] - \overset{(*E,C)}{\to} C \\
| \downarrow (d,C) \quad | \downarrow \quad (d,C) \\
C - \overset{(C)}{\to} C
\end{array}
\]
(see Example 4.9.9(2)) is fully faithful, and thus preserves, reflects, and creates direct universal arrows:
\begin{itemize}
  \item a cone $\pi : K \to r : E \to C$ is universal if and only if the $C$-arrow $\pi_d : K \to r$ is iso (see Proposition 6.2.5);
  \item for a functor $K : E \to C$ and an iso $C$-arrow $u : d \cdot K \to r$, there is a unique cone $\pi : K \to r : E \to C$ with $\pi_d = u$, given by the family of $C$-arrows $\pi_e = (v_e \cdot K) \circ u$, one for each $e \in |E|$, with $v_e$ the unique $E$-arrow $e \to d$, and this unique cone is a colimit of $K$.
\end{itemize}

Corollary 9.1.20.
\begin{itemize}
  \item If a category $E$ has an initial object $d$, then any functor $K : E \to C$ has a limit $\pi : d : K \to K : *E \to C$, given by the family of $C$-arrows $\pi_e = K \cdot v_e$, one for each $e \in |E|$, with $v_e$ the unique $E$-arrow $d \to e$.
  \item If a category $E$ has a terminal object $d$, then any functor $K : E \to C$ has a colimit $\pi : K \to K : d : E \to C$, given by the family of $C$-arrows $\pi_e = v_e \cdot K$, one for each $e \in |E|$, with $v_e$ the unique $E$-arrow $e \to d$.
\end{itemize}

Proof. The cell $(d,C)$ in Remark 9.1.19(2) creates a limit of $K$ from the identity $d : K \to K : d$.

Theorem 9.1.21.
\begin{itemize}
  \item If $E$ has an initial object $d$, then the identity functor $E \to E$ has a limit $v : d \to E$, given by the unique $E$-arrow $v_e : d \to e$ for each $e \in |E|$. Conversely, if the identity functor $E \to E$ has a limit $v : d \to E$, then $d$ is an initial object.
  \item If $E$ has a terminal object $d$, then the identity functor $E \to E$ has a colimit $v : E \to d$, given by the unique $E$-arrow $v_e : e \to d$ for each $e \in |E|$. Conversely, if the identity functor $E \to E$ has a colimit $v : E \to d$, then $d$ is a terminal object.
\end{itemize}

Proof. The first assertion follows from Corollary 9.1.20. Now suppose that the identity functor $E \to E$ has a limit $v : d \to E$. We just need to show that $f = v_e$ for any $E$-arrow $f : d \to e$. By the
naturality $\nu$, the triangle

\[
\begin{array}{c}
d \\ \downarrow \nu_d \\ d \\
\end{array}
\begin{array}{c}
e \\ \downarrow f \\ e \\
\end{array}
\begin{array}{c}
d \\ \downarrow \nu_e \\ d \\
\end{array}
\]

commutes for every $E$-arrow $f : d \to e$; in particular, the triangle

\[
\begin{array}{c}
d \\ \downarrow \nu_d \\ d \\
\end{array}
\begin{array}{c}
e \\ \downarrow \nu_e \\ e \\
\end{array}
\begin{array}{c}
d \\ \downarrow \nu_e \\ d \\
\end{array}
\]

commutes for every $e \in |E|$. The commutativity of the second triangle shows that $\nu_d \circ \nu = \nu$, and we have $\nu_d = 1_d$ by the uniqueness of factorization. The desired identity now follows from the commutativity of the first triangle.

\[\square\]

### 9.2. General adjoint functor theorem

**Definition 9.2.1.** Given a category $E$, a set $D$ of the objects in $E$ is called

- [jointly] weakly initial in $E$ if to every object $e \in |E|$ there is an $E$-arrow $d \to e$ with $d \in D$.
- [jointly] weakly terminal in $E$ if to every object $e \in |E|$ there is an $E$-arrow $e \to d$ with $d \in D$.

**Proposition 9.2.2.** Let $D$ be a set of objects in a category $E$.

- If $D$ is left cofinal, then $D$ is weakly initial. The converse holds if $E$ is complete.
- If $D$ is right cofinal, then $D$ is weakly terminal. The converse holds if $E$ is cocomplete.

**Proof.** The first assertion is obvious. Now suppose that $E$ is complete and $D \subseteq |E|$ is weakly initial. Given $e \in |E|$, we need to show that the right module $D(E)e : D \to *$ is connected, where $D$ is the full subcategory of $E$ generated by $D$. Let $f_0 : d_0 \to e$ and $f_1 : d_1 \to e$ be two $E$-arrows with $d_0, d_1 \in D$. Since $E$ is complete, there is a pullback

\[
\begin{array}{c}
d_0 \\ \downarrow \nu_d \\ d \\
\end{array}
\begin{array}{c}
e \\ \downarrow f_0 \\ e \\
\end{array}
\begin{array}{c}
d_1 \\ \downarrow \nu_d \\ d \\
\end{array}
\]

. Then, since $D$ is weakly initial, there is an arrow $h : d \to r$ with $d \in D$, yielding a commutative diagram

\[
\begin{array}{c}
d \\ \downarrow h \\ r \\
\end{array}
\begin{array}{c}
\begin{array}{c}
d_0 \\ \downarrow \nu_d \\ d \\
\end{array}
\begin{array}{c}
e \\ \downarrow f_0 \\ e \\
\end{array}
\begin{array}{c}
d_1 \\ \downarrow \nu_d \\ d \\
\end{array}
\end{array}
\]

. Hence $f_0$ and $f_1$ are connected in $D(E)e$.

\[\square\]

**Theorem 9.2.3.**

- A complete category $E$ has an initial object if and only if it has a small weakly initial set.
- A cocomplete category $E$ has a terminal object if and only if it has a small weakly terminal set.

**Proof.** If $E$ has an initial object $d$, then it has a small weakly initial set $\{d\}$. Conversely, suppose that $E$ has a small weakly initial set $D$ and let $D$ be the full subcategory of $E$ generated by $D$. Then $D$ is left cofinal in $E$ by Proposition 9.2.2. Since $E$ is complete and $D$ is small, the inclusion $D \to E$ has a limit $\pi : r \to D$, and this limit extends to a limit $\pi : r \to E$ of the identity $E \to E$ (see Remark 9.1.15(2)). Hence $E$ has an initial object by Theorem 9.1.21.

\[\square\]
Definition 9.2.4.

- Given a left module $M: \ast \to A$, a weakly initial set in the comma category $\mathcal{C}/A$ is called a solution set of $M$; that is, a solution set of $M$ is a set $\{s_i: \ast \to a_i\}$ of $M$-arrows such that every $M$-arrow $m: \ast \to a$ factors through some $s_i$ along an $A$-arrow $h: a_i \to a$ as shown in

$$\begin{array}{ccc}
* & \xrightarrow{s_i} & a_i \\
m & \downarrow & \downarrow h \\
\ast & \to & a
\end{array}$$

- Given a right module $M$, a weakly terminal set in the comma category $\mathcal{C}/A$ is called a solution set of $M$; that is, a solution set of $M$ is a set $\{s_i: x_i \to \ast\}$ of $M$-arrows such that every $M$-arrow $m: x \to \ast$ factors through some $s_i$ along an $X$-arrow $h: x \to x_i$ as shown in

$$\begin{array}{ccc}
x_i & \xleftarrow{s_i} & \ast \\
h & \downarrow & \downarrow \uparrow m \\
x & \to & x_i
\end{array}$$

Theorem 9.2.5.

- A left module $M: \ast \to A$ over a complete category $A$ is representable if and only if $M$ is continuous and has a small solution set.

- A right module $M: X \to \ast$ over a cocomplete category $X$ is representable if and only if $M$ is cocontinuous and has a small solution set.

Proof. Suppose that $M$ is representable. Then $M$ is continuous by Corollary 7.5.9, and a unit $u$ of $M$ yields a small solution set $\{u\}$. Conversely, suppose that $M$ is continuous and has a small solution set. Then the comma category $\mathcal{C}/A$ is complete by Corollary 7.6.3, and has an initial object by Theorem 9.2.3. Hence $M$ is representable by Proposition 6.1.4.

Corollary 9.2.6.

- A module $M: X \to A$ with $A$ complete is representable if and only if for every object $x \in |X|$ the left module $x\{M\}: \ast \to A$ is continuous and has a small solution set.

- A module $M: X \to A$ with $X$ cocomplete is corepresentable if and only if for every object $a \in |A|$ the right module $\{M\} a: X \to \ast$ is cocontinuous and has a small solution set.

Proof. Since a module $M: X \to A$ is representable if and the left module $x\{M\}: \ast \to A$ is representable for every $x \in |X|$ (see Corollary 6.4.11), the assertion follows from Theorem 9.2.5.

Theorem 9.2.7. (General Adjoint Functor Theorem).

- A functor $G: A \to X$ with $A$ complete has a left adjoint if and only if the following conditions hold:
  1. $G$ is continuous;
  2. for each object $x \in |X|$, the left module $x\{X\} G: \ast \to A$ has a small solution set, i.e. a small set $\{s_i: x \to G a_i\}$ of $X$-arrows such that every $X$-arrow $f: x \to G a_i$ factors through some $s_i$ along an $A$-arrow $h: a_i \to a$ as shown in

$$\begin{array}{ccc}
x & \xleftarrow{s_i} & G a_i \\
f & \downarrow & \downarrow h \\
x \to & G a_i & \to a
\end{array}$$

- A functor $F: X \to A$ with $X$ cocomplete has a right adjoint if and only if the following conditions hold:
  1. $F$ is cocontinuous;
(2) For each object \(a \in \|A\|\), the right module \(F(A) a : X \to *\) has a small solution set, i.e. a small set \(\{s_i : x_i : F \to a\}\) of \(A\)-arrows such that every \(A\)-arrow \(f : x : F \to a\) factors through some \(s_i\) along an \(X\)-arrow \(h : x \to x_i\) as shown in

\[
\begin{array}{ccc}
  x & \xrightarrow{h} & x_i \\
  \downarrow & & \downarrow \\
  x & \xrightarrow{h \cdot f} & a \\
  \xrightarrow{m} & & \xrightarrow{s_i} \\
  M & \xrightarrow{h \cdot m} & a
\end{array}
\]

Proof. By Remark 8.3.2(2), a functor \(G : A \to X\) has a left adjoint iff the module \((X) G : X \to A\) is representable. By Corollary 9.2.6, this is the case iff for every \(x \in \|X\|\) the left module \(x(X) G : * \to A\) is continuous and has a small solution set. By Corollary 7.5.8, \(G\) is continuous iff \(x(X) G\) is continuous for every \(x \in \|X\|\). The assertion now follows.

9.3. Epimorphisms and Monomorphisms

Definition 9.3.1.

(1) An arrow \(m : * \to a\) in a left module \(M : * \to A\) is called epic (or an epimorphism) if for any parallel pair of \(A\)-arrows \(h, h' : a \to b\), \(m \circ h = m \circ h'\) implies that \(h = h'\).

(2) An arrow \(m : x \to a\) in a module \(M : X \to A\) is called

- epic (or an epimorphism) if for any parallel pair of \(A\)-arrows \(h, h' : a \to b\), \(m \circ h = m \circ h'\) implies that \(h = h'\).
- monic (or a monomorphism) if for any parallel pair of \(X\)-arrows \(h, h' : y \to x\), \(h \circ m = h' \circ m\) implies that \(h = h'\).

(3) An arrow \(m : x \to a\) in a category \(C\) is called

- epic (or an epimorphism) if for any parallel pair of \(C\)-arrows \(h, h' : a \to b\), \(m \circ h = m \circ h'\) implies that \(h = h'\).
- monic (or a monomorphism) if for any parallel pair of \(C\)-arrows \(h, h' : y \to x\), \(h \circ m = h' \circ m\) implies that \(h = h'\).

Remark 9.3.2.

(1) An arrow \(m : x \to a\) in a category \(C\) is epic (resp. monic) if and only if it is an epimorphism (resp. monomorphism) in the hom endomodule \((C) : C \to C\).

(2) An arrow \(m : x \to a\) in a module \(M : X \to A\) is

- epic if and only if it is an epimorphism in the left module \(x(M) : * \to A\),
- monic if and only if it is a monomorphism in the right module \(M a : X \to *\),

and an arrow \(m : x \to a\) in a category \(C\) is

- epic if and only if it is an epimorphism in the left module \(x(C) : * \to C\),
- monic if and only if it is a monomorphism in the right module \(C a : C \to *\).

Proposition 9.3.3.

- Given a left module \(M : * \to A\),

  (1) an \(M\)-arrow \(m : * \to a\) is epic if and only if it is an epimorphism in the collage category \([M]\).

  (2) an \(A\)-arrow \(f : a \to b\) is epic if and only if it is an epimorphism in the collage category \([M]\).

- Given a right module \(M : X \to *\),

  (1) an \(M\)-arrow \(m : x \to *\) is monic if and only if it is a monomorphism in the collage category \([M]\).
(2) an $X$-arrow $f : y \to x$ is monic if and only if it is a monomorphism in the collage category $[\mathcal{M}]$.

Proof. Obvious by the construction of a collage.

Remark 9.3.4. By Proposition 9.3.3, the properties that hold for epimorphisms (resp. monomorphisms) in a category are also enjoyed by epimorphisms (resp. monomorphisms) in a module.

Proposition 9.3.5.

- Let $\mathcal{M} : \ast \to A$ be a left module. Given an $\mathcal{M}$-arrow $m : \ast \to a$ and an $A$-arrow $f : a \to b$,
  1. if $m$ and $f$ are epic, so is $m \circ f$;
  2. if $m \circ f$ is epic, so is $f$.

- Let $\mathcal{M} : X \to \ast$ be a right module. Given an $\mathcal{M}$-arrow $m : x \to \ast$ and an $X$-arrow $f : y \to x$,
  1. if $m$ and $f$ are monic, so is $f \circ m$;
  2. if $f \circ m$ is monic, so is $f$.

Proof. See [AHS09] 7.34 and 7.41 (see also Remark 9.3.4).

Proposition 9.3.6. Let $\mathcal{M} : X \to A$ be a module.

- If $u : x \to r$ is a direct universal $\mathcal{M}$-arrow, then an $\mathcal{M}$-arrow $m : x \to a$ is epic if and only if its adjunct $u \downarrow m : r \to a$ along $u$ (see Remark 6.2.2(3)) is epic.

- If $u : r \to a$ is an inverse universal $\mathcal{M}$-arrow, then an $\mathcal{M}$-arrow $m : x \to a$ is monic if and only if its adjunct $m / u : x \to r$ along $u$ (see Remark 6.2.2(3)) is monic.

Proof. $m : x \to a$ is epic if it is an epimorphism in the left module $x(\mathcal{M}) : \ast \to A$ and $u \downarrow m : r \to a$ is epic if it is an epimorphism in the left module $r(A) : \ast \to A$. But the left module isomorphism $u(\mathcal{M}!A) : r(A) \to x(\mathcal{M}) : \ast \to A$ maps $u \downarrow m$ to $m$.

Note. We list below some properties of monomorphisms in a category.

Proposition 9.3.7.

1. If a monomorphism $m : x \to a$ has a section (i.e. if there exists an arrow $h : a \to x$ such that $h \circ m = 1$), then $m$ is an isomorphism.
2. Monomorphisms are pullback stable.
3. A continuous function preserves monomorphisms.

Proof.

1. See [AHS09] 7.36.
2. See [AHS09] 11.18.
3. See [AHS09] 13.5.

Definition 9.3.8.

- An epic arrow $m : \ast \to a$ in a left module $\mathcal{M} : \ast \to A$ is called extremally epic (or an extremal epimorphism) if it does not factor through a proper monic $A$-arrow; that is, if it satisfies the following extremal condition: $m = m \circ u$ with $u$ a monic $A$-arrow implies that $u$ is an isomorphism.

- A monic arrow $m : x \to \ast$ in a right module $\mathcal{M} : X \to \ast$ is called extremally monic (or an extremal monomorphism) if it does not factor through a proper epic $X$-arrow; that is, if it satisfies the following extremal condition: $m = u \circ m'$ with $u$ an epic $X$-arrow implies that $u$ is an isomorphism.

Remark 9.3.9. An extremal epimorphism (resp. monomorphism) in a two sided module and in a category is defined similarly.

- An arrow $m : x \to a$ in a module $\mathcal{M} : X \to A$ is extremally epic if and only if it is an extremal epimorphism in the left module $x(\mathcal{M}) : \ast \to A$, and an arrow $m : x \to a$ in a category $C$ is extremally epic if and only if it is an extremal epimorphism in the left module $x(C) : \ast \to C$. 

An arrow \( m : x \rightarrow a \) in a module \( M : X \rightarrow A \) is extremally monic if and only if it is an extremal monomorphism in the right module \( (M)a : X \rightarrow \ast \), and an arrow \( m : x \rightarrow a \) in a category \( C \) is extremally monic if and only if it is an extremal monomorphism in the right module \( (C)a : C \rightarrow \ast \).

**Proposition 9.3.10.** A unit of a left (resp. right) module is extremally epic (resp. monic).

**Proof.** Let \( u \) be a unit of a left module \( M : \ast \rightarrow A \). Clearly \( u \) is epic. Now consider a factorization \( u = v \circ k \) with \( k \) a monic \( A \)-arrow and let \( u \downarrow v \) be the adjunct of \( v \) along \( u \) as shown in

\[
\begin{array}{c}
\ast \\
\downarrow \quad \downarrow \quad \downarrow \\
\bullet \\
\end{array}
\]

Since \( u \circ u \downarrow v \circ k = v \circ k = u \), we have \( u \downarrow v \circ k = 1 \) by the universality of \( u \). Hence \( k \) is an isomorphism by Proposition 9.3.7(1).

**Note.** The following is a restatement of Proposition 9.3.10 in terms of universal arrows.

**Proposition 9.3.11.** A direct (resp. inverse) universal arrow is extremally epic (resp. monic).

**Proof.** Since a direct universal arrow \( u : x \rightarrow r \) in a module \( M : X \rightarrow A \) is the same thing as a unit of the left module \( x(M) : \ast \rightarrow A \) (see Remark 6.2.2(2)), and \( u : x \rightarrow r \) is extremally epic in \( M \) if it is so in \( x(M) \) (see Remark 9.3.9), the assertion follows from Proposition 9.3.10.

**Proposition 9.3.12.**

- Given a left module \( M : \ast \rightarrow A \), if \( A \) is complete and \( M \) is continuous, then any \( M \)-arrow satisfying the extremal condition is epic.
- Given a right module \( M : X \rightarrow \ast \), if \( X \) is cocomplete and \( M \) is cocontinuous, then any \( M \)-arrow satisfying the extremal condition is monic.

**Proof.** Suppose that an \( M \)-arrow \( m \) satisfies the extremal condition. To show that \( m \) is epic, let \( h \) and \( h' \) be a parallel pair of \( A \)-arrows such that \( m \circ h = m \circ h' \). We need to show that \( h = h' \), and for this, it suffices to show that the pair \( (h,h') \) has an equalizer given by an isomorphism. Since \( A \) is complete, \( (h,h') \) has an equalizer \( u \), so it remains to show that \( u \) is iso. Since \( M \) is continuous, \( u \) remains to be an equalizer of \( (h,h') \) in the collage category \( [M] \) (see Remark 7.5.11). Hence there is a unique \( M \)-arrow \( m/u \), the adjunct of \( m \) along \( u \), making the diagram

\[
\begin{array}{c}
m/u \\
\downarrow \quad \downarrow \\
m \\
\end{array}
\]

commute. Now that \( u \) is monic (because it is an equalizer) and \( m \) satisfies the extremal condition, we see that \( u \) is iso as required.

# 9.4. Subobjects

**Definition 9.4.1.**

- A subobject \( s \) in a right module \( M : X \rightarrow \ast \) is a pair \( s = (s,s) \) consisting of an object \( s \in \|X\| \) and a monic \( M \)-arrow \( s : s \rightarrow \ast \).
- A quotient object \( s \) in a left module \( M : \ast \rightarrow A \) is a pair \( s = (s,s) \) consisting of an object \( s \in \|A\| \) and an epic \( M \)-arrow \( s : \ast \rightarrow s \).
Remark 9.4.2.
(1) A subobject and its components are denoted using the same letter.
(2) We regard a subobject in a right module $\mathcal{M} : \mathbf{X} \to \ast$ as an object in the comma category $[\mathcal{M}]$ and say that
   a) subobjects $s$ and $t$ are isomorphic when they are so in $[\mathcal{M}]$; that is, when there is an iso $\mathbf{X}$-arrow $u : s \to t$ making the triangle
   \[
   \begin{array}{ccc}
   s & \to & t \\
   \downarrow^{u} & & \downarrow^{t} \\
   * & \to & *
   \end{array}
   \]
   commute;
   b) two sets $S$ and $T$ of subobjects are equivalent when any subobject in $S$ is isomorphic to some subobject in $T$, and vice versa;
   c) a set $S$ of subobjects is essentially small when it is equivalent to a small set of subobjects.
(3) We write $s \cong t$ if subobjects $s$ and $t$ are isomorphic. Assuming the axiom of choice, the following conditions are equivalent for a set $S$ of subobjects:
   a) $S$ is essentially small;
   b) $S$ has a small subset equivalent to $S$;
   c) the quotient set $S/ \cong$ is small.
(4) A subobject $s$ in a right module $\mathcal{M} : \mathbf{X} \to \ast$ is called extremal if the $\mathcal{M}$-arrow $s : s \to \ast$ is extremally monic. Dually, a quotient object $s$ in a left module $\mathcal{M} : \ast \to \mathbf{A}$ is called extremal if the $\mathcal{M}$-arrow $s : \ast \to s$ is extremally epic.

Definition 9.4.3. Let $\mathbf{C}$ be a category and $c$ be an object in $\mathbf{C}$.
- A subobject of $c$ is a subobject of the right module $(c) : \mathbf{C} \to \ast$; that is, a subobject of $c$ is a pair $s = (s, s)$ consisting of an object $s \in \left[ \mathbf{C} \right]$ and a monic $\mathbf{C}$-arrow $s : s \to c$.
- A quotient object of $c$ is a quotient object of the left module $c (c) : \ast \to \mathbf{C}$; that is, a quotient object of $c$ is a pair $s = (s, s)$ consisting of an object $s \in \left[ \mathbf{C} \right]$ and an epic $\mathbf{C}$-arrow $s : c \to s$.

Proposition 9.4.4.
- A continuous function $H : \mathbf{C} \to \mathbf{B}$ preserves subobjects; that is, $H$ sends each subobject of $c \in \left[ \mathbf{C} \right]$ to a subobject of $c : H \in \left[ \mathbf{B} \right]$.
- A cocontinuous function $H : \mathbf{C} \to \mathbf{B}$ preserves quotient objects; that is, $H$ sends each quotient object of $c \in \left[ \mathbf{C} \right]$ to a quotient object of $c : H \in \left[ \mathbf{B} \right]$.

Proof. Immediate from Proposition 9.3.7(3).

Definition 9.4.5.
- Let $S = \{s_i : s_i \to \ast\}$ be a set of subobjects in a right module $\mathcal{M} : \mathbf{X} \to \ast$. We say that an $\mathcal{M}$-arrow $m : x \to \ast$ factors through $S$ if for each $s_i \in S$ there exists a (necessarily unique) $\mathbf{X}$-arrow $\alpha_i : x \to s_i$ making the diagram
  \[
  \begin{array}{ccc}
  \mathbf{X} & \to & \ast \\
  \downarrow^{m} & & \downarrow^{s_i} \\
  s_i & \to & *
  \end{array}
  \]
  commute.
- Let $S = \{s_i : \ast \to s_i\}$ be a set of quotient objects in a left module $\mathcal{M} : \ast \to \mathbf{A}$. We say that an $\mathcal{M}$-arrow $m : \ast \to a$ factors through $S$ if for each $s_i \in S$ there exists a (necessarily unique) $\mathbf{A}$-arrow $\alpha_i : s_i \to a$ making the diagram
  \[
  \begin{array}{ccc}
  \ast & \to & \mathbf{A} \\
  \downarrow^{m} & & \downarrow^{\alpha_i} \\
  s_i & \to & a
  \end{array}
  \]
  commute.
Remark 9.4.6. If an \(\mathcal{M}\)-arrow \(m : x \rightarrow *\) factors through \(S = \{s_i : s_i \rightarrow *\}\), then it determines a unique discrete cone \((\alpha_i : x \rightarrow s_i)\) from \(m\) to \(S\) in the comma category \([\mathcal{M}]\).

Definition 9.4.7.
- If \(S\) is a set of subobjects in a right module \(\mathcal{M} : X \rightarrow *\), its intersection is an \(\mathcal{M}\)-arrow \(r : r \rightarrow *\) which is terminal among those that factors through \(S\); that is:
  1. \(r\) factors through \(S\),
  2. if an \(\mathcal{M}\)-arrow \(m : x \rightarrow *\) factors through \(S\), then it uniquely factors through \(r : r \rightarrow *\).
- If \(S\) is a set of quotient objects in a left module \(\mathcal{M} : * \rightarrow A\), its union is an \(\mathcal{M}\)-arrow \(r : * \rightarrow r\) which is initial among those that factors through \(S\); that is:
  1. \(r\) factors through \(S\),
  2. if an \(\mathcal{M}\)-arrow \(m : * \rightarrow a\) factors through \(S\), then it uniquely factors through \(r : * \rightarrow r\).

Remark 9.4.8.
1. The unique factorization requirement in the definition implies that any intersection \(r : r \rightarrow *\) of \(S\) is monic, i.e. a subobject in \(\mathcal{M}\).
2. By Remark 9.4.6, an intersection of \(S\) is identified with a product of \(S\) in the comma category \([\mathcal{M}]\). For example, an intersection \(r\) of subobjects \(s_0\) and \(s_1\) is given by a product diagram

\[
\begin{array}{ccc}
s_0 & \xrightarrow{\pi_0} & * \\
\downarrow{\pi_1} & & \downarrow{r} \\
S_1 & \xrightarrow{\rightarrow} & s_1
\end{array}
\]

in \([\mathcal{M}]\) (\(\pi_0\) and \(\pi_1\) are also monic by Proposition 9.3.5).
3. Note that, given a pair of subobjects \((s_0, s_1)\) of an object \(c\) in a category \(C\), a commutative diagram as in

\[
\begin{array}{ccc}
r & \xrightarrow{\pi_1} & S_1 \\
\downarrow{\pi_0} & & \downarrow{r} \\
S_0 & \xrightarrow{s_0} & c \leftarrow s_1
\end{array}
\]

is a pullback diagram of \(s_0 \xrightarrow{s_0} c \leftarrow s_1\) in \(C\) if and only if it is a product diagram of \(s_0 \xrightarrow{s_0} c\) and \(s_1 \xrightarrow{s_1} c\) in the comma category \([(C)c]\). Hence an intersection of \(s_0\) and \(s_1\) is identified with a pullback of \(s_0 \xrightarrow{s_0} c \leftarrow s_1\). This is generalized for arbitrary set \(S\) of subobjects of \(c \in [C]\); an intersection of \(S\) is identified with a limit, multiple pullback, of \(S\).

Proposition 9.4.9.
- Two equivalent sets of subobjects have same intersections.
- Two equivalent sets of quotient objects have same unions.

Proof. Let \(S\) and \(T\) be two sets of subobjects and suppose that they are equivalent. Clearly an \(\mathcal{M}\)-arrow \(m : * \rightarrow a\) factors through \(S\) if and only if it factors through \(T\). Hence \(S\) and \(T\) have same intersections.

Proposition 9.4.10.
- If a category \(C\) is complete, then any small set of subobjects of an object \(c \in [C]\) has an intersection. If a functor \(H : C \rightarrow B\) is continuous, then it preserves subobjects and any intersection of a small set of subobjects.
- If a category \(C\) is cocomplete, then any small set of quotient objects of an object \(c \in [C]\) has a union. If a functor \(H : C \rightarrow B\) is cocontinuous, then it preserves quotient objects and any union of a small set of quotient objects.

Proof. Immediate since an intersection is given as a limit (see Remark 9.4.8(3)). (We have already seen in Proposition 9.4.4 that a continuous functor preserves subobjects.)
9.5. Epi-mono factorizations

Definition 9.4.11.
- A right module $\mathcal{M} : X \to *$ is called well-powered (resp. extremally well-powered) if the set of subobjects (resp. extremal subobjects) in $\mathcal{M}$ is essentially small.
- A left module $\mathcal{M} : * \to A$ is called well-copowered (resp. extremally well-copowered) if the set of quotient objects (resp. extremal quotient objects) in $\mathcal{M}$ is essentially small.

Remark 9.4.12. If $S$ denotes the set of subobjects (resp. extremal subobjects) of $\mathcal{M}$, then, by Remark 9.4.2(3), $\mathcal{M}$ is well-powered (resp. extremally well-powered) if and only if the quotient set $S/\equiv$ is small.

Definition 9.4.13. A category $\mathcal{C}$ is called
- well-powered (resp. extremally well-powered) if the set of subobjects (resp. extremal subobjects) of any $c \in \mathcal{C}$ is essentially small.
- well-copowered (resp. extremally well-copowered) if the set of quotient objects (resp. extremal quotient objects) of any $c \in \mathcal{C}$ is essentially small.

Remark 9.4.14. A category $\mathcal{C}$ is
- well-powered (resp. extremally well-powered) if for every $c \in \mathcal{C}$ the right module $(C) : C \to *$ is well-powered (resp. extremally well-powered).
- well-copowered (resp. extremally well-copowered) if for every $c \in \mathcal{C}$ the left module $c(C) : * \to C$ is well-copowered (resp. extremally well-copowered).

Proposition 9.4.15.
- If a category $\mathcal{C}$ is complete and well-powered, then any set of subobjects of an object $c \in \mathcal{C}$ has an intersection. If a functor $H : \mathcal{C} \to \mathcal{B}$ is continuous, then it preserves subobjects and intersections.
- If a category $\mathcal{C}$ is cocomplete and well-copowered, then any set of quotient objects of an object $c \in \mathcal{C}$ has a union. If a functor $H : \mathcal{C} \to \mathcal{B}$ is cocontinuous, then it preserves quotient objects and unions.

Proof. Since $\mathcal{C}$ is well-powered, any set $S$ of subobjects of $c$ has a small set $S'$ of subobjects of $c$ equivalent to $S$, and, since any functor preserves isomorphisms, the images of $S$ and $S'$ under $H$ are also equivalent. Now, since equivalent sets of subobjects have same intersections (see Proposition 9.4.9), the assertion is reduced to Proposition 9.4.10.

9.5. Epi-mono factorizations

Definition 9.5.1.
- A left module $\mathcal{M} : * \to A$ is said to have (extremal-epi,mono)-factorizations when every $\mathcal{M}$-arrow $m$ factors as $m = p \circ s$ with $p$ an extremally epic $\mathcal{M}$-arrow and $s$ a monic $A$-arrow.
- A right module $\mathcal{M} : X \to *$ is said to have (epi,extremal-mono)-factorizations when every $\mathcal{M}$-arrow $m$ factors as $m = s \circ p$ with $p$ an extremally monic $\mathcal{M}$-arrow and $s$ an epic $X$-arrow.

Proposition 9.5.2.
- Let $\mathcal{M} : * \to A$ be a left module and suppose that $A$ is complete and $\mathcal{M}$ is continuous. Then (extremal-epi,mono)-factorizations are essentially unique; that is, if $p_0 \circ s_0 = p_1 \circ s_1$ are two (extremal-epi,mono)-factorizations of an $\mathcal{M}$-arrow, then there exists a (necessarily unique) isomorphism $u$ making the diagram commute.
Let $\mathcal{M} : \mathbf{X} \to *$ be a right module and suppose that $\mathbf{X}$ is cocomplete and $\mathcal{M}$ is cocontinuous. Then (epi, extremal-mono)-factorizations are essentially unique; that is, if $s_0 \circ p_0 = s_1 \circ p_1$ are two (epi, extremal-mono)-factorizations of an $\mathcal{M}$-arrow; then there exists a (necessarily unique) isomorphism $u$ making the diagram commute.

**Proof.** Let

be a pullback of $(s_0, s_1)$ in $\mathbf{A}$. Since $\mathcal{M}$ is continuous, this pullback remains to be a pullback in the collage category $[\mathcal{M}]$ (see Remark 7.5.11). Hence there exists a unique $\mathcal{M}$-arrow $m$, the adjunct of $(p_0, p_1)$ along $(u_0, u_1)$, making the diagram commute. Since monomorphisms are pullback stable, $u_0$ and $u_1$ are monic, and hence they are isomorphisms by the extremal epicity of $p_0$ and $p_1$. Now $u := u_0^{-1} \circ u_1$ gives a desired isomorphism.

**Theorem 9.5.3.**

- Let $\mathcal{M} : * \to *$ be a left module and suppose that $\mathbf{A}$ is complete and $\mathcal{M}$ is continuous. Under this condition, if $\mathbf{A}$ is well-powered, then $\mathcal{M}$ has essentially unique (extremal-epi, mono)-factorizations.

- Let $\mathcal{M} : \mathbf{X} \to *$ be a right module and suppose that $\mathbf{X}$ is cocomplete and $\mathcal{M}$ is cocontinuous. Under this condition, if $\mathbf{X}$ is well-copowered, then $\mathcal{M}$ has essentially unique (epi, extremal-mono)-factorizations.

**Proof.** The essential uniqueness of factorizations follows from Proposition 9.5.2. Now let $m : * \to a$ be an $\mathcal{M}$-arrow and consider all the possible factorizations $m = m_k \circ s_k$ with $s_k$ monic. By Proposition 9.4.15, there exists an intersection $r : r \to a$ of all $s_k$. Since $\mathcal{M}$ is continuous, by the second assertion of Proposition 9.4.15, it preserves this intersection. Hence $m$ factors through $r$ as $m = n \circ r$ for some (necessarily unique) $\mathcal{M}$-arrow $n : * \to r$. The proof is complete if we show that $n$ is extremally epic. For this, by Proposition 9.3.12, it suffices to show that $n$ satisfies the extremal condition. Let $n = n' \circ u$ be a factorization of $n$ with $u$ monic. We need to show that $u$ is an isomorphism. Since $m = n' \circ (u \circ r)$ and $u \circ r$ is monic, there exists a unique $\mathbf{A}$-arrow $v$ making the diagram commute. Since $r = v \circ u \circ r$ and $r$ is monic, we have $1 = v \circ u$. Hence $u$ is an isomorphism by Proposition 9.3.7(1) as required.

**Definition 9.5.4.** A category $\mathbf{C}$ is said to have
9.6. Generators

Definition 9.6.1. Let $E$ be a category and $M : X \to A$ be a module.

- A cone $\alpha : K \rightrightarrows a : E^* \rightrightarrows M$ is said to be [jointly] epic if for any pair of parallel $A$-arrows $h, h' : a \to b$, $\alpha_e \circ h = \alpha_e \circ h'$ for every $e \in |E|$ implies that $h = h'$.
- A cone $\alpha : x \rightrightarrows K : *E \rightrightarrows M$ is said to be [jointly] monic if for any pair of parallel $X$-arrows $h, h' : y \to x$, $h \circ \alpha_e = h' \circ \alpha_e$ for every $e \in |E|$ implies that $h = h'$.

Remark 9.6.2. A cone $\alpha : K \rightrightarrows a : E^* \rightrightarrows M$ is epic if and only if $\alpha$ is an epic $(E^*, M)$-arrow (see Definition 4.6.5). Dually, a cone $\alpha : x \rightrightarrows K : *E \rightrightarrows M$ is monic if and only if $\alpha$ is a monic $(*E, M)$-arrow.

Proposition 9.6.3.

- If $\pi : K \rightrightarrows r : E^* \rightrightarrows M$ is a universal cone, then a cone $\alpha : K \rightrightarrows a : E^* \rightrightarrows M$ is epic if and only if its adjunct $\pi \alpha : r \rightrightarrows a$ along $\pi$ is epic.
- If $\pi : r \rightrightarrows K : *E \rightrightarrows M$ is a universal cone, then a cone $\alpha : x \rightrightarrows K : *E \rightrightarrows M$ is monic if and only if its adjunct $\alpha \pi : x \rightrightarrows r$ along $\pi$ is monic.

Proof. By Remark 9.6.2, this is an instance of Proposition 9.3.6 where $M$ is given by $(E^*, M)$ (resp. $(*E, M)$).

Definition 9.6.4. A set $E$ of objects in a category $C$ is said to

- generate $C$ if for any parallel pair of $C$-arrows $h, h' : c \to d$, $h \neq h'$ implies that there is an $x \in E$ and a $C$-arrow $f : x \to c$ with $h \circ f \neq h' \circ f$.
- cogenerate $C$ if for any parallel pair of $C$-arrows $h, h' : d \to c$, $h \neq h'$ implies that there is an $x \in E$ and a $C$-arrow $f : c \to x$ with $h \circ f \neq h' \circ f$.

Remark 9.6.5. Let $C$ be a category. If $c$ is an object in $C$ and $E$ is a set of objects in $C$, we denote by $[E, c]$ the set of all $C$-arrows $f : x \to c$ with $x \in E$ (to be precise\(^2\), the set of all pairs $(x, f)$ with $f \in x(C)$ and $x \in E$). Indexed by itself, the set $[E, c]$ is seen as a discrete cone (see Remark 4.6.4(4)) from the family of objects $(\text{dom}(f))_{f \in [E, c]}$ to $c$. Dually, we denote by $[c, E]$ the set of all $C$-arrows $f : c \to x$ with $x \in E$; $[c, E]$ is seen as a discrete cone from $c$ to the family of objects $(\text{cod}(f))_{f \in [E, c]}$. With this notation and the terminology introduced in Definition 9.6.1, Definition 9.6.4 is stated more succinctly as:

\(^2\)Recall that we do not require pairwise disjointness of hom-sets.
Theorem 9.6.6.

- Let $M : \star \to A$ be a left module and suppose that $A$ is complete and $M$ is continuous. Under this condition, if $A$ has a small cogenerating set, then $M$ is extremally well-copowered.
- Let $M : X \to \star$ be a right module and suppose that $X$ is cocomplete and $M$ is cocontinuous. Under this condition, if $X$ has a small generating set, then $M$ is extremally well-powered.

**Proof.** We will use the notation in Remark 9.6.5. Let $E$ be a small cogenerating set of $A$ and denote by $\{*, E\}$ the set of all $M$-arrows $m : \star \to a$ with $a \in E$. Let $\mathcal{P} \{*, E\}$ denote the set of all subsets of $\{*, E\}$; since $E$ is small, so is $\mathcal{P} \{*, E\}$. Given an extremal quotient object $s$ of $M$, define $[s] = \mathcal{P} \{*, E\}$ by $[s] = \{ s \circ f | f \in [s, E]\}$. By the smallness of $\mathcal{P} \{*, E\}$ and noting Remark 9.4.12, the proof is complete if we show that, given two extremal quotient objects $s$ and $t$, $[s] = [t]$ iff $s \cong t$.

If $s$ and $t$ are isomorphic, i.e., if there is an iso $A$-arrow $u : t \to s$ such that $s = t \circ u$, then, by the bijectiveness of $\mathcal{P}$, we have

$$[s] = \{ s \circ f | f \in [s, E]\} = \{ t \circ u \circ f | f \in [s, E]\} = \{ t \circ g | g \in [t, E]\} = [t].$$

Now suppose that $[s] = [t]$ and let $I = [s] = [t]$. Since $s : \star \to s$ and $t : \star \to t$ are epic, the assignments $f \mapsto s \circ f : [s, E] \to I$ and $f \mapsto t \circ f : [t, E] \to I$ are bijective. Hence the discrete cones $[s, E]$ and $[t, E]$ are written as $\sigma = (s_i : s \to a_i)_{i \in I}$ and $\tau = (t_i : t \to a_i)_{i \in I}$ with $s \circ \sigma_i = t \circ \tau_i$ for each $i \in I$. Let $\pi = (\pi_i : r \to a_i)_{i \in I}$ be a product diagram of the family of object $(a_i)_{i \in I}$ in $A$. Since $M$ is continuous, $\pi$ remains to be a product diagram in the collage category $\mathcal{M}$ (see Remark 7.5.11). Hence there exists a unique $M$-arrow $m : \star \to r$, the adjunct of $s \circ \sigma = t \circ \tau$ along $\pi$, making the diagram commute. Since $\sigma = [s, E]$ and $\tau = [t, E]$ are monic by the definition of a cogenerating set, so are $\sigma/\pi$ and $\tau/\pi$ by Proposition 9.6.3. Hence $s \circ \sigma/\pi$ and $t \circ \tau/\pi$ are (extremal-epi,mono) -factorizations of $m$, and by the essential uniqueness of such factorizations (see Proposition 9.5.2), $s$ and $t$ are isomorphic as required. 

**Corollary 9.6.7.**

- A complete category with a small cogenerating set is extremally well-copowered.
- A cocomplete category with a small generating set is extremally well-powered.

**Proof.** Let $C$ be a complete category with a small cogenerating set. By Remark 9.4.14, it suffices to show that for any $c \in [C]$, the left module $c(C) : \star \to C$ is extremally well-copowered. But this follows from Theorem 9.6.6. 

9.7. Special adjoint functor theorem

**Theorem 9.7.1.**

- If a category $A$ is complete, well-powered and with a small cogenerating set, then a left module $M : \star \to A$ is representable if and only if $M$ is continuous.
- If a category $X$ is cocomplete, well-copowered and with a small generating set, then a right module $M : X \to \star$ is representable if and only if $M$ is cocontinuous.
9.7. Special adjoint functor theorem

Proof. If $\mathcal{M}$ is representable, then $\mathcal{M}$ is continuous by Corollary 7.5.9. Conversely, suppose that $\mathcal{M}$ is continuous. Then $\mathcal{M}$ has (extremal-epi,mono)-factorizations by Theorem 9.5.3. This implies that the set of extremal quotient objects in $\mathcal{M}$ forms a solution set of $\mathcal{M}$, and this set is essentially small by Theorem 9.6.6. Hence $\mathcal{M}$ is representable by Theorem 9.2.5. ☐

Corollary 9.7.2. Let $\mathcal{M} : \mathbf{X} \to \mathbf{A}$ be a module.

- If a category $\mathbf{A}$ is complete, well-powered and with a small cogenerating set, then $\mathcal{M}$ is representable if and only if for every object $x \in |\mathbf{X}|$ the left module $x(M) : * \to \mathbf{A}$ is continuous.
- If a category $\mathbf{X}$ is cocomplete, well-copowered and with a small generating set, then $\mathcal{M}$ is corepresentable if and only if for every object $a \in |\mathbf{A}|$ the right module $(M)a : \mathbf{X} \to *$ is cocontinuous.

Proof. Since a module $\mathcal{M} : \mathbf{X} \to \mathbf{A}$ is representable iff the left module $x(M) : * \to \mathbf{A}$ is representable for every $x \in |\mathbf{X}|$ (see Corollary 6.4.11), the assertion follows from Theorem 9.7.1. ☐

Theorem 9.7.3. (Special Adjoint Functor Theorem).

- If a category $\mathbf{A}$ is complete, well-powered and with a small cogenerating set, then a functor $G : \mathbf{A} \to \mathbf{X}$ has a left adjoint if and only if $G$ is continuous.
- If a category $\mathbf{X}$ is cocomplete, well-copowered and with a small generating set, then a functor $F : \mathbf{X} \to \mathbf{A}$ has a right adjoint if and only if $F$ is cocontinuous.

Proof. By Remark 8.3.2(2), a functor $G : \mathbf{A} \to \mathbf{X}$ has a left adjoint iff the module $(\mathbf{X})G : \mathbf{X} \to \mathbf{A}$ is representable. By Corollary 9.7.2, this is the case iff for every object $x \in |\mathbf{X}|$ the left module $x(\mathbf{X})G : * \to \mathbf{A}$ is continuous. By Corollary 7.5.8, this is the case iff $G$ is continuous. ☐

Corollary 9.7.4.

- If a category $\mathbf{C}$ is complete, well-powered and with a small cogenerating set, then $\mathbf{C}$ is cocomplete as well.
- If a category $\mathbf{C}$ is cocomplete, well-copowered and with a small generating set, then $\mathbf{C}$ is complete as well.

Proof. Since $\mathbf{C}$ is cocomplete iff for every small category $\mathbf{E}$ the diagonal functor $|!_{\mathbf{E}} : \mathbf{C}|$ has a left adjoint (see Remark 7.3.5), and since $|!_{\mathbf{E}} : \mathbf{C}|$ is continuous by Theorem 7.3.17, the assertion follows from Theorem 9.7.3. ☐

Corollary 9.7.5.

- If a category $\mathbf{C}$ is complete, well-powered and with a small cogenerating set, then $\mathbf{C}$ has an initial object.
- If a category $\mathbf{C}$ is cocomplete, well-copowered and with a small generating set, then $\mathbf{C}$ has a terminal object.

Proof. Immediate from Corollary 9.7.4 since any cocomplete category has an initial object. ☐

Remark 9.7.6. Conversely, Theorem 9.7.3 follows from Corollary 9.7.5. [ML98] takes this route to tackle the special adjoint functor theorem.
10. Collages and Commas (continued)

10.1. Cylinder modules

In this section, we define (Definition 10.1.1) and study the following modules:

- \( \uparrow \mathrm{CYL} : \mathrm{CAT} \to \mathrm{CLG} \)
- \( \uparrow \mathrm{CYL} : \mathrm{CAT} \to \mathrm{MOD} \)
- \( \downarrow \mathrm{CYL} : \mathrm{COM} \to \mathrm{CAT} \)

Although we give separate definitions for \( \uparrow \mathrm{CYL} : \mathrm{CAT} \to \mathrm{MOD} \) and \( \uparrow \mathrm{CYL} : \mathrm{CAT} \to \mathrm{CLG} \), they are regarded as the same thing (Remark 10.1.2) under the identification \( \mathrm{CLG} \cong \mathrm{MOD} \) and treated as such (cf. Remark 3.1.15(1)) throughout the section.

Definition 10.1.1.

(1) The module \( \uparrow \mathrm{CYL} : \mathrm{CAT} \to \mathrm{CLG} \) is defined in the following way:

- a) a \( \uparrow \mathrm{CYL} \)-arrow from a category \( E \) to a collage \( M : X \to A \), written \( \alpha : S \to T : E \to M \), is a triple \((S, \alpha, T)\) consisting of a functor \( S : E \to X \), a second functor \( T : E \to A \), and a natural transformation from \( S \circ M_0 \) to \( M_1 \circ T \).

\[
\begin{array}{ccc}
S & \downarrow \varepsilon & T \\
X & \alpha & A \\
\end{array}
\]

- b) for a functor \( H : D \to E \) and a \( \uparrow \mathrm{CYL} \)-arrow \( \alpha : S \to T : E \to M \) as in

\[
\begin{array}{ccc}
D & \downarrow H & E \\
X & \alpha & A \\
\end{array}
\]

their composite is the \( \uparrow \mathrm{CYL} \)-arrow \( H \circ \alpha : H \circ S \to T \circ H : D \to M \) with the natural transformation \( H \circ \alpha : H \circ S \circ M_0 \to M_1 \circ T \circ H : D \to [M] \) given by the usual composition of a functor and a natural transformation.

- c) for a \( \uparrow \mathrm{CYL} \)-arrow \( \alpha : S \to T : E \to M \) and a collage cell \( \Phi : P \to Q : M \to N \) as in

\[
\begin{array}{ccc}
E & \downarrow S & T \\
X & \alpha & A \\
\end{array}
\]

\[
\begin{array}{ccc}
P & \downarrow \phi & Q \\
Y & \downarrow \gamma & B \\
\end{array}
\]

their composite is the \( \uparrow \mathrm{CYL} \)-arrow \( \alpha \circ \phi : S \circ P \to Q \circ T : E \to N \) with the natural transformation \( \alpha \circ \phi : S \circ P \circ N_0 \to M_1 \circ Q \circ T : E \to [N] \) defined by

\[
\alpha \circ \phi = \alpha \circ [\phi]
\]

the usual composite of a natural transformation and a functor.

(2) The module \( \uparrow \mathrm{CYL} : \mathrm{CAT} \to \mathrm{MOD} \) is defined in the following way:
a) a ↑CYL-arrow from a category $E$ to a module $M : X \to A$, written $\alpha : S \triangleright T : E \triangleright M$, is a cylinder

```
  E
 / \  α
S  T  ↓M
X----A
```

b) for a functor $H : D \to E$ and a ↑CYL-arrow $\alpha : S \triangleright T : E \triangleright M$ as in

```
  D
  H
↓  ↓
S  E
  α
X----M
  ↓A
```

their composite is the ↑CYL-arrow $H \circ \alpha : H \circ S \triangleright T \circ H : D \triangleright M$, the usual composite of a functor and a cylinder (see Definition 4.3.23).

c) for a ↑CYL-arrow $\alpha : S \triangleright T : E \triangleright M$ and a module cell $\Phi : P \triangleright Q : M \to N$ as in

```
  E
  α
S  T  ↓M
X----A
  ↓P
Y----Q
  ↓Φ
  ↓N
  ↓B
```

their composite is the ↑CYL-arrow $\alpha \circ \Phi : S \circ P \triangleright T \circ Q \circ E \triangleright N$, the usual composite of a cylinder and a cell (see Definition 4.3.13).

(3) The module ↓CYL : COM ⇢ CAT is defined in the following way:

a) a ↓CYL-arrow from a comma $K : X \ CGSize{\rightarrow} A$ to a category $E$, written $\alpha : S \ CGSize{\rightarrow} T : K \ CGSize{\rightarrow} E$, is a triple $(S, \alpha, T)$ consisting of a functor $S : X \to E$, a second functor $T : A \to E$, and a natural transformation $\alpha : S \to T$ from $K_0 \circ S$ to $T \circ K_1$.

b) for a comma cell $\Phi : P \ CGSize{\rightarrow} Q : J \rightarrow K$ and a ↓CYL-arrow $\alpha : S \ CGSize{\rightarrow} T : K \ CGSize{\rightarrow} E$ as in

```
  Y
  \Phi
↓  ↓
S  E
  α
X----A
  ↓P
  ↓Φ
  ↓N
  ↓B
```

their composite is the ↓CYL-arrow $\Phi \circ \alpha : P \circ S \ CGSize{\rightarrow} T \circ Q \ CGSize{\rightarrow} J \ CGSize{\rightarrow} E$ with the natural transformation $\Phi \circ \alpha : J_0 \circ P \circ S \ CGSize{\rightarrow} T \circ Q \circ J_1 : [J] \ CGSize{\rightarrow} E$ defined by

$\Phi \circ \alpha = [\Phi] \circ \alpha$

the usual composite of a functor and a natural transformation.

c) for a ↓CYL-arrow $\alpha : S \ CGSize{\rightarrow} T : K \ CGSize{\rightarrow} E$ and a functor $H : E \ CGSize{\rightarrow} D$ as in

```
  E
  α
S  T  ↓M
X----A
  ↓H
  ↓D
```

with the natural transformation $H \circ \alpha : K_0 \circ S \ CGSize{\rightarrow} T \circ H \circ E \ CGSize{\rightarrow} D$ defined by

$H \circ \alpha = [H] \circ \alpha$
10.1. Cylinder modules 291

, their composite is the \( \downarrow \text{CYL} \)-arrow \( \alpha \circ H : S \circ H \Rightarrow H \circ T : K \Rightarrow D \) with the natural transformation \( \alpha \circ H : K_0 \circ S \circ H \Rightarrow H \circ T \circ K_1 : [K] \Rightarrow D \) given by the usual composition of a natural transformation and a functor.

**Remark 10.1.2.** By Remark 4.3.4(4), the identity

\[
\begin{array}{ccc}
\text{CAT} & \downarrow \text{CYL} & \text{MOD} \\
\downarrow 1 & \downarrow 1 & \downarrow \text{Id} \\
\text{CAT} & \downarrow \text{CLG} & \\
\end{array}
\]

holds.

**Definition 10.1.3.**

(1) The unit cylinder of a module \( \mathcal{M} : X \Rightarrow A \) is the cylinder

\[
\begin{array}{ccc}
\mathcal{M}_0 & \downarrow \mathcal{M} & \mathcal{M}_1 \\
\downarrow 0 \circ \mathcal{M} & \downarrow \mathcal{M} & \downarrow 1 \circ \mathcal{M} \\
X & \downarrow \mathcal{M} & A \\
\end{array}
\]

defined by

\[
[1^1_{\mathcal{M}}]_m = m
\]

for \( m \) an arrow of \( \mathcal{M} \).

(2) The unit cylinder of a comma \( K : X \Rightarrow A \) is the cylinder

\[
\begin{array}{ccc}
K_0 & \downarrow K & K_1 \\
\downarrow 0 \circ K & \downarrow K & \downarrow 1 \circ K \\
X & \downarrow K & A \\
\end{array}
\]

defined by

\[
[1^1_{K}]_k = k
\]

for \( k \) an object of \( [K] \).

**Remark 10.1.4.**

(1) By Remark 4.3.4(4), the unit cylinders \( 1_{\mathcal{M}}^1 \) and \( 1_{K}^1 \) are also written as

\[
\begin{array}{ccc}
\mathcal{M}_0 & \downarrow \mathcal{M} & \mathcal{M}_1 \\
\downarrow 0 \circ \mathcal{M} & \downarrow \mathcal{M} & \downarrow 1 \circ \mathcal{M} \\
X & \downarrow \mathcal{M} & A \\
\end{array}
\]

using the collages of \( \mathcal{M} \) and \( K^1 \) (recall from Remark 3.2.24(2) that \( [K] \) denotes the collage category of \( K^1 \)).

(2) The unit cylinder \( 1_{\mathcal{M}}^1 \) forms a \( \uparrow \text{CYL} \)-arrow \( 1_{\mathcal{M}}^1 : \mathcal{M}_0 \Rightarrow \mathcal{M}_1 : [\mathcal{M}] \Rightarrow \mathcal{M} \).

(3) The unit cylinder \( 1_{K}^1 \) forms a \( \downarrow \text{CYL} \)-arrow \( 1_{K}^1 : K_0 \Rightarrow K_1 : [K] \Rightarrow K \).

**Proposition 10.1.5.**

(1) Given a collage cell \( \Phi : P \Rightarrow Q : \mathcal{M} \Rightarrow \mathcal{N} \), the two compositions

\[
\begin{array}{ccc}
P & \downarrow \Phi & Q \\
\downarrow 0 \circ \Phi & \downarrow \Phi & \downarrow 1 \circ \Phi \\
Y & \downarrow \Phi & B \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \downarrow \Phi & A \\
\downarrow 0 \circ \Phi & \downarrow \Phi & \downarrow 1 \circ \Phi \\
Y & \downarrow \Phi & B \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{M}_0 & \downarrow \mathcal{M} & \mathcal{M}_1 \\
\downarrow 0 \circ \mathcal{M} & \downarrow \mathcal{M} & \downarrow 1 \circ \mathcal{M} \\
X & \downarrow \mathcal{M} & A \\
\end{array}
\]
where $[\Phi]:[M] \to [N]$ is the comma functor of $\Phi$ (see Remark 3.2.20(2)), yield the same natural transformations $[M] \to [N]$; that is, the square

$$
\begin{array}{c}
[M] \xrightarrow{1_M^i} [M] \\
[\Phi] \downarrow \downarrow \downarrow [\Phi] \\
[N] \xrightarrow{1_N^i} [N]
\end{array}
$$

commutes.

(2) Given a comma cell $\Phi:P \rightsquigarrow Q:J \to K$, the two compositions

$$
\begin{array}{c}
Y \xleftarrow{J_0} [J] \xrightarrow{J_1} B \\
P \downarrow \quad \quad \quad \quad \downarrow \\
X \xleftarrow{K_0} [K] \xrightarrow{K_1} A
\end{array}
$$

, where $[\Phi]:[J] \to [K]$ is the collage functor of $\Phi$ (see Remark 3.2.24(2)), yield the same natural transformations $[J] \to [K]$; that is, the square

$$
\begin{array}{c}
[J] \xrightarrow{1_J^i} [J] \\
[\Phi] \downarrow \downarrow \downarrow [\Phi] \\
[K] \xrightarrow{1_K^i} [K]
\end{array}
$$

commutes.

Proof. Immediate from the definitions of the unit cylinders $1_M^i$ and $1_K^i$, and the constructions of the comma functor $[\Phi]$ and the collage functor $[\Phi]$. □

Proposition 10.1.6.

(1) For a module $M:X \to A$, the unit cylinder $1_M^i$ (see Remark 10.1.4(1)) is given by the composition

$$
\begin{array}{c}
M_0 \xrightarrow{[M]} [M] \\
\downarrow \downarrow \downarrow \downarrow [\epsilon_M] \\
M_1 \xrightarrow{[M]} [M]
\end{array}
$$

; that is, the triangle

$$
\begin{array}{c}
[M] \xrightarrow{1_M^i} [M] \\
[\epsilon_M] \downarrow \downarrow \downarrow [\epsilon_M] \\
[M]
\end{array}
$$

commutes, where $1_M^i$ is the unit cylinder of the comma $M^i$ and $\epsilon_M$ is the isomorphism in Theorem 3.2.25.
(2) For a comma \( K : X \to A \), the unit cylinder \( 1^i_K \) (see Remark 10.1.4(1)) is given by the composition

\[
\begin{array}{c}
\xymatrix{
X & [K] \\
\eta_K \
|_K \ar[uu] & K_1 \\
K_0 \
\}
\end{array}
\]

; that is, the triangle

\[
\begin{array}{c}
\xymatrix{
[K_1] & \eta_K \
|_K \
|_K \ar[uu] & K_0 \\

}
\end{array}
\]

commutes, where \( \eta_K \) is the isomorphism in Theorem 3.2.25 and \( 1^i_K \) is the unit cylinder of the module \( K^1 \).

Proof. Immediate from the definitions. \( \square \)

Definition 10.1.7.
(1) The comma adjunct of a \( \uparrow \text{CYL} \)-arrow \( \alpha : S \to T : E \to \mathcal{M} \), i.e.

\[
\begin{array}{c}
\xymatrix{
X & \alpha \\
S & E \\
T & A
\end{array}
\]

, is the functor \([\alpha] : E \to [\mathcal{M}]\) defined by

\[
[\alpha] : e = \alpha_e \\
[\alpha] : h = (S \circ h, T \circ h)
\]

for \( e \) an object and \( h \) an arrow of \( E \).

(2) The collage adjunct of a \( \downarrow \text{CYL} \)-arrow \( \alpha : S \to T : K \to E \), i.e.

\[
\begin{array}{c}
\xymatrix{
X & [K] \\
S & E \\
T & A
\end{array}
\]

, is the functor \([\alpha] : [K] \to E\) given by the adjunct (see Theorem 3.1.16) of the cell

\[
\begin{array}{c}
\xymatrix{
X & [K] \\
S & E \\
T & [E] \ar[u]
\end{array}
\]

defined by

\[
[\alpha] : k = \alpha_k
\]

for \( k \) an object of \([K]\).

Remark 10.1.8. Proposition 10.1.9 below justifies the name “adjunct”.
Proposition 10.1.9.

(1) For any \( \uparrow \text{CYL-arrow } \alpha : S \sim T : E \sim M \), the comma adjunct \([\alpha]\) gives the unique functor \( E \to \text{[M]} \) making the diagram

\[
\begin{array}{c}
\text{[M]} \\
\alpha \downarrow \\
E
\end{array} \\
\text{[M]} \sim \text{[M]}
\]

commute; the unit cylinder \( 1^\downarrow_M \) is therefore an inverse universal \( \uparrow \text{CYL-arrow} \).

(2) For any \( \downarrow \text{CYL-arrow } \alpha : S \sim T : K \sim E \), the collage adjunct \([\alpha]\) gives the unique functor \( K \to \text{[E]} \) making the diagram

\[
\begin{array}{c}
K \\
\alpha \downarrow \\
E
\end{array} \\
K \sim K
\]

commute; the unit cylinder \( 1^\uparrow_K \) is therefore a direct universal \( \downarrow \text{CYL-arrow} \).

Proof. The identities

\[
[\alpha] \circ 1^\downarrow_M = \alpha \quad 1^\uparrow_K \circ [\alpha] = \alpha
\]

are verified easily. To prove the uniqueness of \([\alpha]\) and \([\alpha]\), it suffices to prove the identities

\[
[H \circ 1^\downarrow_M] = H \quad [1^\uparrow_K \circ H] = H
\]

for any functor \( H : E \to \text{[M]} \) and any functor \( H : \text{[K]} \to E \). But these are also verified easily.  

Theorem 10.1.10.

(1) The functor \([-] : \text{MOD} \to \text{CAT} \) (see Remark 3.2.20(3)) and the family of unit cylinders \( 1^\downarrow_M : [\text{M}] \to M \), one for each locally small module \( M \), form a counit of the module \( \uparrow \text{CYL} : \text{CAT} \to \text{MOD} \).

(2) The functor \([-] : \text{COM} \to \text{CAT} \) (see Remark 3.2.24(3)) and the family of unit cylinders \( 1^\uparrow_K : K \to [K] \), one for each locally small comma \( K \), form a unit of the module \( \downarrow \text{CYL} : \text{COM} \to \text{CAT} \).

Proof. The family of unit cylinders \( 1^\downarrow_M \) (resp. \( 1^\uparrow_K \)) satisfies the naturality condition by Proposition 10.1.5, and each unit cylinder is universal as we have seen in Proposition 10.1.9.

Definition 10.1.11. The unit cylinder of a category \( E \) is the cylinder

\[
\begin{array}{c}
E \\
\downarrow 1 \downarrow [1_E] \\
\downarrow (E)
\end{array} \\
E \sim E
\]

defined by

\[
[1_E]_e = 1_e
\]

for \( e \in [E] \), i.e. by the identity natural transformation \( 1_E \to 1_E \).

Proposition 10.1.12. For any functor \( H : D \to E \), the square

\[
\begin{array}{c}
D \sim \text{[D]} \\
H \downarrow \\
E \sim \text{[E]}
\end{array}
\]

commutes.
Proof. Easily verified. \( \square \)

**Proposition 10.1.13.** For any locally small category \( E \), the unit cylinder \([1_E]\) forms a direct universal \( \uparrow \text{CYL-arrow} \).

**Proof.** The family of module isomorphisms \( \Psi^E_M : (E, M) \to (\langle E \rangle, \mathcal{M}) \) in Corollary 5.5.5, one for each locally small module \( \mathcal{M} \), gives a representation of the left slice of \( \uparrow \text{CYL} \) at \( E \). The unit corresponding to this representation is given by the inverse image of identity cell \( \langle E \rangle \to \langle E \rangle \) under \( \Psi^E_{\langle E \rangle} \), but it is immediately seen that this is nothing but the unit cylinder \([1_E]\) of \( E \). \( \square \)

**Theorem 10.1.14.** The embedding \( (-) : \text{CAT} \to \text{MOD} \) (see Theorem 1.2.30) and the family of unit cylinders \([1_E] : E \to \langle E \rangle\), one for each locally small category \( E \), form a unit of the module \( \uparrow \text{CYL} : \text{CAT} \to \text{MOD} \).

**Proof.** Immediate from Proposition 10.1.12 and Proposition 10.1.13. \( \square \)

**Theorem 10.1.15.** There is a canonical adjunction between the functor \( [-] : \text{MOD} \to \text{CAT} \) and the embedding \( (-) : \text{CAT} \to \text{MOD} \).

**Proof.** This follows by applying Theorem 8.3.13 to the counit and unit of the module \( \uparrow \text{CYL} \) given in Theorem 10.1.10(1) and Theorem 10.1.14. \( \square \)

**Remark 10.1.16.** The functor \( [-] : \text{MOD} \to \text{CAT} \) is thus a right adjoint of the embedding \( (-) : \text{CAT} \to \text{MOD} \) (cf. Remark 3.1.17).

### 10.2. Equivalence \( \text{COM} \simeq \text{CLG} \)

In this section, we define (Definition 10.2.1) and study the following modules:

- \( \downarrow \text{CYL} : \text{COM} \to \text{CLG} \)
- \( \uparrow \text{CYL} : \text{COM} \to \text{MOD} \)

Although we give separate definitions for them, the two modules are regarded as the same thing under the identification \( \text{CLG} \simeq \text{MOD} \). After Remark 10.2.2(3) we will deal with only the module \( \downarrow \text{CYL} : \text{COM} \to \text{CLG} \); however, any result in this section stated for \( \downarrow \text{CYL} : \text{COM} \to \text{CLG} \) also holds with \( \text{CLG} \) changed to \( \text{MOD} \).

**Definition 10.2.1.**

1. The module \( \downarrow \text{CYL} : \text{COM} \to \text{CLG} \) is defined in the following way:
   a) a \( \downarrow \text{CYL} \)-arrow from a comma \( K : Y \to B \) to a collage \( M : X \to A \), written \( \alpha : S \Rightarrow T : K \Rightarrow M \), is a triple \((S, \alpha, T)\) consisting of a functor \( S : Y \to X \), a second functor \( T : B \to A \), and a natural transformation
   \[
   Y \leftarrow K \leftarrow [K] \leftarrow K_1 \rightarrow B
   \]
   \[
   S \downarrow \alpha \downarrow T
   \]
   \[
   X \leftarrow M \leftarrow [M] \leftarrow M_1 \rightarrow A
   \]
   from \( K_0 \circ S \circ M_0 \) to \( M_1 \circ T \circ K_1 \).
   b) for a comma cell \( \Phi : P \Rightarrow Q : J \to K \) and a \( \downarrow \text{CYL} \)-arrow \( \alpha : S \Rightarrow T : K \Rightarrow M \) as in
   \[
   Z \leftarrow J \leftarrow [J] \leftarrow J_1 \rightarrow C
   \]
   \[
   P \downarrow \Phi \downarrow Q
   \]
   \[
   Y \leftarrow K \leftarrow [K] \leftarrow K_1 \rightarrow B
   \]
   \[
   S \downarrow \alpha \downarrow T
   \]
   \[
   X \leftarrow M \leftarrow [M] \leftarrow M_1 \rightarrow A
   \]
10.2. Equivalence $\text{COM} \simeq \text{CLG}$

, their composite is the $\uparrow\text{CYL}$-arrow $\Phi \odot \alpha : P \odot S \rightarrow T \odot Q : J \rightarrow M$ with the natural transformation $\Phi \odot \alpha : J_0 \odot P \odot S \odot M_0 \rightarrow M_1 \odot T \odot Q \odot J_1 : [J] \rightarrow [M]$ defined by

$$\Phi \odot \alpha = [\Phi] \odot \alpha$$

, the usual composite of a functor and a natural transformation.

c) for a $\uparrow\text{CYL}$-arrow $\alpha : S \rightarrow T : K \rightarrow M$ and a module cell $\Phi : P \rightarrow Q : M \rightarrow \mathcal{N}$ as in

\[
\begin{array}{c}
Y \xleftarrow{K_0} [K] \xrightarrow{K_1} B \\
\downarrow \alpha \\
X \xleftarrow{M_0} [M] \xrightarrow{M_1} A \\
\downarrow \Phi \\
Z \xleftarrow{N_0} [N] \xrightarrow{N_1} C
\end{array}
\]

, their composite is the $\uparrow\text{CYL}$-arrow $\alpha \circ \Phi : S \circ P \rightarrow Q \circ T : K \rightarrow \mathcal{N}$ with the natural transformation $\alpha \circ \Phi : K_0 \circ S \circ P \circ N_0 \rightarrow N_1 \circ Q \circ T \circ K_1 : [K] \rightarrow [\mathcal{N}]$ defined by

$$\alpha \circ \Phi = \alpha \circ [\Phi]$$

, the usual composite of a natural transformation and a functor.

(2) The module $\uparrow\text{CYL} : \text{COM} \rightarrow \text{MOD}$ is defined in the following way:

a) a $\uparrow\text{CYL}$-arrow from a comma $K : Y \rightarrow B$ to a module $M : X \rightarrow A$, written $\alpha : S \rightarrow T : K \rightarrow M$, is a triple $(S, \alpha, T)$ consisting of a functor $S : Y \rightarrow X$, a second functor $T : B \rightarrow A$, and a cylinder

\[
\begin{array}{c}
Y \xleftarrow{K_0} [K] \xrightarrow{K_1} B \\
\downarrow S \\
X \xleftarrow{M_0} [M] \xrightarrow{M_1} A \\
\downarrow \alpha \\
\end{array}
\]

from $K_0 \circ S$ to $T \circ K_1$ along $M$.

b) for a comma cell $\Phi : P \rightarrow Q : J \rightarrow K$ and a $\uparrow\text{CYL}$-arrow $\alpha : S \rightarrow T : K \rightarrow M$ as in

\[
\begin{array}{c}
Z \xleftarrow{J_0} [J] \xrightarrow{J_1} C \\
\downarrow P \\
Y \xleftarrow{K_0} [K] \xrightarrow{K_1} B \\
\downarrow S \\
X \xleftarrow{M_0} [M] \xrightarrow{M_1} A \\
\downarrow \alpha \\
\end{array}
\]

, their composite is the $\uparrow\text{CYL}$-arrow $\Phi \odot \alpha : P \odot S \rightarrow T \odot Q : J \rightarrow M$ with the cylinder $\Phi \odot \alpha : J_0 \odot P \odot S \rightarrow T \odot Q \odot J_1 : [J] \rightarrow M$ defined by

$$\Phi \odot \alpha = [\Phi] \odot \alpha$$

, the usual composite of a functor and a cylinder (see Definition 4.3.23).

c) for a $\uparrow\text{CYL}$-arrow $\alpha : S \rightarrow T : K \rightarrow M$ and a module cell $\Phi : P \rightarrow Q : M \rightarrow \mathcal{N}$ as in

\[
\begin{array}{c}
Y \xleftarrow{K_0} [K] \xrightarrow{K_1} B \\
\downarrow S \\
X \xleftarrow{M_0} [M] \xrightarrow{M_1} A \\
\downarrow \Phi \\
Z \xleftarrow{N_0} [N] \xrightarrow{N_1} C \\
\downarrow Q
\end{array}
\]

, their composite is the $\uparrow\text{CYL}$-arrow $\alpha \circ \Phi : S \circ P \rightarrow Q \circ T : K \rightarrow \mathcal{N}$ with the cylinder $\alpha \circ \Phi : K_0 \circ S \circ P \circ \mathcal{N}_0 \rightarrow \mathcal{N}_1 \circ T \circ K_1 : [K] \rightarrow \mathcal{N}$ given by the usual composition of a cylinder and a cell (see Definition 4.3.13).
Remark 10.2.2.
(1) The unit cylinder (see Definition 10.1.3(1)) of a module (resp. collage) \( M : X \to A \) forms a \( \uparrow \text{CYL} \)-arrow \( 1_M : 1_X \sim 1_A : M \sim M \).

(2) The unit cylinder (see Definition 10.1.3(2)) of a comma \( K : X \to A \) forms a \( \downarrow \text{CYL} \)-arrow \( 1_K : 1_X \sim 1_A : K \sim K \).

(3) Since a cylinder
\[
\begin{array}{c}
Y \leftarrow \kappa_0 \rightarrow [K] \rightarrow \kappa_1 \rightarrow B \\
\downarrow s \\
X \leftarrow \alpha \rightarrow A \\
\end{array}
\]
is the same thing as a natural transformation
\[
\begin{array}{c}
Y \leftarrow \kappa_0 \rightarrow [K] \rightarrow \kappa_1 \rightarrow B \\
\downarrow s \\
X \leftarrow M_0 \rightarrow [M] \rightarrow M_1 \rightarrow A \\
\end{array}
\]
(see Remark 4.3.4(4)), the identity
\[
\begin{array}{c}
\text{COM} \Downarrow \downarrow \downarrow \text{MOD} \\
\downarrow 1 \downarrow 1 \\
\text{COM} \Downarrow \downarrow \downarrow \text{CLG} \\
\end{array}
\]
holds.

(4) Each \( \uparrow \text{CYL} \)-arrow \( \alpha : S \sim T : K \sim M \) gives the following arrows:
- \( \uparrow \text{CYL} \)-arrow \( \alpha : \kappa_0 \circ S \sim T \circ \kappa_1 : [K] \sim M \)
- \( \downarrow \text{CYL} \)-arrow \( \alpha : S \circ M_0 \sim M_1 \circ T : K \sim [M] \)
, defining the cells below:
\[
\begin{array}{c}
\text{COM} \Downarrow \downarrow \downarrow \text{CLG} \\
\downarrow [-] \downarrow 1 \\
\text{CAT} \Downarrow \downarrow \downarrow \text{CLG} \\
\end{array}
\]
\[
\begin{array}{c}
\text{COM} \Downarrow \downarrow \downarrow \text{CLG} \\
\downarrow [-] \downarrow 1 \\
\text{COM} \Downarrow \downarrow \downarrow \text{CAT} \\
\end{array}
\]

Definition 10.2.3. Given a \( \uparrow \text{CYL} \)-arrow \( \alpha : S \sim T : K \sim M \), i.e. a natural transformation
\[
\begin{array}{c}
Y \leftarrow \kappa_0 \rightarrow [K] \rightarrow \kappa_1 \rightarrow B \\
\downarrow s \\
X \leftarrow M_0 \rightarrow [M] \rightarrow M_1 \rightarrow A \\
\end{array}
\]
(1) the comma adjunct of \( \alpha \) is the comma cell \( \alpha^\uparrow : S \sim T \circ \kappa : K \sim M \) (depicted as
\[
\begin{array}{c}
Y \leftarrow \kappa_0 \circ \alpha \rightarrow [K] \rightarrow \kappa_1 \circ \alpha \rightarrow B \\
\downarrow s \\
X \leftarrow M_0 \circ \alpha \rightarrow [M] \rightarrow M_1 \circ \alpha \rightarrow A \\
\end{array}
\]
) defined by the comma adjunct \( [\alpha] : [K] \to [M] \) (see Definition 10.1.7(1)) of the \( \uparrow \text{CYL} \)-arrow \( \alpha : \kappa_0 \circ S \sim T \circ \kappa_1 : [K] \sim M \).
(2) the collage adjunct of $\alpha$ is the collage cell $\alpha^\uparrow : \mathcal{S} \leadsto \mathcal{T} : \mathcal{K} \leadsto \mathcal{M}$ (depicted as

\[
\begin{array}{ccc}
Y & \xrightarrow{\mathcal{K}^0} & [\mathcal{K}] \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\mathcal{M}_0} & [\mathcal{M}]
\end{array}
\]

) defined by the collage adjunct $[\alpha] : [\mathcal{K}] \to [\mathcal{M}]$ (see Definition 10.1.7(2)) of the ↓CYL-arrow $\alpha : \mathcal{S} \circ \mathcal{M}_0 \leadsto \mathcal{M}_1 \circ \mathcal{T} : \mathcal{K} \leadsto [\mathcal{M}]$.

**Proposition 10.2.4.** For any ↓CYL-arrow $\alpha : \mathcal{S} \leadsto \mathcal{T} : \mathcal{K} \leadsto \mathcal{M}$,

(1) the comma adjunct $\alpha^\downarrow$ gives the unique comma cell $\mathcal{K} \to \mathcal{M}^\downarrow$ making the diagram

\[
\begin{array}{ccc}
\mathcal{M}^\downarrow & \xrightarrow{1_{\mathcal{M}^\downarrow}} & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{K} & \xrightarrow{\alpha} & \mathcal{M}
\end{array}
\]

commute; the unit cylinder $1_{\mathcal{M}}^\downarrow$ is therefore an inverse universal ↓CYL-arrow.

(2) the collage adjunct $\alpha^\uparrow$ gives the unique collage cell $\mathcal{K}^\uparrow \to \mathcal{M}$ making the diagram

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{1_{\mathcal{K}}^\uparrow} & \mathcal{K}^\uparrow \\
\downarrow & & \downarrow \\
\alpha & \xrightarrow{\alpha^\uparrow} & \mathcal{M}
\end{array}
\]

commute; the unit cylinder $1_{\mathcal{K}}^\uparrow$ is therefore a direct universal ↓CYL-arrow.

**Proof.** By the definitions of $\alpha^\downarrow$ and $\alpha^\uparrow$, this is reduced to Proposition 10.1.9. \(\square\)

**Proposition 10.2.5.**

(1) The functor $\text{COM} \downarrow \text{CLG}$ (see Remark 3.2.20(2)) and the family of unit cylinders $1_{\mathcal{M}}^\downarrow : \mathcal{M}^\downarrow \leadsto \mathcal{M}$, one for each collage $\mathcal{M}$, form a counit of the module ↓CYL;

(2) The functor $\text{COM} \uparrow \text{CLG}$ (see Remark 3.2.24(2)) and the family of unit cylinders $1_{\mathcal{K}}^\uparrow : \mathcal{K} \leadsto \mathcal{K}^\uparrow$, one for each comma $\mathcal{K}$, form a unit of the module ↑CYL.

**Proof.** We have seen in Proposition 10.2.4 the universality of each unit cylinder. It remains to show that the family of unit cylinders $1_{\mathcal{M}}^\downarrow$ (resp. $1_{\mathcal{K}}^\uparrow$) satisfies the naturality condition. But this follows immediately from Proposition 10.1.5. \(\square\)

**Remark 10.2.6.** The counit and unit in Proposition 10.2.5 are depicted as

\[
\begin{array}{ccc}
\text{COM} & \xleftarrow{1_{\text{COM} \downarrow \text{CLG}}} & \text{CLG} \\
\text{COM} & \xrightarrow{1_{\text{COM} \uparrow \text{CLG}}} & \text{CLG}
\end{array}
\]

**Theorem 10.2.7.** There exists an adjoint equivalence

\[
\begin{array}{ccc}
\text{COM} & \xleftarrow{\tau(\eta, \epsilon)} & \text{CLG} \\
\text{COM} & \xrightarrow{1_\text{COM \uparrow \text{CLG}}} & \text{CLG}
\end{array}
\]

with the unit $\eta$ and the counit $\epsilon$ given by the isomorphisms in Theorem 3.2.25.
Proof. This follows by applying Theorem 8.3.13 to the counit and unit of \(\downarrow\mathrm{CYL}\) in Remark 10.2.6, and observing that the isomorphism \(\epsilon_M\) (resp. \(\eta_K\)) in Theorem 3.2.25 gives the adjunct of \(1^1_M\) along \(1^1_M\) (resp. \(1^1_K\) along \(1^1_K\)) by the commutativity of the triangle in Proposition 10.1.6. \(\square\)

**Corollary 10.2.8.** The module \(\downarrow\mathrm{CYL}\) is an equivalence and each unit cylinder is a two-way universal \(\downarrow\mathrm{CYL}\)-arrow.

Proof. Since the functors \(\mathrm{COM} \downarrow \mathrm{CLG}\) and \(\mathrm{COM} \downarrow \mathrm{CLG}\) are equivalences as we have just seen in Theorem 10.2.7, the assertion follows by applying Corollary 8.10.8 to the counit and unit of \(\downarrow\mathrm{CYL}\) in Remark 10.2.6. \(\square\)

### 10.3. Equivalence \([X \downarrow A] \simeq [X \uparrow A]\)

In this section, given a pair of categories \(X\) and \(A\), we define (Definition 10.3.1) and study the following modules:

- \((X \downarrow A) : [X \downarrow A] \rightarrow [X \uparrow A]\)
- \((X \downarrow A) : [X \downarrow A] \rightarrow [X : A]\)

Although we give separate definitions for them, the two modules are regarded as the same thing under the identification \([X \uparrow A] \simeq [X : A]\). After Remark 10.3.2 (3) we will deal with only the module \((X \downarrow A) : [X \downarrow A] \rightarrow [X \uparrow A]\); however, any result in this section stated for \((X \downarrow A) : [X \downarrow A] \rightarrow [X \uparrow A]\) also holds with \([X \uparrow A]\) changed to \([X : A]\).

This section is analogous to the previous section; all definitions and results given for \([X \downarrow A]\) and \([X \uparrow A]\) in this section are the reflections of those given for \(\mathrm{COM}\) and \(\mathrm{CLG}\) in the previous section along the embedding \([X \downarrow A] \hookrightarrow \mathrm{COM}\) and \([X \uparrow A] \hookrightarrow \mathrm{CLG}\) (see Remark 3.2.14(3) and Remark 3.1.6(3)).

**Definition 10.3.1.** Let \(X\) and \(A\) be categories.

1. The module \((X \downarrow A) : [X \downarrow A] \rightarrow [X \uparrow A]\) is defined in the following way:
   a) an \((X \downarrow A)\)-arrow \(\alpha : K \sim \mathcal{M}\) from a comma \(K : X \rightarrow A\) to a collage \(\mathcal{M} : X \rightarrow A\) is given by a natural transformation
      \[
      \begin{array}{ccc}
      X & \xrightarrow{\alpha} & A \\
      \downarrow M_0 & & \downarrow M_1 \\
      [K] & \xleftarrow{K_0} & [M] \\
      \end{array}
      \]
      from \(K_0 \circ M_0\) to \(M_1 \circ K_1\).
   b) for a comma morphism \(\Phi : J \rightarrow K : X \rightarrow A\) and an \((X \downarrow A)\)-arrow \(\alpha : K \sim \mathcal{M}\) as in
      \[
      \begin{array}{ccc}
      X & \xrightarrow{\alpha} & A \\
      \downarrow J_0 & & \downarrow J_1 \\
      [K] & \xleftarrow{K_0} & [M] \\
      \end{array}
      \]
      , their composite is the \((X \downarrow A)\)-arrow \(\Phi \circ \alpha : J \sim \mathcal{M}\) with the natural transformation \(\Phi \circ \alpha : J_0 \circ M_0 \rightarrow M_1 \circ J_1 : [J] \rightarrow [M]\) defined by
      \[
      \Phi \circ \alpha = [\Phi] \circ \alpha
      \]
      , the usual composite of a functor and a natural transformation.
10.3. Equivalence $[\mathbf{X} \downarrow \mathbf{A}] \cong [\mathbf{X} \uparrow \mathbf{A}]$

(c) for an $\langle \mathbf{X} \downarrow \mathbf{A} \rangle$-arrow $\alpha : \mathbb{K} \rightarrow \mathcal{M}$ and a collage morphism $\Phi : \mathcal{M} \rightarrow \mathcal{N} : \mathbf{X} \rightarrow \mathbf{A}$ as in

\[
\begin{array}{c}
\mathbf{X} \xrightarrow{\mathcal{M}_0} \mathcal{M} \xrightarrow{\mathcal{M}_1} \mathbf{A} \\
\end{array}
\]

their composite is the $\langle \mathbf{X} \uparrow \mathbf{A} \rangle$-arrow $\alpha \circ \Phi : \mathbb{K} \sim \mathcal{N}$ with the natural transformation $\alpha \circ \Phi : \mathcal{K}_0 \circ \mathcal{N}_0 \rightarrow \mathcal{N}_1 \circ \mathcal{K}_1 : [\mathbb{K}] \rightarrow [\mathcal{N}]$ defined by

$\alpha \circ \Phi = \alpha \circ \mu$

, the usual composite of a natural transformation and a functor.

(2) The module $\langle \mathbf{X} \downarrow \mathbf{A} \rangle : [\mathbf{X} \downarrow \mathbf{A}] \rightarrow [\mathbf{X} : \mathbf{A}]$ is defined in the following way:

(a) an $\langle \mathbf{X} \downarrow \mathbf{A} \rangle$-arrow $\alpha : \mathbb{K} \rightarrow \mathcal{M}$ from a comma $\mathbb{K} : \mathbf{X} \rightarrow \mathbf{A}$ to a module $\mathcal{M} : \mathbf{X} \rightarrow \mathbf{A}$ is given by a cylinder

\[
\begin{array}{c}
\mathbf{X} \xrightarrow{\mathcal{M}_0} \mathcal{M} \xrightarrow{\mathcal{M}_1} \mathbf{A} \\
\end{array}
\]

(b) for a comma morphism $\Phi : \mathcal{J} \rightarrow \mathbb{K} : \mathbf{X} \rightarrow \mathbf{A}$ and an $\langle \mathbf{X} \downarrow \mathbf{A} \rangle$-arrow $\alpha : \mathbb{K} \rightarrow \mathcal{M}$ as in

\[
\begin{array}{c}
\mathbf{X} \xrightarrow{\mathcal{M}_0} \mathcal{M} \xrightarrow{\mathcal{M}_1} \mathbf{A} \\
\end{array}
\]

their composite is the $\langle \mathbf{X} \downarrow \mathbf{A} \rangle$-arrow $\Phi \circ \alpha : \mathcal{J} \sim \mathcal{M}$ with the cylinder $\Phi \circ \alpha : \mathcal{J}_0 \sim \mathcal{J}_1 : [\mathcal{J}] \sim \mathcal{M}$ defined by

$\Phi \circ \alpha = \mu \circ \alpha$

, the usual composite of a functor and a cylinder (see Definition 4.3.23).

c) for an $\langle \mathbf{X} \uparrow \mathbf{A} \rangle$-arrow $\alpha : \mathbb{K} \sim \mathcal{M}$ and a module morphism $\Phi : \mathcal{M} \rightarrow \mathcal{N} : \mathbf{X} \rightarrow \mathbf{A}$ as in

\[
\begin{array}{c}
\mathbf{X} \xrightarrow{\mathcal{M}_0} \mathcal{M} \xrightarrow{\mathcal{M}_1} \mathbf{A} \\
\end{array}
\]

their composite is the $\langle \mathbf{X} \uparrow \mathbf{A} \rangle$-arrow $\alpha \circ \Phi : \mathbb{K} \sim \mathcal{N}$ with the cylinder $\alpha \circ \Phi : \mathbb{K}_0 \sim \mathbb{K}_1 : [\mathbb{K}] \sim \mathcal{N}$ given by the usual composition of a cylinder and a module morphism (see Definition 4.3.9).

Remark 10.3.2.

(1) The unit cylinder (see Definition 10.1.3(1)) of a module (resp. collage) $\mathcal{M} : \mathbf{X} \rightarrow \mathbf{A}$ forms an $\langle \mathbf{X} \uparrow \mathbf{A} \rangle$-arrow $1_{\mathcal{M}} : \mathcal{M} ! \sim \mathcal{M}$.

(2) The unit cylinder (see Definition 10.1.3(2)) of a comma $\mathbb{K} : \mathbf{X} \rightarrow \mathbf{A}$ forms an $\langle \mathbf{X} \uparrow \mathbf{A} \rangle$-arrow $1_{\mathbb{K}} : \mathbb{K} ! \sim \mathbb{K} !$. 
10.3. Equivalence $[X \downarrow A] \simeq [X \uparrow A]$

(3) Since a cylinder

$$
\begin{array}{c}
\xymatrix{
X & K_0 \\
K_0 & K_1 \\
M_0 & M_1 \\
\ar^{\alpha}_M & \ar_{\alpha \downarrow A} \ar^{\alpha}_A \\
& \ar^{\alpha \uparrow A} \\
& Y
}
\end{array}
$$

is the same thing as a natural transformation

$$
\begin{array}{c}
\xymatrix{
X & K_0 \\
K_0 & K_1 \\
M_0 & M_1 \\
\ar^{\alpha}_M & \ar_{\alpha \downarrow A} \ar^{\alpha}_A \\
& \ar^{\alpha \uparrow A} \\
& Y
}
\end{array}
$$

(see Remark 4.3.4(4)), the identity

$$
[X \downarrow A] \xrightarrow{(X \downarrow A)} [X : A] \\
\uparrow 1 \quad \uparrow 1 \\
[X \downarrow A] \xrightarrow{(X \downarrow A)} [X \uparrow A]
$$

holds.

(4) Each $(X \downarrow A)$-arrow $\alpha : K \sim M$ gives the following arrows:

- $\uparrow$CYL-arrow $\alpha : 1_X \sim 1_A : K \sim M$
- $\downarrow$CYL-arrow $\alpha : K_0 \sim K_1 : [K] \sim [M]$
- $\downarrow$CYL-arrow $\alpha : M_0 \sim M_1 : [K] \sim [M]$

, defining the faithful cells below:

$$
\begin{array}{c}
\xymatrix{
[X \downarrow A] & X \downarrow A & [X \uparrow A] \\
\ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} \\
\ar_{[-]} & \ar_{[-]} & \ar_{[-]} \\
\COM & \COM & \COM
}
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{
[X \downarrow A] & X \downarrow A & [X \uparrow A] \\
\ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} \\
\ar_{[-]} & \ar_{[-]} & \ar_{[-]} \\
\CAT & \CAT & \CAT
}
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{
[X \downarrow A] & X \downarrow A & [X \uparrow A] \\
\ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} \\
\ar_{[-]} & \ar_{[-]} & \ar_{[-]} \\
\COM & \COM & \COM
}
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{
[X \downarrow A] & X \downarrow A & [X \uparrow A] \\
\ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} & \ar_{(X \downarrow A)} \\
\ar_{[-]} & \ar_{[-]} & \ar_{[-]} \\
\CAT & \CAT & \CAT
}
\end{array}
$$

Definition 10.3.3. Given an $(X \downarrow A)$-arrow $\alpha : K \sim M$, i.e. a natural transformation

$$
\begin{array}{c}
\xymatrix{
X & K_0 \\
K_0 & K_1 \\
M_0 & M_1 \\
\ar^{\alpha}_M & \ar_{\alpha \downarrow A} \ar^{\alpha}_A \\
& \ar^{\alpha \uparrow A} \\
& Y
}
\end{array}
$$

(1) the comma adjunct of $\alpha$ is the comma morphism $\alpha \downarrow : K \to M \downarrow : X \to A$ (depicted as

$$
\begin{array}{c}
\xymatrix{
X & K_0 \\
K_0 & K_1 \\
M_0 & M_1 \\
\ar^{\alpha}_M & \ar_{\alpha \downarrow A} \ar^{\alpha}_A \\
& \ar^{\alpha \uparrow A} \\
& Y
}
\end{array}
$$

) defined by the comma adjunct $[\alpha] : [K] \to [M]$ (see Definition 10.1.7(1)) of the $\uparrow$CYL-arrow $\alpha : K_0 \sim K_1 : [K] \sim [M]$.

(2) the collage adjunct of $\alpha$ is the collage morphism $\alpha \downarrow : K \to M : X \to A$ (depicted as

$$
\begin{array}{c}
\xymatrix{
X & K_0 \\
K_0 & K_1 \\
M_0 & M_1 \\
\ar^{\alpha}_M & \ar_{\alpha \downarrow A} \ar^{\alpha}_A \\
& \ar^{\alpha \uparrow A} \\
& Y
}
\end{array}
$$

) defined by the collage adjunct $[\alpha] : [K] \to [M]$ (see Definition 10.1.7(2)) of the $\downarrow$CYL-arrow $\alpha : M_0 \sim M_1 : [K] \sim [M]$. 
Proposition 10.3.4. For any \( \langle X \uparrow A \rangle \)-arrow \( \alpha : K \to M \),

1. the comma adjunct \( \alpha \downarrow \) gives the unique comma morphism \( K \to M \downarrow \) making the diagram

\[
\begin{array}{c}
\xymatrix{
M \ar[r]^{1_M} & M \\
K \ar[u]^\alpha & \ar[l]_\alpha
}\end{array}
\]

commute; the unit cylinder \( 1_M \) is therefore an inverse universal \( \langle X \uparrow A \rangle \)-arrow.

2. the collage adjunct \( \alpha \uparrow \) gives the unique collage morphism \( K \uparrow \to M \) making the diagram

\[
\begin{array}{c}
\xymatrix{
K \ar[r]_{1_K} & K \\
M \ar[u]_\alpha & \ar[l]^{\alpha \uparrow}
}\end{array}
\]

commute; the unit cylinder \( 1_K \) is therefore a direct universal \( \langle X \uparrow A \rangle \)-arrow.

Proof. By the definitions of \( \alpha \downarrow \) and \( \alpha \uparrow \), this is reduced to Proposition 10.1.9.

\[\square\]

Proposition 10.3.5.

1. The functor \( [X \downarrow A] \xrightarrow{\downarrow} [X \uparrow A] \) (see Remark 3.2.20(2)) and the family of unit cylinders \( 1_M : M \to M \downarrow \) one for each collage \( M : X \to A \), form a counit of the module \( \langle X \uparrow A \rangle \);

2. The functor \( [X \downarrow A] \xleftarrow{\uparrow} [X \uparrow A] \) (see Remark 3.2.24(2)) and the family of unit cylinders \( 1_K : K \to K \uparrow \) one for each comma \( K : X \to A \), form a unit of the module \( \langle X \uparrow A \rangle \).

Proof. We have seen in Proposition 10.3.4 the universality of each unit cylinder. It remains to show that the family of unit cylinders \( 1_M \) (resp. \( 1_K \)) satisfies the naturality condition. But this follows immediately from Proposition 10.1.5.

\[\square\]

Remark 10.3.6. The counit and unit in Proposition 10.3.5 are depicted as

\[
\begin{array}{c}
\xymatrix{
[X \downarrow A] \ar[r]^{1_M} & [X \uparrow A] \\
[X \downarrow A] \ar[u]^{(X \downarrow A)} & \ar[l]_{(X \uparrow A)}
}\end{array}
\]

.

Theorem 10.3.7. Given a pair of categories \( X \) and \( A \), there exists an adjoint equivalence

\[
[X \downarrow A] \xrightarrow{\sim} [X \uparrow A]
\]

with the unit \( \eta \) and the counit \( \epsilon \) given by the isomorphisms in Theorem 3.2.25.

Proof. This follows by applying Theorem 8.3.13 to the counit and unit of \( \langle X \uparrow A \rangle \) in Remark 10.3.6, and observing that the isomorphism \( \epsilon_M \) (resp. \( \eta_K \)) in Theorem 3.2.25 gives the adjunct of \( 1_M \) along \( 1_M \) (resp. \( 1_K \) along \( 1_K \)) by the commutativity of the triangle in Proposition 10.1.6.

\[\square\]

Corollary 10.3.8. Given a pair of categories \( X \) and \( A \), the module \( \langle X \uparrow A \rangle \) is an equivalence and each unit cylinder is a two-way universal \( \langle X \uparrow A \rangle \)-arrow.

Proof. Since the functors \( [X \downarrow A] \xrightarrow{\downarrow} [X \uparrow A] \) and \( [X \downarrow A] \xleftarrow{\uparrow} [X \uparrow A] \) are equivalences as we have just seen in Theorem 10.3.7, the assertion follows by applying Corollary 8.10.8 to the counit and unit of \( \langle X \uparrow A \rangle \) in Remark 10.3.6.

\[\square\]
10.4. Equivalences \([X \downarrow] \simeq [X :]\) and \([\downarrow A] \simeq [: A]\)

Note. The following definition is regarded as a special case of Definition 10.1.3 where \(A\) (resp. \(X\)) is the terminal category.

**Definition 10.4.1.**

(1) ▷ The unit cone of a right module \(M : X \rightarrow *\) is the cone

\[
\begin{array}{c}
\xymatrix{ \ast & M \\
X & M \ar[u] & M \\
 & \ast \ar[l] & * \\
}
\end{array}
\]

defined by

\[ [1^1_M]_m = m \]

for \(m\) an arrow of \(M\).

▷ The unit cone of a left module \(M : * \rightarrow A\) is the cone

\[
\begin{array}{c}
\xymatrix{ \ast & M \\
K & M \ar[u] & A \\
 & \ast \ar[l] & * \\
}
\end{array}
\]

defined by

\[ [1^1_M]_m = m \]

for \(m\) an arrow of \(M\).

(2) ▷ The unit cone of a right comma \(K : X \rightarrow *\) is the cone

\[
\begin{array}{c}
\xymatrix{ \ast & K \\
X & K \ar[u] & K \\
 & \ast \ar[l] & * \\
}
\end{array}
\]

defined by

\[ [1^1_K]_k = k \]

for \(k\) an object of \([K]\).

▷ The unit cone of a left comma \(K : * \rightarrow A\) is the cone

\[
\begin{array}{c}
\xymatrix{ \ast & K \\
K & K \ar[u] & A \\
 & \ast \ar[l] & * \\
}
\end{array}
\]

defined by

\[ [1^1_K]_k = k \]

for \(k\) an object of \([K]\).

Note. The following definition is regarded as a special case of Definition 10.3.1 where \(A\) (resp. \(X\)) is the terminal category.

**Definition 10.4.2.**

▷ Given a category \(X\), the module \(\langle X \uparrow \rangle : [X \downarrow] \rightarrow [X :]\) is defined in the following way:

(1) an \(\langle X \uparrow \rangle\)-arrow \(\alpha : K \rightarrow M\) from a right comma \(K : X \rightarrow *\) to a right module \(M : X \rightarrow *\) is given by a cone

\[
\begin{array}{c}
\xymatrix{ \ast & K \\
X & K \ar[u] & M \\
 & \ast \ar[l] & * \\
}
\end{array}
\]
10.4. Equivalences $[X \downarrow] \simeq [X:]$ and $[\downarrow A] \simeq [:A]$

(2) for a right comma morphism $\Phi : J \to K : X \to -$ and an $(X \downarrow)$-arrow $\alpha : K \to M$ as in

\[
\begin{array}{c}
\Phi \\
\downarrow \\
J \\
\downarrow \\
K \\
\downarrow \\
X -- \alpha -- M \\
\downarrow \\
A
\end{array}
\]

, their composite is the $(X \downarrow)$-arrow $\Phi \circ \alpha : J \to M$ with the cone $\Phi \circ \alpha : J \to * : [J] \to M$ defined by $\Phi \circ \alpha = [\Phi] \circ \alpha$

, the usual composite of a functor and a cone.

(3) for an $(X \downarrow)$-arrow $\alpha : K \to M$ and a right module morphism $\Phi : M \to N : X \to -$ as in

\[
\begin{array}{c}
\Phi \\
\downarrow \\
J \\
\downarrow \\
K \\
\downarrow \\
X \leftarrow M \leftarrow \alpha \\
\downarrow \\
A
\end{array}
\]

, their composite is the $(X \downarrow)$-arrow $\alpha \circ \Phi : K \to N$ with the cone $\alpha \circ \Phi : K \to * : [K] \to N$ given by the usual composition of a cone and a module morphism (see Definition 4.6.9).

- Given a category $A$, the module $(\downarrow A) : [\downarrow A] \to [:A]$ is defined in the following way:

(1) an $(\downarrow A)$-arrow $\alpha : K \to M$ from a left comma $K : * \to A$ to a left module $M : * \to A$ is given by a cone

\[
\begin{array}{c}
\alpha \\
\downarrow \\
K \\
\downarrow \\
* -- \alpha -- M \\
\downarrow \\
A
\end{array}
\]

(2) for a left comma morphism $\Phi : J \to K : * \to A$ and an $(\downarrow A)$-arrow $\alpha : K \to M$ as in

\[
\begin{array}{c}
\Phi \\
\downarrow \\
J \\
\downarrow \\
K \\
\downarrow \\
* -- \alpha -- M \\
\downarrow \\
A
\end{array}
\]

, their composite is the $(\downarrow A)$-arrow $\Phi \circ \alpha : J \to M$ with the cone $\Phi \circ \alpha : * \to J : * \to [J] \to M$ defined by $\Phi \circ \alpha = [\Phi] \circ \alpha$

, the usual composite of a functor and a cone.

(3) for an $(\downarrow A)$-arrow $\alpha : K \to M$ and a left module morphism $\Phi : M \to N : * \to A$ as in

\[
\begin{array}{c}
\alpha \\
\downarrow \\
K \\
\downarrow \\
* -- \alpha -- M \\
\downarrow \\
N
\end{array}
\]

, their composite is the $(\downarrow A)$-arrow $\alpha \circ \Phi : K \to N$ with the cone $\alpha \circ \Phi : * \to K : * \to [K] \to N$ given by the usual composition of a cone and a module morphism (see Definition 4.6.9).
Remark 10.4.3.  
(1) The unit cone of a right module $\mathcal{M} : X \rightarrow \ast$ forms an $\langle X \downarrow \rangle$-arrow $1^\uparrow_{\mathcal{M}} : \mathcal{M}^\uparrow \sim \mathcal{M}$. Dually, the unit cone of a left module $\mathcal{M} : \ast \rightarrow A$ forms an $\langle \downarrow A \rangle$-arrow $1^\downarrow_{\mathcal{M}} : \mathcal{M}^\downarrow \sim \mathcal{M}$.  
(2) The unit cone of a right comma $\mathcal{K} : X \rightarrow \ast$ forms an $\langle X \downarrow \rangle$-arrow $1^\uparrow_{\mathcal{K}} : \mathcal{K} \sim \mathcal{K}^\uparrow$. Dually, the unit cone of a left comma $\mathcal{K} : \ast \rightarrow A$ forms an $\langle \downarrow A \rangle$-arrow $1^\downarrow_{\mathcal{K}} : \mathcal{K} \sim \mathcal{K}^\downarrow$.

(3) The following identities hold
\[ \langle X \downarrow \rangle \sim \langle X \downarrow \ast \rangle \quad \langle \downarrow A \rangle \sim \langle \downarrow A \rangle \] 
($\sim$ denotes the canonical isomorphisms), giving canonical isomorphisms
\[ \langle X \downarrow \rangle \cong \langle X \downarrow \ast \rangle \quad \langle \downarrow A \rangle \cong \langle \downarrow A \rangle \]

Proposition 10.4.4.  
(1) The functor $[X \downarrow] \xrightarrow{\sim} [X :]$ (see Remark 3.2.20(1)) and the family of unit cones $1^\uparrow_{\mathcal{M}} : \mathcal{M}^\uparrow \sim \mathcal{M}$, one for each right module $\mathcal{M} : X \rightarrow \ast$, form a counit of the module $(X \downarrow)$;  
- The functor $[\downarrow A] \xrightarrow{\sim} [: A]$ (see Remark 3.2.20(1)) and the family of unit cones $1^\downarrow_{\mathcal{M}} : \mathcal{M}^\downarrow \sim \mathcal{M}$, one for each left module $\mathcal{M} : \ast \rightarrow A$, form a counit of the module $(\downarrow A)$;  
(2) The functor $[X \downarrow] \xrightarrow{\sim} [X :]$ (see Remark 3.2.24(1)) and the family of unit cones $1^\uparrow_{\mathcal{K}} : \mathcal{K} \sim \mathcal{K}^\uparrow$, one for each right comma $\mathcal{K} : X \rightarrow \ast$, form a unit of the module $(X \downarrow)$;  
- The functor $[\downarrow A] \xrightarrow{\sim} [: A]$ (see Remark 3.2.24(1)) and the family of unit cones $1^\downarrow_{\mathcal{K}} : \mathcal{K} \sim \mathcal{K}^\downarrow$, one for each left comma $\mathcal{K} : \ast \rightarrow A$, form a unit of the module $(\downarrow A)$.

Proof. This is a special case of Proposition 10.3.5 where $A$ (resp. $X$) is the terminal category. \[ \square \]

Remark 10.4.5. The counit and unit in Proposition 10.4.4 are depicted as
\[ \xymatrix{ [X \downarrow] \ar[r]^{\downarrow} \ar@{<-}[d]_{1^\downarrow_{\langle X \downarrow \rangle}} & [X :] \ar@{<-}[d]_{1^\uparrow_{\langle X \downarrow \rangle}} \\
[\downarrow A] \ar[r]_{1^\downarrow_{\langle \downarrow A \rangle}} & [: A] } \]

Theorem 10.4.6.  
- Given a category $X$, there exists an adjoint equivalence
\[ \xymatrix{ [X \downarrow] \ar[r]^{\sim} \ar@{<-}[d]_{\tau_{(X \downarrow)}} & [X :] \ar@{<-}[d]_{\tau_{(X :)}} } \]
- Given a category $A$, there exists an adjoint equivalence
\[ \xymatrix{ [\downarrow A] \ar[r]^{\sim} \ar@{<-}[d]_{\tau_{(\downarrow A)}} & [: A] \ar@{<-}[d]_{\tau_{(:, A)}} } \]

Proof. This is a special case of Theorem 10.3.7 where $A$ (resp. $X$) is the terminal category. \[ \square \]
Proof. This is a special case of Corollary 10.3.8 where $\mathbf{A}$ (resp. $\mathbf{X}$) is the terminal category.

Note. Since the module $(\mathbf{X} \downarrow)$ (resp. $(\downarrow \mathbf{A})$) is an equivalence, Theorem 8.11.16 allows the following definition.

Definition 10.4.8. Let $\mathbf{X}$ and $\mathbf{A}$ be categories.

1. The equivalence cells

$$\begin{align*}
\mathbf{X} \downarrow - & \overset{(\mathbf{X} \downarrow)}{\sim} \mathbf{X} : \\
\downarrow & \Downarrow \\
\mathbf{X} \downarrow - & \overset{(\mathbf{X} \downarrow)}{\sim} \mathbf{X} : \\
\downarrow & \Downarrow
\end{align*}$$

are the module morphisms generated by $\mathbf{X} \downarrow$ direct along the counit and unit of $(\mathbf{X} \downarrow)$ (see Remark 10.4.5).

2. The equivalence cells

$$\begin{align*}
\mathbf{A} \downarrow - & \overset{(\mathbf{A} \downarrow)}{\sim} \mathbf{A} : \\
\downarrow & \Downarrow \\
\mathbf{A} \downarrow - & \overset{(\mathbf{A} \downarrow)}{\sim} \mathbf{A} : \\
\downarrow & \Downarrow
\end{align*}$$

are the module morphisms generated by $\mathbf{A} \downarrow$ inverse along the unit cone.

(1) The equivalence cells

$$\begin{align*}
\mathbf{X} \downarrow - & \overset{(\mathbf{X} \downarrow)}{\sim} \mathbf{X} : \\
\downarrow & \Downarrow \\
\mathbf{X} \downarrow - & \overset{(\mathbf{X} \downarrow)}{\sim} \mathbf{X} : \\
\downarrow & \Downarrow
\end{align*}$$

are quasi-inverse to each other defined by

$$\langle \downarrow \rangle = \langle ([\mathbf{X} \downarrow]+[1])^{-1} \rangle \quad \langle \uparrow \rangle = \langle ([\mathbf{X} \downarrow]+[1]) \rangle$$

where $\mathbf{X} \downarrow+[1]$ and $\mathbf{X} \downarrow+[1]$ are the module morphisms generated by $\mathbf{X} \downarrow$ direct along the counit and unit of $(\mathbf{X} \downarrow)$ (see Remark 10.4.5).

(2) The equivalence cells

$$\begin{align*}
\mathbf{A} \downarrow - & \overset{(\mathbf{A} \downarrow)}{\sim} \mathbf{A} : \\
\downarrow & \Downarrow \\
\mathbf{A} \downarrow - & \overset{(\mathbf{A} \downarrow)}{\sim} \mathbf{A} : \\
\downarrow & \Downarrow
\end{align*}$$

are quasi-inverse to each other defined by

$$\langle \downarrow \rangle = \langle ([\mathbf{A} \downarrow]+[1])^{-1} \rangle \quad \langle \uparrow \rangle = \langle ([\mathbf{A} \downarrow]+[1]) \rangle$$

where $\mathbf{A} \downarrow+[1]$ and $\mathbf{A} \downarrow+[1]$ are the module morphisms generated by $\mathbf{A} \downarrow$ direct along the counit and unit of $(\mathbf{A} \downarrow)$ (see Remark 10.4.5).
quasi-inverse to each other are defined by

\[
\langle \uparrow \rangle = ([1^t] 1[: A])^{-1} \quad \langle \downarrow \rangle = [1^t] 1[: A]
\]

where \([1^t] 1[: A]\) and \([1^t] 1[: A]\) are the module morphisms generated by \([: A]\) inverse along the unit and counit of \(\langle \downarrow A \rangle\) (see Remark 10.4.5).

Remark 10.4.9.

(1) \(\rhd\) The cell \(\langle X \downarrow \rangle \to \langle X \downarrow \rangle\) expresses a corepresentation of \(\langle X \downarrow \rangle\) by \(\downarrow\). It sends each \(\langle X \downarrow \rangle\)-arrow \(\alpha : K \rhd M\) to the right comma morphism \(\alpha^1 : K \rightarrow M^1 : X \dashv \ast\), the adjunct of \(\alpha\) along the unit cone of \(M\) as indicated in

\[
\begin{array}{ccc}
M^1 & \xrightarrow{1_M} & M \\
\alpha^1 & \downarrow & \alpha \\
K & \xrightarrow{id} & K \\
\end{array}
\]

; \(\alpha^1\) is given by the functor \([\alpha] : [K] \rightarrow [M]\) defined by

\[
[\alpha] \cdot k = \alpha_k \quad [\alpha] \cdot h = K \cdot h
\]

for \(k\) an object and \(h\) an arrow of \([K]\) (cf. Definition 10.3.3(1)).

(2) \(\rhd\) The cell \(\langle \uparrow A \rangle \to \langle \downarrow A \rangle\) expresses a corepresentation of \(\langle A \uparrow \rangle\) by \(\downarrow\). It sends each \(\langle \uparrow A \rangle\)-arrow \(\alpha : K \rhd M\) to the left comma morphism \(\alpha^1 : K \rightarrow M^1 : \ast \dashv A\), the adjunct of \(\alpha\) along the unit cone of \(M\) as indicated in

\[
\begin{array}{ccc}
M^1 & \xrightarrow{1_M} & M \\
\alpha^1 & \downarrow & \alpha \\
K & \xrightarrow{id} & K \\
\end{array}
\]

; \(\alpha^1\) is given by the functor \([\alpha] : [K] \rightarrow [M]\) defined by

\[
[\alpha] \cdot k = \alpha_k \quad [\alpha] \cdot h = K \cdot h
\]

for \(k\) an object and \(h\) an arrow of \([K]\) (cf. Definition 10.3.3(1)).
10.4. Equivalences $[X ↓] \simeq [X :]$ and $[↓ A] \simeq [: A]$

, i.e. by

$$[\Phi^t]_j = [\Phi] \cdot j$$

for $j$ an object of $[J]$.}

* The cell $(↓ A) \rightarrow (↓ \cdot A)$ sends each left comma morphism $\Phi : J \rightarrow K : * \rightarrow A$ to the $(↓ A)$-arrow $\Phi^t : J \rightarrow K^t$, “the collage transpose of $\Phi$”, given by postcomposition with the unit cone of $K$ as indicated in

\[
\begin{array}{c}
K^t \\
\downarrow \Phi^t \\
\downarrow \Phi \\
J \\
\end{array}
\]

; the collage transpose of a left comma morphism $\Phi : J \rightarrow K : * \rightarrow A$ is thus the cone

\[
\begin{array}{c}
* \\
\Phi^t \\
\downarrow \Phi \\
K^t \\
\downarrow \cdot \\
J \\
\end{array}
\]

defined by the composition

\[
\begin{array}{c}
[J] \\
\downarrow [\Phi] \\
\downarrow [\Phi^t] \\
* \\
\end{array}
\]

, i.e. by

$$[\Phi^t]_j = [\Phi] \cdot j$$

for $j$ an object of $[J]$.}

(3) * The cell $(X ↓) \rightarrow (X :)$ expresses a representation of $(X ↓)$ by $\uparrow$. It sends each $(X ↓)$-arrow $\alpha : K \rightarrow M$ to the right module morphism $\alpha^t : K^t \rightarrow M : X \rightarrow \cdot$, the adjunct of $\alpha$ along the unit cone of $K$ as indicated in

\[
\begin{array}{c}
K^t \\
\downarrow \alpha \\
\downarrow \alpha^t \\
K \\
\end{array}
\]

; $\alpha^t$ is defined by

$$\alpha^t \cdot k = \alpha_k$$

for $k$ an object of $[K]$ (cf. Definition 10.3.3(2)).

* The cell $(A ↓) \rightarrow (\cdot A)$ expresses a representation of $(A ↓)$ by $\uparrow$. It sends each $(A ↓)$-arrow $\alpha : K \rightarrow M$ to the left module morphism $\alpha^t : K^t \rightarrow M : \cdot \rightarrow A$, the adjunct of $\alpha$ along the unit cone of $K$ as indicated in

\[
\begin{array}{c}
K^t \\
\downarrow \alpha \\
\downarrow \alpha^t \\
K \\
\end{array}
\]

; $\alpha^t$ is defined by

$$\alpha^t \cdot k = \alpha_k$$

for $k$ an object of $[K]$ (cf. Definition 10.3.3(2)).
10.4. Equivalences $[X \downarrow] \simeq [X:]$ and $[\downarrow A] \simeq [:A]$

(4) • The cell $(X : \downarrow) \rightarrow (X \downarrow)$ sends each right module morphism $\Phi : M \rightarrow N : X \rightarrow *$ to the $(X \downarrow)$-arrow $\Phi^! : M^! \rightarrow N$, “the comma transpose of $\Phi$”, given by precomposition with the unit cone of $M$ as indicated in

$$
\begin{array}{c}
\xymatrix{M^! \ar[r]^{1^\downarrow_{\bar{M}}} & M \\
\Phi^! \ar@{.>}[rr] & & N}
\end{array}
$$

; the comma transpose of a right module morphism $\Phi : M \rightarrow N : X \rightarrow *$ is thus the cone

$$
\begin{array}{c}
\xymatrix{X \ar@{.>}[r] & [M] \ar[r] & * \\
\Phi^! \ar@{.>}[urr] & ! \ar[ull]
}
\end{array}
$$

defined by the composition

$$
\begin{array}{c}
\xymatrix{X \ar@{.>}[r] & [M] \ar[r] & * \\
\Phi^! \ar@{.>}[urr] & ! \ar[ull]
}
\end{array}
$$

, i.e. by

$$
[\Phi^!]_m = \Phi : m
$$

for $m$ an arrow of $M$.

• The cell $(A : \downarrow) \rightarrow (\downarrow A)$ sends each left module morphism $\Phi : M \rightarrow N : \ast \rightarrow A$ to the $(\downarrow A)$-arrow $\Phi^! : M^! \rightarrow N$, “the comma transpose of $\Phi$”, given by precomposition with the unit cone of $M$ as indicated in

$$
\begin{array}{c}
\xymatrix{M^! \ar[r]^{1^\downarrow_{\bar{M}}} & M \\
\Phi^! \ar@{.>}[rr] & & N}
\end{array}
$$

; the comma transpose of a left module morphism $\Phi : M \rightarrow N : \ast \rightarrow A$ is thus the cone

$$
\begin{array}{c}
\xymatrix{\ast \ar@{.>}[r] & [M] \ar[r] & A \\
\Phi^! \ar@{.>}[urr] & ! \ar[ull]
}
\end{array}
$$

defined by the composition

$$
\begin{array}{c}
\xymatrix{\ast \ar@{.>}[r] & [M] \ar[r] & A \\
\Phi^! \ar@{.>}[urr] & ! \ar[ull]
}
\end{array}
$$

, i.e. by

$$
[\Phi^!]_m = \Phi : m
$$

for $m$ an arrow of $M$. 
11. Extensions

11.1. coYoneda lemma

Definition 11.1.1. Let \( \mathcal{M} : X \to A \) be a module.

- The corepresentable module of the right exponential transpose of \( \mathcal{M} \) is denoted by \( \mathcal{M} \Rightarrow \); that is, the module

\[
\langle \mathcal{M} \Rightarrow \rangle : [X:] \to A
\]

is defined by the composition

\[
[X:] \xrightarrow{(X)} [X:] \xrightarrow{\mathcal{M} \Rightarrow} A
\]

- The representable module of the left exponential transpose of \( \mathcal{M} \) is denoted by \( \Rightarrow \mathcal{M} \); that is, the module

\[
\langle \Rightarrow \mathcal{M} \rangle : X \to [:A]^\sim
\]

is defined by the composition

\[
X \xrightarrow{\mathcal{M}^\sim} [:A]^\sim \xrightarrow{(A)^\sim} [:A]^\sim
\]

Remark 11.1.2.

1. Given a right module \( \mathcal{J} : X \to \ast \) and an object \( a \in \|A\| \), an \( \langle \mathcal{M} \Rightarrow \rangle \)-arrow \( \Phi : \mathcal{J} \Rightarrow a \) is a right module morphism \( \Phi : \mathcal{J} \to \langle \mathcal{M} \rangle a : X \to \ast \), i.e. a conical cell

\[
\begin{array}{ccc}
\ast & \xrightarrow{\Phi} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{M \Rightarrow} & A
\end{array}
\]

This cell is denoted by \( \Phi : \mathcal{J} \Rightarrow a : \mathcal{M} \Rightarrow \), depicted also as

\[
\begin{array}{ccc}
\ast & \xrightarrow{\Phi} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{M \Rightarrow} & A
\end{array}
\]

, and called a conical cell from \( \mathcal{J} \) to \( a \) along \( \mathcal{M} \).

- Given a left module \( \mathcal{J} : \ast \to A \) and an object \( x \in \|X\| \), an \( \langle \Rightarrow \mathcal{M} \rangle \)-arrow \( \Phi : x \Rightarrow \mathcal{J} \) is a left module morphism \( \Phi : \mathcal{J} \to x \langle \mathcal{M} \rangle : \ast \to A \), i.e. a conical cell

\[
\begin{array}{ccc}
\ast & \xrightarrow{\Phi} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{M \Rightarrow} & A
\end{array}
\]

This cell is denoted by \( \Phi : x \Rightarrow \mathcal{J} : \Rightarrow \mathcal{M} \), depicted also as

\[
\begin{array}{ccc}
\ast & \xrightarrow{\Phi} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{M \Rightarrow} & A
\end{array}
\]

, and called a conical cell from \( x \) to \( \mathcal{J} \) along \( \mathcal{M} \).
(2) A conical cell

\[
\begin{array}{c}
X \xrightarrow{M} \ast \\
\downarrow \phi \\
Y \xrightarrow{\ast} \mathcal{N} \xrightarrow{B}
\end{array}
\]

from \(P\) to \(b\) along \(\langle M, \mathcal{N} \rangle\) can be also depicted as a conical cell

\[
\begin{array}{c}
\mathcal{M} \xrightarrow{\ast} b \\
\downarrow \phi \\
X \xrightarrow{\mathcal{M}} \mathcal{N} \xrightarrow{B}
\end{array}
\]

from \(M\) to \(b\) along \(P\langle \mathcal{N} \rangle\), both being defined by a right module morphism \(\Phi: \mathcal{M} \rightarrow P\langle \mathcal{N} \rangle b: X \rightarrow \ast\).

A conical cell

\[
\begin{array}{c}
\ast \xrightarrow{M} A \\
\downarrow \phi \\
Y \xrightarrow{\ast} \mathcal{N} \xrightarrow{B}
\end{array}
\]

from \(y\) to \(Q\) along \(\langle M, \mathcal{N} \rangle\) can be also depicted as a conical cell

\[
\begin{array}{c}
y \xrightarrow{\ast} \mathcal{M} \\
\downarrow \phi \\
Y \xrightarrow{\mathcal{M}} Q \xrightarrow{A}
\end{array}
\]

from \(y\) to \(M\) along \(\langle \mathcal{N}, Q \rangle\), both being defined by a left module morphism \(\Phi: \mathcal{M} \rightarrow y\langle \mathcal{N} \rangle Q: \ast \rightarrow A\).

(3) With the notation introduced above, the right and left Yoneda morphisms for \(M\) are depicted as

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{M} \ast \\
\downarrow \phi \\
\mathcal{Y} \xrightarrow{\ast} \mathcal{N} \xrightarrow{A}
\end{array}
\]

so that they send each \(M\)-arrows \(m: s \rightarrow t\) to the conical cells

\[
\begin{array}{c}
\langle X \rangle s \xrightarrow{\ast} t \\
\downarrow \phi \\
\langle X \rangle \mathcal{M} \xrightarrow{\ast} \mathcal{A}
\end{array}
\]

With this depiction of Yoneda morphisms, Theorem 5.2.12 is restated as follows:

- for any pair of objects \(s \in \|X\|\) and \(t \in \|A\|\), the assignment \(m \mapsto X|_m\) yields a bijection

\[
s(\mathcal{M}) t \cong \langle X \rangle s(\mathcal{M}^\mathcal{N})(t)
\]

from the set of \(M\)-arrows \(s \rightarrow t\) to the set of conical cells \(\langle X \rangle s \rightarrow t\) along \(M\); moreover, the bijection is natural in \(s\) and \(t\).

- for any pair of objects \(s \in \|X\|\) and \(t \in \|A\|\), the assignment \(m \mapsto m|_A\) yields a bijection

\[
s(\mathcal{M}) t \cong (s)(\mathcal{N}, \mathcal{M})(t(A))
\]

from the set of \(M\)-arrows \(s \rightarrow t\) to the set of conical cells \(s \rightarrow t(A)\) along \(M\); moreover, the bijection is natural in \(s\) and \(t\).

(4) For any module \(\mathcal{M}\),

\[
\langle \mathcal{M}^\mathcal{N} \rangle^\mathcal{N} = \langle \mathcal{N}, \mathcal{M} \rangle^\mathcal{N}
\]

Note. The following definition is a special case of Definition 11.1.1 where \(\mathcal{M}\) is given by the hom of a category.
11.1. coYoneda lemma

Definition 11.1.3.

- The corepresentable module of the right Yoneda functor for a category $X$ is denoted by $X^\dagger$; that is, the module

$$\langle X^\dagger \rangle : [X:] \to X$$

is given by the composition

$$[X:] \xrightarrow{(X)} [X:] \xrightarrow{X^\dagger} X$$

- The representable module of the left Yoneda functor for a category $A$ is denoted by $\\& A$; that is, the module

$$\langle \\& A \rangle : A \to [:A]^\sim$$

is given by the composition

$$A \xrightarrow{\& A} [:A]^\sim \xrightarrow{\sim} [:A]^\sim$$

Remark 11.1.4.

(1) The identities

$$\langle X^\dagger \rangle = \langle \langle X \rangle \rangle \quad \langle \& A \rangle = \langle \& \langle A \rangle \rangle$$

follow from the identities

$$[X] = [\langle X \rangle] \quad [\& A] = [\& \langle A \rangle]$$

(see Definition 2.3.1). Hence the module $X^\dagger$ (resp. $\& A$) is a special instance of a more general module in Definition 11.1.1 where $M$ is given by the hom of $X$ (resp. $A$).

(2) Given a right module $M : X \to *$ and an object $x \in [X]$, an $(X^\dagger)$-arrow $\Phi : M \to x$ is a right module morphism $\Phi : M \to \langle X \rangle x : X \to *$. This right module morphism is denoted by $\Phi : M \to x : X^\dagger$, depicted variously as

, and called a conical cell from $M$ to $x$.

- Given a left module $M : * \to A$ and an object $a \in \|A\|$, an $(\& A)$-arrow $\Phi : a \to M$ is a left module morphism $\Phi : M \to a\langle A \rangle : * \to A$. This left module morphism is denoted by $\Phi : a \to M : \& A$, depicted variously as

, and called a conical cell from $a$ to $M$.

Definition 11.1.5. Let $M : X \to A$ be a module.

- The module

$$\langle M^\dagger \rangle : [X \dagger] \to A$$

is defined by the composition

$$[X \dagger] \xrightarrow{(X!)} [X:] \xrightarrow{M^\dagger} A$$

(see Definition 10.4.2 for $(X \dagger)$).
The module

\[ \langle \ast_{\mathcal{M}} \rangle : X \to [\downarrow A]^\ast \]

is defined by the composition

\[ X \xrightarrow{x_{\mathcal{M}}} [\cdot] : A]^{- \cdot (1)} : A]^{- \cdot [\downarrow A]^\ast} \]

(see Definition 10.4.2 for \((\downarrow A)\)).

**Remark 11.1.6.**

1. Given a right comma \(K : X \to \ast\) and an object \(a \in \|A\|\), an \(\langle M \not\{\rangle\) -arrow \(\alpha : K \to a\) is a cone

\[
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{1} & \ast \\
\downarrow \alpha & & \downarrow a \\
X - \xrightarrow{\alpha} & M \to A
\end{array}
\]

2. Given a left comma \(* \xleftarrow{1} [K]\) from \(X\) to the left comma fibration \(\mathbb{K} : [\mathbb{K}] \to A\) along \(\mathcal{M}\).

\[
\begin{array}{ccc}
X - \xrightarrow{\alpha} & M \to A
\end{array}
\]

3. The module \(\langle \ast_{\mathcal{M}} \rangle : X \to [\downarrow A]^\ast\) is represented by the functor \([\cdot] : A \to [\downarrow A]^\ast\), the left comma exponential transpose of \(\mathcal{M}\) (see Definition 3.2.27); indeed, a module isomorphism

\[ \langle \mathcal{M} \not\{\rangle \cong \langle [\downarrow A] \rangle \]

is obtained from the equivalence cell in Definition 10.4.8(1) by the pasting composition

\[
\begin{array}{ccc}
[X \downarrow] & \xrightarrow{(-1)} & [X] : \xleftarrow{\mathcal{M}} \ast \\
\downarrow 1 & & \downarrow 1 \\
[X \downarrow] & \xrightarrow{(-1)} & [X \downarrow] : \xleftarrow{\mathcal{M} \not\{\rangle} A
\end{array}
\]

(see Remark 1.2.32).

4. The modules \(\langle \ast_{\mathcal{M}} \rangle : X \to [\downarrow A]^\ast\) is represented by the functor \([\cdot] : A \to [\downarrow A]^\ast\), the right comma exponential transpose of \(\mathcal{M}\) (see Definition 3.2.27); indeed, a module isomorphism

\[ \langle \ast_{\mathcal{M}} \rangle \cong [\cdot] \langle \downarrow A \rangle\]

is obtained from the equivalence cell in Definition 10.4.8(1) by the pasting composition

\[
\begin{array}{ccc}
X & \xrightarrow{x_{\mathcal{M}}} & [\cdot] : A]^{- \cdot (1)} : A]^{- \cdot [\downarrow A]^\ast} \\
\downarrow 1 & & \downarrow 1 \\
X & \xrightarrow{x_{\mathcal{M}}} & [\cdot] [\downarrow A]^{- \cdot [\downarrow A]^\ast} \\
\downarrow 1 & & \downarrow 1 \\
X & \xrightarrow{x_{\mathcal{M}}} & [\downarrow A]^{- \cdot [\downarrow A]^\ast}
\end{array}
\]

(see Remark 1.2.32).
quasi-inverse to each other is obtained from the pair of equivalence cells in Definition 10.4.8(2) by the pasting compositions

\[
\begin{array}{c}
\text{[X]: } \xrightarrow{(X)} [X] \\ \downarrow \downarrow \\
\text{[X]: } \xrightarrow{M} A \\
\end{array}
\quad
\begin{array}{c}
\text{[X]: } \xrightarrow{(X)} [X] \\ \downarrow \downarrow \\
\text{[X]: } \xrightarrow{M} A
\end{array}
\]

. By this construction, we see that the equivalence \( \langle M \varnothing \rangle \simeq \langle M \varnothing \rangle \) is natural in \( M \). The cell \( \downarrow_M \) sends each conical cell

\[
\begin{array}{c}
\xrightarrow{\mathcal{J}} \\
\phi \\
\text{[J]} \\
\phi^i \\
\text{[J]} \\
\Delta x \\
\end{array}
\]

\[
\xrightarrow{\phi} \xrightarrow{\phi^i} \xrightarrow{\Delta x}
\]

to its comma transpose, i.e. the cone

\[
\begin{array}{c}
\xrightarrow{\mathcal{J}} \\
\phi \\
\text{[J]} \\
\phi^i \\
\text{[J]} \\
\Delta x \\
\end{array}
\]

(cf. Remark 10.4.9(4)).

- The modules \( \langle \_M \rangle: X \rightarrow [\_A]^\sim \) and \( \langle \_M \rangle: X \rightarrow [\downarrow \_A]^\sim \) are equivalent. Indeed, a pair of equivalence cells

\[
\begin{array}{c}
\xrightarrow{\_M} [\_A]^\sim \\
\downarrow \downarrow \\
\xrightarrow{\_M} [\downarrow \_A]^\sim \\
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\_M} [\downarrow \_A]^\sim \\
\downarrow \downarrow \\
\xrightarrow{\_M} [\_A]^\sim
\end{array}
\]

quasi-inverse to each other is obtained from the pair of equivalence cells in Definition 10.4.8(2) by the pasting compositions

\[
\begin{array}{c}
\xrightarrow{\_M} [\_A]^\sim \\
\downarrow \downarrow \\
\xrightarrow{\_M} [\downarrow \_A]^\sim \\
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\_M} [\downarrow \_A]^\sim \\
\downarrow \downarrow \\
\xrightarrow{\_M} [\_A]^\sim
\end{array}
\]

. By this construction, we see that the equivalence \( \langle \_M \rangle \simeq \langle \_M \rangle \) is natural in \( M \). The cell \( \downarrow_M \) sends each conical cell

\[
\begin{array}{c}
\xrightarrow{\mathcal{J}} \\
\phi \\
\text{[J]} \\
\phi^i \\
\text{[J]} \\
\Delta x \\
\end{array}
\]

\[
\xrightarrow{\phi} \xrightarrow{\phi^i} \xrightarrow{\Delta x}
\]

to its comma transpose, i.e. the cone

\[
\begin{array}{c}
\xrightarrow{\mathcal{J}} \\
\phi^i \\
\text{[J]} \\
\phi^i \\
\text{[J]} \\
\Delta x
\end{array}
\]

(cf. Remark 10.4.9(4)).

(4) The equivalence in (3) above (to be precise, a special case where \( M \) is given by the hom of a category) is called the coYoneda lemma in [ML98]. The composition of the equivalence cell \( \downarrow_M \langle M \varnothing \rangle \rightarrow \langle M \varnothing \rangle \) and the right Yoneda morphism in Remark 11.1.2(3) yields a fully faithful cell

\[
\begin{array}{c}
\xrightarrow{\mathcal{J}} \\
[A] \\
\downarrow \downarrow \\
\xrightarrow{\_M \varnothing} [A]
\end{array}
\]

\[
\xrightarrow{\phi} \xrightarrow{(X)} [A] \\
\downarrow \downarrow \\
\xrightarrow{\_M \varnothing} [A]
\]
(see Definition 3.2.29 for $X$) making the diagram

$$
\begin{array}{c}
\left(\mathcal{M}: X \to A\right) \downarrow
\end{array}
$$

commute up to isomorphism. The cell $\left(\mathcal{M}: X \to A\right)$ establishes a bijection from the set of $\mathcal{M}$-arrows $s \to t$ to the set of cones $X \downarrow s \to t$ along $\mathcal{M}$. It may be more appropriate to call $\left(\mathcal{M}: X \to \mathbf{A}\right)$ "coYoneda" instead; in any case, however, it is the equivalence $\left(\mathbf{M} \subseteq \mathbf{A}\right) \cong\left(\mathbf{M} \subseteq \mathbf{A}\right)$ that plays a critical role in Section 11.2, providing a way to transform a weighted limit to a conical limit.

**Definition 11.1.7.** Let $\mathbf{E}$ be a category and $\mathcal{M}: X \to A$ be a module.

- The corepresentable module of the right action of $\mathcal{M}$ on $[\mathbf{E}, \mathbf{A}]$ is denoted by $\mathbf{M} \subseteq \mathbf{E}$; that is, the module

$$
\left(\mathbf{M} \subseteq \mathbf{E}\right): [X: \mathbf{E}] \to [\mathbf{E}, \mathbf{A}]
$$

is defined by the composition

$$
[X: \mathbf{E}] \overleftarrow{(\mathbf{X}: \mathbf{E})} [X: \mathbf{E}] \overrightarrow{\mathbf{M}: \mathbf{E} \rightarrow \mathbf{A} \mathbf{E} \rightarrow \mathbf{A}} [\mathbf{E}, \mathbf{A}]
$$

- The representable module of the left action of $\mathcal{M}$ on $[\mathbf{E}, X]$ is denoted by $\mathbf{E} \subseteq \mathbf{M}$; that is, the module

$$
\left(\mathbf{E} \subseteq \mathbf{M}\right): [\mathbf{E}, X] \to [\mathbf{E}, \mathbf{A}]^{-1}
$$

is defined by the composition

$$
[E, X] \overleftarrow{\mathbf{E} \subseteq \mathbf{M}} [\mathbf{E}, \mathbf{A}]^{-1} \overrightarrow{[\mathbf{E}, \mathbf{A}]^{-1}} [\mathbf{E}, \mathbf{A}]^{-1}
$$

**Remark 11.1.8.**

1. Given a module $\mathcal{J}: X \to \mathbf{E}$ and a functor $K: \mathbf{E} \to \mathbf{A}$, an $\left(\mathbf{M} \subseteq \mathbf{E}\right)$-arrow $\Phi: \mathcal{J} \to K$ is a module morphism $\Phi: \mathcal{J} \to \left(\mathcal{M}\right) K: X \to \mathbf{E}$, i.e. a cell

$$
\begin{array}{c}
\xymatrix{X \ar[r]^{\mathcal{J}} & \mathbf{E} \\
\ar[ur]_{\Phi} & \mathbf{K} \\
\ar[rr]_{\mathcal{M}} & X & \mathbf{A}}
\end{array}
$$

This cell is denoted by $\Phi: \mathcal{J} \to K: \mathbf{M} \subseteq \mathbf{E}$, depicted also as

$$
\begin{array}{c}
\xymatrix{\mathcal{J} \ar[r] & \mathbf{E} \\
\Phi \ar[ur]_{\mathcal{M}} & \mathbf{K} \\
X & \mathbf{A}}
\end{array}
$$

and called a cell from $\mathcal{J}$ to $K$ along $\mathcal{M}$.

2. Given a module $\mathcal{J}: \mathbf{E} \to \mathbf{A}$ and a functor $K: \mathbf{E} \to X$, an $\left(\mathbf{E} \subseteq \mathbf{M}\right)$-arrow $\Phi: K \to \mathcal{J}$ is a module morphism $\Phi: \mathcal{J} \to K\left(\mathcal{M}\right): \mathbf{E} \to \mathbf{A}$, i.e. a cell

$$
\begin{array}{c}
\xymatrix{\mathbf{E} \ar[r]^{\mathcal{J}} & \mathbf{A} \\
\Phi \ar[ur]_{\mathcal{M}} & \mathbf{K} \\
\ar[rr]_{\mathcal{1}} & X & \mathbf{A}}
\end{array}
$$

and called a cell from $\mathcal{J}$ to $K$ along $\mathcal{M}$. 
This cell is denoted by $\Phi : K \to J : E \otimes M$, depicted also as

$$\begin{array}{c}
\xymatrix{
E \\
X \ar[r]_{\Phi} & M \ar[r] & A
}
\end{array}$$

and called a cell from $K$ to $J$ along $M$.

(2) A cell

$$\begin{array}{c}
\xymatrix{
X \ar[r]_{M} & A \\
Y \ar[r]_{P(\mathcal{N})} & B
}
\end{array}$$

from $P$ to $Q$ along $(\mathcal{M}, \mathcal{N})$, i.e. a module morphism $\Phi : M \to P(\mathcal{N}) Q : X \to A$, can be also depicted as a cell

$$\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]_{\Phi} & Q \\
X \ar[r]_{P(\mathcal{N})} & B
}
\end{array}$$

from $\mathcal{M}$ to $Q$ along $P(\mathcal{N})$ or a cell

$$\begin{array}{c}
\xymatrix{
P \ar[r]_{\Phi} & \mathcal{M} \\
X \ar[r]_{\mathcal{N}Q} & A
}
\end{array}$$

from $P$ to $\mathcal{M}$ along $(\mathcal{N}) Q$.

(3) The right exponential transpose of a cell

$$\begin{array}{c}
\xymatrix{
\mathcal{N} \ar[r]_{\delta} & \mathcal{M} \\
X \ar[r]_{\Phi} & A
}
\end{array}$$

, i.e. a module morphism $\Phi : \mathcal{J} \to (\mathcal{M}) K : X \to E$, is a natural transformation

$$\begin{array}{c}
\xymatrix{
\mathcal{J} \ar[r]_{\delta} & \mathcal{M} \\
X \ar[r]_{\Phi} & A
}
\end{array}$$

from $\mathcal{J} \delta$ to $[\mathcal{M}] \delta K$ (see Proposition 2.1.6), i.e. a cylinder

$$\begin{array}{c}
\xymatrix{
\mathcal{J} \ar[r]_{\delta} & \mathcal{M} \\
X \ar[r]_{\Phi} & A
}
\end{array}$$

. The right exponential transposition $\Phi \mapsto \Phi \delta$ yields an iso cell

$$\begin{array}{c}
\xymatrix{
[X : E] & E \ar[l]_{\Phi \delta} \\
& A
}\end{array}$$

, giving a canonical isomorphism

$$\langle \mathcal{M} \otimes E \rangle \cong \langle E, \langle \mathcal{M} \rangle \rangle$$
The slice of $\Phi$ at $e \in \|E\|$ is the conical cell

$$\xymatrix{X \ar[rr]^{e \cdot K \cdot e} & & \Phi \ar[rr]^{\cdot \Phi} & & M \ar[rr]^{\cdot \Phi} & & A}$$

given by the component of the cylinder $\Phi \rho$ at $e$.

The left exponential transpose of a cell

$$\xymatrix{E \ar[rr]^{\rho} & & \Phi \ar[rr]^{\Phi} & & M \ar[rr]^{\Phi} & & A}$$

, i.e. a module morphism $\Phi : J \to K(M) : E \to A$, is a natural transformation

$$\xymatrix{X \ar[rr]^{\rho \cdot \Phi} & & \rho \cdot [\cdot \Phi] \ar[rr]^{\rho \cdot [\cdot \Phi]} & & [\cdot \Phi]}$$

from $K \circ [\cdot M]$ to $\cdot J$ (see Proposition 2.1.6), i.e. a cylinder

$$\xymatrix{E \ar[rr]^{\rho \cdot \Phi} & & \rho \cdot [\cdot \Phi] \ar[rr]^{\rho \cdot [\cdot \Phi]} & & [\cdot \Phi]}$$

. The left exponential transposition $\Phi \mapsto \cdot \Phi$ yields an iso cell

$$\xymatrix{[E,X] \ar[rr]^{(E \cdot M)} & & [E : A] \ar[rr]^{\cdot M} & & [E, [\cdot A]]}$$

$$\xymatrix{[E,X] \ar[rrrr]_{(E, \cdot M)} & & \ldots & & \ar[rrrr]_{\cdot M} & & \ldots & & \ar[rrrr]_{[E, [\cdot A]]}}$$

giving a canonical isomorphism

$$\langle E \cdot M \rangle \cong \langle E, \cdot M \rangle$$

. The slice of $\Phi$ at $e \in \|E\|$ is the conical cell

$$\xymatrix{X \ar[rr]^{e \cdot K \cdot e} & & \Phi \ar[rr]^{e(\Phi)} & & M \ar[rr]^{\cdot \Phi} & & A}$$

given by the component of the cylinder $\cdot \Phi$ at $e$.

4) For any category $E$ and any module $M$,

$$\langle M \cdot E \rangle \cong \langle E^{-} \cdot M \rangle$$

5) The module $\langle M \cdot E \rangle : [X : ] \to A$ in Definition 11.1.1 is identified with $\langle M \cdot * \rangle : [X : *] \to [*, A]$; that is, the module $\langle M \cdot E \rangle$ is regarded as a special instance of a more general $\langle M \cdot E \rangle$ where $E$ is the terminal category. Dually, the module $\langle * \cdot M \rangle : [X : ] \to [*, A]$ is identified with $\langle * \cdot M \rangle : [*, X] \to [*, A]^{-}$.

Note. The following definition is a special case of Definition 11.1.7 where $M$ is given by the hom of a category.
Definition 11.1.9.

- Given categories $\mathbf{A}$ and $\mathbf{X}$, the corepresentable module of the right general Yoneda functor for $[\mathbf{A}, \mathbf{X}]$ is denoted by $\mathbf{X} \not
\mathbf{A}$; that is, the module

$$\langle \mathbf{X} \not \mathbf{A} \rangle : [\mathbf{X} : \mathbf{A}] \to [\mathbf{A}, \mathbf{X}]$$

is given by the composition

$$[\mathbf{X} : \mathbf{A}] \xrightarrow{\langle \mathbf{X} \not \mathbf{A} \rangle} [\mathbf{X} : \mathbf{A}] \xrightarrow{\mathbf{X} \not \mathbf{A}} [\mathbf{A}, \mathbf{X}]$$

- Given categories $\mathbf{X}$ and $\mathbf{A}$, the representable module of the left general Yoneda functor for $[\mathbf{X}, \mathbf{A}]$ is denoted by $\mathbf{X} \not \mathbf{A}$; that is, the module

$$\langle \mathbf{X} \not \mathbf{A} \rangle : [\mathbf{X}, \mathbf{A}] \to [\mathbf{X} : \mathbf{A}]$$

is given by the composition

$$[\mathbf{X}, \mathbf{A}] \xrightarrow{\mathbf{X} \not \mathbf{A}} [\mathbf{X} : \mathbf{A}] \xrightarrow{\langle \mathbf{X} \not \mathbf{A} \rangle} [\mathbf{X}, \mathbf{A}]$$

Remark 11.1.10.

1. The identities

$$\langle \mathbf{X} \not \mathbf{A} \rangle = \langle \langle \mathbf{X} \not \mathbf{A} \rangle \rangle \quad \langle \mathbf{X} \not \mathbf{A} \rangle = \langle \mathbf{X} \not \langle \mathbf{A} \rangle \rangle$$

follow from the identities

$$[\mathbf{X} \not \mathbf{A}] = [\langle \mathbf{X} \not \mathbf{A} \rangle] \quad [\mathbf{X} \not \mathbf{A}] = [\langle \mathbf{X} \not \langle \mathbf{A} \rangle \rangle]$$

(see Definition 2.3.7). Hence the module $\mathbf{X} \not \mathbf{A}$ (resp. $\mathbf{X} \not \mathbf{A}$) is a special instance of a more general module in Definition 11.1.7 where $\mathcal{M}$ is given by the hom of $\mathbf{X}$ (resp. $\mathbf{A}$).

2. Given a module $\mathcal{M} : \mathbf{X} \to \mathbf{A}$ and a functor $\mathbf{K} : \mathbf{A} \to \mathbf{X}$, an $(\mathbf{X} \not \mathbf{A})$-arrow $\Phi : \mathcal{M} \to \mathcal{K}$ is a module morphism $\Phi : \mathcal{M} \to [\langle \mathbf{X} \rangle \mathbf{K} : \mathbf{X} \to \mathbf{A}]$. This module morphism is denoted by $\Phi : \mathcal{M} \to \mathcal{K}$; $\mathbf{X} \not \mathbf{A}$, depicted variously as

$$\xymatrix{ \mathbf{X} \ar@{<-}[r]^{\mathcal{M}} & \mathbf{A} \ar@{<-}[r]_{\mathbf{K}} & \mathbf{X} \ar@{<-}[d]_{\Phi} } \quad \xymatrix{ \mathbf{X} \ar@{<-}[r]^{\mathcal{M}} \ar@{<-}[d]_{\mathcal{K}} & \mathbf{A} \ar@{<-}[r]_{\Phi} \ar@{<-}[d]_{\mathbf{K}} & \mathbf{X} }$$

, and called a cell from $\mathcal{M}$ to $\mathbf{K}$.

- Given a module $\mathcal{M} : \mathbf{X} \to \mathbf{A}$ and a functor $\mathbf{K} : \mathbf{X} \to \mathbf{A}$, an $(\mathbf{X} \not \mathbf{A})$-arrow $\Phi : \mathcal{K} \to \mathcal{M}$ is a module morphism $\Phi : \mathcal{K} \to \mathcal{M} : \mathbf{X} \not \mathbf{A}$, depicted variously as

$$\xymatrix{ \mathbf{X} \ar@{<-}[r]^{\mathcal{M}} & \mathbf{A} \ar@{<-}[r]_{\mathbf{K}} & \mathbf{X} \ar@{<-}[d]_{\Phi} } \quad \xymatrix{ \mathbf{X} \ar@{<-}[r]^{\mathcal{M}} \ar@{<-}[d]_{\mathcal{K}} & \mathbf{A} \ar@{<-}[r]_{\Phi} \ar@{<-}[d]_{\mathbf{K}} & \mathbf{X} }$$

, and called a cell from $\mathbf{K}$ to $\mathcal{M}$.
11.2. Weighted limits

Note. Definition 11.2.1 and Definition 11.2.9 give two definitions of universal conical cells. We will see that they are equivalent in Theorem 11.2.11.

Definition 11.2.1.
- A conical cell \( \xymatrix{ J \ar[r]^-{r} & \ast \ar[r]^-{\Phi} & X } \) is called universal if it is a direct universal \( (\mathcal{M} \mathcal{C}) \)-arrow (see Definition 11.1.1). Given a right module \( J : X \to \ast \), a universal conical cell \( \xymatrix{ J \ar[r]^-{r} & \ast \ar[r]^-{\Phi} & X } \) or the pair \( (r, \Phi) \), or the object \( r \) itself, is called a colimit of \( J \) along \( \mathcal{M} \).
- A conical cell \( \xymatrix{ \ast \ar[r]^-{r} & J \ar[r]^-{\Phi} & X } \) is called universal if it is an inverse universal \( (\mathcal{C} \mathcal{M}) \)-arrow (see Definition 11.1.1). Given a left module \( J : \ast \to A \), a universal conical cell \( \xymatrix{ J \ar[r]^-{r} & \ast \ar[r]^-{\Phi} & X } \) or the pair \( (r, \Phi) \), or the object \( r \) itself, is called a limit of \( J \) along \( \mathcal{M} \).

Remark 11.2.2. A conical cell \( \xymatrix{ J \ar[r]^-{r} & \ast \ar[r]^-{\Phi} & \mathcal{M} } \) is universal if and only if to every conical cell \( \Phi : J \to a : \mathcal{M} \) there is a unique \( A \)-arrow \( \Phi' : \ast \to a \) such that \( \Phi = \Phi' \circ \Phi \). Dually, a conical cell \( \Phi : r \to J : \mathcal{M} \) is universal if and only if to every conical cell \( \Phi : x \to J : \mathcal{M} \) there is a unique \( X \)-arrow \( \Phi' : x \to r \) such that \( \Phi = \Phi' \circ \Phi \).

Proposition 11.2.3.
- A conical cell

\[
\begin{array}{ccccc}
J & \ar[r]^-{r} & \ast & \ar[r]^-{\Phi} & X \\
\downarrow{\Phi} & & \downarrow{\Gamma} & & \downarrow{\Delta r} \\
X & \ar[r]^-{r} & \ast & \ar[r]^-{\Phi} & A \\
\end{array}
\]

(see Remark 11.1.6(3)) is a universal cone.
- A conical cell

\[
\begin{array}{ccccc}
J & \ar[r]^-{r} & \ast & \ar[r]^-{\Phi} & X \\
\downarrow{\Phi} & & \downarrow{\Gamma} & & \downarrow{\Delta r} \\
X & \ar[r]^-{r} & \ast & \ar[r]^-{\Phi} & A \\
\end{array}
\]

(see Remark 11.1.6(3)) is a universal cone.

Proof. By Theorem 8.11.5, the equivalence cell \( \downarrow_{\mathcal{M}}: (\mathcal{M} \mathcal{C}) \to (\mathcal{M} \mathcal{C}) \) in Remark 11.1.6(3) preserves and reflects direct universal arrows.

Theorem 11.2.4. Let \( \mathcal{M} : X \to A \) be a module.
- An \( \mathcal{M} \)-arrow \( u : x \to r \) is direct universal if and only if the conical cell

\[
\begin{array}{ccccc}
\ast & \ar[r]^-{u} & x & \ar[r]^-{r} & \ast \\
\downarrow{\Phi} & & \downarrow{\Gamma} & & \downarrow{\Delta r} \\
X & \ar[r]^-{r} & \ast & \ar[r]^-{\Phi} & A \\
\end{array}
\]

(see Remark 11.1.2(3)) is universal.
11.2. Weighted limits

- An $\mathcal{M}$-arrow $u : r \sim a$ is inverse universal if and only if the conical cell

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^{u} & \mathcal{M} \ar[r]^{a(A)} & A }
\end{array}
\]

(see Remark 11.1.2(3)) is universal.

**Proof.** Apply Proposition 6.2.23 to the fully faithful cell in Remark 11.1.2(3). 

**Note.** The following definition is a special case of Definition 11.2.1 where $\mathcal{M}$ is given by the hom of a category.

**Definition 11.2.5.**

- A conical cell $\begin{array}{c}
\xymatrix{ \mathcal{M} \ar[r]^{r} & X \ar[r]^{a} & X }
\end{array}$ is called universal if it is a direct universal $\langle X \mathcal{M} \rangle$-arrow (see Definition 11.1.3). Given a right module $\mathcal{M} : X \rightarrow \star$, a universal conical cell $\Upsilon : \mathcal{M} \sim r : X \mathcal{M}$ or the pair $(r, \Upsilon)$, or the object $r$ itself, is called a colimit of $\mathcal{M}$ in $X$.

- A conical cell $\begin{array}{c}
\xymatrix{ A \ar[r]^{r} & \mathcal{M} \ar[r]^{a} & A }
\end{array}$ is called universal if it is an inverse universal $\langle \mathcal{M} \mathcal{A} \rangle$-arrow (see Definition 11.1.3). Given a left module $\mathcal{M} : \star \rightarrow A$, a universal conical cell $\Upsilon : r \sim \mathcal{M} : \mathcal{A}$ or the pair $(r, \Upsilon)$, or the object $r$ itself, is called a limit of $\mathcal{M}$ in $A$.

**Remark 11.2.6.** A colimit of a right module $\mathcal{M} : X \rightarrow \star$ in $X$ is the same thing as a colimit of $\mathcal{M}$ along the hom endomodule $\langle X \mathcal{M} \rangle$. Dually, a limit of a left module $\mathcal{M} : \star \rightarrow A$ in $A$ is the same thing as a limit of $\mathcal{M}$ along the hom endomodule $\langle \mathcal{M} \mathcal{A} \rangle$.

**Proposition 11.2.7.**

- A representation $(r, \Upsilon)$ of a right module $\mathcal{M} : X \rightarrow \star$ gives a colimit $\begin{array}{c}
\xymatrix{ \mathcal{M} \ar[r]^{r} & X \ar[r]^{a} & X }
\end{array}$ of $\mathcal{M}$ in $X$ (cf. Remark 2.3.4(2)).

- A representation $(r, \Upsilon)$ of a left module $\mathcal{M} : \star \rightarrow A$ gives a limit $\begin{array}{c}
\xymatrix{ A \ar[r]^{r} & \mathcal{M} \ar[r]^{a} & A }
\end{array}$ of $\mathcal{M}$ in $A$ (cf. Remark 2.3.4(2)).

**Proof.** Since $\Upsilon : \mathcal{M} \rightarrow \langle X \mathcal{M} \rangle r : X \rightarrow \star$ is an isomorphism and the Yoneda functor $X \mathcal{M}$ is fully faithful, $\Upsilon$ is a direct universal $\langle X \mathcal{M} \rangle$-arrow by Corollary 6.2.17.

**Remark 11.2.8.** The converse does not hold in general. For example, consider the conical cell

\[
\begin{array}{c}
\xymatrix{ \{0,1\} \ar[r]^{1} & \star \ar[r]^{1} & \star }
\end{array}
\]

given by the unique function $!$ from the set $\{0,1\}$ to the hom $\langle \star \rangle$ of the terminal category. The cell, being the only cell $\{0,1\} \rightarrow \langle \star \rangle$, forms a colimit of $\{0,1\}$ but not a representation.
Definition 11.2.9.

- A conical cell \( E \rightarrow J \rightarrow * \) is called universal if it is a direct universal \( \langle J, M \rangle \)-arrow (see Definition 1.2.8). Given a functor \( K : E \rightarrow X \), a universal conical cell \( \Upsilon : K \sim r : J \rightarrow M \) or the pair \((r, \Upsilon)\), or the object \( r \) itself, is called a colimit of \( K \) weighted by \( J \) (or \( J \)-weighted colimit of \( K \)) along \( M \), with the object \( r \) denoted by \( \prod J K \).

- A conical cell \( * \rightarrow J \rightarrow E \) is called universal if it is an inverse universal \( \langle J, M \rangle \)-arrow (see Definition 1.2.8). Given a functor \( K : E \rightarrow A \), a universal conical cell \( \Upsilon : r \sim K : J \rightarrow M \) or the pair \((r, \Upsilon)\), or the object \( r \) itself, is called a limit of \( K \) weighted by \( J \) (or \( J \)-weighted limit of \( K \)) along \( M \), with the object \( r \) denoted by \( \prod J K \).

Remark 11.2.10.

1. A conical cell \( \Phi : K \sim r : J \rightarrow M \) is universal if and only if to every conical cell \( \Phi : K \sim a : J \rightarrow M \) there is a unique \( A \)-arrow \( \Upsilon / \Phi : r \rightarrow a \) such that \( \Phi = \Upsilon \circ \Phi / \Upsilon \). Dually, a conical cell \( \Phi : r \sim K : J \rightarrow M \) is universal if and only if to every conical cell \( \Phi : x \sim K : J \rightarrow M \) there is a unique \( X \)-arrow \( \Phi / \Phi : x \rightarrow r \) such that \( \Phi = \Phi / \Phi \circ \Phi \).

2. As a special case where \( M \) is given by the hom of a category \( C \),

- a universal conical cell \( E \rightarrow J \rightarrow * \) or the pair \((r, \Upsilon)\), or the object \( r \) itself, is called a \( J \)-weighted colimit of \( K \) in \( C \).

- a universal conical cell \( * \rightarrow J \rightarrow E \) or the pair \((r, \Upsilon)\), or the object \( r \) itself, is called a \( J \)-weighted limit of \( K \) in \( C \).

Theorem 11.2.11.

- A conical cell

\[
\begin{array}{ccc}
E & \rightarrow & J \\
\downarrow \Phi & & \downarrow \Phi \\
K & \rightarrow & M \\
\end{array}
\]

from \( K \) to \( r \) along \( \langle J, M \rangle \) is universal if and only if the conical cell

\[
\begin{array}{ccc}
E & \rightarrow & J \\
\downarrow \Phi & & \downarrow \Phi \\
K & \rightarrow & M \\
\end{array}
\rightarrow
\begin{array}{ccc}
A & \rightarrow & r \\
\end{array}
\]

from \( J \) to \( r \) along \( K \langle M \rangle \) is universal; hence a \( J \)-weighted colimit of \( K \) along \( M \) is the same thing as a colimit of \( J \) along \( K \langle M \rangle \). Conversely, a conical cell

\[
\begin{array}{ccc}
X & \rightarrow & J \\
\downarrow \Phi & & \downarrow \Phi \\
M & \rightarrow & r \\
\end{array}
\]

from \( J \) to \( r \) along \( M \) is universal if and only if the conical cell

\[
\begin{array}{ccc}
X & \rightarrow & J \\
\downarrow \Phi & & \downarrow \Phi \\
M & \rightarrow & r \\
\end{array}
\rightarrow
\begin{array}{ccc}
A & \rightarrow & r \\
\end{array}
\]
from the identity $X \to X$ to $r$ along $(\mathcal{J}, \mathcal{M})$ is universal; hence a colimit of $\mathcal{J}$ along $\mathcal{M}$ is the same thing as a $\mathcal{J}$-weighted colimit of the identity $X \to X$ along $\mathcal{M}$.

- A conical cell

$$
\begin{array}{ccc}
* & \xrightarrow{\mathcal{J}} & E \\
r & \gamma & \downarrow \kappa \\
X & \xrightarrow{\mathcal{M}} & A
\end{array}
$$

from $r$ to $K$ along $(\mathcal{J}, \mathcal{M})$ is universal if and only if the conical cell

$$
\begin{array}{ccc}
r & \xrightarrow{\mathcal{J}} & E \\
x & \xrightarrow{\mathcal{M}K} & A
\end{array}
$$

from $r$ to $\mathcal{J}$ along $(\mathcal{M})K$ is universal; hence a $\mathcal{J}$-weighted limit of $K$ along $\mathcal{M}$ is the same thing as a limit of $\mathcal{J}$ along $(\mathcal{M})K$. Conversely, a conical cell

$$
\begin{array}{ccc}
r & \xrightarrow{\mathcal{J}} & A \\
x & \xrightarrow{\mathcal{M}} & A
\end{array}
$$

from $r$ to $\mathcal{J}$ along $\mathcal{M}$ is universal if and only if the conical cell

$$
\begin{array}{ccc}
* & \xrightarrow{\mathcal{J}} & A \\
r & \gamma & \downarrow \iota \\
X & \xrightarrow{\mathcal{M}} & A
\end{array}
$$

from $r$ to the identity $A \to A$ along $(\mathcal{J}, \mathcal{M})$ is universal; hence a limit of $\mathcal{J}$ along $\mathcal{M}$ is the same thing as a $\mathcal{J}$-weighted limit of the identity $A \to A$ along $\mathcal{M}$.

**Proof.** The left slice of the module $(\mathcal{J}, \mathcal{M})$ at $K$ and the left slice of the module $(\mathcal{K}, \mathcal{M}) \not\supset$ at $\mathcal{J}$ are the same left module over $A$ given by

$$a \mapsto (\mathcal{J}) (E : (\mathcal{K}, \mathcal{M}) a)$$

Hence $\Upsilon : K \to r$ is a direct universal $(\mathcal{J}, \mathcal{M})$-arrow iff $\Upsilon : \mathcal{J} \to r$ is a direct universal $((\mathcal{K}, \mathcal{M}) \not\supset)$-arrow.

The left slice of the module $\mathcal{M} \not\supset$ at $\mathcal{J}$ and the left slice of the module $(\mathcal{J}, \mathcal{M})$ at $1_X$ are the same left module over $A$ given by

$$a \mapsto (\mathcal{J}) (X : ((\mathcal{M}) a))$$

Hence $\Upsilon : \mathcal{J} \to r$ is a direct universal $(\mathcal{M} \not\supset)$-arrow iff $\Upsilon : 1_X \to r$ is a direct universal $(\mathcal{J}, \mathcal{M})$-arrow. \qed

**Theorem 11.2.12.**

- A conical cell

$$
\begin{array}{ccc}
E & \xrightarrow{\mathcal{J}} & * \\
\kappa & \gamma & \downarrow r \\
X & \xrightarrow{\mathcal{M}} & A
\end{array}
$$

is universal if and only if its comma transpose

$$
\begin{array}{ccc}
E & \xleftarrow{\mathcal{J}^!} & [\mathcal{J}] \\
\kappa & \gamma^i & \downarrow \Delta r \\
X & \xrightarrow{\mathcal{M}^r} & A
\end{array}
$$

is a universal cone.
A conical cell

\[
\begin{array}{ccc}
E & \xrightarrow{r} & \ast \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

is universal if and only if its comma transpose

\[
\begin{array}{ccc}
[\mathcal{J}] & \xrightarrow{\mathcal{J}^i} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

\[
\begin{array}{ccc}
\Delta r & \downarrow & \mathcal{J}^i \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

is a universal cone.

Proof. Since the conical cell

\[
\begin{array}{ccc}
E & \xrightarrow{\mathcal{J}} & \ast \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta r} & A
\end{array}
\]

is universal iff the conical cell

\[
\begin{array}{ccc}
E & \xrightarrow{\mathcal{J}^i} & \ast \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

along the composite module \( K(\mathcal{M}) \) is universal (see Theorem 11.2.11), and since the cone

\[
\begin{array}{ccc}
E & \xrightarrow{\mathcal{J}^i} & [\mathcal{J}] \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

universal iff the cone

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma^i} & [\mathcal{J}] \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

along the composite module \( K(\mathcal{M}) \) is universal (see Theorem 7.1.3), the assertion is reduced to Proposition 11.2.3.

Remark 11.2.13. By Theorem 11.2.12 and by the bijectiveness of comma transposition, we see that

- a \( \mathcal{J} \)-weighted colimit of \( K \) along \( \mathcal{M} \) is the same thing as a colimit of \( \mathcal{J}^i \circ K \) along \( \mathcal{M} \).
- a \( \mathcal{J} \)-weighted limit of \( K \) along \( \mathcal{M} \) is the same thing as a limit of \( K \circ \mathcal{J}^i \) along \( \mathcal{M} \).

Limits thus subsume weighted limits. We will see that the converse is also the case in Theorem 11.2.15.

Corollary 11.2.14. Let \( \mathcal{J} \) and \( \mathcal{M} \) be modules and \( K \) be a functor as in Theorem 11.2.12. If \( \mathcal{M} \) is cocomplete (resp. complete) and \( \mathcal{J} \) is small, then \( K \) has a \( \mathcal{J} \)-weighted colimit (resp. limit) along \( \mathcal{M} \).

Proof. By Remark 11.2.13, it suffices to show that \( \mathcal{J}^i \circ K : [\mathcal{J}] \to X \) has a colimit along \( \mathcal{M} \). But since the smallness of \( \mathcal{J} \) guarantees the smallness of the comma category \([\mathcal{J}]\), this holds by the cocompleteness of \( \mathcal{M} \).

Note. The bijective correspondence between cones and conical cells stated in Corollary 5.5.7 gives the following result.

Theorem 11.2.15.

- A cone

\[
\begin{array}{ccc}
E & \xrightarrow{\mu} & \ast \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

is universal if and only if the conical cell

\[
\begin{array}{ccc}
E & \xrightarrow{\Delta E} & \ast \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mu} & A
\end{array}
\]

is universal.
11.2. Weighted limits

- A cone

\[
\begin{array}{c}
A \\
\downarrow \mu \\
X \rightarrow M \rightarrow A
\end{array}
\]

is universal if and only if the conical cell

\[
\begin{array}{c}
* \rightarrow \Delta A \\
\downarrow \mu \\
E \rightarrow K \\
\downarrow \mu \\
X \rightarrow M \rightarrow A
\end{array}
\]

is universal.

**Proof.** Immediate by the isomorphism in Corollary 5.5.7.

**Remark 11.2.16.** By Theorem 11.2.15 and by the bijectiveness of the assignment \( \mu \mapsto (\mu) \), we see that given a module \( M : X \rightarrow A \),

- a colimit of a functor \( K : E \rightarrow X \) along \( M \) is the same thing as an \( (\Delta E^*) \)-weighted colimit of \( K \) along \( M \).
- a limit of a functor \( K : E \rightarrow A \) along \( M \) is the same thing as an \( (* \Delta E^*) \)-weighted limit of \( K \) along \( M \).

Weighted limits thus subsume limits, and vice versa as we saw in Remark 11.2.13.

**Definition 11.2.17.** A cell \( X \rightarrow M \rightarrow A \) is said to

\[
\begin{array}{c}
P \phi Q \\
Y \rightarrow \tilde{N} \rightarrow B
\end{array}
\]

- preserve (reflect, create) colimits weighted by a right module \( J : E \rightarrow * \) if the postcomposition cell

\[
\begin{array}{c}
[E, X] \rightarrow J, M \\
\downarrow \phi \phi \\
[E, Y] \rightarrow \tilde{J}, \tilde{M}
\end{array}
\]

(see Definition 1.2.23) preserves (reflects, creates) direct universal arrows.
- preserve (reflect, create) limits weighted by a left module \( J : * \rightarrow E \) if the postcomposition cell

\[
\begin{array}{c}
*[X] \rightarrow J, M \\
\downarrow \phi \phi \\
*[Y] \rightarrow \tilde{J}, \tilde{M}
\end{array}
\]

(see Definition 1.2.23) preserves (reflects, creates) inverse universal arrows.

**Remark 11.2.18.**

(1) Recalling the definition of weighted colimits (resp. limits) and the definition of the postcomposition cell, Definition 11.2.17 can be stated in elementary terms as follows: \( \Phi \) is said to

- **a) preserve**
  - \( J \)-weighted colimits if each universal conical cell \( \Upsilon : K \rightarrow r : J \rightarrow M \) yields by composition with \( \Phi \) a universal conical cell \( \Upsilon \circ \Phi : K \circ P \rightarrow Q \circ r : J \rightarrow N \).
  - \( J \)-weighted limits if each universal conical cell \( \Upsilon : r \rightarrow K : J \rightarrow M \) yields by composition with \( \Phi \) a universal conical cell \( \Upsilon \circ \Phi : P \circ r \rightarrow Q \circ K : J \rightarrow N \).

- **b) reflect**
  - \( J \)-weighted colimits if a conical cell \( \Upsilon : K \rightarrow r : J \rightarrow M \) is universal whenever the conical cell \( \Upsilon \circ \Phi : K \circ P \rightarrow Q \circ r : J \rightarrow N \) is universal.
11.2. Weighted limits

- \( J \)-weighted limits if a conical cell \( \Upsilon : r \to K : J \to M \) is universal whenever the conical cell \( \Upsilon \circ \Phi : r : P \to Q \circ K : J \to N \) is universal.

c) create

- \( J \)-weighted colimits if for every functor \( K : E \to X \) and for every universal conical cell \( \Theta : K \circ P \to s : J \to N \) there is exactly one conical cell \( \Upsilon : K \to r : J \to M \) with \( \Upsilon \circ \Phi = \Theta \), and if this \( \Upsilon \) is universal.

- \( J \)-weighted limits if for every functor \( K : E \to A \) and for every universal conical cell \( \Theta : s \to Q \circ K : J \to N \) there is exactly one conical cell \( \Upsilon : r \to K : J \to M \) with \( \Upsilon \circ \Phi = \Theta \), and if this \( \Upsilon \) is universal.

(2) A cell \( \Phi \) is said to

- preserve (reflect, create) weighted colimits if it preserves (reflects, creates) \( J \)-weighted colimits for any right module \( J \).

- preserve (reflect, create) weighted limits if it preserves (reflects, creates) \( J \)-weighted colimits for any left module \( J \).

**Proposition 11.2.19.** A cell \( \Phi \) preserves (reflects, creates) weighted colimits (resp. weighted limits) if and only if \( \Phi \) preserves (reflects, creates) colimits (resp. limits).

**Proof.** (\( \Rightarrow \)) By Theorem 11.2.15 and since the isomorphism in Corollary 5.5.7 is natural in \( M \), for any category \( E \), if a cell \( \Phi : M \to N \) preserves (reflects, creates) colimits weighted by \( \Delta_E \ast \), then it preserves (reflects, creates) colimits over \( E \).

(\( \Leftarrow \)) By Theorem 11.2.12 and since the equivalence \( (M \mathcal{G}) \simeq (M \mathcal{E}) \) is natural in \( M \) (see Remark 11.1.6(3)), for any right module \( J \), if a cell \( \Phi : M \to N \) preserves (reflects, creates) colimits over the comma category \([J]\), then it preserves (reflects, creates) colimits weighted by \( J \).

**Theorem 11.2.20.**

- For any right module \( M : X \to \ast \), the following conditions are equivalent:
  1. \( M \) is representable;
  2. \( M \) preserves colimits and \( M \) has a colimit (in the sense of Definition 11.2.5) in \( X \);
  3. \( M \) preserves colimits and the right comma fibration \( M^1 : [M] \to X \) has a colimit in \( X \).

When these conditions hold, a conical cell \( \xymatrix{ M \ar[r]^-r & X } \) is universal if and only if the right module morphism \( \Upsilon : M \to (X) r \) is iso; that is, if and only if the pair \((r, \Upsilon)\) forms a representation of \( M \).

- For any left module \( M : \ast \to A \), the following conditions are equivalent:
  1. \( M \) is representable;
  2. \( M \) preserves limits and \( M \) has a limit (in the sense of Definition 11.2.5) in \( A \);
  3. \( M \) preserves limits and the left comma fibration \( M^1 : [M] \to A \) has a limit in \( A \).

When these conditions hold, a conical cell \( \xymatrix{ A \ar[r]^-r & A } \) is universal if and only if the left module morphism \( \Upsilon : M \to r(A) \) is iso; that is, if and only if the pair \((r, \Upsilon)\) forms a representation of \( M \).

**Proof.**

(2) \( \Leftarrow \) (3) By Proposition 11.2.3.
(1) \( \Rightarrow \) (3) \( M \) preserves colimits by Corollary 7.5.9. Since the comma category \([M]\) has a terminal object by Proposition 6.1.4, \( M^1 : [M] \to X \) has a colimit by Corollary 9.1.20.
(3) \( \Rightarrow \) (1) \( M^1 : [M] \to X \) creates a colimit by Corollary 7.6.2; in particular, \( M^1 \) creates a colimit of the identity \([M] \to [M]\) from its own colimit. Hence the comma category \([M]\) has a terminal object by Theorem 9.1.21, and \( M \) is representable by Proposition 6.1.4.
The “if” part of the second assertion is already seen in Proposition 11.2.7. Since a universal conical cell \( \Upsilon: \mathcal{M} \rightrightarrows X \) is the same thing as a direct universal \( (X \leftarrow \Upsilon)_\rightharpoonup \) arrow, the “only if” part follows by applying Theorem 6.2.9 to the right Yoneda functor \( X \leftarrow \_ \).

## 11.3. Extensions

**Note.** Definition 11.3.1 and Definition 11.3.12 give two definitions of universal cells. We will see that they are equivalent in Theorem 11.3.14.

**Definition 11.3.1.**

- A cell \( \begin{tikzcd} J \arrow[r, shift left=0.5em] & E \arrow[l, shift right=0.5em] \end{tikzcd} \rightrightarrows \mathcal{M} \) along \( \mathcal{M} \) is called
  
  (1) universal if it is a direct universal \( (\mathcal{M} \leftarrow E)_\rightharpoonup \) arrow (see Definition 11.1.7);
  
  (2) pointwise universal if each slice

\[
\begin{array}{c}
\mathcal{X} \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{A} \\
\mathcal{E} \rightrightarrows \mathcal{R} \rightrightarrows \mathcal{A}
\end{array}
\]

(see Remark 11.1.8(3)) is a universal conical cell (see Definition 11.2.1), i.e. a direct universal \( (\mathcal{M} \leftarrow) \) arrow.

Given a module \( \mathcal{J}: X \rightrightarrows E \), a universal (resp. pointwise universal) cell \( \Upsilon: \mathcal{J} \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{E} \) or the pair \( (\mathcal{R}, \Upsilon) \), or the functor \( \mathcal{R} \) itself, is called an extension (resp. pointwise extension) of \( \mathcal{J} \) direct along \( \mathcal{M} \).

- A cell \( \begin{tikzcd} \mathcal{R} \arrow[r, shift left=0.5em] & \mathcal{E} \arrow[l, shift right=0.5em] \end{tikzcd} \rightrightarrows \mathcal{M} \) along \( \mathcal{M} \) is called

\[
\begin{array}{c}
\mathcal{X} \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{A} \\
\mathcal{E} \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{A}
\end{array}
\]

(see Remark 11.1.8(3)) is a universal conical cell (see Definition 11.2.1), i.e. an inverse universal \( (\mathcal{M} \leftarrow) \) arrow.

Given a module \( \mathcal{J}: E \rightrightarrows A \), a universal (resp. pointwise universal) cell \( \Upsilon: \mathcal{R} \rightrightarrows \mathcal{J}: \mathcal{E} \rightrightarrows \mathcal{M} \) or the pair \( (\mathcal{R}, \Upsilon) \), or the functor \( \mathcal{R} \) itself, is called an extension (resp. pointwise extension) of \( \mathcal{J} \) inverse along \( \mathcal{M} \).

**Remark 11.3.2.** A cell \( \Phi: \mathcal{J} \rightrightarrows \mathcal{R}: \mathcal{M} \rightrightarrows \mathcal{E} \) is universal if and only if to every cell \( \Phi: \mathcal{J} \rightrightarrows \mathcal{K}: \mathcal{M} \rightrightarrows \mathcal{E} \) there is a unique natural transformation \( \Upsilon \downarrow \Phi: \mathcal{R} \rightrightarrows \mathcal{K} \) such that \( \Phi = \Upsilon \circ \Upsilon \downarrow \Phi \). Dually, a cell \( \Phi: \mathcal{R} \rightrightarrows \mathcal{J}: \mathcal{E} \rightrightarrows \mathcal{M} \) is universal if and only if to every cell \( \Phi: \mathcal{K} \rightrightarrows \mathcal{J}: \mathcal{E} \rightrightarrows \mathcal{M} \) there is a unique natural transformation \( \Phi \downarrow \Upsilon: \mathcal{K} \rightrightarrows \mathcal{R} \) such that \( \Phi = \Phi \downarrow \Upsilon \circ \Upsilon \).

**Note.** The isomorphism in Remark 11.1.8(3) yields the following.

**Proposition 11.3.3.**

- A cell

\[
\begin{array}{c}
\mathcal{J} \rightrightarrows \mathcal{E} \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{A} \\
\mathcal{X} \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{A}
\end{array}
\]

is universal (resp. pointwise universal) if and only if its right exponential transpose

\[
\begin{array}{c}
\mathcal{J} \rightrightarrows \mathcal{E} \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{A} \\
[X:] \rightrightarrows \mathcal{M} \rightrightarrows \mathcal{A}
\end{array}
\]
forms a left Kan lift (resp. pointwise left Kan lift) of $J \vartriangleright$ along $M \vartriangleright$, i.e. if and only if the cylinder

\[
\begin{array}{c}
J \\
\vartriangleright \\
\downarrow \\
M \\
\vartriangleright \\
\downarrow \\
\uparrow
\end{array}
\xrightarrow{E}
\begin{array}{c}
X \\
\downarrow \\
M \\
\downarrow \\
\uparrow \\
A
\end{array}
\]

is direct universal (resp. pointwise direct universal).

- A cell

\[
\begin{array}{c}
\downarrow \\
\vartriangleleft \\
\downarrow \\
\vartriangleleft \\
\uparrow
\end{array}
\xrightarrow{E}
\begin{array}{c}
R \\
\downarrow \\
J \\
\downarrow \\
M \\
\downarrow \\
\uparrow
\end{array}
\xrightarrow{X}
\begin{array}{c}
\downarrow \\
\vartriangleleft \\
\downarrow \\
\vartriangleleft \\
\uparrow
\end{array}
\xrightarrow{A}
\]

is universal (resp. pointwise universal) if and only if its left exponential transpose

\[
\begin{array}{c}
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\uparrow
\end{array}
\xrightarrow{E}
\begin{array}{c}
R \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\uparrow
\end{array}
\xrightarrow{X}
\begin{array}{c}
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\uparrow
\end{array}
\xrightarrow{[: A]^-}
\]

forms a right Kan lift (resp. pointwise right Kan lift) of $\vartriangleright J$ along $\vartriangleright M$, i.e. if and only if the cylinder

\[
\begin{array}{c}
J \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\uparrow
\end{array}
\xrightarrow{E}
\begin{array}{c}
R \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\uparrow
\end{array}
\xrightarrow{X}
\begin{array}{c}
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\uparrow
\end{array}
\xrightarrow{[: A]^-}
\]

is inverse universal (resp. pointwise inverse universal).

**Proof.** By the isomorphism in Remark 11.1.8(3), $\Upsilon$ is universal iff the cylinder $\Upsilon \vartriangleright$ is direct universal. Since the slice of $\Upsilon$ at each $e \in \|E\|$ is given by the component of $\Upsilon \vartriangleright$ at $e$, $\Upsilon$ is pointwise universal iff the cylinder $\Upsilon \vartriangleright$ is pointwise direct universal.

**Remark 11.3.4.** By Proposition 11.3.3 and by the bijectiveness of exponential transposition, we see that

- an extension (resp. pointwise extension) of $J$ direct along $M$ is the same thing as a left Kan lift (resp. pointwise left Kan lift) of $J \vartriangleright$ along $M \vartriangleright$, i.e. a lift (resp. pointwise lift) of $J \vartriangleright$ direct along $M \vartriangleright$.
- an extension (resp. pointwise extension) of $J$ inverse along $M$ is the same thing as a right Kan lift (resp. pointwise right Kan lift) of $\vartriangleright J$ along $\vartriangleright M$, i.e. a lift (resp. pointwise lift) of $\vartriangleright J$ inverse along $\vartriangleright M$.

Hence lifts (resp. pointwise lifts), in fact Kan lifts (resp. pointwise Kan lifts), subsume extensions (resp. pointwise extensions). We will see in Corollary 11.3.18 that the converse is also the case.

**Proposition 11.3.5.** A pointwise universal cell is universal.

**Proof.** This is reduced to Proposition 6.5.8 by the equivalence of conditions in Proposition 11.3.3.

**Theorem 11.3.6.** Let $J$ and $M$ be modules as in Definition 11.3.1.

- If there is a family of universal conical cells

\[
\begin{array}{c}
\downarrow \\
\vartriangleleft \\
\downarrow \\
\vartriangleleft \\
\uparrow
\end{array}
\xrightarrow{E}
\begin{array}{c}
R \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\uparrow
\end{array}
\xrightarrow{X}
\begin{array}{c}
\downarrow \\
\vartriangleleft \\
\downarrow \\
\vartriangleleft \\
\uparrow
\end{array}
\xrightarrow{[: A]^-}
\]

, one for each object $e \in \|E\|$, then there is a unique cell

\[
\begin{array}{c}
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\uparrow
\end{array}
\xrightarrow{E}
\begin{array}{c}
R \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\downarrow \\
\uparrow
\end{array}
\xrightarrow{X}
\begin{array}{c}
\downarrow \\
\vartriangleright \\
\downarrow \\
\vartriangleright \\
\uparrow
\end{array}
\xrightarrow{: A}
\]

such that $r_e = R \cdot e$ and $(\Upsilon) e = \Upsilon e$, and $\Upsilon$ is pointwise universal.
If there is a family of universal conical cells

\[
\xymatrix{
& e(J) \\
X & & M \\
& A
}
\]

, one for each object \( e \in \|E\| \), then there is a unique cell

\[
\xymatrix{
& E \\
X & & M \\
& A
}
\]

such that \( e : R = r_e \) and \( e(\Upsilon) = \Upsilon_e \), and \( \Upsilon \) is pointwise universal.

Proof. By the equivalence of conditions in Proposition 11.3.3, this is reduced to an instance of Theorem 6.5.10 where \( M \) is given by the module \( M \nabla \) and \( K \) is given by the functor \( J \nabla \).

Theorem 11.3.7. Consider a cell \( \Upsilon \) as in Definition 11.3.1 and let \( D \) be an essentially wide subcategory of \( E \). Then \( \Upsilon \) is pointwise universal if and only if its right (resp. left) slice at each \( d \in \|D\| \) is universal.

Proof. Since the slice of \( \Upsilon \) at each \( e \in \|E\| \) is given by the component of \( \Upsilon \nabla \) at \( e \) (see Remark 11.1.8(3)), the assertion is reduced to Theorem 6.5.18.

Note. The following definition is a special case of Definition 11.3.1 where \( M \) is given by the hom of a category.

Definition 11.3.8.

- A cell \( \xymatrix{
X & \ar[l]_R \\
A & \ar[l]_Y \\
& M
} \) is called

  (1) universal if it is a direct universal \( (X \nabla A) \)-arrow (see Definition 11.1.9);
  (2) pointwise universal if each slice

\[
\xymatrix{
& \ar[r]_{(M)a} & \ar[r]_{(\Upsilon)a} & \ar[r]_{R:a} & \\
& \ar[r]_{(X)} & \ar[r]_{(\Upsilon)} & \ar[r]_{A} & \\
& \ar[r]_{A} & \ar[r]_{A} & \ar[r]_{A}
}
\]

is a universal conical cell (see Definition 11.2.5), i.e. a direct universal \( (X \nabla \) -arrow.

Given a module \( M : X \to A \), a universal (resp. pointwise universal) cell \( \Upsilon : J \nabla R : M \nabla E \) or the pair \( (R, \Upsilon) \), or the functor \( R \) itself, is called a left extension (resp. pointwise left extension) of \( M \).

- A cell \( \xymatrix{
A & \ar[l]_R \\
X & \ar[l]_Y \\
& M
} \) is called

  (1) universal if it is an inverse universal \( (X \nabla A) \)-arrow (see Definition 11.1.9);
  (2) pointwise universal if each slice

\[
\xymatrix{
& \ar[r]_{x(M)} & \ar[r]_{x(\Upsilon)} & \ar[r]_{x(R)} & \\
& \ar[r]_{x(A)} & \ar[r]_{x(A)} & \ar[r]_{x(A)} & \\
& \ar[r]_{A} & \ar[r]_{A} & \ar[r]_{A}
}
\]

is a universal conical cell (see Definition 11.2.5), i.e. an inverse universal \( (X \nabla A) \)-arrow.

Given a module \( M : X \to A \), a universal (resp. pointwise universal) cell \( \Upsilon : R \nabla M : X \nabla A \) or the pair \( (R, \Upsilon) \), or the functor \( R \) itself, is called a right extension (resp. pointwise right extension) of \( M \).
11.3. Extensions

Remark 11.3.9. A left extension (resp. pointwise left extension) of a module \( M : X \to A \) is the same thing as an extension (resp. pointwise extension) of \( M \) direct along the hom endomodule \( (X) \). Dually, a right extension (resp. pointwise right extension) of a module \( M : X \to A \) is the same thing as an extension (resp. pointwise extension) of \( M \) inverse along the hom endomodule \( (A) \).

Proposition 11.3.10. Let \( M : X \to A \) be a module.

1. A corepresentation \((R, \Upsilon)\) of \( M \) gives a pointwise left extension of \( M \) (cf. Remark 2.3.11(3)).

2. A representation \((R, \Upsilon)\) of \( M \) gives a pointwise right extension of \( M \) (cf. Remark 2.3.11(3)).

Proof. By definition, \( \Upsilon \) is a pointwise left extension of \( M \) iff for each \( a \in \|A\| \), the slice

\[
\xymatrix{ (M)_a 
\ar[r]^-* \ar[d]_-{\tau} & \ar[r]^-{R\cdot a} & X \ar[d]^-{(X)} \ar[r]^-{\Upsilon} & A \ar[d]^-{(A)} }
\]

is a universal conical cell, and by Proposition 2.3.12, \((R, \Upsilon)\) is a corepresentation of \( M \) iff for each \( a \in \|A\| \), \((R\cdot a, (\Upsilon)_a)\) is a representation of \((M)_a\). The assertion is thus reduced to Proposition 11.2.7. \( \square \)

Remark 11.3.11. The converse does not hold as we saw in Remark 11.2.8.

Definition 11.3.12.

1. A cell \( \xymatrix{ D \ar[r]^-J & E \ar[r]^-{\kappa} & \ar[r]^-{\tau} & A \ar[r]^-{R} & X \ar[r]^-{\Upsilon} & A \ar[r]^-{M} & \ast \ar[r]^-{\eta} & \ast } \) along \( \langle J, M \rangle \) is called

(1) direct universal if it is a direct universal \( \langle J, M \rangle \)-arrow (see Definition 1.2.8);

(2) pointwise direct universal if each right slice

\[
\xymatrix{ D \ar[r]^-{(J) e} \ar[d]_-{\kappa} & \ar[r]^-{\tau e} & \ar[r]^-{R e} & X \ar[r]^-{\Upsilon} & A \ar[r]^-{M} & \ast \ar[r]^-{\eta} & \ast }
\]

(see Definition 2.1.8) is a universal conical cell (see Definition 11.2.9), i.e. a direct universal \( \langle (J) e, M \rangle \)-arrow.

Given a functor \( K : D \to X \), a direct universal (resp. pointwise direct universal) cell \( \Upsilon : K \to R : J \to M \) or the pair \((R, \Upsilon)\), or the functor \( R \) itself, is called an extension (resp. pointwise extension) of \( K \) direct along \( \langle J, M \rangle \).

2. A cell \( \xymatrix{ E \ar[r]^-J & D \ar[r]^-{\kappa} & \ar[r]^-{\tau} & A \ar[r]^-{M} & \ast \ar[r]^-{\eta} & \ast } \) along \( \langle J, M \rangle \) is called

(1) inverse universal if it is an inverse universal \( \langle J, M \rangle \)-arrow (see Definition 1.2.8);

(2) pointwise inverse universal if each left slice

\[
\xymatrix{ \ast \ar[r]^-{e(J)} \ar[d]_-{\kappa} & \ar[r]^-{\tau e} & \ar[r]^-{e\tau} & X \ar[r]^-{\Upsilon} & A \ar[r]^-{M} & \ast \ar[r]^-{\eta} & \ast }
\]

(see Definition 2.1.8) is a universal conical cell (see Definition 11.2.9), i.e. an inverse universal \( \langle e(J), M \rangle \)-arrow.
Given a functor \( K : D \to A \), an inverse universal (resp. pointwise inverse universal) cell \( \Upsilon : R \to K : J \to M \) or the pair \((R, \Upsilon)\), or the functor \( R \) itself, is called an extension (resp. pointwise extension) of \( K \) inverse along \((J, M)\).

**Remark 11.3.13.**

(1) A cell \( \Phi : K \to L : J \to M \) is direct universal if and only if to every cell \( \Phi : K \to L : J \to M \) there is a unique natural transformation \( \Upsilon : \Theta \) such that \( \Phi = \Theta \circ \Upsilon \). Dually, a cell \( \Phi : R \to K : J \to M \) is inverse universal if and only if to every cell \( \Phi : L \to K : J \to M \) there is a unique natural transformation \( \Phi = \Phi \circ \Upsilon \).

(2) As a special case where \( M \) is given by the hom of a category \( C \), a direct universal (resp. pointwise universal) cell \( D \to E \) or the pair \((R, \Upsilon)\), or the functor \( R \) itself, is called an extension (resp. pointwise extension) of \( K \) direct along \( J \).

**Theorem 11.3.14.**

- A cell

\[
\begin{array}{c}
D \xrightarrow{J} E \\
\downarrow K \downarrow \Upsilon \downarrow R \\
X \xrightarrow{M} A
\end{array}
\]

from \( K \) to \( R \) along \((J, M)\) is direct universal (resp. pointwise direct universal) if and only if the cell

\[
\begin{array}{c}
D \xrightarrow{J} E \\
\downarrow \Upsilon \downarrow K \downarrow M \\
A
\end{array}
\]

from \( J \) to \( R \) along \( K(M) \) is universal (resp. pointwise universal); hence an extension (resp. pointwise extension) of \( K \) direct along \((J, M)\) is the same thing as an extension (resp. pointwise extension) of \( J \) direct along \( K(M) \). Conversely, a cell

\[
\begin{array}{c}
X \xrightarrow{J} E \\
\downarrow \Upsilon \downarrow M \\
A
\end{array}
\]

from \( J \) to \( R \) along \( M \) is universal (resp. pointwise universal) if and only if the cell

\[
\begin{array}{c}
X \xrightarrow{J} E \\
\downarrow \Upsilon \downarrow R \\
A
\end{array}
\]

from the identity \( X \to X \) to \( R \) along \((J, M)\) is direct universal (resp. pointwise direct universal); hence an extension (resp. pointwise extension) of \( J \) direct along \( M \) is the same thing as an extension (resp. pointwise extension) of the identity \( X \to X \) direct along \((J, M)\).
from $\mathbb{R}$ to $\mathbb{K}$ along $(\mathcal{J}, \mathcal{M})$ is inverse universal (resp. pointwise inverse universal) if and only if the cell

$$\begin{array}{c}
\mathbb{R} \xymatrix{
\mathcal{E} & \\
\mathcal{J} & \mathcal{D}
}
\end{array}$$

from $\mathbb{R}$ to $\mathcal{J}$ along $\mathcal{M}|\mathbb{K}$ is universal (resp. pointwise universal); hence an extension (resp. pointwise extension) of $\mathbb{K}$ inverse along $(\mathcal{J}, \mathcal{M})$ is the same thing as an extension (resp. pointwise extension) of $\mathcal{J}$ inverse along $\mathcal{M}|\mathbb{K}$. Conversely, a cell

$$\begin{array}{c}
\mathbb{R} \xymatrix{
\mathcal{E} & \\
\mathcal{J} & \mathcal{A}
}
\end{array}$$

from $\mathbb{R}$ to $\mathcal{J}$ along $\mathcal{M}$ is universal (resp. pointwise universal) if and only if the cell

$$\begin{array}{c}
\mathbb{R} \xymatrix{
\mathcal{E} & \\
\mathcal{J} & \mathcal{A}
}
\end{array}$$

from $\mathbb{R}$ to the identity $\mathcal{A} \to \mathcal{A}$ along $(\mathcal{J}, \mathcal{M})$ is inverse universal (resp. pointwise inverse universal); hence an extension (resp. pointwise extension) of $\mathcal{J}$ inverse along $\mathcal{M}$ is the same thing as an extension (resp. pointwise extension) of the identity $\mathcal{A} \to \mathcal{A}$ inverse along $(\mathcal{J}, \mathcal{M})$.

Proof. The left slice of the module $(\mathcal{J}, \mathcal{M})$ at $\mathbb{K}$ and the left slice of the module $(\mathcal{K}(\mathcal{M})) \not\!\setminus \!\mathcal{E}$ at $\mathcal{J}$ are the same left module over $[\mathcal{E}, \mathcal{A}]$ given by

$$\mathbb{R} \mapsto (\mathcal{J})(\mathcal{D} : \mathcal{E})(\mathcal{K}(\mathcal{M}) \mathcal{R})$$

. Hence the cell $\mathcal{D} \xymatrix{
\mathcal{J} & \mathcal{E} & \\
\mathcal{R} & \mathcal{A} & \\
\mathcal{M} & \mathcal{A}
}$ is direct universal iff the cell $\mathcal{D} \xymatrix{
\mathcal{J} & \mathcal{E} & \\
\mathcal{R} & \mathcal{A} & \\
\mathcal{M} & \mathcal{A}
}$ is universal. Since, by

Theorem 11.2.11, the slice $\mathcal{D} \xymatrix{
\mathcal{J} & \mathcal{E} & \\
\mathcal{R} & \mathcal{A} & \\
\mathcal{M} & \mathcal{A}
}$ of the cell $\mathcal{D} \xymatrix{
\mathcal{J} & \mathcal{E} & \\
\mathcal{R} & \mathcal{A} & \\
\mathcal{M} & \mathcal{A}
}$ at $\mathcal{e} \in [\mathcal{E}]$ is universal iff the slice $\mathcal{D} \xymatrix{
\mathcal{J} & \mathcal{E} & \\
\mathcal{R} & \mathcal{A} & \\
\mathcal{M} & \mathcal{A}
}$ at $\mathcal{e} \in [\mathcal{E}]$ is universal, the pointwise version follows.

The left slice of the module $\mathcal{M} \not\!\setminus \!\mathcal{E}$ at $\mathcal{J}$ and the left slice of the module $(\mathcal{J}, \mathcal{M})$ at $1\mathcal{X}$ are the same left module over $[\mathcal{E}, \mathcal{A}]$ given by

$$\mathbb{R} \mapsto (\mathcal{J})(\mathcal{X} : \mathcal{E})(\mathcal{M} \mathcal{R})$$

. Hence the cell $\mathcal{J} \xymatrix{
\mathcal{E} & \\
\mathcal{R} & \mathcal{A}
}$ is universal iff the cell $\mathcal{J} \xymatrix{
\mathcal{E} & \\
\mathcal{R} & \mathcal{A}
}$ is direct universal. Since, by

Theorem 11.2.11, the slice $\mathcal{J} \xymatrix{
\mathcal{E} & \\
\mathcal{R} & \mathcal{A}
}$ of the cell $\mathcal{J} \xymatrix{
\mathcal{E} & \\
\mathcal{R} & \mathcal{A}
}$ at $\mathcal{e} \in [\mathcal{E}]$ is universal iff the slice $\mathcal{J} \xymatrix{
\mathcal{E} & \\
\mathcal{R} & \mathcal{A}
}$ at $\mathcal{e} \in [\mathcal{E}]$ is universal, the pointwise version follows. $\square$
Note. Theorem 11.3.15 and Theorem 11.3.6 are reduced to each other by the equivalence of conditions in Theorem 11.3.14 and that in Theorem 11.2.11.

**Theorem 11.3.15.** Let \( J \) and \( M \) be modules and \( K \) be a functor as in Definition 11.3.12.

- If there is a family of universal conical cells
  
  \[
  \begin{array}{c}
  \text{D} \xrightarrow{(J)_{e}} * \\
  \text{K} \xrightarrow{\tau_{e}} \text{r}_{e} \\
  \text{X} \xrightarrow{\mathcal{M}} A
  \end{array}
  \]

  , one for each object \( e \in \| \mathcal{E} \| \), then there is a unique cell
  
  \[
  \begin{array}{c}
  \text{D} \xrightarrow{J} \text{E} \\
  \text{K} \xrightarrow{\tau} \text{r} \\
  \text{X} \xrightarrow{\mathcal{M}} A
  \end{array}
  \]

  such that \( \text{r}_{e} = \text{R} \cdot e \) and \( \langle \mathcal{T} \rangle e = \tau_{e} \), and \( \mathcal{T} \) is pointwise direct universal.

- If there is a family of universal conical cells
  
  \[
  \begin{array}{c}
  * \xrightarrow{e(J)} \text{D} \\
  \text{r}_{e} \xrightarrow{\tau_{e}} \text{r} \\
  \text{X} \xrightarrow{\mathcal{M}} A
  \end{array}
  \]

  , one for each object \( e \in \| \mathcal{E} \| \), then there is a unique cell
  
  \[
  \begin{array}{c}
  \text{E} \xrightarrow{J} \text{D} \\
  \text{r} \xrightarrow{\tau} \text{r} \\
  \text{X} \xrightarrow{\mathcal{M}} A
  \end{array}
  \]

  such that \( e \cdot \text{R} = \text{r}_{e} \) and \( e \langle \mathcal{T} \rangle = \tau_{e} \), and \( \mathcal{T} \) is pointwise direct universal.

**Proof.** As noted above, this is reduced to Theorem 11.3.6. \( \square \)

**Note.** The axiom of choice is used in the proof of the following.

**Corollary 11.3.16.** Let \( J \) and \( M \) be modules and \( K \) be a functor as in Definition 11.3.12. If \( M \) is cocomplete (resp. complete) and \( D \) is small, then \( K \) has a pointwise extension direct (resp. inverse) along \( (J, M) \).

**Proof.** By Corollary 11.2.14, \( K \) has a colimit along \( M \) weighted by \( (J)_{e} \) for any \( e \in \| \mathcal{E} \| \), and by Theorem 11.3.15, a family of universal conical cells \( \tau_{e} : K \sim r_{e} : (J)_{e} \to M \), one chosen for each \( e \in \| \mathcal{E} \| \), extends to a pointwise direct universal cell \( \tau : K \sim R : J \to M \). \( \square \)

**Theorem 11.3.17.**

- Consider a cylinder
  
  \[
  \begin{array}{c}
  \text{D} \xrightarrow{F} \text{E} \\
  \text{K} \xrightarrow{\mu} \text{r} \\
  \text{X} \xrightarrow{\mathcal{M}} A
  \end{array}
  \]

  weighted by a fully faithful functor \( F \). If the cell
  
  \[
  \begin{array}{c}
  \text{D} \xrightarrow{F(E)} \text{E} \\
  \text{K} \xrightarrow{\mu E} \text{r} \\
  \text{X} \xrightarrow{\mathcal{M}} A
  \end{array}
  \]

  generated by \( E \) inverse along \( \mu \) (see Example 5.3.5(3)) is pointwise direct universal, so is \( \mu \). The converse holds if \( F \) is fully faithful and essentially surjective, i.e. if \( F \) is an equivalence.
Consider a cylinder
\[ \begin{array}{c}
E \xleftarrow{\mu} D \\
\downarrow R \\
X \xrightarrow{\mu} A
\end{array} \]
weighted by a fully faithful functor \( F \). If the cell
\[ \begin{array}{c}
E \xrightarrow{(E)F} D \\
\downarrow R \mu \\
X \xrightarrow{\mu} A
\end{array} \]
generated by \( E \) direct along \( \mu \) (see Example 5.3.5(3)) is pointwise inverse universal, so is \( \mu \). The converse holds if \( F \) is fully faithful and essentially surjective, i.e. if \( F \) is an equivalence.

**Proof.** By the equivalence of conditions in Theorem 11.3.14, this follows as a special case of the lemma below where \( \mathcal{M} \) is given by the composite module \( K \mathcal{M} \). (Conversely, the lemma is a special case of the theorem where \( K \) is an identity.)

**Lemma.** Consider a cylinder
\[ \begin{array}{c}
X \xrightarrow{F} E \\
\downarrow 1 \\
X \xrightarrow{\mu} A
\end{array} \]
weighted by a fully faithful functor \( F \). If the cell
\[ \begin{array}{c}
X \xrightarrow{F(E)} E \\
\downarrow 1 \mu|E \\
X \xrightarrow{\mu} A
\end{array} \]
generated by \( E \) inverse along \( \mu \) is pointwise universal, so is \( \mu \). The converse holds if \( F \) is fully faithful and essentially surjective, i.e. if \( F \) is an equivalence.

**Proof.** The first assertion follows immediately from the claim below. The second assertion follows from the claim and Theorem 11.3.7.

**Claim.** For any \( x \in \| X \| \), the right slice
\[ \begin{array}{c}
\xrightarrow{\langle A \rangle R} \xrightarrow{\langle A \rangle \mu|E} \xrightarrow{R \circ F \cdot x} \xrightarrow{\langle A \rangle x} \xrightarrow{\langle A \rangle R} \xrightarrow{\langle A \rangle \mu|E} \xrightarrow{R \circ F \cdot x}
\end{array} \]
of \( \mu|E \) at \( F \cdot x \) is a universal conical cell iff the the component \( \mu_x : x \sim R \circ F \cdot x \) of \( \mu \) at \( x \) is a direct universal \( \mathcal{M} \)-arrow.

**Proof.** Depict \( \mu \) as \( \begin{array}{c}
\xrightarrow{\langle A \rangle R} \xrightarrow{\langle A \rangle \mu F} \xrightarrow{R \circ F \cdot x} \xrightarrow{\langle A \rangle x} \xrightarrow{\langle A \rangle R} \xrightarrow{\langle A \rangle \mu F} \xrightarrow{R \circ F \cdot x}
\end{array} \), and apply Proposition 5.3.6 and Corollary 5.3.12 to these cylinders respectively. Then we have a commutative diagram
\[ \begin{array}{c}
\xrightarrow{(F)x} \\
\xrightarrow{(X)|x} \xrightarrow{\langle A \rangle \mu|E} \\
\xrightarrow{(X)|x} \xrightarrow{\langle A \rangle \mu|E} \\
\xrightarrow{(X)|x} \xrightarrow{\langle A \rangle \mu|E} \\
\xrightarrow{(X)|x} \xrightarrow{\langle A \rangle \mu|E}
\end{array} \]
(the upper right triangle commutes by Proposition 5.3.6, and the lower left triangle commutes by Corollary 5.3.12). Since $F$ is fully faithful, $\langle F \rangle (x)$ is iso. Hence $\langle \mu \rangle (E) (F \cdot x) : F (E) (F \cdot x) \sim R \cdot F \cdot x$ is a direct universal $\langle M \rangle$-arrow iff so is $X \langle X \rangle \cdot (X) \cdot x \sim R \cdot F \cdot x$, and by Theorem 11.2.4, this is the case iff $\mu : x \sim R \cdot F \cdot x$ is a direct universal $\langle M \rangle$-arrow.

**Note.** The bijective correspondence between cylinders and cells stated in Corollary 5.5.5 gives the following result.

**Corollary 11.3.18.**

- A cylinder

\[
\begin{array}{c}
K \\
E \\
\downarrow R \\
X \\
\mu \\
\downarrow M \\
A
\end{array}
\]

is direct universal (resp. pointwise direct universal) if and only if so is the cell

\[
\begin{array}{c}
E \\
\downarrow \langle E \rangle \\
K \\
\downarrow (\mu) \\
X \\
\downarrow M \\
A
\end{array}
\]

- A cylinder

\[
\begin{array}{c}
R \\
E \\
\downarrow K \\
X \\
\mu \\
\downarrow M \\
A
\end{array}
\]

is inverse universal (resp. pointwise inverse universal) if and only if so is the cell

\[
\begin{array}{c}
E \\
\downarrow \langle E \rangle \\
R \\
\downarrow (\mu) \\
X \\
\downarrow M \\
A
\end{array}
\]

**Proof.** By the isomorphism in Corollary 5.5.5, the cylinder $\mu$ is universal iff so is the cell $\langle \mu \rangle$. The pointwise version follows from Theorem 11.3.17 by replacing the functor $F : D \to E$ with the identity $E \to E$.

**Remark 11.3.19.** By Corollary 11.3.18 and by the bijectiveness of the assignment $\mu \mapsto \langle \mu \rangle$, we see that given a module $\langle M \rangle : X \to A$,

- a lift (resp. pointwise lift) of a functor $K : E \to X$ direct along $\langle M \rangle$ is the same thing as an extension (resp. pointwise extension) of $K$ direct along $\langle \langle E \rangle, \langle M \rangle \rangle$.
- a lift (resp. pointwise lift) of a functor $K : E \to A$ inverse along $\langle M \rangle$ is the same thing as an extension (resp. pointwise extension) of $K$ inverse along $\langle \langle E \rangle, \langle M \rangle \rangle$.

Extensions (resp. pointwise extensions) thus subsume lifts (resp. pointwise lifts), and vice versa as we saw in Remark 11.3.4.

**Theorem 11.3.20.** Let $\langle M \rangle : X \to A$ be a module.

- The following conditions are equivalent:
  1. $\langle M \rangle$ is corepresentable;
  2. each right slice $\langle M \rangle a : X \to *$ preserves colimits and $\langle M \rangle$ has a pointwise left extension (see Definition 11.3.8).
When these conditions hold, a cell \( X \rightarrow \langle X \rangle R \) is pointwise universal if and only if the module morphism \( \Upsilon : M \rightarrow \langle X \rangle R \) is iso; that is, if and only if the pair \((R, \Upsilon)\) forms a corepresentation of \( M \).

- The following conditions are equivalent:
  1. \( M \) is representable;
  2. each left slice \( x(M) : * \rightarrow A \) preserves limits and \( M \) has a pointwise right extension (see Definition 11.3.8).

When these conditions hold, a cell \( X \rightarrow M \rightarrow \langle \langle A \rangle \rangle \rightarrow A \) is pointwise universal if and only if the module morphism \( \Upsilon : M \rightarrow R \langle \langle A \rangle \rangle \) is iso; that is, if and only if the pair \((R, \Upsilon)\) forms a representation of \( M \).

Proof. By Corollary 6.4.11, \( M \) is corepresentable iff each right slice \( \langle M \rangle a : X \rightarrow \ast \) is representable. By Theorem 11.3.6, \( M \) has a pointwise left extension iff each right slice \( \langle M \rangle a : X \rightarrow \ast \) has a colimit. The first assertion is thus reduced to Theorem 11.2.20. The “if” part of the second assertion is already seen in Proposition 11.3.10. Since pointwise universal cell is universal (see Proposition 11.3.5) and a universal cell \( \Upsilon : M \rightarrow R \langle \langle A \rangle \rangle \rightarrow A \) is the same thing as a direct universal \( \langle \langle \langle A \rangle \rangle \rangle \rightarrow A \)-arrow, the “only if” part follows by applying Theorem 6.2.9 to the right general Yoneda functor \( X \rightarrow A \).

Theorem 11.3.21. Let \( \Phi \) be a cell from a module \( M \) to a module \( N \).

- Suppose that \( \Phi : M \rightarrow N \) has a right adjoint (see Definition 8.9.1). Then it preserves direct extensions; that is, each direct universal cell \( \Upsilon : J \rightarrow M \) along \( M \) yields by composition with \( \Phi \) a direct universal cell \( \Upsilon \circ \Phi : J \rightarrow N \) along \( N \).

- Suppose that \( \Phi : M \rightarrow N \) has a left adjoint (see Definition 8.9.1). Then it preserves inverse extensions; that is, each inverse universal cell \( \Upsilon : J \rightarrow M \) along \( M \) yields by composition with \( \Phi \) an inverse universal cell \( \Upsilon \circ \Phi : J \rightarrow N \) along \( N \).

Proof. Since a direct universal cell \( \Upsilon : J \rightarrow M \) is the same thing as a direct universal \( \langle J, M \rangle \)-arrow, the assertion is equivalent to saying that if a cell \( \Phi \) has a right adjoint, then the postcomposition cell \( \langle J, \Phi \rangle \) (see Definition 1.2.23) preserves direct universal arrows. But since \( \langle J, \Phi \rangle \) has a right adjoint as well by Theorem 8.9.3, this follows from Theorem 8.9.5.

11.4. Kan extensions

Note. The isomorphism in Theorem 5.5.3 allows the following definition.

Definition 11.4.1. Given a pair of functors \( C \leftarrow F \rightarrow D \rightarrow E \),

- a natural transformation

\[
\begin{array}{ccc}
D & \xrightarrow{F} & E \\
\mu \downarrow & \mu \downarrow & \mu \downarrow \\
C & \xrightarrow{(C)} & C
\end{array}
\]

from \( K \rightarrow R \circ F \) or the pair \((R, \mu)\), or the functor \( R \) itself, is called a left Kan extension (resp. pointwise left Kan extension) of \( K \) along \( F \) if the cell

\[
\begin{array}{ccc}
D & \xrightarrow{F(E)} & E \\
\mu \downarrow & \mu \downarrow & \mu \downarrow \\
C & \xrightarrow{(C)} & C
\end{array}
\]

generated by \( E \) inverse along \( \mu \) is direct universal (resp. pointwise direct universal).
11.4. Kan extensions

- a natural transformation

\[
\begin{array}{c}
E \xrightarrow{F} D \\
\mu \downarrow \quad \downarrow K \\
C \xrightarrow{(C)} C
\end{array}
\]

from \( F \circ R \) to \( K \) or the pair \((R, \mu)\), or the functor \( R \) itself, is called a right Kan extension (resp. pointwise right Kan extension) of \( K \) along \( F \) if the cell

\[
\begin{array}{c}
E \xrightarrow{(E)F} D \\
\mu \downarrow \quad \downarrow K \\
C \xrightarrow{(C)} C
\end{array}
\]

generated by \( E \) direct along \( \mu \) is inverse universal (resp. pointwise inverse universal).

Remark 11.4.2.

(1) By the isomorphism in Theorem 5.5.3, a natural transformation \( \mu : K \to R \circ F \) forms a direct universal arrow \( \mu : K \Rightarrow R \) of the module \( F \Rightarrow C \) (see Remark 4.5.4(3)) if and only if the cell \( \mu \downarrow E \) is direct universal. Hence a (not necessarily pointwise) left Kan extension of \( K \) along \( F \) may be defined, as in the literature, more directly as a direct universal natural transformation \( \alpha \) that \( \mu = \mu \circ [F \circ \alpha] \) is the pasting composite of \( \mu \) and \( \alpha \). Dually, a right Kan extension of \( K \) along \( F \) may be defined as an inverse universal natural transformation \( \alpha \) such that \( \alpha = [F \circ \alpha] \circ \mu \).

Proposition 11.4.3. Given functors as in Definition 11.4.1,

- the following conditions are equivalent:
  (1) \( R \) is a left Kan extension (resp. pointwise left Kan extension) of \( K \) along \( F \);
  (2) \( R \) is an extension (resp. pointwise extension) \( D \xrightarrow{F(E)} E \) of \( K \) direct along the representable module \( F(E) \);
  (3) \( R \) is an extension (resp. pointwise extension) \( E \xrightarrow{(E)F} D \) of \( K \) inverse along the corepresentable module \( (E)F \);

- the following conditions are equivalent:
  (1) \( R \) is a right Kan extension (resp. pointwise right Kan extension) of \( K \) along \( F \);
  (2) \( R \) is an extension (resp. pointwise extension) \( E \xrightarrow{(E)F} D \) of \( K \) inverse along the corepresentable module \( (E)F \);
  (3) \( R \) is an extension (resp. pointwise extension) \( E \xrightarrow{(E)F} D \) of \( K \) inverse along the corepresentable module \( (E)F \).
11.4. Kan extensions

Proof.

(1) \(\Rightarrow\) (2) By definition.

(1) \(\Leftarrow\) (2) If \(R\) and \(\Upsilon\) form an extension (resp. pointwise extension) of \(K\) direct along \(F\), then \(R\) and \(\Upsilon|E\) (see Theorem 5.5.3) form a left Kan extension (resp. pointwise left Kan extension) of \(K\) along \(F\).

(2) \(\Leftrightarrow\) (3) This is Theorem 11.3.14.

Remark 11.4.4. A Kan extension is thus regarded as a special instance of an extension as defined in Definition 11.3.12 where \(J\) is representable and \(M\) is the hom of a category, or as a special instance of an extension as defined in Definition 11.3.1 where both \(J\) and \(M\) are representable.

Proposition 11.4.5. A pointwise Kan extension is a Kan extension.

Proof. This follows from Proposition 11.3.5 on noting Remark 11.4.4.

Theorem 11.4.6. If \(D\) is small and \(C\) is cocomplete (resp. complete), then any functor \(K: D \to C\) has a pointwise left (resp. right) Kan extension along any functor \(F: D \to E\).

Proof. By the equivalence of (1) and (2) in Proposition 11.4.3, and since the completeness of a category is the same thing as the completeness of its hom (see Remark 7.3.7), this is a special case of Corollary 11.3.16 where \(J\) is given by the representable module \(F/E\) (resp. the corepresentable module \(F\/E\)) and \(M\) is given by the hom endomodule \(C/E\).

Theorem 11.4.7. If \(F: D \to E\) is a fully faithful functor, then a pointwise left (resp. right) Kan extension of a functor \(K: D \to C\) along \(F\) is a natural isomorphism. The converse holds if \(F\) is fully faithful and essentially surjective, i.e. if \(F\) is an equivalence.

Proof. By the definition of a pointwise Kan extension, this is a special case of Theorem 11.3.17 where \(M\) is given by the hom of \(C\), noting that a natural isomorphism \(K \to R \delta F\) in \(C\) is the same thing as a pointwise universal cylinder \(K \rightarrow R \delta F\) along the hom \(\langle C\rangle\) (see Proposition 6.5.9).

Theorem 11.4.8.

- For a functor \(F: X \to A\) the following conditions are equivalent:
  (1) \(F\) has a right adjoint;
  (2) \(F\) preserves colimits and the identity \(X \to X\) has a pointwise left Kan extension along \(F\).

When these conditions hold, a natural transformation \(\eta: 1_X \to G \delta F\) forms a pointwise left Kan extension of the identity \(X \to X\) along \(F\) if and only if \(\eta\) is the unit of an adjunction \(\eta\)

- For a functor \(G: A \to X\) the following conditions are equivalent:
  (1) \(G\) has a left adjoint;
  (2) \(G\) preserves limits and the identity \(A \to A\) has a pointwise right Kan extension along \(G\).

When these conditions hold, a natural transformation \(\epsilon: G \delta F \to 1_A\) forms a pointwise right Kan extension of the identity \(A \to A\) along \(G\) if and only if \(\epsilon\) is the counit of an adjunction \(\epsilon\).

Proof. We see that this faithfully translates into an instance of Theorem 11.3.20 where \(M\) is given by the representable module \(F\langle A\rangle\), noting that
(1) a right adjoint of \( F \) is the same thing as a corepresentation of the module \( F(A) \) (see Remark 8.3.2(2));
(2) \( F \) preserves colimits iff for each \( a \in \|A\| \) the right module \( F(A) a : X \to * \) preserves colimits (see Corollary 7.5.8);
(3) a pointwise left Kan extension of the identity \( X \to X \) along \( F \) is the same thing as a pointwise left extension of the module \( F(A) \) (see Proposition 11.4.3);
(4) a natural transformation \( \eta : 1_X \to G \circ F \) is the unit of an adjunction \( G \dashv F \) iff the module morphism \( \eta A : F(A) \to (X) G \) is iso (see Proposition 8.3.8), and by definition, \( \eta \) forms a pointwise left Kan extension iff the cell \( F(A) \xrightarrow{\eta A} A \xrightarrow{\eta} (X) \xrightarrow{G} X \) is pointwise universal.

Note. The following is a special case of Theorem 11.3.21.

**Theorem 11.4.9.**

- Suppose that a functor \( H : C \to B \) has a right adjoint. Then it preserves left Kan extensions; that is, each left Kan extension \( \mu : K \to R \circ F \) of a functor \( K : D \to C \) along a functor \( F : D \to E \) yields by composition with \( H \) a left Kan extension \( \mu \circ H : K \circ H \to H \circ R \circ F \) of \( K \circ H : D \to B \) along \( F \).
- Suppose that a functor \( H : C \to B \) has a left adjoint. Then it preserves right Kan extensions; that is, each right Kan extension \( \mu : F \circ R \to K \) of a functor \( K : D \to C \) along a functor \( F : D \to E \) yields by composition with \( H \) a right Kan extension \( \mu \circ H : F \circ R \circ H \to H \circ K \) of \( H \circ K : D \to B \) along \( F \).

**Proof.** Since a left Kan extension in \( C \) along \( F \) is the same thing as a direct universal \( (F \uparrow C) \)-arrow (see Remark 11.4.2(1)), the assertion is equivalent to saying that if a functor \( H \) has a left adjoint, then the postcomposition cell \( (F \uparrow H) \) (see Remark 4.5.8(3)) preserves direct universal arrows. Since an adjoint of \( H \) is the same thing as an adjoint of its hom \( (H) \) (see Remark 8.9.2(2)), \( (F \uparrow H) = (F \uparrow (H)) \) has a right adjoint as well by Corollary 8.9.4. Hence the assertion follows from Theorem 8.9.5.

**Problem.** In Remark 11.3.4, we saw that pointwise Kan lifts subsume pointwise extensions, hence in particular pointwise Kan extensions. Does the converse hold? Do pointwise Kan extensions subsume pointwise Kan lifts?

### 11.5. Density

**Definition 11.5.1.**

1. A module \( M : X \to A \) is called
   - dense if its right exponential transpose \( [M \rightarrow] : A \to [X:] \) is fully faithful.
   - codense if its left exponential transpose \( [\leftarrow M] : X \to [:A] \) is fully faithful.
2. A functor \( F : D \to E \) is called
   - dense if its representable module \( F(E) : D \to E \) is dense.
   - codense if its corepresentable module \( (E) F : E \to D \) is codense.
3. A subcategory \( D \) of a category \( E \) is called
   - dense in \( E \) if the inclusion \( D \to E \) is dense.
   - codense in \( E \) if the inclusion \( D \to E \) is codense.

**Proposition 11.5.2.** Given a module \( M : X \to A \),
- the following conditions are equivalent:
  1. \( M \) is dense;
(2) the identity $\mathcal{M} \to \mathcal{M}$ forms a pointwise universal cell

\[
\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]^{1_{\mathcal{M}}} & \mathcal{M} \\
X \ar[u]^{1_{\mathcal{M}}} & A \ar[u]_{1_{\mathcal{M}}} \\
}
\end{array}
\]

(3) for every $a \in \|A\|$, the identity $\langle \mathcal{M} \rangle a \to \langle \mathcal{M} \rangle a$ forms a universal conical cell

\[
\begin{array}{c}
\xymatrix{
\langle \mathcal{M} \rangle a \ar[r]^{a} & \langle \mathcal{M} \rangle a \\
X \ar[u]^{1_{\langle \mathcal{M} \rangle a}} & A \ar[u]_{1_{\langle \mathcal{M} \rangle a}} \\
}
\end{array}
\]

(4) for every $a \in \|A\|$, the unit cone

\[
\begin{array}{c}
\xymatrix{
\mathcal{M} & \langle \mathcal{M} \rangle a \ar[l]^{\Delta a} \\
X \ar[u]^{1_{\langle \mathcal{M} \rangle a}} & A \ar[u]_{1_{\langle \mathcal{M} \rangle a}} \\
}
\end{array}
\]

(see Definition 10.4.1 and Remark 3.2.28) of $\langle \mathcal{M} \rangle a$ forms a universal cone.

- the following conditions are equivalent:

  (1) $\mathcal{M}$ is codense;
  
  (2) the identity $\mathcal{M} \to \mathcal{M}$ forms a pointwise universal cell

\[
\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]^{1_{\mathcal{M}}} & \mathcal{M} \\
X \ar[u]^{1_{\mathcal{M}}} & A \ar[u]_{1_{\mathcal{M}}} \\
}
\end{array}
\]

(3) for every $x \in \|X\|$, the identity $x \langle \mathcal{M} \rangle \to x \langle \mathcal{M} \rangle$ forms a universal conical cell

\[
\begin{array}{c}
\xymatrix{
\langle \mathcal{M} \rangle a \ar[r]^{a} & \langle \mathcal{M} \rangle a \\
x \ar[u]^{1_{\langle \mathcal{M} \rangle a}} & X \ar[u]_{1_{\langle \mathcal{M} \rangle a}} \\
}
\end{array}
\]

(4) for every $x \in \|X\|$, the unit cone

\[
\begin{array}{c}
\xymatrix{
\Delta a & x \langle \mathcal{M} \rangle \ar[l]^{\Delta x} \\
X \ar[u]^{1_{\langle \mathcal{M} \rangle a}} & A \ar[u]_{1_{\langle \mathcal{M} \rangle a}} \\
}
\end{array}
\]

(see Definition 10.4.1 and Remark 3.2.28) of $x \langle \mathcal{M} \rangle$ forms a universal cone.

Proof.
(1) $\iff$ (2) By Proposition 11.3.3, the cell

\[
\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]^{1_{\mathcal{M}}} & \mathcal{M} \\
X \ar[u]^{1_{\mathcal{M}}} & A \ar[u]_{1_{\mathcal{M}}} \\
}
\end{array}
\]

is pointwise universal iff its right exponential transpose

\[
\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]^{1_{\mathcal{M}}} & \mathcal{M} \\
X \ar[u]^{1_{\mathcal{M}}} & A \ar[u]_{1_{\mathcal{M}}} \\
}
\end{array}
\]

forms a pointwise left Kan lift of $\mathcal{M} \mathcal{X}$ along $\mathcal{M} \mathcal{X}$. By Theorem 6.6.3, this is the case iff $\mathcal{M} \mathcal{X}$ is fully faithful.

(2) $\iff$ (3) Evident by the definition of pointwise universality.

(3) $\iff$ (4) Since the unit cone $1_{\langle \mathcal{M} \rangle a}$ is the comma transpose of the identity $1_{\langle \mathcal{M} \rangle a}$ (see Remark 10.4.9(4)), this is immediate from Proposition 11.2.3.

\[\Box\]

**Theorem 11.5.3.** The right (resp. left) Yoneda module is dense (resp. codense).

**Proof.** Immediate from Proposition 5.1.3. \[\Box\]

**Corollary 11.5.4.** The right (resp. left) Yoneda functor is dense (resp. codense).

**Proof.** Since the Yoneda module is represented by the Yoneda functor (see Theorem 5.2.17), the assertion follows from Theorem 11.5.3. \[\Box\]

**Note.** If $F$ in Corollary 6.2.17 is dense (resp. codense), then we have the following.

**Theorem 11.5.5.** Let $F : D \to E$ be functor.
If \( \mathcal{F} \) is fully faithful and dense, then an \( \mathbf{E} \)-arrow \( u : r : \mathcal{F} \rightarrow e \) is iso if and only if it is universal from \( \mathcal{F} \) to \( e \), i.e. the \( \mathcal{F}(\mathbf{E}) \)-arrow \( u : r \Rightarrow e \) is inverse universal.

If \( \mathcal{F} \) is fully faithful and codense, then an \( \mathbf{E} \)-arrow \( u : e \rightarrow F : \mathcal{F} \) is iso if and only if it is universal from \( e \) to \( \mathcal{F} \), i.e. the \( \mathcal{F}(\mathbf{E}) \)-arrow \( u : e \Rightarrow r \) is direct universal.

Proof. By Example 5.2.9(2), the diagram

\[
\begin{array}{ccc}
\mathcal{D} \mathcal{F} r & \mathcal{F}(\mathcal{E}) \mathcal{F} r = \mathcal{F}(\mathcal{E})(F : r) \\
\mathcal{D} \mathcal{F} e & \mathcal{F}(\mathcal{E}) e \equiv \mathcal{F}(\mathcal{E}) e
\end{array}
\]

commutes. Since \( \mathcal{F} \) is fully faithful, \( (\mathcal{F}) \mathcal{r} \) is iso. Hence \( \mathcal{D} \mathcal{F} u \) is iso iff \( \mathcal{F}(\mathcal{E}) u \) is iso. But \( \mathcal{D} \mathcal{F} u \) is iso iff \( u \) is inverse universal by the definition of inverse universality, and \( \mathcal{F}(\mathcal{E}) u \) is iso iff \( u \) is iso because \( (\mathcal{F}(\mathcal{E})) \mathcal{r} \) is fully faithful by the denseness of \( \mathcal{F} \).

\[\square\]

Theorem 11.5.6.

1. If a module \( \mathcal{M} : X \rightarrow A \) is dense and has a representation \((R, \Upsilon)\), then for any \( a \in \parallel A \) the composition

\[
\begin{array}{ccc}
[\langle M \rangle a] & 1 & * \\
\parallel M a & \downarrow & \parallel a \\
X - \rightarrow M - \rightarrow A \\
R & \Upsilon & \downarrow 1 \\
A - \rightarrow \langle A \rangle - \rightarrow A
\end{array}
\]

yields a universal cone \( [M \downarrow a] \Rightarrow R \Rightarrow a \) in \( A \).

2. If a module \( \mathcal{M} : X \rightarrow A \) is codense and has a corepresentation \((R, \Upsilon)\), then for any \( x \in \parallel X \) the composition

\[
\begin{array}{ccc}
* & \leftarrow [x \langle M \rangle] & 1 \\
\downarrow x & \downarrow [x \parallel M] & \downarrow [x \parallel M] \\
X - \rightarrow M - \rightarrow A \\
\uparrow 1 & \Upsilon & \downarrow R \\
X - \rightarrow \langle X \rangle - \rightarrow X
\end{array}
\]

yields a universal cone \( x \Rightarrow R \Rightarrow [x \downarrow M] \) in \( X \).

Proof. The unit cone \( 1_{\parallel M} a \) is universal by Proposition 11.5.2, and \( \Upsilon \) preserves it by Theorem 7.3.13.

\[\square\]

Corollary 11.5.7.

1. Any right module is a colimit of representables. Specifically, given a right module \( \mathcal{M} : X \rightarrow * \), the composition

\[
\begin{array}{ccc}
[\mathcal{M}] & 1 & * \\
\parallel \mathcal{M} & \downarrow 1_{\mathcal{M}} & \downarrow \mathcal{M} \\
X - \rightarrow (X - \rightarrow *) - \rightarrow [X :] \\
x \Rightarrow X & \Rightarrow X & \downarrow 1 \\
[\mathcal{M}] - \rightarrow [\mathcal{M}] - \rightarrow [X :]
\end{array}
\]

of the unit cone of \( \mathcal{M} \) and the Yoneda representation yields a universal cone \( \parallel \mathcal{M} \Rightarrow [X \Rightarrow X] \Rightarrow \mathcal{M} \) in \( [X :] \).
Any left module is a colimit of representables. Specifically, given a left module \( \mathcal{M} : \ast \rightarrow A \), the composition
\[
\ast \leftarrow \mathcal{M} \\
\mathcal{M} \downarrow \downarrow \\
\lbrack [A] \rbrack^\sim \end{matrix}
\]
of the unit cone of \( \mathcal{M} \) and the Yoneda corepresentation yields a universal cone \( \mathcal{M}^\sim \circ [\bowtie A] \sim \mathcal{M} \) in \( \lbrack [A] \rbrack \).

Proof. Since the Yoneda module \( X \rightrightarrows \) is dense (see Theorem 11.5.3), and since \((X \rightrightarrows) (\mathcal{M}) = \mathcal{M}\) (see Proposition 5.1.3), the assertion follows by applying Theorem 11.5.6 to the Yoneda representation.

\[\square\]

Theorem 11.5.8.

- Given functors \( F : X \rightarrow A \) and \( R : [X:] \rightarrow A \), the following conditions are equivalent:
  1. \( R \) is a pointwise left Kan extension
  \[
  X \xrightarrow{X^\rightrightarrows} [X:] \\
  F \downarrow \downarrow \\
  A \leftarrow \langle A \rangle \rightarrow A
  \]
of \( F \) along the right Yoneda functor \( X^\rightrightarrows \);
  2. \( R \) is a pointwise extension
  \[
  X \xrightarrow{X^\rightrightarrows} [X:] \\
  F \downarrow \downarrow \\
  A \leftarrow \langle A \rangle \rightarrow A
  \]
of \( F \) direct along the right Yoneda module \( X^\rightrightarrows \);
  3. \( R \) is a pointwise extension
  \[
  X \xrightarrow{X^\rightrightarrows} [X:] \\
  F \downarrow \downarrow \\
  A \leftarrow \langle F(A) \rangle \rightarrow A
  \]
of the right Yoneda module \( X^\rightrightarrows \) direct along the representable module \( F(A) : X \rightarrow A \);
  4. \( R \) is a left adjoint
  \[
  [X:] \xrightarrow{(F(A))^\rightrightarrows} A \\
  R \downarrow \downarrow \\
  \]
  of the right exponential transpose of the representable module \( F(A) : X \rightarrow A \).

- Given functors \( G : A \rightarrow X \) and \( R : [A]^- \rightarrow X \), the following conditions are equivalent:
  1. \( R \) is a pointwise right Kan extension
  \[
  [A]^\sim \xrightarrow{\bowtie A} A \\
  R \downarrow \downarrow \\
  X \leftarrow \langle (X) \rangle \rightarrow X
  \]
of \( G \) along the left Yoneda functor \( \bowtie A \).
(2) \( R \) is a pointwise extension

\[
X \xrightarrow{\gamma} \frac{\gamma}{R} \rightarrow A
\]

of \( G \) inverse along the left Yoneda module \( \gamma \). A;

(3) \( R \) is a pointwise extension

\[
X \xrightarrow{\gamma} \frac{\gamma}{R} \rightarrow A
\]

of the left Yoneda module \( \gamma \) inverse along the corepresentable module \( (X)G : X \to A \);

(4) \( R \) is a right adjoint

\[
X \xrightarrow{\gamma} \frac{\gamma}{R} \rightarrow A
\]

of the left exponential transpose of the corepresentable module \( (X)G : X \to A \).

Proof. (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) Since \( (X,) \equiv [X,] (X) \) (see Theorem 5.2.17), this follows from Proposition 11.4.3.

(3) \( \Leftrightarrow \) (4) By Remark 11.3.4, and since \( (X,) \) \( = 1 \) (see Proposition 5.1.3), \( R \) is a pointwise extension of \( X, \) direct along \( F(A) \) iff \( R \) is a pointwise left Kan lift

\[
\begin{array}{c}
\eta \\
\downarrow \\
A
\end{array}
\xrightarrow{\gamma} \frac{\gamma}{R} \rightarrow A
\]

of the identity \( [X:] \to [X:] \) along \( (F(A)) \), and by the equivalence of (2) and (5) in Proposition 8.3.8, this is the case iff \( \eta \) is the unit of an adjunction \( [(F(A))] \Leftrightarrow R: [X:] \to A \).

\[\square\]

Corollary 11.5.9.

- If a functor \( F : X \to A \) has for each right module \( M : X \to \star \) an \( M \)-weighted colimit \( \xrightarrow{M} \),

\[
\begin{array}{c}
\eta \\
\downarrow \\
A
\end{array}
\xrightarrow{\gamma} \frac{\gamma}{R_M} \rightarrow A
\]

then \( F \) has a pointwise left Kan extension \( R : [X:] \to A \) along the right Yoneda functor \( [X,] : X \to [X:] \) with \( R_M = R^\star M \).

- If a functor \( G : A \to X \) has for each left module \( M : \star \to A \) an \( M \)-weighted limit

\[
\begin{array}{c}
\eta \\
\downarrow \\
A
\end{array}
\xrightarrow{\gamma} \frac{\gamma}{R_M} \rightarrow A
\]

then \( G \) has a pointwise right Kan extension \( R : [:A,] \to X \) along the left Yoneda functor \( [:A,] : A \to [:A,] \) with \( R_M = R^\star M \).

Proof. By the equivalence of (1) and (2) in Theorem 11.5.8, the problem translates into showing that \( F \) has a pointwise extension \( R : [X:] \to A \) direct along the right Yoneda module \( (X,) : X \to [X:] \) with \( R_M = R^\star M \). But since \( (X,) (M) = M \) for any right module \( M : X \to \star \) (see Proposition 5.1.3), this follows from Theorem 11.3.15.

\[\square\]

Theorem 11.5.10.

- If \( F : X \to A \) is a functor with \( X \) small and \( A \) cocomplete, then \( F \) has a pointwise left Kan extension

\[
\begin{array}{c}
\eta \\
\downarrow \\
A
\end{array}
\xrightarrow{\gamma} \frac{\gamma}{R} \rightarrow A
\]

following properties:
(1) $R$ is cocontinuous;

(2) $F \cong R \mathcal{O}[X.]$; that is, if a functor $S : [X.] \to A$ has the properties above, then $S \cong R$.

- If $G : A \to X$ is a functor with $A$ small and $X$ complete, then $G$ has a pointwise right Kan extension $[\mathcal{O}A] \xrightarrow{\lambda} A \xrightarrow{\eta} X$ along the left Yoneda functor $\mathcal{O}A$, and $R$ is characterized by the following properties:

(1) $R$ is continuous;

(2) $[\mathcal{O}A] \circ R \cong G$; that is, if a functor $S : [\mathcal{O}A] \to X$ has the properties above, then $S \cong R$.

Proof. By Theorem 11.5.6, a pointwise left Kan extension $(R, \mu)$ exists. Since $R$ is given by a left adjoint of the functor $\mathcal{O}[F(A)] : A \to [X.]$ by Theorem 11.5.8, $R$ is cocontinuous by Corollary 8.9.8 ($[X.]$ is cocomplete by Corollary 7.4.3). Since the Yoneda functor is fully faithful, $\mu$ is a natural isomorphism by Theorem 11.4.7. Now suppose that a functor $S : [X.] \to A$ is cocontinuous and has a natural isomorphism $\nu : F \to S \mathcal{O}[X.]$. We will show that the adjunct of $\nu$ along $\mu$, i.e. the unique natural transformation $\mu \mathcal{O}\nu : R \to S$ with $\nu = \mu \mathcal{O}\nu ([X.] \circ \mu \mathcal{O}\nu)$ (see Remark 11.4.2(2)), is a natural isomorphism. For this, it suffices to show that the component of $\mu \mathcal{O}\nu$ at each right module $M : X \to \ast$ is an isomorphism. By Corollary 11.5.7, there is a universal cone $\pi : M \mathcal{O}[X.] \to M$, and the composite $\pi \mathcal{O}\mu \mathcal{O}\nu : M \mathcal{O}[X.] \circ R \to M \mathcal{O}S$ factors as

$$M \mathcal{O}[X.] \xrightarrow{\text{adjunct}} M \mathcal{O}R \xrightarrow{\text{universal}} M \mathcal{O}S.$$

Since $R$ and $S$ are cocontinuous, $\pi \mathcal{O}R$ and $\pi \mathcal{O}S$ are universal cones, and since $\mu$ and $\nu$ are natural isomorphisms, so is $M \mathcal{O}[X.] \circ \pi \mathcal{O}\mu \mathcal{O}\nu = M \mathcal{O}R \circ \pi \mathcal{O}S$; hence $M \mathcal{O}R \mathcal{O}S$ is an isomorphism by Proposition 6.3.3(2).

**Corollary 11.5.11.** If $F : D \to E$ is a functor between small categories, then the precomposition functor $[F.] : [E.] \to [D.]$ (resp. $[: F.] : [E.] \to [: D.]$) has both left and right adjoints.

**Proof.** Consider the Yoneda functors as in

$$\begin{array}{ccc}
D & \xrightarrow{D\mathcal{O} [\circ]} & [D.] \\
\downarrow F & & \\
E & \xrightarrow{E\mathcal{O} [\circ]} & [E.]
\end{array}$$

Since $D$ is small and $[E.]$ is cocomplete (see Corollary 7.4.3), $F \mathcal{O}[E.]$ has a pointwise left Kan extension

$$\begin{array}{ccc}
D & \xrightarrow{D\mathcal{O} [\circ]} & [D.] \\
\downarrow F & & \\
E & \xrightarrow{E\mathcal{O} [\circ]} & [E.]
\end{array}$$

along $D\mathcal{O}$ by Theorem 11.5.10, and $R$ is a left adjoint of the functor $(F \mathcal{O}[E.])_{\mathcal{O}}$ by Theorem 11.5.8. But

$$
\langle [F \mathcal{O}[E.]] \circ (E\mathcal{O}) \rangle_{\mathcal{O}} = \langle F \circ (E\mathcal{O}) \rangle_{\mathcal{O}} \cong \langle F \rangle_{\mathcal{O}}_{\mathcal{O}}^{\ast} = [F.] \circ (E\mathcal{O})_{\mathcal{O}}^{\ast} = [F.]^{\ast}^{\ast}
$$
(*1 by Theorem 5.2.17; *2 by Proposition 2.1.7; *3 by Proposition 5.1.3). Hence \([F:]\) has a left adjoint, namely \(R\). Now consider the composition

\[
\begin{array}{ccc}
E & \xrightarrow{E\triangleright} & [E:] \\
\downarrow_{[E\triangleright] \circ [F]} & & \downarrow_{[F]} \\
[D:] & \rightarrow & [D] \\
\end{array}
\]

Since \([F:]\) is cocontinuous (see Corollary 7.4.4), \([F:]\) is a pointwise left Kan extension of \([E\triangleright] \circ [F:]\) along \(E\triangleright\) by Theorem 11.5.10, and thus has a right adjoint by Theorem 11.5.8. \qed
12. Ends

12.1. Ends

**Definition 12.1.1.** Let $E$ be a category and $M : X \to A$ be a module.

- A cylinder $* \to E^\times E$ is called universal if it is an inverse universal $(E, M)$-arrow (see Definition 4.4.5). Given a bifunctor $K : E^\times E \to A$, a universal cylinder $\pi : r \rightsquigarrow K : E \rightsquigarrow M$ or the pair $(r, \pi)$, or the object $r$ itself, is called an end of $K$ along $M$, with the object $r$ denoted by $\prod_K E$ (or just by $\prod K$).

- A cylinder $E^\times E \to *$ is called universal if it is a direct universal $(E, M)$-arrow (see Definition 4.4.5). Given a bifunctor $K : E^\times E \to X$, a universal cylinder $\pi : K \rightsquigarrow r : E \rightsquigarrow M$ or the pair $(r, \pi)$, or the object $r$ itself, is called a coend of $K$ along $M$, with the object $r$ denoted by $\int_K E$ (or just by $\int K$).

**Remark 12.1.2.**

1. A cylinder $\pi : r \rightsquigarrow K : E \rightsquigarrow M$ is universal if and only if to every cylinder $\alpha : x \rightsquigarrow K : E \rightsquigarrow M$ there is a unique $X$-arrow $\alpha/\pi : x \to r$ such that $\alpha = \alpha/\pi \circ \pi$. Dually, a cylinder $\pi : K \rightsquigarrow r : E \rightsquigarrow M$ is universal if and only if to every cylinder $\alpha : K \rightsquigarrow a : E \rightsquigarrow M$ there is a unique $A$-arrow $\pi/\alpha : r \to a$ such that $\alpha = \pi \circ \pi/\alpha$.

2. As a special case where $M$ is the hom of a category $C$,

- an extraordinary natural transformation $\pi : r \rightsquigarrow K : E \to C$ from an object $r \in \|C\|$ to a bifunctor $K : E^\times E \to C$ is called universal if it is an inverse universal $(E, C)$-arrow (see Remark 4.4.6(4)); given a bifunctor $K : E^\times E \to C$, a universal extraordinary natural transformation $\pi : r \rightsquigarrow K : E \to C$ or the pair $(r, \pi)$, or the object $r$ itself, is called an end of $K$ in $C$.

- an extraordinary natural transformation $\pi : K \rightsquigarrow r : E \to C$ from a bifunctor $K : E^\times E \to C$ to an object $r \in \|C\|$ is called universal if it is a direct universal $(E, C)$-arrow (see Remark 4.4.6(4)); given a bifunctor $K : E^\times E \to C$, a universal extraordinary natural transformation $\pi : K \rightsquigarrow r : E \to C$ or the pair $(r, \pi)$, or the object $r$ itself, is called a coend of $K$ in $C$.

**Note.** The bijective correspondence between cylinders and conical cells stated in Theorem 5.5.10 gives the following result.

**Theorem 12.1.3.** Let $E$ be a category and $M : X \to A$ be a module.

- A cylinder

$$
\begin{array}{c}
* \\
\downarrow r \\
X \\
\end{array} 
= \begin{array}{c}
E^\times E \\
\downarrow \pi \\
\downarrow K \\
M \\
A \\
\end{array}
$$

is universal if and only if the conical cell

$$
\begin{array}{c}
\{E\} \\
\downarrow \langle E \rangle \\
X \\
\end{array} 
= \begin{array}{c}
E^\times E \\
\downarrow \langle \pi \rangle \\
\downarrow K \\
M \\
A \\
\end{array}
$$

is universal.
A cylinder

\[
\begin{array}{c}
\mathbb{E}^* \times \mathbb{E} \quad \pi \\
\downarrow \quad \downarrow r
\end{array}
\]

\[ X \rightarrow \mathbb{M} \rightarrow A \]

is universal if and only if the conical cell

\[
\begin{array}{c}
\mathbb{E}^* \times \mathbb{E} \ (\mathbb{E}^*) \quad \ast \\
\downarrow \quad \downarrow r
\end{array}
\]

\[ X \rightarrow \mathbb{M} \rightarrow A \]

is universal.

Proof. Immediate by the isomorphism in Theorem 5.5.10.

Remark 12.1.4. By Theorem 12.1.3 and by the bijectiveness of the assignment \( \pi \mapsto \langle \pi \rangle \), we see that given a module \( \mathcal{M} : X \rightarrow A \),

- an end of a bifunctor \( K : \mathbb{E}^* \times \mathbb{E} \rightarrow A \) along \( \mathcal{M} \) is the same thing as an \( \langle \mathbb{E} \rangle \)-weighted limit of \( K \) along \( \mathcal{M} \).
- a coend of a bifunctor \( K : \mathbb{E}^* \times \mathbb{E} \rightarrow X \) along \( \mathcal{M} \) is the same thing as an \( \langle \mathbb{E}^* \rangle \)-weighted colimit of \( K \) along \( \mathcal{M} \).

Weighted limits thus subsume ends.

**Theorem 12.1.5.** Let \( \mathbb{E} \) be a category and \( \mathcal{M} : X \rightarrow A \) be a module.

- A cone \( \pi : r \Rightarrow K : * \mathbb{E} \Rightarrow \mathcal{M} \) is universal if and only if the cylinder \( \pi : r \Rightarrow K \circ [!_{\mathbb{E}^*} \times \mathbb{E}] : \mathbb{E} \Rightarrow \mathcal{M} \) (see Remark 4.6.4(2)) is universal.
- A cone \( \pi : K \Rightarrow r : \mathbb{E}^* \Rightarrow \mathcal{M} \) is universal if and only if the cylinder \( \pi : [!_{\mathbb{E}^*} \times \mathbb{E}] \circ K \Rightarrow r : \mathbb{E} \Rightarrow \mathcal{M} \) (see Remark 4.6.4(2)) is universal.

Proof. Since a cone \( \pi : r \Rightarrow K : * \mathbb{E} \Rightarrow \mathcal{M} \) is universal iff it is an inverse universal \( \langle * \mathbb{E}, \mathcal{M} \rangle \)-arrow, and the cylinder \( \pi : r \Rightarrow K \circ [!_{\mathbb{E}^*} \times \mathbb{E}] : \mathbb{E} \Rightarrow \mathcal{M} \) is universal iff it is an inverse universal \( \langle \mathbb{E}, \mathcal{M} \rangle \)-arrow, the assertion follows by applying Theorem 6.2.14 to the composite in Theorem 4.6.33.

Remark 12.1.6. Theorem 12.1.5 says that given a module \( \mathcal{M} : X \rightarrow A \),

- a limit of a functor \( K : \mathbb{E} \rightarrow A \) along \( \mathcal{M} \) is the same thing as an end of the bifunctor \( K \circ [!_{\mathbb{E}^*} \times \mathbb{E}] : \mathbb{E}^* \times \mathbb{E} \rightarrow A \) along \( \mathcal{M} \).
- a colimit of a functor \( K : \mathbb{E} \rightarrow X \) along \( \mathcal{M} \) is the same thing as a coend of the bifunctor \( [!_{\mathbb{E}^*} \times \mathbb{E}] \circ K : \mathbb{E}^* \times \mathbb{E} \rightarrow X \) along \( \mathcal{M} \).

A limit is thus regarded as an end with a dummy variable; recalling Remark 12.1.4 and Remark 11.2.13, we now see that ends, limits, and weighted limits subsume one another.

**Definition 12.1.7.** A cell \( X \rightarrow \mathcal{M} \rightarrow A \) is said to

\[
\begin{array}{c}
X \rightarrow \mathcal{M} \rightarrow A \\
\downarrow \quad \downarrow r
\end{array}
\]

\[ X \rightarrow \mathcal{M} \rightarrow A \]

- preserve (reflect, create) ends over a category \( \mathbb{E} \) if the postcomposition cell \( \langle \mathbb{E}, \Phi \rangle \) (see Definition 4.4.15) preserves (reflects, creates) inverse universal arrows.
- preserve (reflect, create) coends over a category \( \mathbb{E} \) if the postcomposition cell \( \langle \mathbb{E}, \Phi \rangle \) (see Definition 4.4.15) preserves (reflects, creates) direct universal arrows.
Remark 12.1.8.

(1) Recalling the definition of ends (resp. coends) and the definition of the postcomposition cell, Definition 12.1.7 can be stated in elementary terms as follows: \( \Phi \) is said to

a) preserve

- ends over \( \mathbf{E} \) if each universal cylinder \( \pi : \mathbf{r} \Rightarrow K : \mathbf{E} \Rightarrow \mathcal{M} \) yields by composition with \( \Phi \) a universal cylinder \( \pi \circ \Phi : \mathbf{r} \Rightarrow Q \circ \Phi K : \mathbf{E} \Rightarrow \mathcal{N} \).

- coends over \( \mathbf{E} \) if each universal cylinder \( \pi : K \Rightarrow r : \mathbf{E} \Rightarrow \mathcal{M} \) yields by composition with \( \Phi \) a universal cylinder \( \pi \circ \Phi : K \circ \Phi P \Rightarrow Q \circ \Phi r : \mathbf{E} \Rightarrow \mathcal{N} \).

b) reflect

- ends over \( \mathbf{E} \) if a cylinder \( \pi : r \Rightarrow K : \mathbf{E} \Rightarrow \mathcal{M} \) is universal whenever the cylinder \( \pi \circ \Phi : r \Rightarrow Q \circ \Phi K : \mathbf{E} \Rightarrow \mathcal{N} \) is universal.

- coends over \( \mathbf{E} \) if a cylinder \( \pi : K \Rightarrow r : \mathbf{E} \Rightarrow \mathcal{M} \) is universal whenever the cylinder \( \pi \circ \Phi : K \circ \Phi P \Rightarrow Q \circ \Phi r : \mathbf{E} \Rightarrow \mathcal{N} \) is universal.

c) create

- ends over \( \mathbf{E} \) if for every bifunctor \( K : \mathbf{E} \times \mathbf{E} \Rightarrow \mathbf{A} \) and for every universal cylinder \( \kappa : s \Rightarrow Q \circ \Phi K : \mathbf{E} \Rightarrow \mathcal{N} \) there is exactly one cylinder \( \pi : r \Rightarrow K : \mathbf{E} \Rightarrow \mathcal{M} \) with \( \pi \circ \Phi = \kappa \), and if this \( \pi \) is universal.

- coends over \( \mathbf{E} \) if for every bifunctor \( K : \mathbf{E} \times \mathbf{E} \Rightarrow \mathbf{X} \) and for every universal cylinder \( \kappa : K \circ \Phi P \Rightarrow s : \mathbf{E} \Rightarrow \mathcal{N} \) there is exactly one cylinder \( \pi : K \Rightarrow r : \mathbf{E} \Rightarrow \mathcal{M} \) with \( \pi \circ \Phi = \kappa \), and if this \( \pi \) is universal.

(2) A cell \( \Phi \) is said to

- preserve (reflect, create) ends, if it preserves (reflects, creates) ends over any category.

- preserve (reflect, create) coends, if it preserves (reflects, creates) coends over any category.

Proposition 12.1.9. A cell \( \Phi \) preserves (reflects, creates) ends (resp. coends) if and only if \( \Phi \) preserves (reflects, creates) limits (resp. colimits).

Proof. \((\Rightarrow)\) The “only if” part says that for any category \( \mathbf{E} \), if the cell \( \langle \mathbf{E}, \Phi \rangle \) preserves (reflects, creates) inverse universal arrows, then the cell \( \langle \Phi \mathbf{E}, \Phi \rangle \) does the same. But this follows by applying Proposition 6.2.24 to the pasting composition in Corollary 4.6.34.

\((\Leftarrow)\) Because of Proposition 11.2.19, showing this is equivalent to showing that if \( \Phi \) preserves (reflects, creates) weighted limits, then \( \Phi \) preserves (reflects, creates) ends. But by Theorem 12.1.3 and since the isomorphism in Theorem 5.5.10 is natural in \( \mathcal{M} \), for any category \( \mathbf{E} \), if a cell \( \Phi : \mathcal{M} \Rightarrow \mathcal{N} \) preserves (reflects, creates) limits weighted by \( \langle \mathbf{E} \rangle \), then it preserves (reflects, creates) ends over \( \mathbf{E} \).

\( \square \)

12.2. Ends with parameters

Definition 12.2.1. Let \( \mathbf{E} \) and \( \mathbf{D} \) be categories and \( \mathcal{M} : \mathbf{X} \rightarrow \mathbf{A} \) be a module.

- A cylinder \( \ast 
\begin{array}{c|c}
\mathbf{R} & \sim \\
\mathbf{W} & \sim \\
\mathbf{D}^\times \mathbf{D} & \sim \\
\end{array}
\mathcal{K}
\begin{array}{c|c}
\mathbf{E}, \mathbf{X} & \Rightarrow \\
\mathcal{E} \mathcal{M} & \Rightarrow \\
\mathbf{E}, \mathbf{A} & \Rightarrow \\
\end{array}
\)

(1) universal if it is an inverse universal \( \langle \mathbf{D}, \langle \mathbf{E}, \mathcal{M} \rangle \rangle \)-arrow (see Definition 4.4.5);

(2) pointwise universal if each slice

\[
\begin{array}{c|c|c|c}
\ast & \mathbf{D}^\times \mathbf{D} & \sim \\
\mathbf{R} & \mathbf{W} & \mathcal{K} \\
\mathbf{E}, \mathbf{X} & \ast \langle \mathbf{E}, \mathcal{M} \rangle & \ast \mathbf{E}, \mathbf{A} \\
\mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{A} \\
\end{array}
\]

(see Remark 4.7.8(2)) is a universal cylinder, i.e. an inverse universal \( \langle \mathbf{D}, \mathcal{M} \rangle \)-arrow.
Given a bifunctor $K : D^\sim \times D \to [E, A]$, a pointwise universal cylinder $\omega : R \to K : D \to (E, M)$ or the pair $(R, \omega)$, or the functor $R$ itself, is called a pointwise end of $K$ along $(E, M)$, with the functor $R$ denoted by $\prod^E D K$ (or just by $\prod^E K$).

- A cylinder $D^\sim \times D \overset{\omega}{\to} [E, A]$ along the module $(E, M)$ (see Definition 4.3.5) is called
  (1) universal if it is a direct universal $(D, (E, M))$-arrow (see Definition 4.4.5);
  (2) pointwise universal if each slice

```
      * D^\sim \times D
    \downarrow \omega \downarrow K
    [E, X] \to (E, M) \to [E, A]
```

(see Remark 4.7.8(2)) is a universal cylinder, i.e. a direct universal $(D, M)$-arrow.

Given a bifunctor $K : D^\sim \times D \to [E, X]$, a pointwise universal cylinder $\omega : K \to R : D \to (E, M)$ or the pair $(R, \omega)$, or the functor $R$ itself, is called a pointwise coend of $K$ along $(E, M)$, with the functor $R$ denoted by $\prod^E D K$ (or just by $\prod^E K$).

**Proposition 12.2.2.**

- The following conditions are equivalent:
  (1) an extraordinary cylinder

```
    * D^\sim \times D
  \downarrow \omega \downarrow K
  [E, X] \to (E, M) \to [E, A]
```

is universal (resp. pointwise universal);

(2) its transpose

```
  E --- E --- E
  \downarrow \omega^\perp \downarrow K^\perp
  X \to (D, M) \to [D^\sim \times D, A]
```

(see Definition 4.7.7) is an inverse universal (resp. pointwise inverse universal) ordinary cylinder.

- The following conditions are equivalent:
  (1) an extraordinary cylinder

```
    D^\sim \times D
  \downarrow \omega \downarrow R
  [E, X] \to (E, M) \to [E, A]
```

is universal (resp. pointwise universal);

(2) its transpose

```
  E --- E --- E
  \downarrow \omega^\perp \downarrow R
  [D^\sim \times D, X] \to (D, M) \to A
```

(see Definition 4.7.7) is a direct universal (resp. pointwise direct universal) ordinary cylinder.

**Proof.** By the isomorphism in Remark 4.7.8(1), $\omega$ is universal iff $\omega^\perp$ is inverse universal. Since the slice of $\omega$ at each $e \in \parallel E \parallel$ is given by the component of $\omega^\perp$ at $e$, $\omega$ is pointwise universal iff $\omega^\perp$ is pointwise inverse universal. 

$\square$
Remark 12.2.3. By Proposition 12.2.2 and by the bijectiveness of transposition, we see that given a module \( \mathcal{M} : X \to A \),

- an end (resp. pointwise end) of a bifunctor \( K : D^\to \times D \to [E, A] \) along the module \( (E, \mathcal{M}) \) is the same thing as a lift (resp. pointwise lift) of \( K^\to : E \to [D^\to \times D, A] \) inverse along the module \( (D, \mathcal{M}) \).
- a coend (resp. pointwise coend) of a bifunctor \( K : D^\to \times D \to [E, X] \) along the module \( (E, \mathcal{M}) \) is the same thing as a lift (resp. pointwise lift) of \( K^\to : E \to [D^\to \times D, X] \) direct along the module \( (D, \mathcal{M}) \).

Proposition 12.2.4. A pointwise universal cylinder is universal.

Proof. This is reduced to Proposition 6.5.8 by the equivalence of (1) and (2) in Proposition 12.2.2.

Definition 12.2.5. Let \( E \) and \( D \) be categories and \( \mathcal{M} : X \to A \) be a module.

- A bicylinder \( \begin{array}{ccc} E & \times D^\to \times D & \to A \\ {\mathcal{R}} \downarrow & \omega & \downarrow {\mathcal{K}} \\ X & \to \mathcal{M} & \to A \end{array} \) is called
  1. universal if it is an inverse universal \( (E \times D^\to, \mathcal{M}) \)-arrow (see Remark 4.7.2(1));
  2. pointwise universal if each left slice

\[
\begin{array}{ccc}
X & \to \mathcal{M} & \to A \\
\mathcal{R}(e) & \downarrow & \omega_e \\
* & \to D^\to \\
K^\to(e) & \downarrow & {\mathcal{K}}^\to(e)
\end{array}
\]

(see Definition 4.7.5) is a universal cylinder, i.e. an inverse universal \( (D, \mathcal{M}) \)-arrow.

Given a bifunctor \( K : E \times D^\to \times D \to A \), a pointwise universal bicylinder \( \omega : \mathcal{R} \to K : E \times D \to \mathcal{M} \) or the pair \( (\mathcal{R}, \omega) \), or the functor \( \mathcal{R} \) itself, is called an \( E \)-parameterized end of \( K \) along \( \mathcal{M} \), with the functor \( \mathcal{R} \) denoted by \( \prod^E_\mathcal{M} K \) (or just by \( \prod^E K \)).

- A bicylinder \( \begin{array}{ccc} E \times D^\to \times D & \to E \\ {\mathcal{K}} \downarrow & \omega & \downarrow {\mathcal{R}} \\ X & \to \mathcal{M} & \to A \end{array} \) is called
  1. universal if it is a direct universal \( (E \times D^\to, \mathcal{M}) \)-arrow (see Remark 4.7.2(1));
  2. pointwise universal if each left slice

\[
\begin{array}{ccc}
X & \to \mathcal{M} & \to A \\
\mathcal{K}(e) & \downarrow & \omega_e \\
D^\to & \to * \\
K^\to(e) & \downarrow & {\mathcal{R}}(e)
\end{array}
\]

(see Definition 4.7.5) is a universal cylinder, i.e. a direct universal \( (D, \mathcal{M}) \)-arrow.

Given a bifunctor \( K : E \times D^\to \times D \to X \), a pointwise universal bicylinder \( \omega : \mathcal{K} \to R : E \times D \to \mathcal{M} \) or the pair \( (\mathcal{R}, \omega) \), or the functor \( \mathcal{R} \) itself, is called an \( E \)-parameterized coend of \( K \) along \( \mathcal{M} \), with the functor \( \mathcal{R} \) denoted by \( \prod^E_\mathcal{M} K \) (or just by \( \prod^E K \)).

Proposition 12.2.6.

- The following conditions are equivalent:
  1. a bicylinder

\[
\begin{array}{ccc}
E & \times D^\to \times D & \to A \\ {\mathcal{R}} \downarrow & \omega & \downarrow {\mathcal{K}} \\ X & \to \mathcal{M} & \to A
\end{array}
\]

is universal (resp. pointwise universal);
(2) its right exponential transpose
\[
\begin{array}{c}
\ast \\
\mathcal{K} \\
[\mathcal{E}, \mathcal{X}] \\
\end{array}
\xrightarrow{\mathcal{R}}
\begin{array}{c}
\mathcal{D} \times \mathcal{D} \\
\mathcal{K} \\
[\mathcal{E}, \mathcal{A}] \\
\end{array}
\]

(see Definition 4.7.5) is a universal (resp. pointwise universal) extraordinary cylinder;
(3) its left exponential transpose
\[
\begin{array}{c}
\mathcal{E} \\
\mathcal{R} \\
\mathcal{X} \\
\end{array}
\xrightarrow{\mathcal{K}^{-1}}
\begin{array}{c}
\mathcal{E} \\
\mathcal{K} \\
[\mathcal{D} \times \mathcal{D}, \mathcal{A}] \\
\end{array}
\]

(see Definition 4.7.5) is an inverse universal (resp. pointwise inverse universal) ordinary cylinder.

The following conditions are equivalent:
(1) a bicylinder
\[
\begin{array}{c}
\mathcal{E} \times \mathcal{D} \times \mathcal{D} \\
\mathcal{K} \\
\mathcal{X} \\
\end{array}
\xrightarrow{\mathcal{R}}
\begin{array}{c}
\mathcal{E} \\
\omega \\
\mathcal{A} \\
\end{array}
\]
is universal (resp. pointwise universal);
(2) its right exponential transpose
\[
\begin{array}{c}
\mathcal{D} \times \mathcal{D} \\
\mathcal{K}^{-1} \\
[\mathcal{E}, \mathcal{X}] \\
\end{array}
\xrightarrow{\mathcal{R}}
\begin{array}{c}
\ast \\
\mathcal{K} \\
[\mathcal{E}, \mathcal{A}] \\
\end{array}
\]

(see Definition 4.7.5) is a universal (resp. pointwise universal) extraordinary cylinder;
(3) its left exponential transpose
\[
\begin{array}{c}
\mathcal{E} \\
\mathcal{K}^{-1} \\
[\mathcal{D} \times \mathcal{D}, \mathcal{X}] \\
\end{array}
\xrightarrow{\mathcal{K}^{-1}}
\begin{array}{c}
\mathcal{E} \\
\mathcal{K} \\
\mathcal{A} \\
\end{array}
\]

(see Definition 4.7.5) is a direct universal (resp. pointwise direct universal) ordinary cylinder.

**Proof.** By the isomorphisms in Remark 4.7.6, \(\omega\) is universal iff \(\omega^{-1}\) is universal iff \(\omega^{-1}\) is inverse universal. Since the left slice of \(\omega\) at each \(e \in \parallel \mathcal{E} \parallel\) is given by the component of \(\omega^{-1}\) at \(e\), \(\omega\) is pointwise universal iff \(\omega^{-1}\) is pointwise inverse universal. By Proposition 12.2.2, \(\omega^{-1}\) is pointwise universal iff \([\omega^{-1}]^T\) is pointwise inverse universal. But \([\omega^{-1}]^T = \omega^{-1}\) by the commutative diagram in Remark 4.7.8(1). \(\square\)

**Remark 12.2.7.** By Proposition 12.2.6 and by the bijectiveness of exponential transposition, we see that given a module \(\mathcal{M} : \mathcal{X} \rightarrow \mathcal{A}\),
- the following are the same thing:
  (1) an \(\mathcal{E}\)-parameterized end of a bifunctor \(\mathcal{K} : \mathcal{E} \times \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{A}\) along \(\mathcal{M}\);
  (2) a pointwise end of \(\mathcal{K}^{-1} : \mathcal{D} \times \mathcal{D} \rightarrow [\mathcal{E}, \mathcal{A}]\) along the module \(\langle \mathcal{E}, \mathcal{M} \rangle\);
  (3) a pointwise lift of \(\mathcal{K}^{-1} : \mathcal{E} \rightarrow [\mathcal{D} \times \mathcal{D}, \mathcal{A}]\) inverse along the module \(\langle \mathcal{D}, \mathcal{M} \rangle\).
- the following are the same thing:
  (1) an \(\mathcal{E}\)-parameterized coend of a bifunctor \(\mathcal{K} : \mathcal{E} \times \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{X}\) along \(\mathcal{M}\);
  (2) a pointwise coend of \(\mathcal{K}^{-1} : \mathcal{D} \times \mathcal{D} \rightarrow [\mathcal{E}, \mathcal{X}]\) along the module \(\langle \mathcal{E}, \mathcal{M} \rangle\);
  (3) a pointwise lift of \(\mathcal{K}^{-1} : \mathcal{E} \rightarrow [\mathcal{D} \times \mathcal{D}, \mathcal{X}]\) direct along the module \(\langle \mathcal{D}, \mathcal{M} \rangle\).

**Proposition 12.2.8.** A pointwise universal bicylinder is universal.
Proof. This is reduced to Proposition 6.5.8 by the equivalence of (1) and (3) in Proposition 12.2.6.

Theorem 12.2.9. (Parameter Theorem for ends). Let M be a module and K be a bifunctor as in Definition 12.2.5.

- Suppose that for each object e ∈ [E], the bifunctor K∗(e) : D × D → A has an end along M and a universal cylinder ωe : r_e ∼ K∗(e) : D ∼ M is chosen. Then there is a unique functor R : E → X with R(e) = r_e such that ω := ([ω_e]_d(e,d)∈[E×D]) forms a bicylinder R ∼ K : E × D ∼ M, and R and ω form an E-parameterized end of K along M.

- Suppose that for each object e ∈ [E], the bifunctor K∗(e) : D × D → X has a coend along M and a universal cylinder ωe : K∗(e) ∼ r_e : D ∼ M is chosen. Then there is a unique functor R : E → A with R(e) = r_e such that ω := ([ω_e]_d(e,d)∈[E×D]) forms a bicylinder K ∼ R : E × D ∼ M, and R and ω form an E-parameterized coend of K along M.

Proof. By the equivalence of (1) and (3) in Proposition 12.2.6, this is reduced to an instance of Theorem 6.5.10 where M is given by the module (D,M).

Theorem 12.2.10. Let M : X → A be a module.

- Suppose that a bifunctor K : E × E × D × D → A has an [E × E]-parameterized end

\[
\begin{array}{ccc}
E × E & \xrightarrow{\kappa} & E × E \times D × D \\
\downarrow{\omega} & & \downarrow{K} \\
X & \in_{\mathcal{M}} & A
\end{array}
\]

along M. Then an end of K (regarded as a bifunctor \([E × D]^\sim × [E × D] → A\)) along M exists if and only if an end of R in X exists. If this is the case, then for any universal cylinder \(\lambda : \kappa ∼ K\) along M, there is a unique universal cylinder \(\pi : \kappa ∼ R\) in X such that the diagram

\[
\begin{array}{ccc}
\pi_e \downarrow & & \kappa(e,e,d,d) \\
\lambda_e,(e,d) \xrightarrow{\kappa} & K(e,e,d,d) \\
R(e,e) \xrightarrow{\rho} & X
\end{array}
\]

commutes for every \((e,d) ∈ [E × D]\).

- Suppose that a bifunctor K : E × E × D × D → X has an [E × E]-parameterized coend

\[
\begin{array}{ccc}
E × E × D × D & \xrightarrow{\kappa} & E × E \\
\downarrow{\omega} & & \downarrow{R} \\
X & \in_{\mathcal{M}} & A
\end{array}
\]

along M. Then a coend of K (regarded as a bifunctor \([E × D]^\sim × [E × D] → X\)) along M exists if and only if a coend of R in A exists. If this is the case, then for any universal cylinder \(\lambda : K ∼ R\) along M, there is a unique universal cylinder \(\pi : R ∼ \rho\) in A such that the diagram

\[
\begin{array}{ccc}
\pi_e \downarrow & & \rho \\
\lambda_e,(e,d) \xrightarrow{\kappa} & K(e,e,d,d) \\
R(e,e) \xrightarrow{\rho} & X
\end{array}
\]

commutes for every \((e,d) ∈ [E × D]\).

Proof. Given a family of X-arrows \(\alpha_e : x → R(e,e)\) indexed by the objects \(e ∈ [E]\), let \(\alpha ∘ \omega\) denote the family of M-arrows \(\alpha_e ∘ \omega_{e,e,d} : x → K(e,e,d,d)\) indexed by the objects \((e,d) ∈ [E × D]\). The proof is complete if we show that the assignment \(\alpha → \alpha ∘ \omega\) yields an isomorphism from the right module \((E,X)(R) : X → *\) (the right slice at R of the module \((E,X) : X → [E × E, X]\)) to the right module \((E × D,M)(K) : X → *\) (the right slice at K of the module \((E × D,M) : X → [[E × D]^\sim × [E × D], A]\)). But this follows from the claim below.
Claim.
(1) The composite $\alpha \circ \omega$ forms a cylinder $x \sim K$ if and only if $\alpha$ forms a cylinder $x \sim R$.
(2) For any cylinder $\xi : x \sim K$ along $\mathcal{M}$, there is a unique cylinder $\alpha : x \sim R$ in $X$ such that $\xi = \alpha \circ \omega$.
(3) For an $X$-arrow $f : y \to x$ and a cylinder $\alpha : x \to R$ in $X$, the associative law

$$f \circ [\alpha \circ \omega] = [f \circ \alpha] \circ \omega$$

holds.

Proof.
(1) Clearly $\alpha_e \circ \omega_{e,e,d}$ is natural in $d$ for each $e \in \|E\|$. By Theorem 4.1.4, it thus suffices to show that for each $d \in \|D\|$, $\alpha_{e'} \circ \omega_{e,e,d}$ is natural in $e$ iff so is $\alpha_e$. For this, let $h : e \to e'$ be an $E$-arrow and consider the diagram

$$\begin{array}{ccc}
x & \xrightarrow{\alpha_e} & R(e,e) \xrightarrow{\omega_{e,e,d}} K(e,e,d,d) \\
\downarrow \alpha_{e'} & & \downarrow \alpha(e,h) \\
R(e',e') & \xrightarrow{\omega_{e',e',d}} & R(e,e') \xrightarrow{\omega_{e,e',d}} K(e',e',d,d)
\end{array}$$

we need to verify that the top left square commutes iff the outer square commutes. By the naturality of $\omega$, the right and bottom trapezoids commute. Hence if the top left square commutes, then the outer square commutes. Now suppose that the outer square commutes. Then a simple diagram chasing shows that

$$\alpha_{e'} \circ R(e,h) \circ \omega_{e,e,d} = \alpha_{e'} \circ R(h,e') \circ \omega_{e,e',d}$$

by the universality of $\omega_{e,e',d}$.

(2) Since $\omega$ is pointwise universal, there is a unique family of $X$-arrows $\alpha_e : x \to R(e,e)$ such that $\alpha_e \circ \omega_{e,e,d} = \xi_{e,d}$ for each $(e,d) \in \|E \times D\|$. By the result above, $\alpha := (\alpha_e)_{e \in \|E\|}$ forms a cylinder $x \sim R$.

(3) Easily verified.

\[\square\]

Corollary 12.2.11. Let $M : X \to A$ be a module.

- Suppose that a bifunctor $K : E^\times E \times D^\times D \to A$ has an $[E^\times E]$-parameterized end and a $[D^\times D]$-parameterized end

$$\begin{array}{ccc}
E^\times E & \xrightarrow{\omega} & E^\times E \times D^\times D \\
\downarrow R & & \downarrow K \\
X & \xrightarrow{M} & A \\
\end{array} \qquad \begin{array}{ccc}
E^\times E \times D^\times D & \xrightarrow{\omega'} & E^\times E \times D^\times D \\
\downarrow R' & & \downarrow K \\
X & \xrightarrow{M} & A \\
\end{array}$$

along $\mathcal{M}$. Then an end of $R$ in $X$ exists if and only if an end of $R'$ in $X$ exists. If this is the case, then there exist a universal cylinder $\pi : r \sim R$ and $\pi' : r \sim R'$ in $X$ such that the diagram

$$\begin{array}{ccc}
r & \xrightarrow{\pi'} & R'(d,d) \\
\downarrow \pi_e & & \downarrow \omega_{e,d,d} \\
R(e,e) & \xrightarrow{\omega_{e,e,d}} & K(e,e,d,d)
\end{array}$$

commutes for every $(e,d) \in \|E \times D\|$.
Suppose that a bifunctor $K : E^\times E \times D^\times D \to X$ has an $[E^\times E]$-parameterized coend and a $[D^\times D]$-parameterized coend

\[
\begin{array}{c}
E^\times E \times D^\times D \xrightarrow{\omega} E^\times E \\
\xrightarrow{\kappa} M \xrightarrow{\lambda} A \\
E^\times E \times D^\times D \xrightarrow{\omega'} E^\times E \\
\xrightarrow{\kappa'} M \xrightarrow{\lambda'} A
\end{array}
\]

along $M$. Then a coend of $R$ in $A$ exists if and only if a coend of $R'$ in $A$ exists. If this is the case, then there exist a universal cylinder $\pi : R \to r$ and $\pi' : R' \to r$ in $A$ such that the diagram

\[
\begin{array}{c}
K(e, e, d, d) \xrightarrow{\omega_{e,e,d}} R(e, e) \\
\xrightarrow{\omega_{e,d,d}} R'(d, d) \xrightarrow{\pi_d} r
\end{array}
\]

commutes for every $(e, d) \in \|E \times D\|$.

Proof. Immediate from Theorem 12.2.10.

\(\square\)

Remark 12.2.12.

1. The results in Theorem 12.2.10 and Corollary 12.2.11 are expressed by

\[
\prod_{e \in E} \prod_{d \in D} K \cong \prod_{e \in E} \prod_{d \in D} K \cong \prod_{e \in E} \prod_{d \in D} K \\
\text{resp.} \prod_{e \in E} \prod_{d \in D} K \cong \prod_{e \in E} \prod_{d \in D} K \cong \prod_{e \in E} \prod_{d \in D} K
\]

or, more informatively, by

\[
\prod_{e \in E} \prod_{d \in D} K(e, e, d, d) \cong \prod_{(e, d) \in E \times D} K(e, e, d, e) \cong \prod_{d \in D} \prod_{e \in E} K(e, e, d, d)
\]

(resp.

\[
\prod_{e \in E} \prod_{d \in D} K(e, e, d, d) \cong \prod_{(e, d) \in E \times D} K(e, e, d, d) \cong \prod_{d \in D} \prod_{e \in E} K(e, e, d, d)
\]

), and called the Fubini theorem for ends (resp. coends).

2. Since a limit is regarded as an end with a dummy variable, the result in (1) above includes the corresponding facts for limits: we have the end-limit interchange property

\[
\prod_{e \in E} \prod_{d \in D} K(e, d, d) \cong \prod_{(e, d) \in E \times D} K(e, e, d, d) \cong \prod_{d \in D} \prod_{e \in E} K(e, d, d)
\]

by regarding $K(e, d, d)$ as $K(e, e, d, d)$ dummy in the first variable $e$, and then the limits interchange property

\[
\prod_{e \in E} \prod_{d \in D} K(e, d) \cong \prod_{(e, d) \in E \times D} K(e, e, d) \cong \prod_{d \in D} \prod_{e \in E} K(e, d)
\]

by regarding $K(e, d)$ as $K(e, e, d)$ dummy in the first variable $d$. Dually, we have the coend-colimit interchange property and the colimits interchange property.

12.3. Ends of modules

Theorem 12.3.1. Given a small endomodule $M : E \to E$, i.e. a bifunctor $M : E^\times E \to \text{Set}$ with $E$ small,
an end of $\mathcal{M}$ is given by an equalizer $\pi$ as in

$$
\begin{array}{c}
\Pi_{e'} \mathcal{E} \longrightarrow \pi \\
\Pi e'(\mathcal{M}) \downarrow \Pi e(\mathcal{E}) e' \quad \Pi e(\mathcal{E}) e \\
\end{array}
$$

where

$$
\chi_{e,e'} : \Pi e'(\mathcal{M}) e' \rightarrow [e(\mathcal{E}) e', e(\mathcal{M}) e'] ; m \mapsto [h \mapsto m \circ h]
$$

and

$$
\chi'_{e,e'} : \Pi e(\mathcal{M}) e \rightarrow [e(\mathcal{E}) e', e(\mathcal{M}) e'] ; m \mapsto [h \mapsto h \circ m]
$$

are the functions given by the exponential transpose of the composition between $\mathcal{M}$-arrows and $\mathcal{E}$-arrows.

a coend of $\mathcal{M}$ is given by a coequalizer $\pi$ as in

$$
\begin{array}{c}
\Pi e(\mathcal{M}) e \longrightarrow \pi \\
\Pi e'(\mathcal{M}) e' \downarrow \Pi e(\mathcal{E}) e' \quad \Pi e(\mathcal{E}) e \\
\end{array}
$$

where

$$
\chi_{e,e'} : \Pi e'(\mathcal{M}) e' \times e(\mathcal{E}) e' \rightarrow e(\mathcal{M}) e ; (m, h) \mapsto h \circ m
$$

and

$$
\chi'_{e,e'} : \Pi e(\mathcal{M}) e \times e(\mathcal{E}) e' \rightarrow e'(\mathcal{M}) e' ; (m, h) \mapsto m \circ h
$$

are the functions given by the composition between $\mathcal{M}$-arrows and $\mathcal{E}$-arrows.

Proof. By the claim below, an end of $\mathcal{M}$ is given by a universal fork on $\Pi e(\mathcal{E})(\chi_{e,e'})_{e' \in \mathcal{E}}$ and $\Pi e(\mathcal{E})(\chi'_{e,e'})_{e' \in \mathcal{E}}$, i.e. by an equalizer of them.

Claim. Given a small set $\mathcal{S}$, a family of functions $\alpha_e : \mathcal{S} \rightarrow e(\mathcal{M}) e$, one for each object $e \in \mathcal{E}$, forms a cylinder $\mathcal{S} \sim \mathcal{M}$ if and only if the diagram

$$
\begin{array}{c}
\mathcal{S} \longrightarrow (\alpha_e)_{e \in \mathcal{E}} \\
\mathcal{S} \downarrow \Pi e(\mathcal{E}) e \\
\Pi e'(\mathcal{E}) e' \longrightarrow (\alpha_e)_{e' \in \mathcal{E}} \\
\end{array}
$$

commutes, i.e. if and only if the function $(\alpha_e)_{e \in \mathcal{E}} : \mathcal{S} \rightarrow \Pi e(\mathcal{E}) e$ forms a fork on the functions $
\Pi e(\mathcal{E}) e$ and $\Pi e'(\mathcal{E})(\chi_{e,e'})_{e' \in \mathcal{E}}$.

Proof. The diagram in the claim commutes iff the diagram

$$
\begin{array}{c}
\mathcal{S} \longrightarrow \alpha_e \\
\mathcal{S} \downarrow \Pi e'(\mathcal{E}) e' \\
\Pi e(\mathcal{E}) e \longrightarrow (\alpha_e)_{e' \in \mathcal{E}} \\
\end{array}
$$

is given by a universal fork on $\Pi e(\mathcal{E}) e$ and $\Pi e'(\mathcal{E})(\chi_{e,e'})_{e' \in \mathcal{E}}$.
commutes for each \(e \in \|E\|\), and this diagram commutes iff the diagram

\[
\begin{array}{c}
S \xrightarrow{\alpha_e} e (\mathcal{M}) e \\
\downarrow \alpha_{e'} \\
e' (\mathcal{M}) e' \xrightarrow{\chi_{e,e'}} [e (E) e', e (\mathcal{M}) e']
\end{array}
\]

commutes for each \(e' \in \|E\|\), and this diagram commutes iff the diagram

\[
\begin{array}{c}
S \xrightarrow{\alpha_e} e (\mathcal{M}) e \\
\downarrow \alpha_{e'} \\
e' (\mathcal{M}) e' \xrightarrow{e (\mathcal{M}) h} e (\mathcal{M}) e'
\end{array}
\]

commutes for each \(E\)-arrow \(h : e \to e'\) since for any \(s \in S\),

\[
s \cdot [\alpha_e \circ e (\mathcal{M}) h] = (s \cdot \alpha_e) \circ h = h \cdot [(s \cdot \alpha_e) : \chi_{e,e'}] = h \cdot [s \cdot [\alpha_e \circ \chi_{e,e'}]]
\]

and

\[
s \cdot [\alpha_{e'} \circ h (\mathcal{M}) e] = h \circ (s \cdot \alpha_{e'}) = h \cdot [(s \cdot \alpha_{e'}) : \chi_{e,e'}] = h \cdot [s \cdot [\alpha_{e'} \circ \chi_{e,e'}]]
\]

by the definitions of \(\chi_{e,e'}\) and \(\chi'_{e,e'}\).

\[\square\]

**Corollary 12.3.2.** For a small endomodule \(\mathcal{M} : E \to E\), the set of frames of \(\mathcal{M}\) gives an end of \(\mathcal{M}\) with the universal cylinder

\[
\pi : \prod_E \mathcal{M} \to \mathcal{M} : E \to \text{Set}
\]

defined by

\[
\pi_e (\alpha) = \alpha_e
\]

such that the component of \(\pi\) at \(e \in \|E\|\) sends each frame \(\alpha\) of \(\mathcal{M}\) to its component at \(e\).

**Proof.** This follows from the observation that an element \(\alpha\) of the product \(\prod_{e \in \|E\|} e (\mathcal{M}) e\) is a frame of \(\mathcal{M}\) iff it lies in the equalizer in Theorem 12.3.1. \[\square\]

**Remark 12.3.3.** For a small endomodule \(\mathcal{M} : E \to E\), the symbol \(\prod_E \mathcal{M}\) thus denotes both the set of frames of \(\mathcal{M}\) and an end of \(\mathcal{M}\).

**Definition 12.3.4.** Given a pair of left modules \(\mathcal{M}, \mathcal{N} : * \to E\), the endomodule

\[
\langle \mathcal{M} \circ \mathcal{N} \rangle : E \to E
\]
is defined by the composition

\[
E \xrightarrow{\mathcal{M}} \text{Set} \xrightarrow{(\text{Set})} \text{Set} \xrightarrow{\mathcal{N}} E
\]

where \((\text{Set})\) denotes the hom of the category \(\text{Set}\); that is, for \(e, d \in E\),

\[
e (\mathcal{M} \circ \mathcal{N}) d := ([\mathcal{M} e] (\text{Set}) ([\mathcal{N} d] = ([\mathcal{M} e] (\mathcal{N}) d)
\]

(for \(S, T \in \text{Set}\), we write \([S, T]\) instead of \(S (\text{Set}) T\) (see Remark 1.1.22(6))).

**Remark 12.3.5.** A frame \(\Phi\) of the endomodule \(\langle \mathcal{M} \circ \mathcal{N} \rangle : E \to E\) is the same thing as a natural transformation \(\Phi : \mathcal{M} \to \mathcal{N} : E \to \text{Set}\), i.e, a left module morphism \(\Phi : \mathcal{M} \to \mathcal{N} : * \to E\).
Proposition 12.3.6. Let \( M, N : \ast \to E \) be a pair of left modules. Given a small set \( S \), there is a canonical bijection between the set of cylinders

\[
\begin{array}{ccc}
\ast & \to & E^\ast \times E \\
S & \downarrow \alpha & \downarrow M \circ N \\
\text{Set} & \to & \text{Set}
\end{array}
\]

and the set of conical cells

\[
\begin{array}{ccc}
\ast & \to & M \\
S & \downarrow \alpha & \downarrow N \\
\text{Set} & \to & \text{Set}
\end{array}
\]

, with each component

\((\alpha) e : (M) e \to [S, (N) e]\)

of a cell \((\alpha)\) given by the simple transpose of the component

\([\alpha] e : S \to [(M) e, (N) e]\)

of the corresponding frame \([\alpha]\). Moreover, the bijection is natural in \( S \).

Proof. Immediate since the transposition

\([A, [S, B]] \to [S, [A, B]]\)

is bijective and natural in all three variables.

Remark 12.3.7. Hence an \( M \)-weighted limit of \( N \) in \( \text{Set} \) is the same thing as an end of \( (M \circ N) \) in \( \text{Set} \):

\[
\prod^M N \cong \prod (M \circ N)
\]

Proposition 12.3.8. If \( M, N : \ast \to E \) are small left modules, then an end of the endomodule \( (M \circ N) : E \to E \) exists and is given by the set of module morphisms \( M \to N : \ast \to E \):

\[
\prod E (M \circ N) \cong M (\ast : E) N
\]

Proof. Since \( M \) and \( N \) are small, so is the endomodule \( (M \circ N) \). Hence the set of frames of \( (M \circ N) \) gives an end of \( (M \circ N) \) by Corollary 12.3.2. But the set of frames of \( (M \circ N) \) is the same thing as the set of left module morphisms \( M \to N \) (see Remark 12.3.5).

Theorem 12.3.9. For any left module \( M : \ast \to E \) and any object \( r \in \| E \| \), there is an isomorphism

\[
(M) r \cong \prod_E (r(E) \circ M)
\]

Proof. By Theorem 5.2.10 and Proposition 12.3.8,
Remark 12.3.10.
(1) To apply Proposition 12.3.8, the universe may have to be enlarged temporarily so that \( \mathcal{E} \) becomes small. However, the result holds in the original universe, in which \( \mathcal{M} \) is locally small.
(2) Some authors call the isomorphism in Theorem 12.3.9 the end form of the Yoneda lemma.

Definition 12.3.11. Given a right module \( \mathcal{M} : \mathcal{E} \to \ast \) and a left module \( \mathcal{N} : \ast \to \mathcal{E} \), their product is the endomodule
\[
\langle \mathcal{M} \times \mathcal{N} \rangle : \mathcal{E} \to \mathcal{E}
\]
defined by the composition
\[
\mathcal{E}^\times \times \mathcal{E} \xrightarrow{\mathcal{M} \times \mathcal{N}} \text{Set} \times \text{Set} \xrightarrow{\times} \text{Set}
\]
where \( \text{Set} \times \text{Set} \to \text{Set} \) is the set theoretic product; that is
\[
e(\mathcal{M} \times \mathcal{N}) \ d := e(\mathcal{M}) \times (\mathcal{N}) \ d
\]
for \( e, d \in \mathcal{E} \).

Proposition 12.3.12. Let \( \mathcal{M} : \mathcal{E} \to \ast \) be a right module and \( \mathcal{N} : \ast \to \mathcal{E} \) be a left module. Given a small set \( \mathcal{S} \), there is a canonical bijection between the set of cylinders
\[
\mathcal{E}^\times \times \mathcal{E} \xrightarrow{\mathcal{M} \times \mathcal{N}} \text{Set} \xrightarrow{\times} \text{Set}
\]
and the set of conical cells
\[
\mathcal{E} \xrightarrow{\mathcal{M}} \text{Set} \xrightarrow{\times} \text{Set}
\]
with each component
\[
e(\alpha) : e(\mathcal{M}) \to [(\mathcal{N}) \ e, \mathcal{S}]
\]
of a cell \( \alpha \) given by the exponential transpose of the component
\[
[\alpha]_e : e(\mathcal{M}) \times (\mathcal{N}) \ e \to \mathcal{S}
\]
of the corresponding frame \( [\alpha] \). Moreover, the bijection is natural in \( \mathcal{S} \).

Proof. Immediate since the exponential transposition
\[
[\mathcal{A} \times \mathcal{B}, \mathcal{S}] \xrightarrow{\times} [\mathcal{A}, [\mathcal{B}, \mathcal{S}]]
\]
is bijective and natural in all three variables. \( \square \)

Remark 12.3.13. Hence an \( \mathcal{M} \)-weighted colimit of \( \mathcal{N} \) in \( \text{Set} \) is the same thing as a coend of \( \langle \mathcal{M} \times \mathcal{N} \rangle \) in \( \text{Set} \):
\[
\coprod \mathcal{N} \cong \coprod \langle \mathcal{M} \times \mathcal{N} \rangle
\]
Theorem 12.3.14.

- Given a right module \( \mathcal{M} : \mathbf{E} \to \ast \) and an object \( r \in \mathcal{E} \), there is a canonical universal cylinder

\[
\chi : (\mathcal{M} \times r \mathcal{E}) \to r \mathcal{M}
\]

with each component \( \chi_e : e(\mathcal{M}) \times r \mathcal{E} \to r \mathcal{M} \) given by the assignment \( (m, h) \mapsto h \circ m \); for a set \( S \) and a cylinder \( \alpha : (\mathcal{M} \times r \mathcal{E}) \to S \), the adjunct \( \chi \alpha : r \mathcal{M} \to S \) is given by the partial evaluation of \( \alpha_r : r \mathcal{M} \times r \mathcal{E} \to S \) at \( 1_r \), i.e. by the assignment \( m \mapsto \alpha_r(m, 1_r) \).

- Given a left module \( \mathcal{M} : \ast \to \mathbf{E} \) and an object \( r \in \mathcal{E} \), there is a canonical universal cylinder

\[
\chi : (\mathcal{E} \times \mathcal{M}) \to \mathcal{M} r
\]

with each component \( \chi_e : e \mathcal{E} \times \mathcal{M} \to \mathcal{M} r \) given by the assignment \( (h, m) \mapsto m \circ h \); for a set \( S \) and a cylinder \( \alpha : (\mathcal{E} \times \mathcal{M}) \to S \), the adjunct \( \chi \alpha : \mathcal{M} r \to S \) is given by the partial evaluation of \( \alpha_r : \mathcal{E} r \times \mathcal{M} r \to S \) at \( 1_r \), i.e. by the assignment \( m \mapsto \alpha_r(1_r, m) \).

Proof. The naturality of \( \chi \) follows immediately from the functoriality of \( \mathcal{M} \). To prove the universality of \( \chi \), it suffices to show that \( \chi \alpha \) defined as above gives a unique factorization \( \alpha = \chi \circ \chi \alpha \). Let \( h : r \to e \) be an \( \mathbf{E} \)-arrow and consider the diagram

\[
\begin{array}{ccc}
e(\mathcal{M}) \times r \mathcal{E} & \xrightarrow{1 \times r \mathcal{E}h} & e(\mathcal{M}) \times r \mathcal{E} \\
h(\mathcal{M}) \times 1 & \downarrow & \downarrow \chi_\alpha \\
\mathcal{M} \times r \mathcal{E} & \xrightarrow{\chi_r} & \mathcal{M} r \\
\end{array}
\]

. The inner commutative square maps \( (m, 1_r) \in e(\mathcal{M}) \times r \mathcal{E} \) as follows:

\[
\begin{array}{ccc}
(m, 1_r) & \xrightarrow{1 \times r \mathcal{E}h} & (m, h) \\
h(\mathcal{M}) \times 1 & \downarrow \chi_e & \downarrow \chi_e \\
(h \circ m, 1_r) & \xrightarrow{\chi_r} & h \circ m \\
\end{array}
\]

; hence, by the commutativity of the outer diagram and the definition of \( \chi \alpha \),

\[
[\chi \alpha] (h \circ m) = \alpha_r(h \circ m, 1_r) = \alpha_e(m, h)
\]

. The diagram

\[
\begin{array}{ccc}
e(\mathcal{M}) \times r \mathcal{E} & \xrightarrow{\chi_e} & r \mathcal{M} \\
\downarrow \alpha_e & & \downarrow \chi \alpha \\
\mathcal{M} r & \xrightarrow{\chi_r} & S \\
\end{array}
\]

thus commutes for arbitrary \( e \in \mathcal{E} \); hence \( \alpha = \chi \circ \chi \alpha \). Since \( \chi_r \) maps \( (m, 1_r) \) to \( 1_r \circ m = m \), in order for the diagram

\[
\begin{array}{ccc}
\mathcal{M} \times r \mathcal{E} & \xrightarrow{\chi_r} & \mathcal{M} r \\
\downarrow \alpha_r & & \downarrow \chi \alpha \\
\mathcal{M} r & \xrightarrow{\chi \alpha} & S \\
\end{array}
\]

to commute, \( \chi \alpha \) must map \( m \) to \( \alpha_r(m, 1_r) \); the uniqueness of \( \chi \alpha \) follows. \( \square \)
Corollary 12.3.15.

- For any right module $\mathcal{M} : E \to *$ and any object $e \in \|E\|$, there is an isomorphism
  \[ r(\mathcal{M}) \cong \bigsqcup_E (\mathcal{M} \times r(E)) \]

- For any left module $\mathcal{M} : * \to E$ and any object $r \in \|E\|$, there is an isomorphism
  \[ (\mathcal{M}) r \cong \bigsqcup_E (r(E) r \times \mathcal{M}) \]

Proof. We have shown in Theorem 12.3.14 that there is a canonical universal cylinder $(\mathcal{M} \times r(E)) \to r(\mathcal{M})$. \qed

Remark 12.3.16. Some authors call the isomorphisms in Corollary 12.3.15 the coend form of the Yoneda lemma (cf. Remark 12.3.10(2)). [Lor15] contains a slick proof for the isomorphisms using the Yoneda lemma without constructing a universal cylinder.

Definition 12.3.17. Given a small right module $\mathcal{M} : E \to *$ and a small left module $\mathcal{N} : * \to E$, their tensor product, written $\mathcal{M} \otimes_E \mathcal{N}$ or just $\mathcal{M} \otimes \mathcal{N}$, is the quotient set defined by

\[ \mathcal{M} \otimes_E \mathcal{N} = \bigsqcup_{e \in \|E\|} e(\mathcal{M} \times \mathcal{N}) e \]

, where $\approx$ is the equivalence relation generated by all $(m, n) \approx (m', n')$ such that the diagram

commutes for some $E$-arrow $h$.

Remark 12.3.18. An amalgamation of a right module $\mathcal{M} : E \to *$ and a left module $\mathcal{N} : * \to E$ is a pushout

\[ \begin{array}{ccc}
E & \xrightarrow{\mathcal{N}} & [\mathcal{N}] \\
\mathcal{M} \downarrow & & \downarrow \\
[\mathcal{M}] & \longrightarrow & [\mathcal{M}] +_E [\mathcal{N}]
\end{array} \]

in CAT. For small $\mathcal{M}$ and $\mathcal{N}$, the small category $[\mathcal{M}] +_E [\mathcal{N}]$ is constructed by “pasting” the collage categories $[\mathcal{M}]$ and $[\mathcal{N}]$ together in the obvious way and defining the hom-set between the two endpoints by the tensor product $\mathcal{M} \otimes_E \mathcal{N}$.

Proposition 12.3.19. If $\mathcal{M}$ and $\mathcal{N}$ are small, then a coend of the endomodule $(\mathcal{M} \times \mathcal{N}) : E \to E$ exists and is given by the tensor product of $\mathcal{M}$ and $\mathcal{N}$:

\[ \bigsqcup_E (\mathcal{M} \times \mathcal{N}) \cong \mathcal{M} \otimes_E \mathcal{N} \]

Proof. The equivalence relation $\approx$ in Definition 12.3.17 is given by the coequalizer $\pi$ as in

\[ \begin{array}{ccc}
\bigsqcup_{e \in \|E\|} (\mathcal{M} \times \mathcal{N}) e \times e (\mathcal{E}) e' & \xrightarrow{\bigsqcup_{e \in \|E\|} (\chi_{e,e'})_{e \in \|E\|}} & \bigsqcup_{e \in \|E\|} e(\mathcal{M} \times \mathcal{N}) e' \\
\bigsqcup_{e \in \|E\|} (\chi_{e,e'})_{e \in \|E\|} & \downarrow & \pi \\
\bigsqcup_{e \in \|E\|} e(\mathcal{M} \times \mathcal{N}) e & \xrightarrow{\pi} & \mathcal{M} \otimes_E \mathcal{N}
\end{array} \]

, where $\chi_{e,e'}$ denotes the function $e' (\mathcal{M} \times \mathcal{N}) e \times e (\mathcal{E}) e' \to e(\mathcal{M} \times \mathcal{N}) e$ given by the composition $((m', n), h) \mapsto (h \circ m', n)$, and $\chi_{e,e'}$ denotes the function $e' (\mathcal{M} \times \mathcal{N}) e \times e (\mathcal{E}) e' \to e' (\mathcal{M}) e'$ given by the composition $((m', n), h) \mapsto (m', n \circ h)$. The assertion thus follows from Theorem 12.3.1. \qed
Remark 12.3.20. Given a small right module \( \mathcal{M} : E \to \ast \), there are bijections
\[
((E)r)(E):(M) \cong r(M) \cong \mathcal{M} \otimes_E r(E)
\]
, dually, for a small left module \( \mathcal{M} : \ast \to E \),
\[
(r(E)) (\cdot E):(M) \cong (M) r \cong (E) r \otimes_E \mathcal{M}
\]
; the bijections on the left are the Yoneda lemma, and the bijections on the right follow from Corollary 12.3.15 and Proposition 12.3.19. Some authors call the latter bijections the coYoneda lemma (cf. Remark 11.1.6(4)).

Theorem 12.3.21. A left small module \( \mathcal{N} : \ast \to E \) has a pointwise left Kan extension along the right Yoneda functor \( E \rightharpoonup \), given by the tensor product as shown:

\[
\begin{array}{ccc}
E & \xrightarrow{E \rightharpoonup} & [E:] \\
\mathcal{N} \downarrow & \mu & \downarrow \otimes \mathcal{N} \\
Set & \xrightarrow{\otimes \mathcal{N}} & \Set
\end{array}
\]

Moreover, \( \mu \) is a natural isomorphism.

Proof. By Corollary 11.5.9, it suffices to show that for any right module \( \mathcal{M} : E \to \ast \), \( \mathcal{M} \otimes \mathcal{N} \) gives an \( \mathcal{M} \)-weighted colimit of \( \mathcal{N} \). But since a \( \mathcal{M} \)-weighted colimit of \( \mathcal{N} \) is the same thing as a coend of \( \langle \mathcal{M} \times \mathcal{N} \rangle \) (see Remark 12.3.13), this follows from Proposition 12.3.19. Since the Yoneda functor \( E \rightharpoonup \) is fully faithful, the second assertion follows from Theorem 11.4.7. \( \Box \)
A. List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Usage</th>
<th>Meaning</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \circ )</td>
<td>( f \circ g )</td>
<td>composite of ( f ) and ( g )</td>
<td>0.2, 0.9</td>
</tr>
<tr>
<td>( \circ )</td>
<td>( g \circ f )</td>
<td>composite of ( f ) and ( g )</td>
<td>0.2, 0.9</td>
</tr>
<tr>
<td>( : )</td>
<td>( c : F )</td>
<td>value of ( F ) at ( c )</td>
<td>0.5, 0.9</td>
</tr>
<tr>
<td>( \cdot )</td>
<td>( \cdot c )</td>
<td>value of ( F ) at ( c )</td>
<td>0.5, 0.9</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( \sim C )</td>
<td>opposite category of ( C )</td>
<td>0.6</td>
</tr>
<tr>
<td>( f^\sim )</td>
<td>( b \to a )</td>
<td>opposite of ( f : a \to b )</td>
<td>0.6</td>
</tr>
<tr>
<td>( \vee )</td>
<td>( \vee )</td>
<td>opposite module of ( M )</td>
<td>1.1.34</td>
</tr>
<tr>
<td>( \triangleright )</td>
<td>( \triangleright K )</td>
<td>right exponential transpose of ( K )</td>
<td>0.11</td>
</tr>
<tr>
<td>( \triangleright )</td>
<td>( \triangleright \alpha )</td>
<td>right exponential transpose of ( \alpha )</td>
<td>4.7.5, 4.8.5</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \downarrow K )</td>
<td>left exponential transpose of ( K )</td>
<td>0.11</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \downarrow \alpha )</td>
<td>left exponential transpose of ( \alpha )</td>
<td>4.7.5, 4.8.5</td>
</tr>
<tr>
<td>( \uparrow )</td>
<td>( \uparrow K )</td>
<td>simple transpose of ( K )</td>
<td>0.11</td>
</tr>
<tr>
<td>( \uparrow )</td>
<td>( \uparrow \alpha )</td>
<td>simple transpose of ( \alpha )</td>
<td>4.7.7, 4.8.7</td>
</tr>
<tr>
<td>( ! )</td>
<td>( !_E )</td>
<td>unique functor ( E \to )</td>
<td>0.17</td>
</tr>
<tr>
<td>( ! )</td>
<td>( !_E \times D )</td>
<td>projection ( E \times D \to D )</td>
<td>0.17</td>
</tr>
<tr>
<td>( ! )</td>
<td>( E \times !_D )</td>
<td>projection ( E \times D \to D )</td>
<td>0.17</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( \Delta_D c )</td>
<td>constant functor ( c )</td>
<td>0.18</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( \Delta_E )</td>
<td>constant right module ( c )</td>
<td>1.1.29</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( \Delta_E^* )</td>
<td>constant left module ( c )</td>
<td>1.1.29</td>
</tr>
<tr>
<td>( \parallel )</td>
<td>( \parallel C \parallel )</td>
<td>set of objects of ( C )</td>
<td>0.3</td>
</tr>
<tr>
<td>( \to )</td>
<td>( M : X \to \ast )</td>
<td>right module ( M ) over ( X )</td>
<td>1.1.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( K : X \to \ast )</td>
<td>right comma ( K ) over ( X )</td>
<td>3.2.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( M : \ast \to A )</td>
<td>left module ( M ) over ( A )</td>
<td>1.1.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( K : \ast \to A )</td>
<td>left comma ( K ) over ( A )</td>
<td>3.2.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( M : X \to A )</td>
<td>module ( M ) from ( X ) to ( A )</td>
<td>1.1.11</td>
</tr>
<tr>
<td>( \to )</td>
<td>( K : X \to A )</td>
<td>comma ( K ) from ( X ) to ( A )</td>
<td>3.2.7</td>
</tr>
<tr>
<td>( \to )</td>
<td>( M : X \to A )</td>
<td>morphism ( M \to N ) of right modules</td>
<td>1.1.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( M : X \to A )</td>
<td>morphism ( M \to N ) of left modules</td>
<td>1.1.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( M : X \to A )</td>
<td>morphism ( M \to N ) of modules</td>
<td>1.1.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( S \to M : E \to \ast )</td>
<td>cone from a set ( S ) to a right module ( M )</td>
<td>4.10.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( S \to M : E \to \ast )</td>
<td>cone from a set ( S ) to a left module ( M )</td>
<td>4.10.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( L \to M : E \to A )</td>
<td>wedge from a right module ( L ) to a module ( M )</td>
<td>4.10.1</td>
</tr>
<tr>
<td>( \to )</td>
<td>( L \to M : E \to A )</td>
<td>wedge from a left module ( L ) to a module ( M )</td>
<td>4.10.1</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( \sim x )</td>
<td>arrow in a right module</td>
<td>1.1.7</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( \sim )</td>
<td>arrow in a left module</td>
<td>1.1.7</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( \sim x )</td>
<td>arrow in a two-sided module</td>
<td>1.1.17</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( [\cdot] )</td>
<td>category of right modules over ( X )</td>
<td>1.1.3</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( [\cdot] A )</td>
<td>category of left modules over ( A )</td>
<td>1.1.3</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( [\cdot] K )</td>
<td>precomposition with ( K )</td>
<td>1.1.3</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( [\cdot] K )</td>
<td>precomposition with ( K )</td>
<td>1.1.3</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( [\cdot] A )</td>
<td>category of modules ( X \to A )</td>
<td>1.1.13</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( [\cdot] T )</td>
<td>category of modules ( S \times T )</td>
<td>1.1.13</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( [\cdot] M )</td>
<td>category of cells ( M \to N )</td>
<td>1.3.2</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( (C) )</td>
<td>hom of a category ( C )</td>
<td>1.1.21</td>
</tr>
<tr>
<td>Symbol</td>
<td>Usage</td>
<td>Meaning</td>
<td>Reference</td>
</tr>
<tr>
<td>--------</td>
<td>-------</td>
<td>---------</td>
<td>-----------</td>
</tr>
<tr>
<td>(H)</td>
<td>hom of a functor H</td>
<td>1.2.27</td>
<td></td>
</tr>
<tr>
<td>(τ)</td>
<td>hom of a natural transformation τ</td>
<td>1.3.5</td>
<td></td>
</tr>
<tr>
<td>(X) x</td>
<td>representable right module of x</td>
<td>2.3.2</td>
<td></td>
</tr>
<tr>
<td>a (A)</td>
<td>representable left module of a</td>
<td>2.3.2</td>
<td></td>
</tr>
<tr>
<td>(X) G</td>
<td>corepresentable module of G : A → X</td>
<td>2.3.8</td>
<td></td>
</tr>
<tr>
<td>F (A)</td>
<td>representable module of F : X → A</td>
<td>2.3.8</td>
<td></td>
</tr>
<tr>
<td>⟨.,⟩</td>
<td>hom of category [A, B]</td>
<td>1.1.22</td>
<td></td>
</tr>
<tr>
<td>(J, M)</td>
<td>module of cells J → M</td>
<td>1.2.8</td>
<td></td>
</tr>
<tr>
<td>(J, Φ)</td>
<td>postcomposition with Φ</td>
<td>1.2.23</td>
<td></td>
</tr>
<tr>
<td>(E, M)</td>
<td>module of cylinders E ⊸ M</td>
<td>4.3.5, 4.4.5</td>
<td></td>
</tr>
<tr>
<td>(E, Φ)</td>
<td>postcomposition with Φ</td>
<td>4.3.15, 4.4.15</td>
<td></td>
</tr>
<tr>
<td>(H, M)</td>
<td>precomposition with H</td>
<td>4.3.27</td>
<td></td>
</tr>
<tr>
<td>⟨.⟩</td>
<td>hom of category [X:]</td>
<td>1.1.22</td>
<td></td>
</tr>
<tr>
<td>: A</td>
<td>hom of category [: A]</td>
<td>1.1.22</td>
<td></td>
</tr>
<tr>
<td>(X : A)</td>
<td>hom of category [X : A]</td>
<td>1.1.22</td>
<td></td>
</tr>
<tr>
<td>[M]</td>
<td>collage category of a collage M</td>
<td>3.1.1</td>
<td></td>
</tr>
<tr>
<td>Φ</td>
<td>collage functor of a collage morphism/cell Φ</td>
<td>3.1.3, 3.1.5</td>
<td></td>
</tr>
<tr>
<td>[K]</td>
<td>collage category of a comma K</td>
<td>3.2.24</td>
<td></td>
</tr>
<tr>
<td>[K]</td>
<td>comma category of a comma K</td>
<td>3.2.7</td>
<td></td>
</tr>
<tr>
<td>[Φ]</td>
<td>comma functor of a comma morphism/cell Φ</td>
<td>3.2.9, 3.2.13</td>
<td></td>
</tr>
<tr>
<td>[M]</td>
<td>comma category of a module Morphism/cell Φ</td>
<td>3.2.17</td>
<td></td>
</tr>
<tr>
<td>[M]</td>
<td>comma category of a module M</td>
<td>3.2.15</td>
<td></td>
</tr>
<tr>
<td>*</td>
<td>module of cones *E ⊸ M</td>
<td>4.6.5</td>
<td></td>
</tr>
<tr>
<td>(E*, M)</td>
<td>module of cones E* ⊸ M</td>
<td>4.6.5</td>
<td></td>
</tr>
<tr>
<td>(E*, Φ)</td>
<td>postcomposition with Φ</td>
<td>4.6.15</td>
<td></td>
</tr>
<tr>
<td>(*H, M)</td>
<td>precomposition with H</td>
<td>4.6.27</td>
<td></td>
</tr>
<tr>
<td>(H*, M)</td>
<td>precomposition with H</td>
<td>4.6.27</td>
<td></td>
</tr>
<tr>
<td>(E × *D, M)</td>
<td>module of wedges E × *D ⊸ M</td>
<td>4.8.3</td>
<td></td>
</tr>
<tr>
<td>(E × D*, M)</td>
<td>module of wedges E × D* ⊸ M</td>
<td>4.8.3</td>
<td></td>
</tr>
<tr>
<td>(X : *E)</td>
<td>module of wedges X ⊸ *E</td>
<td>4.10.1</td>
<td></td>
</tr>
<tr>
<td>(E* : A)</td>
<td>module of wedges E* ⊸ A</td>
<td>4.10.1</td>
<td></td>
</tr>
<tr>
<td>↑</td>
<td>category of collages X ⊸ A</td>
<td>3.1.4</td>
<td></td>
</tr>
<tr>
<td>[X ↑ A]</td>
<td>category of right collages over X</td>
<td>3.1.4</td>
<td></td>
</tr>
<tr>
<td>↑ A</td>
<td>category of left collages over A</td>
<td>3.1.4</td>
<td></td>
</tr>
<tr>
<td>[M ↑ N]</td>
<td>category of collage cells M → N</td>
<td>3.1.8</td>
<td></td>
</tr>
<tr>
<td>K↑</td>
<td>module of K</td>
<td>3.2.21</td>
<td></td>
</tr>
<tr>
<td>1↑ k</td>
<td>unit cylinder of K</td>
<td>10.4.1</td>
<td></td>
</tr>
<tr>
<td>φ↑</td>
<td>collage transpose of φ</td>
<td>10.4.9</td>
<td></td>
</tr>
<tr>
<td>↓</td>
<td>category of right commas over X</td>
<td>3.2.4</td>
<td></td>
</tr>
<tr>
<td>↓ A</td>
<td>category of left commas over A</td>
<td>3.2.4</td>
<td></td>
</tr>
<tr>
<td>[X ↓ A]</td>
<td>category of commas X → A</td>
<td>3.2.10</td>
<td></td>
</tr>
<tr>
<td>M↓</td>
<td>comma of M</td>
<td>3.2.17</td>
<td></td>
</tr>
<tr>
<td>M↓ a</td>
<td>right comma of M at a</td>
<td>3.2.28</td>
<td></td>
</tr>
<tr>
<td>x↓ M</td>
<td>left comma of M at x</td>
<td>3.2.28</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Usage</td>
<td>Meaning</td>
<td>Reference</td>
</tr>
<tr>
<td>--------</td>
<td>-------</td>
<td>---------</td>
<td>-----------</td>
</tr>
<tr>
<td>X↓x</td>
<td></td>
<td>right comma of X at x</td>
<td>3.2.30</td>
</tr>
<tr>
<td>a↓A</td>
<td></td>
<td>left comma of A at a</td>
<td>3.2.28</td>
</tr>
<tr>
<td>1_M</td>
<td></td>
<td>unit cylinder of M</td>
<td>10.1.3</td>
</tr>
<tr>
<td>M</td>
<td></td>
<td>unit cone of M</td>
<td>10.4.1</td>
</tr>
<tr>
<td>φ^i</td>
<td></td>
<td>comma transpose of φ</td>
<td>10.4.9</td>
</tr>
<tr>
<td>↓</td>
<td>(X ↓ A)</td>
<td>cylinder module [X ↓ A] -&gt; [X ↑ A]</td>
<td>10.3.1</td>
</tr>
<tr>
<td>↓</td>
<td>(X ↓)</td>
<td>cone module [X ↓] -&gt; [X ↑]</td>
<td>10.4.2</td>
</tr>
<tr>
<td>↓</td>
<td>(↓ A)</td>
<td>cone module [↓ A] -&gt; ↓A</td>
<td>10.4.2</td>
</tr>
<tr>
<td>M→</td>
<td></td>
<td>right exponential transpose of M</td>
<td>2.1.1</td>
</tr>
<tr>
<td>M→E</td>
<td></td>
<td>right action of M : X → A on [E, A]</td>
<td>2.2.1</td>
</tr>
<tr>
<td>X→</td>
<td></td>
<td>right Yoneda functor for X</td>
<td>2.3.1</td>
</tr>
<tr>
<td>X→A</td>
<td></td>
<td>right general Yoneda functor for [A, X]</td>
<td>2.3.7</td>
</tr>
<tr>
<td>&lt;</td>
<td>M&lt;</td>
<td>left exponential transpose of M</td>
<td>2.1.1</td>
</tr>
<tr>
<td>E&lt;</td>
<td>M</td>
<td>left action of M : X → A on [E, X]</td>
<td>2.2.1</td>
</tr>
<tr>
<td>≪</td>
<td>A</td>
<td>left Yoneda functor for A</td>
<td>2.3.1</td>
</tr>
<tr>
<td>X≪A</td>
<td></td>
<td>left general Yoneda functor for [X, A]</td>
<td>2.3.7</td>
</tr>
<tr>
<td>≪</td>
<td>M&lt;</td>
<td>right comma exponential transpose of M</td>
<td>3.2.27</td>
</tr>
<tr>
<td>X≪</td>
<td>M</td>
<td>right comma exponential transpose of M</td>
<td>3.2.29</td>
</tr>
<tr>
<td>≪</td>
<td>M&lt;</td>
<td>left comma exponential transpose of M</td>
<td>3.2.27</td>
</tr>
<tr>
<td>≪</td>
<td>A</td>
<td>left comma exponential transpose of M</td>
<td>3.2.29</td>
</tr>
<tr>
<td>≪</td>
<td>X≈</td>
<td>right Yoneda module for X</td>
<td>5.1.1</td>
</tr>
<tr>
<td>X≪A</td>
<td></td>
<td>right general Yoneda module for [A, X]</td>
<td>5.1.6</td>
</tr>
<tr>
<td>≪</td>
<td>A</td>
<td>left Yoneda module for A</td>
<td>5.1.1</td>
</tr>
<tr>
<td>X≪A</td>
<td></td>
<td>left general Yoneda module for [X, A]</td>
<td>5.1.6</td>
</tr>
<tr>
<td>≪</td>
<td>M&lt;</td>
<td>corepresentable module of M ≪</td>
<td>11.1.1</td>
</tr>
<tr>
<td>X≪</td>
<td>M&lt;</td>
<td>corepresentable module of X ≪</td>
<td>11.1.3</td>
</tr>
<tr>
<td>M≳</td>
<td>E</td>
<td>corepresentable module of M ≳E</td>
<td>11.1.7</td>
</tr>
<tr>
<td>X≳</td>
<td>E</td>
<td>corerepresentable module of X ≳E</td>
<td>11.1.9</td>
</tr>
<tr>
<td>≪</td>
<td>M</td>
<td>representable module of ≪M</td>
<td>11.1.1</td>
</tr>
<tr>
<td>≪</td>
<td>A</td>
<td>representable module of ≪A</td>
<td>11.1.3</td>
</tr>
<tr>
<td>E≈</td>
<td>M</td>
<td>representable module of E ≈M</td>
<td>11.1.7</td>
</tr>
<tr>
<td>E≈</td>
<td>A</td>
<td>representable module of E ≈A</td>
<td>11.1.9</td>
</tr>
<tr>
<td>≪</td>
<td>M</td>
<td>corepresentable module of M ≪</td>
<td>11.1.5</td>
</tr>
<tr>
<td>≪</td>
<td>M</td>
<td>representable module of ≪M</td>
<td>11.1.5</td>
</tr>
<tr>
<td>▼</td>
<td>(F ▼ M)</td>
<td>module of cylinders left weighted by F</td>
<td>4.5.3</td>
</tr>
<tr>
<td>▼</td>
<td>(F ▼)</td>
<td>postcomposition with F</td>
<td>4.5.7</td>
</tr>
<tr>
<td>▼</td>
<td>(F ▼ M)</td>
<td>module of cylinders right weighted by F</td>
<td>4.5.3</td>
</tr>
<tr>
<td>▼</td>
<td>(F ▼)</td>
<td>postcomposition with F</td>
<td>4.5.7</td>
</tr>
<tr>
<td>↑</td>
<td>(X↑M)</td>
<td>right hom of M : X → A</td>
<td>5.2.1</td>
</tr>
<tr>
<td>↑</td>
<td>(X↑M)</td>
<td>right Yoneda morphism for M</td>
<td>5.2.5</td>
</tr>
<tr>
<td>↑</td>
<td>(X↑M)</td>
<td>right general Yoneda morphism for (E, M)</td>
<td>5.3.3</td>
</tr>
<tr>
<td>X↑m</td>
<td></td>
<td>module morphism generated by X direct along m</td>
<td>5.2.4</td>
</tr>
<tr>
<td>X↑a</td>
<td></td>
<td>module morphism generated by X direct along α</td>
<td>5.3.2</td>
</tr>
<tr>
<td>X↑</td>
<td></td>
<td>Yoneda representation</td>
<td>5.2.19</td>
</tr>
<tr>
<td>X↑A</td>
<td></td>
<td>general Yoneda representation</td>
<td>5.3.19</td>
</tr>
<tr>
<td>↑</td>
<td>(M↑A)</td>
<td>left hom of M : X → A</td>
<td>5.2.1</td>
</tr>
<tr>
<td>↑</td>
<td>(M↑A)</td>
<td>left Yoneda morphism for M</td>
<td>5.2.5</td>
</tr>
<tr>
<td>E⇒</td>
<td>(M↑A)</td>
<td>left general Yoneda morphism for (E, M)</td>
<td>5.3.3</td>
</tr>
<tr>
<td>m↑A</td>
<td></td>
<td>module morphism generated by A inverse along m</td>
<td>5.2.4</td>
</tr>
<tr>
<td>Symbol</td>
<td>Usage</td>
<td>Meaning</td>
<td>Reference</td>
</tr>
<tr>
<td>--------</td>
<td>-------</td>
<td>---------</td>
<td>-----------</td>
</tr>
<tr>
<td>$\alpha \uparrow A$</td>
<td></td>
<td>module morphism generated by $A$ inverse along $\alpha$</td>
<td>5.3.2</td>
</tr>
<tr>
<td>$\uparrow A$</td>
<td></td>
<td>Yoneda corepresentation</td>
<td>5.2.19</td>
</tr>
<tr>
<td>$X \uparrow A$</td>
<td></td>
<td>general Yoneda corepresentation</td>
<td>5.3.19</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>$X \downarrow$</td>
<td>inverse of $X \uparrow$</td>
<td>5.3.13</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>$\downarrow A$</td>
<td>inverse of $\uparrow A$</td>
<td>5.3.13</td>
</tr>
<tr>
<td>$\rightarrow M \rightarrow N$</td>
<td></td>
<td>right symmetric cell from $M$ to $N$</td>
<td>8.1.1</td>
</tr>
<tr>
<td>$X \rightarrow A$</td>
<td></td>
<td>right symmetric cell from $X$ to $A$</td>
<td>8.3.1</td>
</tr>
<tr>
<td>$\rightarrow M \rightarrow N$</td>
<td></td>
<td>left symmetric cell from $M$ to $N$</td>
<td>8.1.1</td>
</tr>
<tr>
<td>$X \rightarrow A$</td>
<td></td>
<td>left symmetric cell from $X$ to $A$</td>
<td>8.3.1</td>
</tr>
<tr>
<td>$\top M \top N$</td>
<td></td>
<td>module of right symmetric cells $M \rightarrow N$</td>
<td>8.1.5</td>
</tr>
<tr>
<td>$X \top A$</td>
<td></td>
<td>module of right symmetric cells $X \rightarrow A$</td>
<td>8.4.1</td>
</tr>
<tr>
<td>$\downarrow M \downarrow N$</td>
<td></td>
<td>module of left symmetric cells $M \rightarrow N$</td>
<td>8.1.5</td>
</tr>
<tr>
<td>$X \downarrow A$</td>
<td></td>
<td>module of left symmetric cells $X \rightarrow A$</td>
<td>8.4.1</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>$\Pi E M$</td>
<td>set of frames of $M : E \rightarrow E$</td>
<td>4.1.1</td>
</tr>
<tr>
<td></td>
<td>$\Pi E \Phi$</td>
<td>postcomposition with $\Phi$</td>
<td>4.1.5</td>
</tr>
<tr>
<td></td>
<td>$\Pi H$</td>
<td>precomposition with $H$</td>
<td>4.1.8</td>
</tr>
<tr>
<td></td>
<td>$\Pi E_* M$</td>
<td>set of frames of $M : * \rightarrow E$</td>
<td>4.2.1</td>
</tr>
<tr>
<td></td>
<td>$\Pi E_+ M$</td>
<td>set of frames of $M : E \rightarrow *$</td>
<td>4.2.1</td>
</tr>
<tr>
<td></td>
<td>$\Pi E_+ \Phi$</td>
<td>postcomposition with $\Phi$</td>
<td>4.2.7</td>
</tr>
<tr>
<td></td>
<td>$\Pi H_+ \Phi$</td>
<td>precomposition with $H$</td>
<td>4.2.11</td>
</tr>
<tr>
<td></td>
<td>$\Pi H$</td>
<td>precomposition with $H$</td>
<td>4.2.11</td>
</tr>
<tr>
<td></td>
<td>$\Pi K$</td>
<td>limit of $K$</td>
<td>7.1.1</td>
</tr>
<tr>
<td></td>
<td>$\Pi^E K$</td>
<td>$E$-parameterized limit of $K$</td>
<td>7.2.8</td>
</tr>
<tr>
<td></td>
<td>$\Pi^J K$</td>
<td>$J$-weighted limit of $K$</td>
<td>11.2.9</td>
</tr>
<tr>
<td></td>
<td>$\square K$</td>
<td>colimit of $K$</td>
<td>7.1.1</td>
</tr>
<tr>
<td></td>
<td>$\square^E K$</td>
<td>$E$-parameterized colimit of $K$</td>
<td>7.2.8</td>
</tr>
<tr>
<td></td>
<td>$\square^J K$</td>
<td>$J$-weighted colimit of $K$</td>
<td>11.2.9</td>
</tr>
<tr>
<td>$/\to m/u$</td>
<td></td>
<td>adjunct of $m$ inverse along $u$</td>
<td>6.1.2</td>
</tr>
<tr>
<td>$\setminus$</td>
<td>$u/m$</td>
<td>adjunct of $m$ direct along $u$</td>
<td>6.1.2</td>
</tr>
<tr>
<td>$\triangleright$</td>
<td>$p \triangleright a$</td>
<td>exponential of $p$ and $a$</td>
<td>8.8.1</td>
</tr>
<tr>
<td>$\circ$</td>
<td>$M \circ N$</td>
<td>pointwise exponential of $M, N : * \rightarrow E$</td>
<td>12.3.4</td>
</tr>
<tr>
<td>$\times$</td>
<td>$M \times N$</td>
<td>pointwise product of $M : E \rightarrow *$ and $N : * \rightarrow E$</td>
<td>12.3.11</td>
</tr>
<tr>
<td>$\otimes$</td>
<td>$M \otimes E N$</td>
<td>tensor product of $M : E \rightarrow *$ and $N : * \rightarrow E$</td>
<td>12.3.17</td>
</tr>
</tbody>
</table>
Bibliography


[Lor15] Fosco Loregian, This is the (co)end, my only (co)friend. arXiv:1501.02503, 2015.


