Categories and Modules

Takahiro Kato

March 12, 2015
ABSTRACT. Modules (also known as profunctors or distributors) and morphisms among them subsume categories and functors and provide more general and abstract framework to explore the theory of structures. In this book we generalize and redevelop the basic notions and results of category theory using this framework of modules.

Topics

Chapter 1 introduces modules and cells among them. A module $M : X \to A$ from a category $X$ to $A$ is a functor of the form $M : X^{op} \times A \to \text{Set}$, assigning a set of “arrows” to each pair of objects $x \in X$ and $a \in A$. A cell from a module $M$ to $N$ sends each arrow of $M$ to an arrow of $N$. The hom-functor of a category $C$ forms an endomodule $(C) : C \to C$ called the hom of $C$, and the arrow function of a functor $F : C \to D$ forms a cell $(F) : (C) \to (D)$ from the hom of $C$ to the hom of $D$. Modules and cells thus subsume categories and functors in this way and set up a more general and conceptual framework to explore the structure of mathematics. Presheaves and copresheaves are called right and left modules respectively in this book and studied as special instances of modules.

Chapter 2 discusses the action of a module on its domain and codomain, the operation that yields the Yoneda embedding functor in the case of a hom endomodule. The chapter introduces an important class of modules called representable, which each functor produces by the composition with the hom of its codomain.

Chapter 3 presents two variants of modules, namely collages and commas, which are special sorts of cospans and spans between two categories. We will establish an isomorphism between the category of modules and the category of collages, and, later in Chapter 9, construct an adjoint equivalence between the category of commas and the category of collages. Two forgetful functors from $\text{MOD}$ (the category of modules and cells) to $\text{CAT}$ are defined through construction of collages and commas, and it is shown that they form left and right adjoints of the embedding $\text{CAT} \hookrightarrow \text{MOD}$ given by the hom operation $C \hookrightarrow (C)$.

Chapter 4 introduces the notion of frames of a module. A cylindrical frame of an endomodule abstracts a natural transformation between two functors, and a conical frame of a right (resp. left) module abstracts a cone between an object and a functor. Inner and outer cylinders, which are, so to speak, natural and extranatural transformations spanning a module, are defined using cylindrical frames. Likewise, cones are defined along a module using conical frames.

Chapter 5 succeeds Chapter 2 and discusses the actions of the domain and codomain of a module on its arrows and frames. It is shown, as a generalization of the Yoneda embedding, that these actions embed a module $X \to A$ in (the hom of) the category of right modules (i.e. presheaves) over $X$ and the category of left module (i.e. copresheaves) over $A$. The Yoneda lemma is presented in a general form to state that the morphisms from the representable module of a functor $F : X \to A$ to an arbitrary module $M : X \to A$ correspond one-to-one with the cylinders defined between $F$ and $M$. Using the lemma, we show a variety of bijective correspondences between frames and cells.

Chapters 6, 7, 8, and 10 explore the fundamental concepts of category theory in the framework of modules. Universals are defined along a module and we study if they are preserved by a cell. The embedding of a module in the category of presheaves (resp. copresheaves) makes the definition of a universal simple: an arrow of a module is universal if its image under the embedding is an isomorphism. Lifts, of which Kan lifts are special instances, are defined as universal cylinders, and extensions, of which Kan extensions are special instances, are defined as universal cells. We introduce the notion of pointwise lifts and show that the
notion subsumes that of pointwise extensions and vice versa. The concept of an adjoint is extended to a cell. We show that a cell preserves universals if it has adjoints, and deduce RAPL (right adjoints preserve limits) as a corollary of this general mechanism. The concept of density is also generalized for modules. In fact, density is most naturally defined with modules. The fact "every presheaf is a colimit of representables" is proved using the concept of density to show how the concept works in the framework of modules.

Advice on reading

The exposition is carried out within the framework of set-enriched 1-dimensional categories. Chapter 0 presents the notations and some elementary facts of category theory used in the sequel.

Size issues are not treated rigorously. The specification of a universe is almost always implicit; a universe $\mathcal{U}$ is chosen so that given categories and modules become locally $\mathcal{U}$-small, i.e. all hom-sets become $\mathcal{U}$-small. We just say "small" (resp. "large") instead of "$\mathcal{U}$-small" (resp. "$\mathcal{U}$-large") for $\mathcal{U}$ chosen implicitly. A category or a module constructed is called large if it is not, in general, locally small. The choice of a universe is fixed in each context.

This book is made up of sequences of "Definition" - "Proposition" - "Theorem" - "Corollary", with each optionally preceded by "Note" and followed by "Remark". Each "Proposition" is tightly associated with the preceding "Definition" and states some straightforward consequences of it. Two statements dual to each other are indicated by the symbols $\triangleright$ and $\triangleleft$. Proofs are given for the assertions indicated by $\triangleright$.

Parentheses and brackets serve mainly as punctuation to enhance readability; parentheses are used to delimit objects and arrows, whereas square (resp. angle) brackets are used to delimit categories and functors (resp. modules and cells).
# Contents

0. Preliminaries .......................................................... 6

1. Modules and Cells .................................................. 10
   1.1. Modules ................................................................ 10
   1.2. Cells .................................................................. 21
   1.3. Cell morphisms ................................................ 32

2. Slicing and Action ...................................................... 34
   2.1. Slicing of modules ............................................. 34
   2.2. Action of modules ............................................. 39
   2.3. Yoneda functors and representations .................... 41

3. Collages and Commas ................................................ 46
   3.1. Collages ................................................................ 46
   3.2. Commas ................................................................ 52

4. Frames ................................................................. 65
   4.1. Cylindrical frames ............................................. 65
   4.2. Conical frames ................................................ 68
   4.3. Inner cylinders ................................................ 73
   4.4. Outer cylinders ................................................ 87
   4.5. Weighted cylinders ............................................ 91
   4.6. Cones ................................................................ 97
   4.7. Bicylinders and wedges ..................................... 108
   4.8. Cones and wedges in a category ......................... 117
   4.9. Cones and wedges in Set ................................... 128

5. Yoneda Lemma .......................................................... 136
   5.1. Yoneda modules ............................................... 136
   5.2. Yoneda morphisms ............................................. 140
   5.3. Yoneda morphisms for cylinders ......................... 151
   5.4. Yoneda morphisms for cones .............................. 168
   5.5. Correspondences between frames and cells .......... 173

6. Universals ............................................................ 187
   6.1. Units of one-sided modules .............................. 187
   6.2. Universal arrows ............................................. 190
   6.3. Units of two-sided modules ............................. 198
   6.4. Universal cylinders ......................................... 203
   6.5. Kan lifts ......................................................... 206
0. Preliminaries

1. Given a universe $\mathcal{U}$, $\textbf{Set}_\mathcal{U}$, $\textbf{Cat}_\mathcal{U}$, and $\textbf{CAT}_\mathcal{U}$ denote the categories of $\mathcal{U}$-small sets, $\mathcal{U}$-small categories, and locally $\mathcal{U}$-small categories respectively. By $\textbf{Set}$ we mean $\textbf{Set}_\mathcal{U}$ for $\mathcal{U}$ chosen implicitly, and similarly for $\textbf{Cat}$ and $\textbf{CAT}$.

2. A category is defined in terms of hom-sets. Pairwise disjointness of hom-sets is not required. Given a category $C$, an element $f$ of a hom-set $\text{hom}_C(a,b)$ or a triple $(a,f,b)$ such that $f \in \text{hom}_C(a,b)$ is called an arrow of $C$ (or a $C$-arrow) and written $f : a \to b$, $a \xrightarrow{f} b$, or just $f$ if its domain and codomain are understood or unimportant. The composite of arrows $f : a \to b$ and $g : b \to c$ is written as

$$f \circ g = g \circ f$$

3. The set of objects of a category $C$ is denoted by $\|C\|$ and the object function of a functor $F$ is denoted by $\|F\|$.

4. Italic small letters $a$, $b$, $c$, ... vary over objects and arrows. When we write $c \in C$ for a category $C$, $c$ stands for an object or arrow of $C$.

5. The evaluation of a functor $F : C \to B$ at $c \in C$ is written as

$$c : F = F(c) = F \cdot c$$

, and the component of a natural transformation $\tau : F \to G : C \to B$ at an object $c \in \|C\|$ is written as

$$c : \tau = \tau_c = \tau \cdot c$$

6. The opposite of a category $C$ is denoted by $C^\circ$, and the opposite of a functor $F : C \to B$ is denoted by $F^\circ : C^\circ \to B^\circ$, or often by $F : C^\circ \to B^\circ$ using the same name as its original counterpart.

7. Given categories $E$ and $D$, the product, coproduct, and functor categories are denoted by $E \times D$, $E + D$ and $[E,D]$ respectively. The terminal category and its sole object are denoted by $*$; the interval category is denoted by $2$ and its objects are denoted by $0$ and $1$.

8. Italic capital letters $F$, $G$, ... vary over both functors and natural transformations. When we write $F \in [E,D]$, $F : E \to D$, or $E \xrightarrow{F} D$ for categories $E$ and $D$, $F$ stands for an object or an arrow of the functor category $[E,D]$, i.e. a functor or a natural transformation.
9. Let \( E, D, \) and \( C \) be categories. The composite of \( F : E \to D \) and \( G : D \to C \) is written as
\[
F \circ G = G \circ F
\]
. The composition forms a bifunctor
\[
(F, G) \mapsto F \circ G : [E, D] \times [D, C] \to [E, C]
\]
.

10. The bifunctorial operation \([-,-]\) on \( \text{CAT}^{-} \times \text{CAT} \) is defined in the following way:
a) for a pair of categories \( X \) and \( A, [X, A] \) is the category of functors \( X \to A \);
b) for a functor \( P : X \to Y \) and a category \( A, [P, A] : [Y, A] \to [X, A] \) is the precomposition functor defined by
\[
F : [P, A] = P \circ F
\]
for \( F \in [Y, A] \);
c) for a functor \( Q : A \to B \) and a category \( X, [X, Q] : [X, A] \to [X, B] \) is the postcomposition functor defined by
\[
F : [X, Q] = F \circ Q
\]
for \( F \in [X, A] \).

11. The right and left exponential transposes of a bifunctor \( K : E \times D \to C \) are denoted by
\[
K^r : D \to [E, C] \quad K^s : E \to [D, C]
\]
, and the (simple) transpose of a functor \( K : E \to [D, C] \) is denoted by \( K^s : D \to [E, C] \). These transpositions form iso functors as indicated in the following commutative diagram:

12. By the obvious isomorphism \([*, D]\cong D\), a functor (resp. natural transformation) \(* \to D\) is identified with an object (resp. arrow) of \( D\). Given categories \( E \) and \( D \) and given an object \( d \in \|D\|\), the product functor \( E \times d \) as in
\[
\begin{array}{ccc}
E & \xleftarrow{\times} & E \times d & \xrightarrow{\times} & * \\
E \downarrow & & E \times d \downarrow & & d \\
E & \xleftarrow{\times} & E \times d & \xrightarrow{\times} & D
\end{array}
\]
yields the section
\[
E \times d : E \to E \times D
\]
of \( E \times D \) at \( d \) under the identification \( E \times * \cong E \).
13. Given categories $E$ and $C$, the evaluation

$$(e, F) \mapsto e \cdot F : E \times [E, C] \to C$$

is identified with the composition

$$(e, F) \mapsto e \circ F : [\ast, E] \times [E, C] \to [\ast, C]$$

Given an object $e \in \|E\|$, the precomposition functor

$$[e, C] : [E, C] \to C$$

"evaluation at $e"$, takes each $F \in [E, C]$ and yields the composite

$$\ast \xrightarrow{e} E \xrightarrow{F} C$$

i.e. the evaluation $e \cdot F$. Note that the diagram

$$
\begin{array}{ccc}
[D, [E, C]] & \xleftarrow{\gamma} & [E, [D, C]] \\
\downarrow & & \downarrow \\
[D, [e, C]] & \longrightarrow & [e, [D, C]] \\
& & \\
\downarrow & & \downarrow \\
[D, C] & & [D, C]
\end{array}
$$

commutes; that is, the evaluation

$$\ast \xrightarrow{e} E \xrightarrow{K} [D, C]$$

of $K$ at $e$ is given by the composition

$$D \xrightarrow{K^T} [E, C] \xrightarrow{[e, C]} C$$

14. Given categories $E$, $D$, $C$, and an object $d \in \|D\|$, the precomposition functor

$$[E \times d, C] : [E \times D, C] \to [E, C]$$

"partial evaluation at $d"$, takes each $K \in [E \times D, C]$ and yields the composite

$$E \xrightarrow{E \times d} E \times D \xrightarrow{K} C$$

the right slice of $K$ at $d$. Note that the diagram

$$
\begin{array}{ccc}
[E \times D, C] & \xleftarrow{\alpha} & [D, [E, C]] \\
\downarrow & & \downarrow \\
[E \times d, C] & \longrightarrow & [d, [E, C]] \\
& & \\
\downarrow & & \downarrow \\
[E, C] & & [d, [E, C]]
\end{array}
$$

commutes; that is, the partial evaluation of $K : E \times D \to C$ at $d$ is the same thing as the evaluation

$$\ast \xrightarrow{d} D \xrightarrow{K^\ast} [E, C]$$

of the right exponential transpose of $K$ at $d$. 
15. Given a category $D$, the unique functor $D \to \ast$ is denoted by $\Delta_D$ or just $\Delta$. Given categories $E$ and $D$, the product functor $E \times \Delta_D$ as in

$$
\begin{array}{ccc}
E & \xleftarrow{\cdot} & E \times D \\
\downarrow & & \downarrow \Delta_D \\
E & \xleftarrow{\cdot} & E \times \ast
\end{array}
$$

yields the projection $E \times \Delta_D : E \times D \to E$ under the identification $E \times \ast \cong E$.

16. Given categories $D$, $E$, and $C$, the $D$-ary diagonal functor of the functor category $[E, C]$ is the precomposition functor

$$[E \times \Delta_D, C] : [E, C] \to [E \times D, C]$$

sending each functor $F : E \to C$ to the composite bifunctor

$$E \times D \xrightarrow{E \times \Delta_D} E \xrightarrow{F} C$$

"$F$ duplicated across $D$". As a special case, the $D$-ary diagonal functor

$$[\Delta_D, C] : C \to [D, C]$$

of a category $C$ sends each object $c \in \|C\|$ to the constant functor $\Delta c : D \to C$ given by the composition

$$D \xrightarrow{\Delta_D} \ast \xrightarrow{c} C$$

. Note that the diagrams

$$
\begin{array}{ccc}
[E \times \Delta_D, C] & \xrightarrow{\cdot} & [\Delta_D, [E, C]] \\
\downarrow & & \downarrow \\
[E \times D, C] & \xleftarrow{\cdot} & [D, [E, C]]
\end{array}
\quad
\begin{array}{ccc}
[E \times \Delta_D, C] & \xrightarrow{\cdot} & [E, [\Delta_D, C]] \\
\downarrow & & \downarrow \\
[E \times D, C] & \xleftarrow{\cdot} & [E, [D, C]]
\end{array}
$$

commute.
1. Modules and Cells

1.1. Modules

Definition 1.1.1.

- A right module $M$ over a category $X$, written $M : X \to \ast$, is a functor $M : X^\to \to \text{Set}$. Given a pair of right modules $M, N : X \to \ast$, a morphism $\Phi$ from $M$ to $N$, written $\Phi : M \to N : X \to \ast$, is a natural transformation $\Phi : M \to N : X^\to \to \text{Set}$.

- A left module $M$ over a category $A$, written $M : \ast \to A$, is a functor $M : A \to \text{Set}$. Given a pair of left modules $M, N : \ast \to A$, a morphism $\Phi$ from $M$ to $N$, written $\Phi : M \to N : \ast \to A$, is a natural transformation $\Phi : M \to N : A \to \text{Set}$.

Remark 1.1.2.

1. The notation $[X^\to, \text{Set}]$ (see Preliminaries(10)) is abbreviated to $[X]$:
   a) for a category $X$, $[X]$ denotes the category of right modules over $X$;
   b) for a functor $K : E \to X$, $[K] : [X] \to [E]$ denotes the precomposition functor given by the assignment $M \mapsto K \circ M$.

   If $X$ is a small category, then the category $[X]$ is locally small. The assignment $K \mapsto [K]$ is contravariant functorial.

- The notation $[A, \text{Set}]$ (see Preliminaries(10)) is abbreviated to $[A]$:
   a) for a category $A$, $[A]$ denotes the category of left modules over $A$;
   b) for a functor $K : E \to A$, $[K] : [A] \to [E]$ denotes the precomposition functor given by the assignment $M \mapsto K \circ M$.

   If $A$ is a small category, then the category $[A]$ is locally small. The assignment $K \mapsto [K]$ is contravariant functorial.

2. By definition,

   $[A] = [A^\to]$.

   ; a left module over $A$ is the same thing as a right module over the opposite category $A^\to$.

3. If $M : X \to \ast$ is a right module, then the image of an object (resp. arrow) $x \in X$ under the functor $M : X^\to \to \text{Set}$ is written $(x) \langle M \rangle$ or just $x \langle M \rangle$. For an object $x \in [X]$, the set $x \langle M \rangle$ is called the hom-set at $x$. An element $m$ of a hom-set $x \langle M \rangle$ or a pair $(x, m)$ such that $m \in x \langle M \rangle$ is called an arrow of $M$ (or an $M$-arrow) and written $m : x \to \ast$, $x \xrightarrow{m} \ast$, or just $m$ if its domain is understood or unimportant.
1. Modules and Cells

- If $M : * \to A$ is a left module, then the image of an object (resp. arrow) $a \in A$ under the functor $M : A \to \text{Set}$ is written $(M)(a)$ or just $(M)a$. For an object $a \in |A|$, the set $(M)a$ is called the hom-set at $a$. An element $m$ of a hom-set $(M)a$ or a pair $(m, a)$ such that $m \in (M)a$ is called an arrow of $M$ (or an $M$-arrow) and written $m : * \to a$, $* \sim a$, or just $m$ if its codomain is understood or unimportant.

4.

- If $\Phi : M \to N : X \to *$ is a right module morphism, then the component of the natural transformation $\Phi : M \to N : X^* \to \text{Set}$ at $x \in |X|$ is written $x(\Phi)$. The image of an $M$-arrow $m : x \to *$ under the function $x(\Phi) : x(M) \to x(N)$ is written $m : x(\Phi) = x(\Phi) \cdot m$ or just $m : \Phi = \Phi : m$.

- If $\Phi : M \to N : * \to A$ is a left module morphism, then the component of the natural transformation $\Phi : M \to N : A \to \text{Set}$ at $a \in |A|$ is written $(\Phi)a$. The image of an $M$-arrow $m : * \to a$ under the function $(\Phi)a : (M)a \to (N)a$ is written $m : (\Phi)a = (\Phi)a \cdot m$ or just $m : \Phi = \Phi : m$.

**Proposition 1.1.3.**

- A right module morphism $\Phi : M \to N : X \to *$ is invertible in the category $[X : :]$ if and only if every component $x(\Phi)$ is a bijection.

- A left module morphism $\Phi : M \to N : * \to A$ is invertible in the category $[ : A]$ if and only if every component $(\Phi)a$ is a bijection.

**Proof.** Immediate from the definition on noting that a natural transformation is invertible in a functor category if and only if every its component is invertible. \qed

**Remark 1.1.4.** A right (resp. left) module morphism is called iso if it satisfies the equivalent conditions in Proposition 1.1.3.

**Definition 1.1.5.** A [two-sided] module $M$ from a category $X$ to a category $A$, written $M : X \to A$, is a bifunctor $M : X^* \times A \to \text{Set}$. Given a pair of modules $M,N : X \to A$, a morphism $\Phi$ from $M$ to $N$, written $\Phi : M \to N : X \to A$, is a natural transformation $\Phi : M \to N : X^* \times A \to \text{Set}$.

**Remark 1.1.6.**

1. The notation $[X^* \times A, \text{Set}]$ (see Preliminaries(10)) is abbreviated to $[X : A]$:

   a) for a pair of categories $X$ and $A$, $[X : A]$ denotes the category of modules $X \to A$;

   b) for a pair of functors $S : E \to X$ and $T : D \to A$, $[S : T] : [X : A] \to [E : D]$ denotes the precomposition functor given by the assignment $M \mapsto [S \times T] \circ M$.

   If $X$ and $A$ are small categories, then the category $[X : A]$ is locally small. The assignment $(S, T) \mapsto [S : T]$ is contravariant bifunctorial.

2. By definition,

   $$[X \times A^* :] = [X : A] = [ : X^* \times A]$$

   a two-sided module $M : X \to A$ is the same thing as a right module $M : X \times A^* \to *$ (resp. left module $M : * \to X^* \times A$).
modules and cells

3. If \( M : X \rightarrow A \) is a module, then the image of an object (resp. arrow) \((x, a) \in X^\ast \times A\) under the functor \( M : X^\ast \times A \rightarrow \text{Set} \) is written \((x, a) \in [X] \) or just \(x(M)a\). For a pair of objects \(x \in [X]\) and \(a \in [A]\), the set \(x(M)a\) is called the hom-set from \(x\) to \(a\). An element \(m\) of a hom-set \(x(M)a\) or a triple \((x, m, a)\) such that \(m \in x(M)a\) is called an arrow of \(M\) (or an \(M\)-arrow) and written \(m : x \Rightarrow a\), or just \(m\) if its domain and codomain are understood or unimportant.

4. If \( \Phi : M \rightarrow N : X \rightarrow A \) is a module morphism, then the component of the natural transformation \( \Phi : M \rightarrow N : X \rightarrow \text{Set} \) at \((x, a) \in [X] \times [A]\) is written \(x(\Phi)a\). The image of an \(M\)-arrow \(m : x \Rightarrow a\) under the function \(x(\Phi)a : x(M)a \rightarrow x(N)a\) is written \(m : x(\Phi)a = x(\Phi)a \cdot m\) or just \(m : \Phi = \Phi \cdot m\).

5. The canonical isomorphism \(X^\ast \cong X^\ast \times \ast\) yields a canonical isomorphism
\[
[X] \cong [X : \ast]
\]
natural in \(X\). By this isomorphism, a right module over \(X\) is identified with a two-sided module from \(X\) to the terminal category.

The canonical isomorphism \(A \cong \ast \times A\) yields a canonical isomorphism
\[
[A] \cong [\ast : A]
\]
natural in \(A\). By this isomorphism, a left module over \(A\) is identified with a two-sided module from the terminal category to \(A\).

6. There are obvious isomorphisms
\[
\text{Set} \cong [\ast :] \cong [::] \cong [::]
\]
by which a set is identified with a module \(\ast \rightarrow \ast\) over the terminal category.

7. Given a universe \(\mathcal{U}\), a module \(M : X \rightarrow A\) is called
   a) locally \(\mathcal{U}\)-small if \(X\) and \(A\) are locally \(\mathcal{U}\)-small and all hom-sets of \(M\) are \(\mathcal{U}\)-small; that is, if all hom-sets of \(X, A,\) and \(M\) are \(\mathcal{U}\)-small;
   b) \(\mathcal{U}\)-small if it is locally \(\mathcal{U}\)-small and if \(X\) and \(A\) are \(\mathcal{U}\)-small;
   c) \(\mathcal{U}\)-large if it is not locally \(\mathcal{U}\)-small.

We just say “small” (resp. “large”) instead of “\(\mathcal{U}\)-small” (resp. “\(\mathcal{U}\)-large”) for \(\mathcal{U}\) chosen implicitly.

**Proposition 1.1.7.** A module morphism \(\Phi : M \rightarrow N : X \rightarrow A\) is invertible in the category \([X : A]\) if and only if every component \(x(\Phi)a\) is a bijection.

**Proof.** See the proof of Proposition 1.1.3.

**Remark 1.1.8.** A module morphism is called iso if it satisfies the equivalent conditions in Proposition 1.1.7.

**Definition 1.1.9.**
1. Modules and Cells

1. Let $M : X \to \ast$ be a right module. The composite $g \circ m = m \circ g$ of an $X$-arrow $g : y \to x$ and an $M$-arrow $m : x \rightsquigarrow \ast$, as indicated in

\[
\begin{array}{c}
\ast \\
\downarrow m \\
x \\
\downarrow g \\
y \\
\end{array}
\]

, is the $M$-arrow $y \rightsquigarrow \ast$ defined by

\[g \circ m = g(M) \cdot m\]

, the image of $m$ under the function $g(M) : x(M) \to y(M)$.

- Let $M : \ast \to A$ be a left module. The composite $m \circ f = f \circ m$ of an $M$-arrow $m : \ast \rightsquigarrow a$ and an $A$-arrow $f : a \to b$, as indicated in

\[
\begin{array}{c}
\ast \\
\downarrow m \\
a \\
\downarrow f \\
b \\
\end{array}
\]

, is the $M$-arrow $\ast \rightsquigarrow b$ defined by

\[m \circ f = m : (M) f\]

, the image of $m$ under the function $(M) f : (M) a \to (M) b$.

2. Let $M : X \to A$ be a module.

- The composite $g \circ m = m \circ g$ of an $X$-arrow $g : y \to x$ and an $M$-arrow $m : x \rightsquigarrow a$, as indicated in

\[
\begin{array}{c}
\ast \\
\downarrow m \\
x \\
\downarrow g \\
y \\
\end{array}
\]

, is the $M$-arrow $y \rightsquigarrow a$ defined by

\[g \circ m = g(M) a : m\]

, the image of $m$ under the function $g(M) a : x(M) a \to y(M) a$.

- The composite $m \circ f = f \circ m$ of an $M$-arrow $m : x \rightsquigarrow a$ and an $A$-arrow $f : a \to b$, as indicated in

\[
\begin{array}{c}
\ast \\
\downarrow m \\
a \\
\downarrow f \\
b \\
\end{array}
\]

, is the $M$-arrow $x \rightsquigarrow b$ defined by

\[m \circ f = m : x(M) f\]

, the image of $m$ under the function $x(M) f : x(M) a \to x(M) b$. 
1. Modules and Cells

Remark 1.1.10.

1. A module \( M : X \to A \) thus induces a composition law among the arrows of \( X, A, \) and \( M \). Conversely, a module \( M : X \to A \) may be defined by giving a set of \( M \)-arrows and a composition of them with \( X \)-arrows and \( A \)-arrows satisfying the associativity axiom, i.e. by giving a collage (see Section 3.1) from \( X \) to \( A \).

2. When a two-sided module \( M : X \to A \) is regarded as a right module \( M : X \times A^\sim \to *, \) an \( M \)-arrow \( m : x \sim a \) is written as \( m : (x, a) \sim * \) and the compositions

\[
\begin{array}{c}
\text{y} \\
g \downarrow \\
x \\
\text{m} \\
a
\end{array}
\quad \begin{array}{c}
x \\
m \downarrow f \\
\text{a} \\
\text{b}
\end{array}
\]

are written as

\[
\begin{array}{c}
(y, a) \\
(g, a) \downarrow (g, a) \circ m \\
(x, a) \sim * \\
(x, a) \sim *
\end{array}
\]

- When a two-sided module \( M : X \to A \) is regarded as a left module \( M : * \to X^\sim \times A, \) an \( M \)-arrow \( m : x \sim a \) is written as \( m : * \sim (x, a) \) and the compositions

\[
\begin{array}{c}
\text{y} \\
g \downarrow \\
x \\
\text{m} \\
a
\end{array}
\quad \begin{array}{c}
x \\
m \downarrow f \\
\text{b}
\end{array}
\]

are written as

\[
\begin{array}{c}
* \sim (x, a) \\
\text{m} \circ (g, a) \downarrow (g, a) \\
(y, a)
\end{array}
\quad \begin{array}{c}
* \sim (x, a) \\
\text{m} \circ (x, f) \downarrow (x, f) \\
(x, b)
\end{array}
\]

Definition 1.1.11. The hom of a category \( C \) is the endomodule \( (C) : C \to C \) given by the assignment \( (x, a) \mapsto \text{hom}_C(x, a) \) for \( x, a \in C \).

Remark 1.1.12. For a pair of objects \( a, b \in \|C\|, \) the hom-set \( \text{hom}_C(a, b) \) of a category \( C \) is the same thing as the hom-set \( a(C) b \) of the endomodule \( (C) \). Hereafter a hom-set is written as \( a(C) b \) rather than \( \text{hom}_C(a, b) \); likewise, for an object \( c \in \|C\| \) and a \( C \)-arrow \( f : a \to b, \) the functions

\[
\text{hom}_C(c, f) : \text{hom}_C(c, a) \to \text{hom}_C(c, b) \quad \text{hom}_C(f, c) : \text{hom}_C(b, c) \to \text{hom}_C(a, c)
\]

are written as

\[
c(C)f : c(C)a \to c(C)b \quad f(C)c : b(C)c \to a(C)c
\].
1. Modules and Cells

Notation 1.1.13.

1. In a diagram, a module $\mathcal{M} : X \to A$ and a module morphism $\Phi : \mathcal{M} \to \mathcal{N} : X \to A$ are depicted as

$$
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & A \\
\Phi & \circ & \mathcal{N}
\end{array}
$$

2. Italic capital letters $M, N, \ldots$ vary over both modules and module morphisms. When we write $M \in [X : A], M : X \to A,$ or $X \xrightarrow{M} A, M$ stands for an object or an arrow of the module category $[X : A],$ i.e. a module or a module morphism (cf. Preliminaries(8)).

Definition 1.1.14.

- Given a right module (or module morphism) $M$ and a functor (or natural transformation) $S$ as in

$$
\begin{array}{c}
E \xrightarrow{S} X \xrightarrow{M} \ast
\end{array}
$$

their composite, written $[S] (M)$ or just $S (M),$ is the right module (or module morphism) $E \to \ast$ defined by the composition

$$
\begin{array}{c}
E \xrightarrow{S} X \xrightarrow{M} \ast \to \text{Set}
\end{array}
$$

- Given a left module (or module morphism) $M$ and a functor (or natural transformation) $T$ as in

$$
\begin{array}{c}
\ast \xrightarrow{M} A \xleftarrow{T} D
\end{array}
$$

their composite, written $\langle M \rangle [T]$ or just $\langle M \rangle T,$ is the left module (or module morphism) $\ast \to D$ defined by the composition

$$
\begin{array}{c}
D \xrightarrow{T} A \xleftarrow{M} \text{Set}
\end{array}
$$

Remark 1.1.15.

- The composition

$$(S, M) \mapsto S (M) : [E, X] \times [X :] \to [E :]$$

is functorial in each variable, contravariant in $S$ and covariant in $M.$ If $S : E \to X$ is a functor, then

$$S (M) = S \circ M = M : [S :]$$

for any $M \in [X :].$

- The composition

$$(M, T) \mapsto \langle M \rangle T : [: A] \times [D, A] \to [: D]$$

is functorial in each variable, covariant in both $M$ and $T.$ If $T : D \to A$ is a functor, then

$$\langle M \rangle T = T \circ M = M : [T :]$$

for any $M \in [: A].$
Definition 1.1.16. Given a module (or module morphism) $M$, a functor (or natural transformation) $S$, and a second functor (or natural transformation) $T$, all as in

$$
\begin{array}{c}
E \xrightarrow{S} X \xrightarrow{M} A \xleftarrow{T} D
\end{array}
$$

, their composite, written $[S \langle M \rangle [T]$ or just $S \langle M \rangle T$, is the module (or module morphism) $E \rightarrow D$ defined by the composition

$$
E^\times \times D \xrightarrow{S \times T} X^\times \times A \xrightarrow{M} \text{Set}
$$

.

Remark 1.1.17.

1. The composition

$$(S, M, T) \mapsto S \langle M \rangle T : [E, X] \times [X : A] \times [D, A] \rightarrow [E : D]$$

is functorial in each variable, contravariant in $S$ and covariant in $M$ and $T$. If $S : E \rightarrow X$ and $T : D \rightarrow A$ are functors, then

$$S \langle M \rangle T = [S \times T] \circ M = M : [S : T]$$

(see Remark 1.1.6(1)) for any $M \in [X : A]$.

2. As a special case,

- given

$$
\begin{array}{c}
E \xrightarrow{S} X \xrightarrow{M} A
\end{array}
$$

, their composite $S \langle M \rangle : E \rightarrow A$ is defined by the composition

$$
E^\times \times A \xrightarrow{S \times A} X^\times \times A \xrightarrow{M} \text{Set}
$$

. If $A$ is the terminal category, then the composite $S \langle M \rangle : E \rightarrow *$ coincides with that defined in Definition 1.1.14 under the identification $[X :] \cong [X : *]$ and $[E :] \cong [E : *]$.

- given

$$
\begin{array}{c}
X \xrightarrow{M} A \xleftarrow{T} D
\end{array}
$$

, their composite $\langle M \rangle T : X \rightarrow D$ is defined by the composition

$$
X^\times \times D \xrightarrow{X^\times T} X^\times \times A \xrightarrow{M} \text{Set}
$$

. If $X$ is the terminal category, then the composite $\langle M \rangle T : * \rightarrow D$ coincides with that defined in Definition 1.1.14 under the identification $[: A] \cong [* : A]$ and $[E :] \cong [E : *]$.

Example 1.1.18.

1. Given a module and functors as in

$$
\begin{array}{c}
E \xrightarrow{S} X \xrightarrow{M} A \xleftarrow{T} D
\end{array}
$$
1. Modules and Cells

, their composite is the module $S(M) \rightharpoonup T : E \rightarrow D$ defined by

$e(S(M)T)d = (eS)(M)(T'd)$

for $e \in E$ and $d \in D$. An $S(M) T$-arrow $m : e \rightarrow d$ is given by an $M$-arrow $m : e : S \rightarrow T : d$. For an $E$-arrow $h : e' \rightarrow e$ and an $S(M) T$-arrow $m : e \rightarrow d$, their composite $h \circ m : e' \rightarrow d$ is given by the $M$-arrow $(h : S) \circ m : e' : S \rightarrow T : d$ as indicated in

\[
\begin{array}{ccc}
  \text{e'} & \rightarrow & \text{e} \\
  h & \downarrow & h \circ m \\
  \text{e} & \rightarrow & e : S \rightarrow \rightarrow \rightarrow T : d
\end{array}
\]

; similarly, for a $D$-arrow $h : d \rightarrow d'$ and an $S(M) T$-arrow $m : e \rightarrow d$, their composite $m \circ h : e \rightarrow d'$ is given by the $M$-arrow $m \circ h : e : S \rightarrow T : d'$ as indicated in

\[
\begin{array}{ccc}
  \text{e} & \rightarrow & \text{e'} \\
  h & \downarrow & h \circ m \\
  \text{e} & \rightarrow & e : S \rightarrow \rightarrow \rightarrow T : d'
\end{array}
\]

2. As a special case of above, given a pair of functors as in

\[
E \rightarrow S \rightarrow C \quad \overset{(C)}{\rightarrow} \quad C \rightarrow T \rightarrow D
\]

, their composite is the module $S(C) T : E \rightarrow D$ defined by

$e(S(C)T)d = (eS)(C)(T'd)$

for $e \in E$ and $d \in D$. An $S(C) T$-arrow $f : e \rightarrow d$ is given by an $C$-arrow $f : e : S \rightarrow T : d$. For an $E$-arrow $h : e' \rightarrow e$ and an $S(C) T$-arrow $f : e \rightarrow d$, their composite $h \circ f : e' \rightarrow d$ is given by the $C$-arrow $(h : S) \circ f : e' : S \rightarrow T : d$ as indicated in

\[
\begin{array}{ccc}
  \text{e'} & \rightarrow & \text{e} \\
  h & \downarrow & h \circ f \\
  \text{e} & \rightarrow & e : S \rightarrow \rightarrow \rightarrow T : d
\end{array}
\]

; similarly, for a $D$-arrow $h : d \rightarrow d'$ and an $S(C) T$-arrow $f : e \rightarrow d$, their composite $f \circ h : e \rightarrow d'$ is given by the $C$-arrow $f \circ h : e : S \rightarrow T : d'$ as indicated in

\[
\begin{array}{ccc}
  \text{e} & \rightarrow & \text{e'} \\
  h & \downarrow & h \circ f \\
  \text{e} & \rightarrow & e : S \rightarrow \rightarrow \rightarrow T : d'
\end{array}
\]

3. Given a module morphism and functors as in

\[
E \rightarrow X \overset{M}{\rightarrow} A \rightarrow T \rightarrow D
\]

, their composite is the module morphism $S(\Phi) T : S(M) T \rightarrow S(A) T : E \rightarrow D$ defined by

$e(S(\Phi)T)d = (eS)(\Phi)(T'd)$

for $e \in \|E\|$ and $d \in \|D\|$.
1. Modules and Cells

4. Given a natural transformation, a module, and a functor as in

\[
\begin{array}{c}
E \xrightarrow{S'} X \xrightarrow{M} \mathcal{M} \xrightarrow{S} A \xrightarrow{T} D
\end{array}
\]

their composite is the module morphism \(\sigma(M) T : S(M) T \to S'(M) T : E \to D\) which maps each \(S(M) T\)-arrow \(m : e \sim d\) to the \(S'(M) T\)-arrow \(\sigma_e \circ m : e \sim d\) as indicated in

\[
\begin{array}{c}
e : S' \xrightarrow{\sigma_e} m : S(M) T \xrightarrow{m : S(M) T} T : d
\end{array}
\]

Similarly, given

\[
\begin{array}{c}
E \xrightarrow{S} X \xrightarrow{M} \mathcal{M} \xrightarrow{T} A \xrightarrow{T'} D
\end{array}
\]

their composite is the module morphism \(S(M) \tau : S(M) T \to S(M) T' : E \to D\) which maps each \(S(M) T\)-arrow \(m : e \sim d\) to the \(S(M) T'\)-arrow \(m \circ \tau_d : e \sim d\) as indicated in

\[
\begin{array}{c}
e : S \xrightarrow{m} T : d \xrightarrow{\tau_d} T' : d
\end{array}
\]

5. The evaluation \(x(M)a\) of a module (or module morphism) \(M : X \to A\) at \((x, a) \in X \times A\) is identified with the composition

\[
\begin{array}{c}
* \xrightarrow{x} X \xrightarrow{M} \mathcal{M} \xrightarrow{a} A \xrightarrow{*} *
\end{array}
\]

For a pair of objects \(x \in \|X\|\) and \(a \in \|A\|\), the precomposition functor

\[
[x : a] : [X : A] \to \text{Set}
\]

“evaluation at \((x, a)\)”, sends each module \(M : X \to A\) to the hom-set \(x(M)a\) and sends each module morphism \(\Phi : M \to N : X \to A\) to its component \(x(\Phi)a : x(M)a \to x(N)a\) at \((x, a)\).

6. The partial evaluation of a module \(M : X \times Y \to A \times B\) at \((x, a) \in \|X \times A\|\) is given by the composition

\[
\begin{array}{c}
Y \xrightarrow{x \times Y} X \times Y \xrightarrow{M} \mathcal{M} \xrightarrow{a \times B} A \times B \xrightarrow{b} B
\end{array}
\]

yielding the slice of \(M\) at \((x, a)\), i.e. the module \([x \times Y] (M) [a \times B] : Y \to B\) such that

\[
y([x \times Y](M)[a \times B]) b = (x, y)(M)(a, b)
\]

for \(y \in Y\) and \(b \in B\). The precomposition functor

\[
[x \times Y : a \times B] : [X \times Y : A \times B] \to [Y : B]
\]

“partial evaluation at \((x, a)\)”, sends each module \(M : X \times Y \to A \times B\) to its slice at \((x, a)\).
1. Modules and Cells

7. Given a right module \( M : X \to * \) and a category \( E \), the composition

\[
X \xrightarrow{M} * \xleftarrow{\Delta} E
\]

yields the two-sided module \( \langle M \rangle \Delta_E : X \to E \) by duplicating \( M \) across \( E \). The \( E \)-ary diagonal functor of \( [X:] \) is the precomposition functor

\[
[X : \Delta_E] : [X : ] \to [X : E]
\]

sending each right module \( M : X \to * \) to the composite module \( \langle M \rangle \Delta_E : X \to E \). As a special case, the \( E \)-ary diagonal functor

\[
[: \Delta_E] : \text{Set} \to [:E]
\]

of \( \text{Set} \) sends each set \( S \) to the constant left module \( \langle S \rangle \Delta_E : * \to E \) given by the composition

\[
* \xrightarrow{S} * \xleftarrow{\Delta} E
\]

8. Given a left module \( M : * \to A \) and a category \( E \), the composition

\[
E \xrightarrow{\Delta} * \xrightarrow{M} A
\]

yields the two-sided module \( \Delta_E \langle M \rangle : E \to A \) by duplicating \( M \) across \( E \). The \( E \)-ary diagonal functor of \( [:A] \) is the precomposition functor

\[
[\Delta_E : A] : [:A] \to [E : A]
\]

sending each left module \( M : * \to A \) to the composite module \( \Delta_E \langle M \rangle : E \to A \). As a special case, the \( E \)-ary diagonal functor

\[
[\Delta_E :] : \text{Set} \to [E :]
\]

of \( \text{Set} \) sends each set \( S \) to the constant right module \( \Delta_E \langle S \rangle : E \to * \) given by the composition

\[
E \xrightarrow{\Delta} * \xrightarrow{S} *
\]

8. The join \( X* A : X \to A \) of categories \( X \) and \( A \) is the module given by the composition

\[
X \xrightarrow{\Delta} * \xleftarrow{(+) \to *} \xrightarrow{\Delta} A
\]

where \((*)\) is the hom of the terminal category. For every pair of objects \( x \in [X] \) and \( a \in [A] \), the hom-set \( x(X* A) a \) consists of a single arrow, the identity \( 1_x : * \to * \). As a special case, the join \( X* : X \to * \) is given by the composition

\[
X \xrightarrow{\Delta} * \xleftarrow{(*) \to *}
\]

; dually, the join \( * A : * \to A \) is given by the composition

\[
* \xleftarrow{(*)} * \xrightarrow{\Delta} A
\]
1. Modules and Cells

**Proposition 1.1.19.** If \( \Phi \) in Example 1.1.18(3) is a module isomorphism, so is the composite \( S(\Phi)T \).

*Proof.* Note that \( S(\Phi)T \) is given by the image of \( \Phi \) under the precomposition functor \([S : T]\). Since any functor preserves isomorphisms, \( S(\Phi)T \) is an isomorphism. \( \square \)

**Proposition 1.1.20.**

1. Given

\[
\begin{array}{ccc}
E' & \xrightarrow{S'} & E' \\
\downarrow S & \downarrow S & \downarrow S \\
X & \xrightarrow{M} & A \\
\downarrow T & \downarrow T & \downarrow T \\
D & \xleftarrow{T'} & D'
\end{array}
\]

, the associative law

\[S'(S(M)T)T' = [S' \circ S] (M) [T \circ T']\]

holds.

2. Given

\[
\begin{array}{ccc}
E & \xrightarrow{S} & E \\
\downarrow S & \downarrow S & \downarrow S \\
X & \xrightarrow{M} & A \\
\downarrow T & \downarrow T & \downarrow T \\
D
\end{array}
\]

, the associative law

\[S (\langle M \rangle T) = S (M) T = (S (M)) T\]

holds.

*Proof.*

1. Indeed,

\[
S'(S(M)T)T' = [
[S' \times T'] \circ [[S \times T] \circ M]
\]

\[
= [[S' \times T'] \circ [S \times T]] \circ M
\]

\[
= [[S' \circ S] \times [T' \circ T]] \circ M
\]

\[
= [S' \circ S] \langle M \rangle [T \circ T']
\]

2. By what we have just seen,

\[S (\langle M \rangle T) = S (\langle 1_X \rangle (M) T) = [S \circ 1_X] (M) T = S (M) T\]

and

\[\langle S (M) \rangle T = \langle S (M) [1_A] \rangle T = S (M) [1_A \circ T] = S (M) T\]

\( \square \)

**Definition 1.1.21.** Given a module \( \mathcal{M} : X \to A \), the opposite module \( \mathcal{M}^- : A^- \to X^- \) is defined by the composition

\[
\begin{array}{c}
A \times X^- \xrightarrow{\sim} X^- \times A \\
\downarrow M \\
\text{Set}
\end{array}
\]

, where \( \sim \) denotes the simple transposition \( (a, x) \mapsto (x, a) \); similarly, given a module morphisms \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \), the opposite module morphism \( \Phi^- : \mathcal{M}^- \to \mathcal{N}^- : A^- \to X^- \) is defined by the composition

\[
\begin{array}{c}
A \times X^- \xrightarrow{\sim} X^- \times A \\
\downarrow \Phi^- \\
\text{Set}
\end{array}
\]
1. Modules and Cells

Remark 1.1.22.

1. The iso functor \([X: A] \mapsto [A^- : X^-]\) is given by the assignment \(M \mapsto M^-\) and \(\Phi \mapsto \Phi^-\).

2. For any module \(M\) and any module morphism \(\Phi\),
   \[
   (M^-)^\sim = M \quad (\Phi^-)^\sim = \Phi
   \]
   .

3. For any category \(C\),
   \[
   (C^-)^\sim = (C)^\sim
   \]
   ; that is, the hom of the opposite category is the opposite module of the hom.

4. For any composite module \(S(M) T\),
   \[
   (S(M) T)^\sim = T(M^-) S
   \]
   ; that is, the opposite of the composite
   
   \[
   \begin{array}{c}
   E \xrightarrow{S} X \xrightarrow{M} A \xleftarrow{T} D
   \end{array}
   \]
   is given by the composite
   
   \[
   \begin{array}{cc}
   D^- \xrightarrow{T} A^- \xrightarrow{M^-} X^- \xleftarrow{S} E^-
   \end{array}
   \]
   .

5. The opposite of the module morphisms \(\Phi : M \rightarrow N : X \rightarrow A\) is often denoted by \(\Phi : M^- \rightarrow N^- : A^- \rightarrow X^-\) using the same name as its original counterpart (cf. Preliminaries(6)).

6. The opposite of a right module \(M : X \rightarrow \star\) is the left left module \(M^- : \star \rightarrow X^-\); in fact, \(M\) and \(M^-\) are defined by the same functor
   \[
   M^- = M : X^- \rightarrow \text{Set}
   \]
   .

1.2. Cells

Definition 1.2.1. Let \(M : X \rightarrow A\) and \(N : Y \rightarrow B\) be modules. Given a pair of functors \(P : X \rightarrow Y\) and \(Q : A \rightarrow B\), a module cell (or just a cell) \(\Phi : P \sim Q : M \rightarrow N\), depicted as

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow{P} & \Phi & \downarrow{Q} \\
Y & \xrightarrow{N^-} & B
\end{array}
\]

, is defined by a module morphism \(\Phi : M \rightarrow P \langle N \rangle Q : X \rightarrow A\).

Remark 1.2.2.
1. Modules and Cells

1. A cell $\Phi : P \sim Q : M \to N$ sends each $M$-arrow $m : x \sim a$ to the $N$-arrow $m : x : P \sim Q : a$, the image of $m$ under the function

$$x \langle M \rangle a \xrightarrow{x(\Phi)a} x \langle P \rangle \langle N \rangle Q a = (x : P) \langle N \rangle Q a$$

2. Cells and module morphisms are regarded as special instances of each other. A cell $\Phi : P \sim Q : M \to N$ is thought of as a module morphism from $M$ to the composite module $P \langle N \rangle Q$. Conversely, a module morphism $\Phi : M \to N : X \to A$ is depicted as

$$
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow \Phi \\
X \xrightarrow{N} A
\end{array}
$$

3. The identity module morphism $M \to M$ yields the identity cell

$$
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow 1 \\
X \xrightarrow{N} A
\end{array}
$$

4. Any composition

$$
\begin{array}{c}
X \xrightarrow{P} Y \xrightarrow{N} B \xleftarrow{Q} A
\end{array}
$$

trivially yields a cell

$$
\begin{array}{c}
X \xrightarrow{P \langle N \rangle Q} A \\
\downarrow P \\
Y \xrightarrow{N} B
\end{array}
$$

; a cell

$$
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow 1 \\
Y \xrightarrow{N} B
\end{array}
$$

expresses an identity $M = P \langle N \rangle Q$.

**Definition 1.2.3.**

- Let $M : X \to *$ and $N : Y \to *$ be right modules. Given a functor $P : X \to Y$, a right module cell $\Phi : P \sim * : M \to N$, depicted as

$$
\begin{array}{c}
X \xrightarrow{M} * \\
\downarrow P \\
Y \xrightarrow{N} *
\end{array}
$$

, is defined by a right module morphism $\Phi : M \to P \langle N \rangle : X \to *$. 

22
1. Modules and Cells

Let \( M : \ast \to A \) and \( N : \ast \to B \) be left modules. Given a functor \( Q : A \to B \), a left module cell \( \Phi : \ast \to Q : M \to N \), depicted as

\[
\begin{array}{ccc}
\ast & \xrightarrow{M} & A \\
\downarrow & \Phi & \downarrow Q \\
\ast & \xrightarrow{N} & B
\end{array}
\]

, is defined by a left module morphism \( \Phi : M \to (N)Q : \ast \to A \).

\textit{Remark 1.2.4.} A right (resp. left) module cell in Definition 1.2.3 is regarded as a special instance of a two-sided module cell in Definition 1.2.1 where \( A \) and \( B \) (resp. \( X \) and \( Y \)) are the terminal category under the identification in Remark 1.1.6(5). Conversely, by Remark 1.1.6(2), a two-sided module cell in Definition 1.2.1 is the same thing as a right (resp. left) module cell depicted below:

\[
\begin{array}{ccc}
X \times A & \xrightarrow{M} & \ast \\
P \times Q & \Phi & \downarrow 1 \\
Y \times B & \xrightarrow{N} & \ast
\end{array}
\]

\[
\begin{array}{ccc}
X \times A & \xrightarrow{M} & X \times A \\
P \times Q & \Phi & P \times Q \\
Y \times B & \xrightarrow{N} & Y \times B
\end{array}
\]

\textit{Definition 1.2.5.}

- Let \( M : X \to \ast \) be a right module and \( N : Y \to B \) be a two-sided module. Given a functor \( P : X \to Y \) and an object \( b \in \|B\| \), a right conical cell \( \Phi : P \to b : M \to N \), depicted as

\[
\begin{array}{ccc}
X & \xrightarrow{M} & \ast \\
P & \Phi & \downarrow b \\
Y & \xrightarrow{N} & B
\end{array}
\]

, is defined by a right module morphism \( \Phi : M \to P \langle N \rangle b : X \to \ast \).

- Let \( M : \ast \to A \) be a left module and \( N : Y \to B \) be a two-sided module. Given an object \( y \in \|Y\| \) and a functor \( Q : A \to B \), a left conical cell \( \Phi : y \to Q : M \to N \), depicted as

\[
\begin{array}{ccc}
\ast & \xrightarrow{M} & A \\
y & \Phi & \downarrow Q \\
\ast & \xrightarrow{N} & B
\end{array}
\]

, is defined by a left module morphism \( \Phi : M \to y \langle N \rangle Q : \ast \to A \).

\textit{Remark 1.2.6.}

1. A conical cell in Definition 1.2.5 is regarded as a special instance of a two-sided module cell in Definition 1.2.1 where \( A \) (resp. \( X \)) is the terminal category under the identification in Remark 1.1.6(5).

2. A right (resp. left) module cell in Definition 1.2.3 is regarded as a special instance of a right (resp. left) conical cell in Definition 1.2.5 where \( B \) (resp. \( Y \)) is the terminal category.
1. Modules and Cells

Conversely, a right (resp. left) conical cell in Definition 1.2.5 is the same thing as a right (resp. left) module cell depicted below

\[
\begin{align*}
X &\xrightarrow{\mathcal{J}} M \\
\Phi &\downarrow \quad 1 \\
Y &\xrightarrow{(\mathcal{N})_b} N
\end{align*}
\]

\[
\begin{align*}
* &\xrightarrow{\mathcal{M}} = A \\
\Phi &\downarrow \quad 1 \\
* &\xrightarrow{y(\mathcal{N})} = B
\end{align*}
\]

**Definition 1.2.7.** Given a pair of modules \( \mathcal{J} : E \to D \) and \( \mathcal{M} : X \to A \), the module of cells \( \mathcal{J} \to \mathcal{M} \),

\[
\langle \mathcal{J}, \mathcal{M} \rangle : [E, X] \to [D, A]
\]

is defined by

\[
(S) \langle \mathcal{J}, \mathcal{M} \rangle (T) = (\mathcal{J}) (E : D) (S \langle \mathcal{M} \rangle T)
\]

for \( S \in [E, X] \) and \( T \in [D, A] \), where \( (E : D) \) is the hom of the module category \([E : D]\).

**Remark 1.2.8.**

1. For a pair of functors \( S : E \to X \) and \( T : D \to A \), the set \( (S) \langle \mathcal{J}, \mathcal{M} \rangle (T) \) consists of all cells \( S \sim T : \mathcal{J} \to \mathcal{M} \);

2. If \( \Theta : S \sim T : \mathcal{J} \to \mathcal{M} \) is a cell and \( \sigma : S' \to S : E \to X \) is a natural transformation as in

\[
\begin{align*}
E &\xrightarrow{\mathcal{J}} D \\
S' \xrightarrow{\sigma} S &\xrightarrow{\Theta} T \\
X &\xrightarrow{\mathcal{M}} A
\end{align*}
\]

, then their composite is the cell

\[
\begin{align*}
E &\xrightarrow{\mathcal{J}} D \\
S' \xrightarrow{\sigma \circ \Theta} &\xrightarrow{T} \\
X &\xrightarrow{\mathcal{M}} A
\end{align*}
\]

defined by the module morphism \( \sigma \circ \Theta : \mathcal{J} \to S' \langle \mathcal{M} \rangle T \) given by the composition

\[
\mathcal{J} \xrightarrow{\Theta} S \langle \mathcal{M} \rangle T \xrightarrow{\sigma \langle \mathcal{M} \rangle T} S' \langle \mathcal{M} \rangle T
\]

. By Example 1.1.18(4), the cell \( \sigma \circ \Theta \) sends each \( \mathcal{J} \)-arrow \( j : e \sim d \) to the \( \mathcal{M} \)-arrow

\[
j : (\sigma \circ \Theta) = \sigma_e \circ (j : \Theta) : e : S' \sim T' : d
\]

as indicated in

\[
e : S' \xrightarrow{\sigma_e} T' : d
\]
1. Modules and Cells

- If $\Theta : S \to T : J \to M$ is a cell and $\tau : T \to T' : D \to A$ is a natural transformation as in

$$
\begin{array}{c}
\begin{array}{c}
E \xrightarrow{J} D \\
S \xrightarrow{\theta} T \xrightarrow{\tau} T' \\
X \xrightarrow{M} A
\end{array}
\end{array}
$$

, then their composite is the cell

$$
\begin{array}{c}
\begin{array}{c}
E \xrightarrow{J} D \\
S \xrightarrow{\Theta \circ \tau} T' \\
X \xrightarrow{M} A
\end{array}
\end{array}
$$

defined by the module morphism $\Theta \circ \tau : J \to S(\mathcal{M}) T'$ given by the composition

$$
\begin{array}{c}
\begin{array}{c}
J \xrightarrow{\Theta} S(\mathcal{M}) T \xrightarrow{S(\mathcal{M}) \tau} S(\mathcal{M}) T'
\end{array}
\end{array}
$$

. By Example 1.1.18(4), the cell $\Theta \circ \tau$ sends each $J$-arrow $j : e \sim d$ to the $M$-arrow

$$
\begin{array}{c}
\begin{array}{c}
j : (\Theta \circ \tau) = (j : \Theta) \circ \tau_d : e : S \sim T' : d
\end{array}
\end{array}
$$

as indicated in

$$
\begin{array}{c}
\begin{array}{c}
\xymatrix{ e : S \ar[r]^{j : \Theta} & T' : d \\
& \downarrow_{\tau_d} \\
& T' : d
\end{array}
\end{array}
$$

.

3. If $J$ is small and $\mathcal{M}$ is locally small, then the module $\langle J, \mathcal{M} \rangle$ is locally small.

**Proposition 1.2.9.** Given a module $J$ and a composite module $P \langle N \rangle Q$ as in

$$
\begin{array}{c}
\begin{array}{c}
E \xrightarrow{J} D \\
X \xrightarrow{P \langle N \rangle Q} A \\
Y \xrightarrow{1} B
\end{array}
\end{array}
$$

, the identity

$$
\begin{array}{c}
\begin{array}{c}
[E, X] \xrightarrow{(J, P \langle N \rangle Q)} [D, A] \\
[E, P] \xrightarrow{1} [D, Q] \\
[E, Y] \xrightarrow{(J, N)} [D, B]
\end{array}
\end{array}
$$

, i.e.

$$
\langle J, P \langle N \rangle Q \rangle = [E, P] \langle J, N \rangle [D, Q]
$$

, holds.
1. Modules and Cells

**Proof.** For any \( S \in [E, X] \) and \( T \in [D, A] \),

\[
S\langle J, P \langle N \rangle Q \rangle T = \langle J \rangle \langle E : D \rangle (S \langle P \langle N \rangle Q \rangle T) \\
= \langle J \rangle \langle E : D \rangle ([S \circ P] \langle N \rangle [Q \circ T]) \\
= (S \circ P) \langle J, N \rangle (Q \circ T) \\
= (S : [E, P]) \langle J, N \rangle ([D, Q] \circ T) \\
= (S) \langle [E, P] \langle J, N \rangle [D, Q] \rangle (T)
\]

\( \square \)

**Definition 1.2.10.** If \( \Theta : S \rightarrow T : J \rightarrow M \) is a cell and \( \Phi : M \rightarrow N \) is a module morphism as in

\[
\begin{array}{ccc}
E & \xrightarrow{J} & D \\
\downarrow{S} & \circ & \downarrow{T} \\
X & \xrightarrow{\Phi} & A \\
\end{array}
\]

, then their composite \( \Theta \circ \Phi \) is the cell

\[
\begin{array}{ccc}
E & \xrightarrow{J} & D \\
\downarrow{S} & \circ \Theta \circ \Phi & \downarrow{T} \\
X & \xrightarrow{N} & A \\
\end{array}
\]

defined by the module morphism \( \Theta \circ \Phi : J \rightarrow S \langle N \rangle T \) given by the composition

\[
\begin{array}{ccc}
J & \xrightarrow{\Theta} & S \langle M \rangle T & \xrightarrow{S(\Phi)T} & S \langle N \rangle T \\
\end{array}
\]

. 

**Remark 1.2.11.** See Example 1.1.18(3) for the module morphism \( S \langle \Phi \rangle T \). The cell \( \Theta \circ \Phi \) sends each \( J \)-arrow \( j : e \rightarrow d \) to the \( N \)-arrow

\[
j : (\Theta \circ \Phi) = j : \Theta : \Phi : e : S \rightarrow T : d
\]

, the image of \( j \) under the composite function

\[
e \langle J \rangle d \xrightarrow{e(\Theta)d} e \langle S \langle M \rangle T \rangle d = (e : S) \langle M \rangle (T : d) \xrightarrow{(e \circ S)(\Phi)(T : d)} (e : S) \langle N \rangle (T : d)
\]

. 

**Definition 1.2.12.** Given a module \( J : E \rightarrow D \) and a module morphism \( \Phi : M \rightarrow N : X \rightarrow A \), the module morphism

\[
\langle J, \Phi \rangle : \langle J, M \rangle \rightarrow \langle J, N \rangle : [E, X] \rightarrow [D, A]
\]

, “postcomposition with \( \Phi \)”, is defined by

\[
(S \langle J, \Phi \rangle T) = \langle J \rangle \langle E : D \rangle (S \langle \Phi \rangle T)
\]

for each pair of functors \( S : E \rightarrow X \) and \( T : D \rightarrow A \).
1. Modules and Cells

Remark 1.2.13.

1. The module morphism \( \langle J, \Phi \rangle \) maps each cell \( \Theta : S \to T : J \to M \) to the cell \( \Theta \circ \Phi : S \to T : J \to \mathcal{N} \) defined in Definition 1.2.10.

2. The assignment \( \Phi \to \langle J, \Phi \rangle \) is functorial; indeed the functor

\[
\langle J, - \rangle : [X : A] \to [[E, X] : [D, A]]
\]

is defined by

\[
S \langle J, M \rangle T = (J) (E : D) (S \langle M \rangle T)
\]

for \( S \in [E, X] \), \( T \in [D, A] \), and \( M \in [X : A] \).

Note. By Remark 1.2.2(2), the following definition is regarded as a special case of Definition 1.2.10 and vice versa.

Definition 1.2.14. Given a pair of cells as in

\[
\begin{array}{ccc}
E & \xrightarrow{J} & D \\
S & \xrightarrow{\Theta} & T \\
X & \xrightarrow{M} & A \\
P & \xrightarrow{\Phi} & Q \\
Y & \xrightarrow{\mathcal{N}} & B \\
\end{array}
\]

, their composite \( \Theta \circ \Phi = \Phi \circ \Theta \) is the cell

\[
\begin{array}{ccc}
E & \xrightarrow{J} & D \\
S \circ P & \xrightarrow{\Theta \circ \Phi} & Q \circ T \\
Y & \xrightarrow{\mathcal{N}} & B \\
\end{array}
\]

defined by the module morphism \( \Theta \circ \Phi : J \to [S \circ P] \langle \mathcal{N} \rangle [Q \circ \mathcal{T}] \) given by the composition

\[
J \xrightarrow{\Theta \circ \Phi} S \langle M \rangle T \xrightarrow{S(\Phi)T} S \langle \mathcal{N} \rangle Q \mathcal{T} = [S \circ P] \langle \mathcal{N} \rangle \langle Q \circ \mathcal{T} \rangle
\]

.

Remark 1.2.15. The cell \( \Theta \circ \Phi \) sends each \( J \)-arrow \( j : e \to d \) to the \( \mathcal{N} \)-arrow

\[
j : (\Theta \circ \Phi) = j : \Theta \circ \Phi : e : S : P \to Q : T : d
\]

, the image of \( j \) under the composite function

\[
\begin{array}{ccc}
e \langle J \rangle d & \xrightarrow{e(\Theta)d} & e \langle S \langle M \rangle T \rangle d = (e : S) \langle M \rangle (T : d) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & (e : S) \langle P \langle \mathcal{N} \rangle \mathcal{T} \rangle d = (e : S : P) \langle \mathcal{N} \rangle (Q : T : d)
\end{array}
\]
1. Modules and Cells

**Proposition 1.2.16.** Modules and cells among them form a category with the composition given in Definition 1.2.14 and the identities given in Remark 1.2.2(3).

**Proof.** The only non-trivial part is the verification of the associativity of the composition. Consider cells as in

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow \Phi & & \downarrow Q \\
X' & \xrightarrow{M'} & A' \\
\downarrow \Phi' & & \downarrow Q' \\
X'' & \xrightarrow{M''} & A'' \\
\downarrow \Phi'' & & \downarrow Q'' \\
X''' & \xrightarrow{M'''} & A'''
\end{array}
\]

The composites \((\Phi \circ \Phi') \circ \Phi''\) and \(\Phi \circ (\Phi' \circ \Phi'')\) are defined by the module morphisms \((\Phi \circ \Phi') \circ \Phi'' = [\Phi \circ (\Phi' \circ \Phi'')]\circ [Q' \circ Q]\) and \(\Phi \circ (\Phi' \circ \Phi'') = [Q' \circ Q]\) respectively. But, by the functoriality (see Remark 1.1.17(1)) and associativity (see Proposition 1.1.20) of the composition, we have

\[
(\Phi \circ \Phi') \circ \Phi'' = [\Phi \circ (\Phi' \circ \Phi'')]\circ [Q' \circ Q] = \Phi \circ (\Phi' \circ \Phi'') = [Q' \circ Q]
\]

\qed}

**Remark 1.2.17.**

1. Given a universe \(\mathcal{U}\), \(\text{MOD}_{\mathcal{U}}\) denotes the category consisting of all locally \(\mathcal{U}\)-small modules and all cells among them; \(\text{MOD}_{\mathcal{U}}\) is \(\mathcal{U}\)-large. By \(\text{MOD}\) we mean \(\text{MOD}_{\mathcal{U}}\) for \(\mathcal{U}\) chosen implicitly.

2. Given a pair of categories \(X\) and \(A\), there is a canonical embedding \([X : A] \rightarrow \text{MOD}\), identical on objects, given by the obvious arrow function (see Remark 1.2.2(2)). The embedding is not, in general, full.

**Proposition 1.2.18.** A cell \(\Phi : P \rightsquigarrow Q : \mathcal{M} \rightarrow \mathcal{N}\) is invertible in the category \(\text{MOD}\) if and only if the functors \(P\) and \(Q\) are iso and the module morphism \(\Phi : \mathcal{M} \rightarrow P\langle \mathcal{N} \rangle Q\) is iso.

**Proof.** If \(\Phi : P \rightsquigarrow Q : \mathcal{M} \rightarrow \mathcal{N}\) has an inverse \(\Psi : S \rightsquigarrow T : \mathcal{N} \rightarrow \mathcal{M}\), then \(P\) (resp. \(Q\)) and \(S\) (resp. \(T\)) are inverse to each other and \(\Phi : \mathcal{M} \rightarrow P\langle \mathcal{N} \rangle Q\) and \(P\langle \Psi \rangle Q : P\langle \mathcal{N} \rangle Q \rightarrow \mathcal{M}\) are inverse to each other. Conversely, if \(P\), \(Q\), and \(\Phi : \mathcal{M} \rightarrow P\langle \mathcal{N} \rangle Q\) are iso, the inverse of \(\Phi : P \rightsquigarrow Q : \mathcal{M} \rightarrow \mathcal{N}\) is given by the cell \(\Psi : S \rightsquigarrow T : \mathcal{N} \rightarrow \mathcal{M}\) with \(S = P^{-1}\), \(T = Q^{-1}\), and the module morphism \(\Psi : \mathcal{N} \rightarrow S\langle \mathcal{M} \rangle T\) defined by \(\Psi = [P^{-1}\langle \Phi^{-1}\rangle][Q^{-1}]\). \(\Box\)

**Definition 1.2.19.** A cell \(\Phi : P \rightsquigarrow Q : \mathcal{M} \rightarrow \mathcal{N}\) is called

1. iso if it satisfies the equivalent conditions in Proposition 1.2.18;
2. fully faithful if the module morphism \(\Phi : \mathcal{M} \rightarrow P\langle \mathcal{N} \rangle Q\) is iso.

**Remark 1.2.20.** In Proposition 1.2.27 we will see the relation between the notion of fully faithfulness for cells and that for functors.
1. Modules and Cells

Note. Proposition 1.2.9 allows the following definition.

**Definition 1.2.21.** Given a module $\mathcal{J} : \mathcal{E} \to \mathcal{D}$ and a cell

\[ X \xrightarrow{-\ M\ } A \]
\[ P \downarrow \Phi \downarrow Q \]
\[ Y \xrightarrow{-\ N\ } B \]

, the cell

\[ [E, X] \xrightarrow{(\mathcal{J}, M)} [D, A] \]
\[ [E, P] \xrightarrow{(\mathcal{J}, \Phi)} [D, Q] \]
\[ [E, Y] \xrightarrow{(\mathcal{J}, N)} [D, B] \]

"postcomposition with $\Phi", is defined by the postcomposition module morphism

\[ (\mathcal{J}, \mathcal{M}) \xrightarrow{(\mathcal{J}, \Phi)} (\mathcal{J}, P \langle N \rangle Q) = [E, P] (\mathcal{J}, N)[D, Q] \]

with $\Phi : \mathcal{M} \to P \langle N \rangle Q$.

**Remark 1.2.22.** The cell $(\mathcal{J}, \Phi)$ sends each cell $\Theta : S \to T : \mathcal{J} \to \mathcal{M}$ to the cell $\Theta \circ \Phi : S \circ P \to Q \circ T : \mathcal{J} \to \mathcal{N}$ defined in Definition 1.2.14.

**Proposition 1.2.23.** The assignment $\Phi \mapsto (\mathcal{J}, \Phi)$ is functorial.

**Proof.** Clearly, the assignment $\Phi \mapsto (\mathcal{J}, \Phi)$ preserves the identities. To verify that it preserves the composition, let $\Phi$ and $\Psi$ be a composable pair of cells and consider the cells $(\mathcal{J}, \Phi)$, $(\mathcal{J}, \Psi)$, and $(\mathcal{J}, \Phi \circ \Psi)$ depicted in

\[ X \xrightarrow{-\ M\ } A \]
\[ P \downarrow \Phi \downarrow Q \]
\[ Y \xrightarrow{-\ N\ } B \]
\[ Z \xrightarrow{-\ L\ } C \]
\[ [E, X] \xrightarrow{-\ M\ } [D, A] \]
\[ [E, P] \xrightarrow{(\mathcal{J}, \Phi)} [D, Q] \]
\[ [E, Y] \xrightarrow{(\mathcal{J}, N)} [D, B] \]
\[ [E, Z] \xrightarrow{(\mathcal{J}, L)} [D, C] \]

. We need to verify that the composition of the cells $(\mathcal{J}, \Phi)$ and $(\mathcal{J}, \Psi)$ yields the cell $(\mathcal{J}, \Phi \circ \Psi)$. First note that $[E, P \circ P'] = [E, P] \circ [E, P']$ and $[D, Q' \circ Q] = [D, Q'] \circ [D, Q]$ by the functoriality of the operations $[E, -]$ and $[D, -]$. The cell $(\mathcal{J}, \Phi \circ \Psi)$ is defined by the module morphism $(\mathcal{J}, \Phi \circ \Psi)[E, P] (\mathcal{J}, \Psi)[D, Q]$ and the cell $(\mathcal{J}, \Phi \circ \Psi)$ is defined by the module morphism $(\mathcal{J}, \Phi \circ P \langle \Psi \rangle Q)$. But, by Remark 1.2.13(2) and Proposition 1.2.9,

\[ (\mathcal{J}, \Phi \circ P \langle \Psi \rangle Q) = (\mathcal{J}, \Phi \circ (\mathcal{J}, P \langle \Psi \rangle Q)) = (\mathcal{J}, \Phi \circ [E, P] \langle \mathcal{J}, \Psi \rangle [D, Q]) \]

. \qed
Remark 1.2.24. Given a small module \( \mathcal{J} : \mathcal{E} \to \mathcal{D} \), the functor 
\[
(\mathcal{J}, -) : \text{MOD} \to \text{MOD}
\]
is defined by the object function \( \mathcal{M} \mapsto (\mathcal{J}, \mathcal{M}) \) and the arrow function \( \Phi \mapsto (\mathcal{J}, \Phi) \), extending the functor \( (\mathcal{J}, -) \) in Remark 1.2.13(2) as shown in
\[
\begin{array}{ccc}
[X: A] & \xrightarrow{(\mathcal{J}, -)} & [[E, X]: [D, A]] \\
\downarrow & & \downarrow \\
\text{MOD} & \xrightarrow{(\mathcal{J}, -)} & \text{MOD}
\end{array}
\]
where \( \hookrightarrow \) denotes the canonical embedding in Remark 1.2.17(2).

Definition 1.2.25. The hom of a functor \( \mathcal{H} : \mathcal{C} \to \mathcal{B} \) is the cell
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{(\mathcal{C})} & \mathcal{C} \\
\mathcal{H} \downarrow & & \mathcal{H} \downarrow \\
\mathcal{B} & \xrightarrow{(\mathcal{B})} & \mathcal{B}
\end{array}
\]
given by the arrow function of \( \mathcal{H} \); that is, for each pair of objects \( x, a \in \|\mathcal{C}\| \), the function
\[
a_{\mathcal{H}}(b) : a(C) \to (a \cdot \mathcal{H})(\mathcal{B}) (\mathcal{H} \cdot b)
\]
is given by
\[
H_{a,b} : \hom_{\mathcal{C}}(a, b) \to \hom_{\mathcal{B}}(\mathcal{H} \cdot a, \mathcal{H} \cdot b), \quad f \mapsto \mathcal{H} \cdot f
\]

Remark 1.2.26.

1. The naturality of the module morphism \( (\mathcal{H}) : (\mathcal{C}) \to \mathcal{H}(\mathcal{B}) \) \( \mathcal{H} : \mathcal{C} \to \mathcal{C} \) follows from the functoriality of \( \mathcal{H} \).

2. Hereafter, given a functor \( \mathcal{H} \), each component of the arrow function of \( \mathcal{H} \) is written as \( a_{\mathcal{H}}(b) \).

Proposition 1.2.27. A functor \( \mathcal{H} : \mathcal{C} \to \mathcal{B} \) is iso (resp. fully faithful) if and only if the hom cell \( (\mathcal{H}) : \mathcal{H} \to \mathcal{H} : (\mathcal{C}) \to (\mathcal{B}) \) is iso (resp. fully faithful) (see Definition 1.2.19).

Proof. Immediate from the definitions.

Theorem 1.2.28. The hom operation defined in Definition 1.1.11 and Definition 1.2.25 embeds \( \text{CAT} \) in \( \text{MOD} \). Specifically, the assignment \( \mathcal{C} \mapsto (\mathcal{C}) \) forms a faithful functor \( \text{CAT} \to \text{MOD} \), injective on objects.

Proof. The verification of the functoriality is straightforward. The faithfulness and the injectiveness on objects are evident.
**Definition 1.2.29.** Given a cell and commutative quadrangles of functors as in

\[
\begin{array}{ccc}
X' & \xrightarrow{R} & X \\ \downarrow{p'} & \phi & \downarrow{Q} \\
Y' & \xrightarrow{S} & Y & \xleftarrow{N} & B & \xleftarrow{G} & B' \\
\end{array}
\]

their pasting composite

\[
\begin{array}{ccc}
X' & \xrightarrow{R(M)F} & A' \\
\downarrow{p'} & \phi & \downarrow{Q'} \\
Y' & \xrightarrow{S(N)G} & B' \\
\end{array}
\]

is defined by module morphism

\[
R\langle \Phi \rangle F : R\langle M \rangle F \to R\langle P\langle N \rangle Q \rangle F = P'\langle S\langle N \rangle G \rangle Q'
\]

given by the composition

\[
\begin{array}{ccc}
X' & \xrightarrow{R} & X \\ \downarrow{1} & \phi & \downarrow{1} \\
X' & \xrightarrow{S} & Y & \xleftarrow{N} & B & \xleftarrow{G} & A' \\
\end{array}
\]

**Remark 1.2.30.** A pasting composition

\[
\begin{array}{ccc}
X' & \xrightarrow{R} & X \\ \downarrow{1} & \phi & \downarrow{1} \\
X' & \xrightarrow{S} & Y & \xleftarrow{N} & B & \xleftarrow{G} & A' \\
\end{array}
\]

with both ends being identities, yields a cell

\[
\begin{array}{ccc}
X' & \xrightarrow{R(M)F} & A' \\
\downarrow{1} & \phi & \downarrow{1} \\
X' & \xrightarrow{S(N)G} & A' \\
\end{array}
\]

, i.e. a module morphism

\[
R\langle \Phi \rangle F : R\langle M \rangle F \to S\langle N \rangle G : X' \to A'.
\]

**Proposition 1.2.31.** In Definition 1.2.29, if \( \Phi \) is fully faithful, so is the cell \( R\langle \Phi \rangle F \).

**Proof.** Immediate from Proposition 1.1.19. \qed

**Proposition 1.2.32.** Cells and commutative quadrangles of functors as in

\[
\begin{array}{ccc}
X' & \xrightarrow{R} & X \\ \downarrow{p'} & \phi & \downarrow{Q} \\
Y' & \xrightarrow{S} & Y & \xleftarrow{N} & B & \xleftarrow{G} & B' \\
\end{array}
\]

\[
\begin{array}{ccc}
Z' & \xrightarrow{T} & Z & \xleftarrow{L} & C & \xleftarrow{H} & C' \\
\end{array}
\]


yield the same cell

\[ R(\Phi \circ \Psi) F = R(\Phi) F \circ S(\Psi) G : P' \circ K' \sim L' \circ Q' : R(M) F \rightarrow T(L) H \]

irrespective of the order of the horizontal and vertical compositions.

**Proof.** The horizontal composition followed by the vertical composition yields the cell \( R(\Phi) F \circ S(\Psi) G \), and the vertical composition followed by the horizontal composition yields the cell \( R(\Phi \circ \Psi) F \), as shown in

\[
\begin{align*}
X' & \xrightarrow{R(M)F} A' \\
P' & \xrightarrow{\Phi} Q' \\
Y' & \xrightarrow{S(\Psi)G} B' \\
K' & \xrightarrow{\Phi \circ \Psi} L' \\
Z' & \xrightarrow{T(L)H} C'
\end{align*}
\]

\[
\begin{align*}
X' & \xrightarrow{R(M)F} A' \\
P' & \xrightarrow{\Phi} Q' \\
Y' & \xrightarrow{S(\Psi)G} B' \\
K' & \xrightarrow{\Phi \circ \Psi} L' \\
Z' & \xrightarrow{T(L)H} C'
\end{align*}
\]

The cell \( R(\Phi) F \circ S(\Psi) G \) is defined by the module morphism \( R(\Phi) F \circ P' \circ S(\Psi) G \) \( Q' \) and the cell \( R(\Phi \circ \Psi) F \) is defined by the module morphism \( R(\Phi \circ P(\Psi) Q) F \). But

\[
R(\Phi \circ P(\Psi) Q) F = R(\Phi) F \circ R(P(\Psi) Q) F
\]

\[
= R(\Phi) F \circ [R \circ P(\Psi) [Q \circ F]]
\]

\[
= R(\Phi) F \circ [P' \circ S(\Psi) [G \circ Q']]
\]

\[
= R(\Phi) F \circ P' \circ S(\Psi) G \circ Q'
\]

\[
\square
\]

### 1.3. Cell morphisms

**Definition 1.3.1.** Given a pair of cells

\[
\begin{align*}
X & \xrightarrow{\Phi} A \\
Y & \xrightarrow{\Psi} B
\end{align*}
\]

\[
\begin{align*}
X & \xrightarrow{\Phi} A \\
Y & \xrightarrow{\Psi} B
\end{align*}
\]

, a morphism from \( \Phi \) to \( \Psi \), written \( \tau : \Phi \rightarrow \Psi : M \rightarrow N \), is a pair \( (\tau^1, \tau^2) \) of natural transformations

\[
\begin{align*}
\tau^1 : P \rightarrow S : X \rightarrow Y \\
\tau^2 : Q \rightarrow T : A \rightarrow B
\end{align*}
\]

such that the quadrangle

\[
\begin{align*}
x : P & \xrightarrow{m \cdot \Phi} Q \cdot a \\
\tau_1^2 & \downarrow \quad \tau_2^2 \\
x : S & \xrightarrow{m \cdot \Psi} T \cdot a
\end{align*}
\]

commutes for every \( M \)-arrow \( m : x \rightarrow a \).
1. Modules and Cells

Remark 1.3.2.

1. The identity morphism of a cell \( \Phi : P \to Q : M \to N \) is given by the pair of the identity natural transformations; that is,
\[
1_\Phi := (1_P, 1_Q)
\]

2. The composition of two cell morphisms \( \tau : \Phi : M \to N \) and \( \sigma : \Psi : M \to N \) is given componentwise; that is
\[
\tau \circ \sigma := (\tau^1 \circ \sigma^1, \tau^2 \circ \sigma^2)
\]

Proposition 1.3.3. Given a pair of modules \( M \) and \( N \), all cells \( M \to N \) and morphisms among them define the cell category \([M : N]\) with the identities and the composition defined in Remark 1.3.2.

Proof. Self explanatory.

Remark 1.3.4. If \( M \) is small and \( N \) is locally small, then the category \([M : N]\) is locally small.

Proposition 1.3.5. A cell morphism \( \tau : \Phi : M \to N \) is invertible in the category \([M : N]\) if and only if \( \tau^1 \) and \( \tau^2 \) are natural isomorphisms.

Proof. Immediate from the definitions.

Remark 1.3.6. A cell morphism is called iso if it satisfies the equivalent conditions in Proposition 1.3.5.

Definition 1.3.7. The hom of a natural transformation \( \tau : F \to G : C \to B \) is the cell morphism \( (\tau) : (F) \to (G) : (C) \to (B) \), given by the pair \((\tau, \tau)\).

Theorem 1.3.8. Given a pair of categories \( C \) and \( B \), the hom operation defined in Definition 1.2.25 and Definition 1.3.7 embeds the functor category \([C, B]\) in the cell category \(([C : (B)]\)). Specifically, the assignment \( H \mapsto (H) \) forms a faithful functor \([C, B] \to [([C : (B)])\), injective on objects.

Proof. The verification of the functoriality is straightforward. The faithfulness and the injectiveness on objects are evident.
2. Slicing and Action

2.1. Slicing of modules

**Definition 2.1.1.** Let $X$ and $A$ be categories.

- The right exponential transposition

$$[X^{-} \times A, \text{Set}] \xrightarrow{\sim} [A, [X^{-}, \text{Set}]]$$

(see Preliminaries(11)) is denoted by

$$[X : A] \xrightarrow{\sim} [A, [X :]]$$

The right exponential transpose of a module $M : X \to A$ is a covariant functor

$$[M^\blacktriangleright] : A \to [X :]$$

and the right exponential transpose of a module morphism $\Phi : M \to N : X \to A$ is a natural transformation

$$[\Phi^\blacktriangleright] : [M^\blacktriangleright] \to [N^\blacktriangleright] : A \to [X :]$$

The evaluation of the functor $M^\blacktriangleright$ at $a \in A$, i.e., the partial evaluation of the module $M$ at $a$, is written $\langle M \rangle (a)$ or just $\langle M \rangle a$. Similarly, the evaluation of the natural transformation $\Phi^\blacktriangleright$ at $a \in \|A\|$, i.e., the partial evaluation of the module morphism $\Phi$ at $a$, is written $\langle \Phi \rangle a$.

- The left exponential transposition

$$[X^{-} \times A, \text{Set}] \xrightarrow{\sim} [X^{-}, [A, \text{Set}]]$$

(see Preliminaries(11)) is denoted by

$$[X : A] \xrightarrow{\sim} [X^{-}, [: A]]$$

or

$$[X : A]^{-} \xrightarrow{\sim} [X, [: A]^{-}]$$

The left exponential transpose of a module $M : X \to A$ is a contravariant functor

$$[\blacktriangleleft M] : X \to [: A]^{-}$$
2. Slicing and Action

and the left exponential transpose of a module morphism $\Phi: \mathcal{M} \to \mathcal{N}: X \to A$ is a natural transformation

$$\left[\chi, \Phi\right]: \left[\chi, \mathcal{N}\right] \to \left[\chi, \mathcal{M}\right]: X \to [\cdot: A]^*$$

The evaluation of the functor $\chi, \mathcal{M}$ at $x \in X$, i.e. the partial evaluation of the module $\mathcal{M}$ at $x$, is written $(x)(\mathcal{M})$ or just $x(\mathcal{M})$. Similarly, the evaluation of the natural transformation $\chi, \Phi$ at $x \in [X]$, i.e. the partial evaluation of the module morphism $\Phi$ at $x$, is written $x(\Phi)$.

Remark 2.1.2.

1. The partial evaluation of a module $\mathcal{M}: X \to A$ at $a \in [A]$ yields the right slice of $\mathcal{M}$ at $a$, i.e. the right module

$$(\mathcal{M})a: X \to *$$

given by

$$x(\langle \mathcal{M} \rangle a) = x(\mathcal{M})a$$

for $x \in X$.

2. The partial evaluation of a module $\mathcal{M}: X \to A$ at $x \in [X]$ yields the left slice of $\mathcal{M}$ at $x$, i.e. the left module

$$x(\mathcal{M}): * \to A$$

given by

$$\langle x(\mathcal{M}) \rangle a = x(\mathcal{M})a$$

for $a \in A$.

2.

- The partial evaluation of $\mathcal{M}$ at an $A$-arrow $f: s \to t$ yields the right module morphism

$$\langle \mathcal{M} \rangle f: \langle \mathcal{M} \rangle s \to \langle \mathcal{M} \rangle t: X \to *$$

which maps each $\mathcal{M}$-arrow $m: x \rightsquigarrow s$ to the $\mathcal{M}$-arrow $m \circ f: x \rightsquigarrow t$ as indicated in

$$x \rightsquigarrow s \quad m \rightsquigarrow f \quad t$$

- The partial evaluation of $\mathcal{M}$ at an $X$-arrow $f: s \to t$ yields the left module morphism

$$f(\mathcal{M}): t(\mathcal{M}) \to s(\mathcal{M}): * \to A$$

which maps each $\mathcal{M}$-arrow $m: t \rightsquigarrow a$ to the $\mathcal{M}$-arrow $f \circ m: s \rightsquigarrow a$ as indicated in

$$s \rightsquigarrow f(\mathcal{M}) \quad f \quad t \rightsquigarrow m \rightsquigarrow a$$

3.
2. Slicing and Action

- The partial evaluation of a module morphism $\Phi : M \to N : X \to A$ at $a \in \parallel A \parallel$ yields the right slice of $\Phi$ at $a$, i.e. the right module morphism
  \[
  (\Phi) a : (M) a \to (N) a : X \to *
  \]
given by
  \[
  x (\Phi) a = x (\Phi) a
  \]
for each $x \in \parallel X \parallel$.

- The partial evaluation of a module morphism $\Phi : M \to N : X \to A$ at $x \in \parallel X \parallel$ yields the left slice of $\Phi$ at $x$, i.e. the left module morphism
  \[
  x (\Phi) : x (M) \to x (N) : * \to A
  \]
given by
  \[
  (x (\Phi)) a = x (\Phi) a
  \]
for each $a \in \parallel A \parallel$.

4. The partial evaluation $(M) a$ of a module $M : X \to A$ at $a \in A$ is identified with the composition
  \[
  X - \xrightarrow{\Delta E} - A \xleftarrow{a} - *
  \]
(cf. Example 1.1.18(5)).

- The partial evaluation $x (M)$ of a module $M : X \to A$ at $x \in X$ is identified with the composition
  \[
  * \xrightarrow{x} X - \xrightarrow{\Delta E} - A
  \]
(cf. Example 1.1.18(5)).

5. For any module $M$,
  \[
  [M^*] = [\times, M^*]
  \]
; that is, the opposite of the right exponential transpose of $M$ is the left exponential transpose of the opposite module $M^*$.

**Example 2.1.3.** The diagonal functors in Example 1.1.18(7) and the exponential transpositions form the following commutative diagrams (cf. Preliminaries(16)):
Proposition 2.1.4. For any module morphism \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \), the following conditions are equivalent:

1. \( \Phi \) is iso;
2. the right module morphism \( \langle \Phi \rangle \mathbf{a} : \langle \mathcal{M} \rangle \mathbf{a} \to \langle \mathcal{N} \rangle \mathbf{a} : X \to * \) is iso for every \( \mathbf{a} \in |A| \);
3. the left module morphism \( x \langle \Phi \rangle : x \langle \mathcal{M} \rangle \to x \langle \mathcal{N} \rangle : * \to A \) is iso for every \( x \in |X| \).

Proof. Since \( x \langle (\Phi) \mathbf{a} \rangle = x \langle (\Phi) \rangle \mathbf{a} = (x \langle \Phi \rangle) \mathbf{a} \) for any \( x \in |X| \) and \( \mathbf{a} \in |A| \), the assertion is immediate on recalling Remark 1.1.4 and Remark 1.1.8. \( \square \)

Proposition 2.1.5.

- Given a module (or module morphism) \( M \) and a functor (or natural transformation) \( K \) as in

\[ X \xymatrix@C+10pt{\ar[r] & \mathcal{M} & \ar[l]^{K} \mathcal{E} } \]

, the right exponential transpose of the composite \( \langle M \rangle \mathcal{K} \) is given by the composition

\[ [X :] \xymatrix@C+10pt{\ar[r]^{M \cdot} & \mathcal{A} & \ar[l]^{K} \mathcal{E} } \]

; that is,

\[ \langle \langle M \rangle \mathcal{K} \rangle \mathcal{K} = [M \cdot] \cdot K \]

.

- Given a module (or module morphism) \( M \) and a functor (or natural transformation) \( K \) as in

\[ \mathcal{E} \xymatrix@C+10pt{\ar[r]^{K} & \mathcal{X} & \ar[l]^{M} \mathcal{A} } \]

, the left exponential transpose of the composite \( \mathcal{K} \langle M \rangle \) is given by the composition

\[ \mathcal{E} \xymatrix@C+10pt{\ar[r]^{K} & \mathcal{X} & \ar[l]^{M} [: \mathcal{A}] } \]

; that is,

\[ \langle \langle \mathcal{K} \langle M \rangle \rangle \rangle \mathcal{E} = K \cdot [\mathcal{K} \langle M \rangle ] \]

.

Proof. For any \( e \in \mathcal{E} \),

\[ \langle \langle (M) \mathcal{K} \rangle \mathcal{K} \rangle \mathcal{K} \cdot e = \langle (M) \mathcal{K} \rangle e = (M) (K \cdot e) = [M \cdot] \cdot (K \cdot e) = \langle [M \cdot] \cdot K \rangle \cdot e \]

.

\( \square \)

Proposition 2.1.6.

- Given a functor \( K \) and a module \( \mathcal{M} \) as in

\[ \mathcal{E} \xymatrix@C+10pt{\ar[r]^{K} & \mathcal{X} & \ar[l]^{M} \mathcal{A} } \]

, the right exponential transpose of the composite module \( \mathcal{K} \langle \mathcal{M} \rangle \) is given by the composition

\[ [E :] \xymatrix@C+10pt{\ar[r]^{[K]} & [X :] \ar[r]^{M \cdot} & \mathcal{A} } \]

; that is,

\[ \langle \langle \mathcal{K} \langle \mathcal{M} \rangle \rangle \mathcal{K} \rangle = [K \cdot] \cdot [M \cdot] \]

.

37
Given a functor \( K \) and a module \( M \) as in
\[
X \xrightarrow{\cdot M} \cdot A \xrightarrow{K} \cdot E
\]
, the left exponential transpose of the composite module \((M)K\) is given by the composition
\[
X \xrightarrow{\cdot M} [:A] \xrightarrow{[K]} [:E]
\]
; that is,
\[
[\cdot (M)K] = [\cdot M] \circ [\cdot K]
\]
.

Proof. For any \( a \in A \),
\[
[(K\langle M \rangle) \cdot a] = (K\langle M \rangle) a = K\langle (M) a \rangle = [K : \cdot (M) a] = [K : \cdot (M \cdot a)] \cdot a
\]
.

Definition 2.1.7. Given a cell
\[
\begin{array}{ccc}
X & \xrightarrow{\cdot M} & \cdot A \\
\downarrow P & & \downarrow \Phi \\
Y & \xrightarrow{\cdot N} & \cdot B
\end{array}
\]

- the right slice of \( \Phi \) at \( a \in \|A\| \) is the right conical cell
\[
\begin{array}{ccc}
X & \xrightarrow{(M)a} & \cdot * \\
\downarrow P & & \downarrow (\Phi)a \\
Y & \xrightarrow{\cdot N} & \cdot B
\end{array}
\]
defined by the right slice of the module morphism \( \Phi : M \to P\langle N \rangle Q : X \to A \) at \( a \), i.e. the right module
\[
(\Phi) a : (M) a \to (P\langle N \rangle Q) a : X \to *
\]
.

- the left slice of \( \Phi \) at \( x \in \|X\| \) is the left conical cell
\[
\begin{array}{ccc}
* & \xrightarrow{x(M)} & \cdot A \\
x : P & & \downarrow x(\Phi) \\
Y & \xrightarrow{\cdot N} & \cdot B
\end{array}
\]
defined by the left slice of the module morphism \( \Phi : M \to P\langle N \rangle Q : X \to A \) at \( x \), i.e. the left module
\[
x(\Phi) : x\langle M \rangle \to x\langle P\langle N \rangle Q \rangle : * \to A
\]
.

Proposition 2.1.8. A cell is fully faithful if and only if every its right (resp. left) slice is fully faithful.

Proof. Immediate from Proposition 2.1.4.
2.2. Action of modules

**Definition 2.2.1.** Let $\mathcal{E}$ be a category and $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ be a module.

- The right action of $\mathcal{M}$ on the functor category $[\mathcal{E}, \mathcal{A}]$ is the covariant functor

\[
[\mathcal{M} \triangleright \mathcal{E}] : [\mathcal{E}, \mathcal{A}] \to [\mathcal{X} : \mathcal{E}]
\]

defined by

\[
[\mathcal{M} \triangleright \mathcal{E}] \cdot K = (\mathcal{M}) K
\]

for $K \in [\mathcal{E}, \mathcal{A}]$.

- The left action of $\mathcal{M}$ on the functor category $[\mathcal{E}, \mathcal{X}]$ is the contravariant functor

\[
[\mathcal{E} \triangleleft \mathcal{M}] : [\mathcal{E}, \mathcal{X}] \to [\mathcal{E} : \mathcal{A}]^\wedge
\]

defined by

\[
K \cdot [\mathcal{E} \triangleleft \mathcal{M}] = K (\mathcal{M})
\]

for $K \in [\mathcal{E}, \mathcal{X}]$.

**Remark 2.2.2.**

1. A module $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ acts on a functor $K : \mathcal{E} \to \mathcal{A}$ by the composition

\[
\mathcal{X} \mathcal{M} \to \mathcal{A} \leftarrow K \mathcal{E}
\]

and yields the module

\[
\langle \mathcal{M} \rangle K : \mathcal{X} \to \mathcal{E}
\]

such that

\[
x (\langle \mathcal{M} \rangle K) e = x (\mathcal{M}) (K \cdot e)
\]

for $x \in \mathcal{X}$ and $e \in \mathcal{E}$.

2. A module $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ acts on a functor $K : \mathcal{E} \to \mathcal{X}$ by the composition

\[
\mathcal{E} \mathcal{K} \to \mathcal{X} \mathcal{M} \to \mathcal{A}
\]

and yields the module

\[
K \langle \mathcal{M} \rangle : \mathcal{E} \to \mathcal{A}
\]

such that

\[
e (K \langle \mathcal{M} \rangle) a = (e \cdot K) (\mathcal{M}) a
\]

for $e \in \mathcal{E}$ and $a \in \mathcal{A}$. 

2.
2. Slicing and Action

- A module $\mathcal{M} : X \to A$ acts on a natural transformation $\tau : S \to T : E \to A$ by the composition

$$X \xrightarrow{\mathcal{M}} A \xleftarrow{\tau} E$$

and yields the module morphism

$$\langle \mathcal{M} \rangle \tau : \langle \mathcal{M} \rangle S \to \langle \mathcal{M} \rangle T : X \to E,$$

which maps each $\langle \mathcal{M} \rangle S$-arrow $m : x \rightsquigarrow e$ to the $\langle \mathcal{M} \rangle T$-arrow $m \circ \tau_e : x \rightsquigarrow e$ as indicated in

$$x \xlongrightarrow{m} S \cdot e \xleftarrow{\tau_e} T \cdot e.$$

- A module $\mathcal{M} : X \to A$ acts on a natural transformation $\tau : S \to T : E \to X$ by the composition

$$E \xrightarrow{\mathcal{M}} X \xleftarrow{\tau} A$$

and yields the module morphism

$$\tau \langle \mathcal{M} \rangle : T \langle \mathcal{M} \rangle \to S \langle \mathcal{M} \rangle : E \to A,$$

which maps each $T \langle \mathcal{M} \rangle$-arrow $m : e \rightsquigarrow a$ to the $S \langle \mathcal{M} \rangle$-arrow $\tau_m \circ \tau_e : e \rightsquigarrow a$ as indicated in

$$e \xlongrightarrow{m} T \cdot e \xleftarrow{\tau_e \circ \tau_m} a.$$

3. By Remark 2.1.2(4),

- the right exponential transpose

$$[\mathcal{M}^\rightharpoonup] : A \to [X:]$$

of a module $\mathcal{M} : X \to A$ is identified with the right action

$$[\mathcal{M}^\rightharpoonup] : [* , A] \to [X : *]$$

of $\mathcal{M}$ on the functor category $[* , A]$.

- the left exponential transpose

$$[* , \mathcal{M}] : X \to [ : A]^\rightharpoonup$$

of a module $\mathcal{M} : X \to A$ is identified with the left action

$$[* , \mathcal{M}] : [* , X] \to [* : A]^\rightharpoonup$$

of $\mathcal{M}$ on the functor category $[* , X]$. 

40
4. For any category $E$ and any module $M: X \to A$,
\[ [M \triangleright E] \cong [E \triangleright M] \]
that is, the opposite of the right action of $M: X \to A$ on $[E, A]$ is (identified with) the left action of the opposite module $M^*: A^* \to X^*$ on $[E^*, A^*]$.

**Theorem 2.2.3.** Given a category $E$ and a module $M: X \to A$, the diagrams

\[
\begin{array}{ccc}
[X : E] & \xrightarrow{\triangleright M} & [E, X] \\
\downarrow \scriptstyle{\Delta_E \triangleright A} & & \downarrow \scriptstyle{\Delta_E \triangleright A} \\
[E, A] & \xrightarrow{M \triangleright E} & [E, X] \\
\end{array}
\]

commute.

*Proof.* Immediate from Proposition 2.1.5. $\square$

**Corollary 2.2.4.** Given a category $E$ and a module $M: X \to A$, the quadrangles

\[
\begin{array}{ccc}
\Delta_E \triangleright A & [E, A] & \xrightarrow{\Delta_E \triangleright A} & [E, A] \\
\downarrow M \triangleright & \downarrow M \triangleright & \downarrow M \triangleright & \\
X & [E, X] & \xrightarrow{\Delta_E \triangleright A} & [E, X] \\
\end{array}
\]

commute.

*Proof.* The diagrams commute by Theorem 2.2.3, and yield the desired commutative quadrangles by Example 2.1.3. $\square$

### 2.3. Yoneda functors and representations

**Definition 2.3.1.**

- The right Yoneda functor for a category $X$ is the functor

\[ [X \triangleright] : X \to [X :] \]

given by the right exponential transpose of the hom $(X) : X \to X$; in short:

\[ [X \triangleright] := [(X) \triangleright] \]
The left Yoneda functor for a category $A$ is the functor
$$\left[ \land A \right] : A \to \left[ : A \right]$$
given by the left exponential transpose of the hom $\langle A \rangle : A \to A$; in short:
$$\left[ \land A \right] := \left[ \land \langle A \rangle \right]$$

**Remark 2.3.2.**
- The right Yoneda functor $\uparrow X$ sends each object $x \in \|X\|$ to the right module
$$\langle X \rangle x : X \to \ast$$
, called the representable right module of $x$, and sends each $X$-arrow $f : s \to t$ to the right module morphism
$$\langle X \rangle f : \langle X \rangle s \to \langle X \rangle t : X \to \ast$$
which maps each $X$-arrow $h : x \to s$ to the $X$-arrow $h \circ f : x \to t$ as indicated in

```
\begin{array}{ccc}
x & \xrightarrow{h} & s \\
\downarrow{h : \langle X \rangle f} & & \downarrow{f} \\
t & \xleftarrow{t} & t
\end{array}
```

(cf. Remark 2.1.2(2)).

- The left Yoneda functor $\land A$ sends each object $a \in \|A\|$ to the left module
$$a \langle A \rangle : \ast \to A$$
, called the representable left module of $a$, and sends each $A$-arrow $f : s \to t$ to the left module morphism
$$f \langle A \rangle : t \langle A \rangle \to s \langle A \rangle : \ast \to A$$
which maps each $A$-arrow $h : t \to a$ to the $A$-arrow $f \circ h : s \to a$ as indicated in

```
\begin{array}{ccc}
S & \xrightarrow{f} & f \langle A \rangle : h \\
\downarrow{f} & & \downarrow{h} \\
t & \xleftarrow{a} & a
\end{array}
```

(cf. Remark 2.1.2(2)).

**Definition 2.3.3.**
- A representation of a right module $M : X \to \ast$ is a pair $(r, \Upsilon)$ consisting of an object $r \in \|X\|$ and a right module isomorphism $\Upsilon : M \cong \langle X \rangle r$.
- A representation of a left module $M : \ast \to A$ is a pair $(r, \Upsilon)$ consisting of an object $r \in \|A\|$ and a left module isomorphism $\Upsilon : M \cong r \langle A \rangle$.

**Remark 2.3.4.**
A right module $M : X \to \ast$ is called representable if it has a representation; that is, if it is isomorphic to "the" representable right module $(X)_r$ for some object $r \in |X|$, which is said to represent $M$.

A left module $M : \ast \to A$ is called representable if it has a representation; that is, if it is isomorphic to "the" representable left module $r (A)$ for some object $r \in |A|$, which is said to represent $M$.

**Definition 2.3.5.** Let $X$ and $A$ be categories.

- The right general Yoneda functor for the functor category $[A, X]$ is the functor
  \[[X \cdot A] : [A, X] \to [X : A]\]
  given by the right action of the hom $(X) : X \to X$ on the functor category $[A, X]$; in short:
  \[[X \cdot A] := [(X) \cdot A]\]

- The left general Yoneda functor for the functor category $[X, A]$ is the functor
  \[[X \cdot A] : [X, A] \to [X : A]^\sim\]
  given by the left action of the hom $(A) : A \to A$ on the functor category $[X, A]$; in short:
  \[[X \cdot A] := [X \cdot (A)]\]

**Remark 2.3.6.**

1. The right general Yoneda functor $X \cdot A$ sends each functor $G : A \to X$ to the module
   
   $(X) G : X \to A$

   , called the corepresentable module of $G$, and sends each natural transformation $\tau : S \to T : A \to X$ to the module morphism
   
   $(X) \tau : (X) S \to (X) T : X \to A$

   which maps each $(X) S$-arrow $h : x \to a$ to the $(X) T$-arrow $h \cdot \tau_a : x \to a$ as indicated in

   \[
   \begin{array}{ccc}
   x & \xrightarrow{h} & S \cdot a \\
   h \cdot (X)\tau & \downarrow & \tau_a \\
   T \cdot a & \end{array}
   \]

   (cf. Remark 2.2.2(2)).

2. The left general Yoneda functor $X \cdot A$ sends each functor $F : X \to A$ to the module

   $F (A) : X \to A$

   , called the representable module of $F$, and sends each natural transformation $\tau : S \to T : X \to A$ to the module morphism

   $\tau (A) : T (A) \to S (A) : X \to A$
2. Slicing and Action

which maps each $T(A)$-arrow $h : x \to a$ to the $S(A)$-arrow $\tau_x \circ h : x \to a$ as indicated in

\[
\begin{array}{c}
\xymatrix{ x : S \\
\tau_x \\
\ar[u]_{\tau(A) \circ h} \\
\ar[d]_h \\
\ar[r]_a & }
\end{array}
\]

(cf. Remark 2.2.2(2)).

2. The right Yoneda functor

\[ [X \rightharpoonup] : X \to [X:] \]

for a category $X$ is identified with the right general Yoneda functor

\[ [X \rightharpoonup \ast] : [\ast, X] \to [X:] \]

for the functor category $[\ast, X]$.

• The left Yoneda functor

\[ [\ast \leftarrow A] : A \to [\ast : A]^\circ \]

for a category $A$ is identified with the left general Yoneda functor

\[ [\ast \leftarrow \ast \leftarrow A] : [\ast, A] \to [\ast : A]^\circ \]

for the functor category $[\ast, A]$.

**Proposition 2.3.7.** Given categories $E$, $X$, and $A$, the quadrangles

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^{[\Delta_{E,X}]} & [E,X] \\
\ar[r]_{X \rightharpoonup} & \ar[d]_{X \rightharpoonup E} \\
[X:] \ar[r]_{X \rightharpoonup \Delta_E} & [X:E] & [E,A] \ar[l]_{[\Delta_{E,A}] \rightharpoonup} \ar[d]_{E \leftarrow A} \ar[r]_{[\Delta_{E,A}] \rightharpoonup} & [\ast : A]^\circ \ar[l]_{[\ast \leftarrow A]} \\
\ar[r] & \ar[r] & & }
\end{array}
\]

commute.

**Proof.** This is a special case of Corollary 2.2.4 where $M$ is given by the hom of $X$ (resp. $A$).

**Definition 2.3.8.** Let $M : X \to A$ be a module.

• A corepresentation of $M$ is a pair $(R, \Upsilon)$ consisting of a functor $R : A \to X$ and a module isomorphism $\Upsilon : M \cong (X)R$.

• A representation of $M$ is a pair $(R, \Upsilon)$ consisting of a functor $R : X \to A$ and a module isomorphism $\Upsilon : M \cong R(A)$.

**Remark 2.3.9.**

1. A module $M : X \to A$ is called

• corepresentable if it has a corepresentation; that is, if it is isomorphic to "the" corepresentable module $(X)R$ for some functor $R : A \to X$, which is said to corepresent $M$.  

44
2. Slicing and Action

- representable if it has a representation; that is, if it is isomorphic to "the" representable module \( R(A) \) for some functor \( R: X \to A \), which is said to represent \( M \).

2. 
- A representation of a right module \( M: X \to * \) is identified with a corepresentation of a two-sided module \( M: X \to A \) where \( A \) is the terminal category.
- A representation of a left module \( M: * \to A \) is identified with a representation of a two-sided module \( M: X \to A \) where \( X \) is the terminal category.

3. 
- A corepresentation of a module \( M: X \to A \) is expressed by a fully faithful cell

\[
\begin{array}{c}
\biguplus \\
\downarrow \Upsilon \\
\biguplus
\end{array}
\]

\[
X \rightarrow^M X \rightarrow^M A
\]

- A representation of a module \( M: X \to A \) is expressed by a fully faithful cell

\[
\begin{array}{c}
\biguplus \\
\downarrow \Upsilon \\
\biguplus
\end{array}
\]

\[
A \rightarrow^M A
\]

4. Example 2.3.10 shows that not all modules are representable.

**Example 2.3.10.** Let \( X \) and \( A \) be discrete categories. A correspondence \( R \) from \( |X| \) to \( |A| \), i.e. a subset of \( |X| \times |A| \), defines a module \( R: X \to A \) by

\[
x(R)a = \begin{cases} \{\ast\} & \text{if } (x,a) \in R \\ \emptyset & \text{otherwise} \end{cases}
\]

\( R \) is representable (resp. corepresentable) if and only if \( R \) is a function \( |X| \to |A| \) (resp. \( |A| \to |X| \)).

**Proposition 2.3.11.** Let \( M: X \to A \) be a module.

- A functor \( R: A \to X \) and a module morphism \( \Upsilon : M \to (X)R : X \to A \) form a corepresentation of \( M \) if and only if for every \( a \in |A| \) the object \( R:a \) and the right module morphism \( (\Upsilon)a : (M)a \to (X)(R:a) : X \to * \) form a representation of the right module \( (M):a \) : \( X \to A \).
- A functor \( X: X \to A \) and a module morphism \( \Upsilon : M \to R(A): X \to A \) form a representation of \( M \) if and only if for every \( x \in |X| \) the object \( x : R \) and the left module morphism \( x(\Upsilon)x(M) \downarrow (x:R)(A): * \to A \) form a representation of the left module \( x(M): * \to A \).

**Proof.** By Proposition 2.1.4, \( \Upsilon : M \to (X)R \) is iso iff

\[
(\Upsilon)a : (M)a \to (X)(R:a) = (X)(R:a)
\]

is iso for every \( a \in |A| \).
3. Collages and Commas

3.1. Collages

Definition 3.1.1.
1. A collage $\mathcal{M}$ from a category $\mathcal{X}$ to a category $\mathcal{A}$, written $\mathcal{M} : \mathcal{X} \to \mathcal{A}$, is defined by a category $\mathcal{M}$, “collage category”, satisfying the following conditions:
   a) the coproduct category $\mathcal{X} + \mathcal{A}$ is a full sub category of $[\mathcal{M}]$ and $\|\mathcal{M}\| = \|\mathcal{X} + \mathcal{A}\|$;
   b) $a([\mathcal{M}]) x = \emptyset$ if $x \in \mathcal{X}$ and $a \in \mathcal{A}$.

Remark 3.1.2.
1. The inclusion of the coproduct category $\mathcal{X} + \mathcal{A}$ into the collage category $[\mathcal{M}]$ is denoted by
   $$\mathcal{M} : \mathcal{X} + \mathcal{A} \to [\mathcal{M}]$$
   or by
   $$\mathcal{X} \xrightarrow{\mathcal{M}_x} [\mathcal{M}] \xleftarrow{\mathcal{M}_a} \mathcal{A}$$

2. A collage $\mathcal{M} : \mathcal{X} \to \ast$ from a category $\mathcal{X}$ to the terminal category is called a right collage over $\mathcal{X}$; the inclusion of $\mathcal{X}$ into the collage category $[\mathcal{M}]$ is denoted by
   $$\mathcal{M} : \mathcal{X} \to [\mathcal{M}]$$

3. A collage $\mathcal{M} : \ast \to \mathcal{A}$ from the terminal category to a category $\mathcal{A}$ is called a left collage over $\mathcal{A}$; the inclusion of $\mathcal{A}$ into the collage category $[\mathcal{M}]$ is denoted by
   $$\mathcal{M} : \mathcal{A} \to [\mathcal{M}]$$

3. A collage $\mathcal{M}$ is called locally small if the collage category $[\mathcal{M}]$ is locally small.
4. Any collage $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ defines the unique functor $[\Delta_{\mathcal{M}}] : [\mathcal{M}] \to \mathbb{2}$ making the diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\mathcal{M}_x} & [\mathcal{M}] & \xrightarrow{\mathcal{M}_a} & \mathcal{A} \\
\Delta_{\mathcal{X}} & & [\Delta_{\mathcal{M}}] & & \Delta_{\mathcal{A}} \\
\ast & \underset{0}{\xleftarrow{}} & \mathbb{2} & \underset{1}{\xrightarrow{}} & \ast \\
\end{array}
$$

commute. Conversely, any functor $H : \mathcal{C} \to \mathbb{2}$ defines a unique collage $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ with $[\mathcal{M}] = \mathcal{C}$ and $[\Delta_{\mathcal{M}}] = H$. 

46
Definition 3.1.3. Given a pair of collages \( \mathcal{M}, \mathcal{N} : X \to A \), a morphism \( \Phi \) from \( \mathcal{M} \) to \( \mathcal{N} \), written \( \Phi : \mathcal{M} \to \mathcal{N} : X \to A \), is defined by a functor \( [\Phi] : [\mathcal{M}] \to [\mathcal{N}] \), “collage functor”, satisfying the following equivalent conditions:

1. \( [\Phi] \) is identity on \( X + A \);
2. the triangle

\[
\begin{array}{ccc}
X + A & \xrightarrow{\Phi} & [\mathcal{N}] \\
\downarrow_{\mathcal{M}} & & \downarrow_{[\Phi]} \\
[\mathcal{M}] & \xrightarrow{[\Phi]} & [\mathcal{N}]
\end{array}
\]

commutes;
3. the two triangles

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi} & A \\
\downarrow_{\mathcal{M}_X} & & \downarrow_{\mathcal{M}_A} \\
[\mathcal{M}] & \xrightarrow{[\Phi]} & [\mathcal{N}]
\end{array}
\]

commute;
4. the triangle

\[
\begin{array}{ccc}
[\mathcal{M}] & \xrightarrow{[\Phi]} & [\mathcal{N}] \\
\downarrow_{[\Delta_M]} & & \downarrow_{[\Delta_N]} \\
2 & \xrightarrow{2} & 2
\end{array}
\]

commutes.

Proposition 3.1.4. Given a pair of categories \( X \) and \( A \), all locally small collages \( X \to A \) and morphisms among them define the category \( [X \uparrow A] \) with the obvious identities and the composition. Indeed, \( [X \uparrow A] \) is fully embedded into the coslice category under \( X + A \).

Proof. Self explanatory.

Definition 3.1.5. Let \( \mathcal{M} : X \to A \) and \( \mathcal{N} : Y \to B \) be collages. Given a pair of functors \( P : X \to Y \) and \( Q : A \to B \), a collage cell \( \Phi : P \to Q : \mathcal{M} \to \mathcal{N} \) is defined by a functor \( [\Phi] : [\mathcal{M}] \to [\mathcal{N}] \), “collage functor”, satisfying the following equivalent conditions:

1. \( [\Phi] \) is identity on \( X + A \);
2. the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{M_X} & [\mathcal{M}] & \xrightarrow{M_A} & A \\
P & \downarrow_{[\Phi]} & \downarrow_{[\Phi]} & \downarrow_{Q} \\
Y & \xleftarrow{N_Y} & [\mathcal{N}] & \xleftarrow{N_B} & B
\end{array}
\]

commutes;
3. Collages and Commas

3. the triangle

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow \Delta_M \\
\downarrow 2 \\
\mathcal{N}
\end{array}
\xrightarrow{\Phi}
\begin{array}{c}
\mathcal{N} \\
\downarrow \Delta_N
\end{array}
\]

commutes.

**Proposition 3.1.6.** All locally small collages and cells among them define the category \(\text{CLG}\) with the obvious identities and the composition. Indeed, there is an obvious isomorphism between \(\text{CLG}\) and the slice category \(\text{CAT}/2\).

**Proof.** Self explanatory. \(\square\)

**Remark 3.1.7.**

1. There is an obvious forgetful functor \([-]: \text{CLG} \to \text{CAT}\), sending each collage \(\mathcal{M}\) to the collage category \([\mathcal{M}]\).

2. Given a pair of categories \(X\) and \(A\), there is a canonical embedding \([X \uparrow A] \to \text{CLG}\), identical on objects, defined by the arrow function \(\Phi \mapsto (\Phi: 1_X \sim 1_A)\). The embedding is not, in general, full.

**Definition 3.1.8.** Given a parallel pair of collage cells \(\Phi: P \sim Q: \mathcal{M} \to \mathcal{N}\) and \(\Psi: S \sim T: \mathcal{M} \to \mathcal{N}\), a morphism from \(\Phi\) to \(\Psi\), written \(\tau: \Phi \to \Psi: \mathcal{M} \to \mathcal{N}\), is defined by a natural transformation \(\tau: [\Phi] \to [\Psi]: [\mathcal{M}] \to [\mathcal{N}]\), “collage natural transformation”.

**Proposition 3.1.9.** Given a pair of collages \(\mathcal{M}\) and \(\mathcal{N}\), all collage cells \(\mathcal{M} \to \mathcal{N}\) and morphisms among them define the category \([\mathcal{M} \uparrow \mathcal{N}]\) with the obvious identities and the composition. Indeed, \([\mathcal{M} \uparrow \mathcal{N}]\) is fully embedded into the functor category \([\mathcal{M}], [\mathcal{N}]\).

**Proof.** Self explanatory. \(\square\)

**Note.** In what follows we show a one-to-one correspondence between modules and collages. In Definition 3.1.10 and Definition 3.1.13, a module and a collage corresponding to each other are given the same name.

**Definition 3.1.10.**

1. Given a module \(M: X \to A\), the corresponding collage is defined by the collage category \([M]\) given in the following way:
   a) the objects of \([M]\) consist of all objects of the coproduct category \(X + A\);
   b) the arrows of \([M]\) consist of all \(X\)-arrows, \(A\)-arrows and \(M\)-arrows:
      - \(x([M])y = x(X)y\) for \(x, y \in |X|\);
      - \(a([M])b = a(A)b\) for \(a, b \in |A|\);
      - \(x([M])a = x(M)a\) for \(x \in |X|\) and \(a \in |A|\);
      - \(a([M])x = \emptyset\) for \(a \in |A|\) and \(x \in |X|\).
   c) the composition law of \([M]\) is those of \(X, A,\) and \(M\) (see Definition 1.1.9).

2. Given a module morphism \(\Phi: M \to N: X \to A\), the corresponding collage morphism is defined by the collage functor \([\Phi]: [M] \to [N]\) given in the following way:
3. Collages and Commas

a) $[\Phi]$ is the identity on $X + A$;

b) $x([\Phi])a = x(\Phi)a$ for $x \in \|X\|$ and $a \in \|A\|$.

3. Given a module cell

$$
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow \Phi \\
Y \xrightarrow{N} B
\end{array}
$$

, the corresponding collage cell

$$
\begin{array}{c}
X \xrightarrow{M_X} [M] \xrightarrow{M_A} A \\
\downarrow [\Phi] \\
Y \xrightarrow{N_Y} [N] \xrightarrow{N_B} B
\end{array}
$$

is defined by the collage functor $[\Phi]: [M] \rightarrow [N]$ given in the following way:

a) $[\Phi]$ is identical to $P + Q$ on $X + Y$;

b) $x([\Phi])a = x(\Phi)a$ for $x \in \|X\|$ and $a \in \|A\|$.

4. Given a module cell morphism $\tau: \Phi \rightarrow \Psi: M \rightarrow N$ between two module cells

$$
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow \Phi \\
Y \xrightarrow{N} B
\end{array} \quad \xrightarrow{\tau} \quad 
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow \Psi \\
Y \xrightarrow{N} B
\end{array}
$$

, the corresponding collage cell morphism is defined by the collage natural transformation $[\tau]: [\Phi] \rightarrow [\Psi]: [M] \rightarrow [N]$ given by:

- $[\tau]_x = \tau^1_x$ for $x \in \|X\|$;
- $[\tau]_a = \tau^2_a$ for $a \in \|A\|$.

Remark 3.1.11. For a right module $M : X \rightarrow \ast$, the corresponding right collage (see Remark 3.1.2(2)) is constructed as above by identifying $M$ with the two-sided module from $X$ to the terminal category. Dually for a left module $M : \ast \rightarrow A$.

Proposition 3.1.12. The module-to-collage correspondence given in Definition 3.1.10 is functorial and defines the following functors:

1. $[X: A] \Downarrow [X \uparrow A]$ for categories $X$ and $A$;

2. $MOD \Downarrow CLG$;


Proof. The functoriality is easily verified. \qed

Definition 3.1.13.
3. Collages and Commas

1. Given a collage $\mathcal{M} : X \to A$, the corresponding module is defined by the composition

$$
X \xrightarrow{M_X} [\mathcal{M}] \xrightarrow{([M])} [\mathcal{M}] \xrightarrow{M_A} A
$$

, where $([M])$ is the hom of the collage category of $\mathcal{M}$.

2. Given a collage morphism $\Phi : \mathcal{M} \to \mathcal{N} : X \to A$, the corresponding module morphism is defined by the pasting composition

$$
X \xrightarrow{M_X} [\mathcal{M}] \xrightarrow{([\mathcal{M}])} [\mathcal{M}] \xrightarrow{M_A} A
$$

(see Remark 1.2.30), where $([\Phi])$ is the hom of the collage functor of $\Phi$.

3. Given a collage cell

$$
\begin{array}{c}
X \xrightarrow{M_X} [\mathcal{M}] \xrightarrow{M_A} A \\
P \downarrow \phi \downarrow Q \\
Y \xrightarrow{N_Y} [\mathcal{N}] \xrightarrow{N_B} B
\end{array}
$$

, the corresponding module cell

$$
\begin{array}{c}
X \xrightarrow{M_X} [\mathcal{M}] \xrightarrow{M_A} A \\
P \downarrow \phi \downarrow Q \\
Y \xrightarrow{N_Y} [\mathcal{N}] \xrightarrow{N_B} B
\end{array}
$$

is defined by the pasting composition

$$
\begin{array}{c}
X \xrightarrow{M_X} [\mathcal{M}] \xrightarrow{([\mathcal{M}])} [\mathcal{M}] \xrightarrow{M_A} A \\
P \downarrow \phi \downarrow Q \\
Y \xrightarrow{N_Y} [\mathcal{N}] \xrightarrow{([\mathcal{N}])} [\mathcal{N}] \xrightarrow{N_B} B
\end{array}
$$

, where $([\Phi])$ is the hom of the collage functor of $\Phi$.

4. Given a collage cell morphism $\tau : \Phi \to \Psi : \mathcal{M} \to \mathcal{N}$ between two collage cells

$$
\begin{array}{c}
X \xrightarrow{M_X} [\mathcal{M}] \xrightarrow{M_A} A \\
P \downarrow \phi \downarrow Q \\
Y \xrightarrow{N_Y} [\mathcal{N}] \xrightarrow{N_B} B
\end{array}
\quad \text{and} \quad
\begin{array}{c}
X \xrightarrow{M_X} [\mathcal{M}] \xrightarrow{M_A} A \\
P \downarrow \phi \downarrow Q \\
Y \xrightarrow{N_Y} [\mathcal{N}] \xrightarrow{N_B} B
\end{array}
$$

, the corresponding module cell morphism is defined by the pair $(M_X \circ [\tau], M_A \circ [\tau])$ of natural transformations, where $[\tau] : ([\Phi]) \to ([\Psi]) : [\mathcal{M}] \to [\mathcal{N}]$ is the collage natural transformation of $\tau$.

Proposition 3.1.14. The collage-to-module correspondence given in Definition 3.1.13 is functorial and defines the following functors:
1. $[X \uparrow A] \xrightarrow{\uparrow} [X : A]$ for categories $X$ and $A$;

2. $\text{CLG} \xrightarrow{\uparrow} \text{MOD};$


Proof. The functoriality is easily verified. \hfill \square

**Theorem 3.1.15.** The corresponding functors

$$
\begin{align*}
[X : A] &\xrightarrow{\uparrow} [X \uparrow A] \\
\text{MOD} &\xrightarrow{\uparrow} \text{CLG} \\
[M : N] &\xrightarrow{\downarrow} [M \uparrow N]
\end{align*}
$$

in Proposition 3.1.12 and Proposition 3.1.14 are isomorphisms inverse to each other.

Proof. Easily verified. \hfill \square

**Remark 3.1.16.**

1. As noted earlier, a module and a collage corresponding to each other are given the same name and freely identified with each other.

2. A module $M : X \to A$ is recovered from its collage by the composition

$$
X \xrightarrow{M_X} [M] \xrightarrow{[M]} [M] \xrightarrow{M_A} A
$$

; that is,

$$
M = M_X([M]) M_A
$$

. This identity yields the cell

$$
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow M_X & & \downarrow M_A \\
[M] & \xrightarrow{[M]} & [M]
\end{array}
$$

called the unit cell of $M$.

**Theorem 3.1.17.** There is a canonical adjunction between the collage operation $M \mapsto [M] : \text{MOD} \to \text{CAT}$ and the hom operation $E \mapsto (E) : \text{CAT} \to \text{MOD}$ with the unit given by the family of unit cells $1_M : M \to ([M])$. Specifically, for each module $M : X \to A$ and each category $E$,

1. the adjunct of a cell

$$
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow S & & \downarrow T \\
E & \xrightarrow{(E)} & E
\end{array}
$$

is given by the functor $F : [M] \to E$ defined in the following way:

a) $F$ is identical to $S + T$ on $X + A$;

b) $x(F) a = x(\Phi) a$ for $x \in [X]$ and $a \in [A]$. 

51
2. the adjunct of a functor $F : [M] \to E$ is given by the cell $\Phi : M \to \langle E \rangle$ defined by the composition

$$
\begin{array}{c}
X \xrightarrow{\cdot} M \xrightarrow{\cdot} A \\
\downarrow M_X \downarrow \downarrow \downarrow M_A \\
[M] \xrightarrow{\cdot} [M] \\
\downarrow F \downarrow \downarrow \downarrow F \\
E \xrightarrow{\cdot} E
\end{array}
$$

*Proof.* It is easily verified that the correspondences are inverse to each other. It remains to prove that the family of the unit cells satisfies the naturality condition. For this we need to see that, given a cell

$$
\begin{array}{c}
X \xrightarrow{\cdot} M \xrightarrow{\cdot} A \\
\downarrow P \downarrow \downarrow \downarrow Q \\
Y \xrightarrow{\cdot} N \xrightarrow{\cdot} B
\end{array}
$$

, the quadrangle

$$
\begin{array}{c}
M \xrightarrow{\cdot} [M] \\
\downarrow \Phi \downarrow \downarrow \downarrow [\Phi] \\
N \xrightarrow{\cdot} [N]
\end{array}
$$

commutes, i.e. the compositions

$$
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{\cdot} M \xrightarrow{\cdot} A \\
\downarrow P \downarrow \downarrow \downarrow Q \\
Y \xrightarrow{\cdot} N \xrightarrow{\cdot} B \\
\downarrow N_X \downarrow \downarrow \downarrow N_A \\
[N] \xrightarrow{\cdot} [N]
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{\cdot} M \xrightarrow{\cdot} A \\
\downarrow P \downarrow \downarrow \downarrow Q \\
Y \xrightarrow{\cdot} N \xrightarrow{\cdot} B \\
\downarrow N_X \downarrow \downarrow \downarrow N_A \\
[N] \xrightarrow{\cdot} [N]
\end{array}
\end{array}
$$

yield the same cell. First note that $P \circ N_X = M_X \circ \Phi$ and $N_A \circ Q = [\Phi] \circ M_A$ by the construction of the collage functor $[\Phi]$. Since $I_N$ and $I_M$ are defined by the identity module morphisms $N \to N$ and $M \to M$, the cells $\Phi \circ I_N$ and $I_M \circ (\Phi)$ are given by the module morphisms $\Phi$ and $M_X (\Phi), M_A$. But $\Phi = M_X (\Phi), M_A$ by Theorem 3.1.15. \qed

*Remark 3.1.18.* The collage operation $M \mapsto [M] : \text{MOD} \to \text{CAT}$ is thus a left adjoint of the hom operation $E \mapsto \langle E \rangle : \text{CAT} \to \text{MOD}$.

### 3.2. Commas

*Definition 3.2.1.*

- A right comma $K$ over a category $X$, written $K : X \to \ast$, consists of a category denoted by $[K]$ and a right comma fibration$^1$ $K : [K] \to X$, i.e. a functor satisfying the following

$^1$A comma fibration is called a discrete fibration in the literature.
condition: for every $X$-arrow $f : x \to K \cdot t$ there is a unique $\llbracket K \rrbracket$-arrow $f^t : s \to t$, the lift of $f$ at $t$, such that $K \cdot f^t = f$.

- A left comma $K$ over a category $A$, written $K : \ast \to A$, consists of a category denoted by $\llbracket K \rrbracket$ and a left comma fibration $K : \llbracket K \rrbracket \to A$, i.e. a functor satisfying the following condition: for every $A$-arrow $f : K \cdot s \to a$ there is a unique $\llbracket K \rrbracket$-arrow $f^s : s \to t$, the lift of $f$ at $s$, such that $K \cdot f^s = f$.

**Remark 3.2.2.**

1. A left comma over a category $A$ is the same thing as a right comma over the opposite category $A^\sim$.

2. The fibre of a right comma $K : X \to \ast$ at $x \in \|X\|$, written $x(\llbracket K \rrbracket)$, is the subcategory of $\llbracket K \rrbracket$ consisting of all objects $t$ with $K \cdot t = x$ and all arrows $h$ with $K \cdot h = 1_x$.

3. The fibre of a left comma $K : \ast \to A$ at $a \in \|A\|$, written $\langle K \rangle a$, is the subcategory of $\llbracket K \rrbracket$ consisting of all objects $s$ with $K \cdot s = a$ and all arrows $h$ with $K \cdot h = 1_a$.

**Definition 3.2.3.**

- A right comma $K : X \to \ast$ is called locally small if $X$ is locally small and all its fibres are small.

- A left comma $K : \ast \to A$ is called locally small if $A$ is locally small and all its fibres are small.

**Proposition 3.2.4.**
3. Collages and Commas

- Given a category \( X \), all locally small right commas over \( X \) and morphisms among them define the category \([X \downarrow] \) with the obvious identities and the composition. Indeed, \([X \downarrow] \) is fully embedded into the slice category over \( X \).

- Given a category \( A \), all locally small left commas over \( A \) and morphisms among them define the category \([\downarrow A] \) with the obvious identities and the composition. Indeed, \([\downarrow A] \) is fully embedded into the slice category over \( A \).

**Proof.** Self explanatory. \( \square \)

**Proposition 3.2.5.**

- The functor \([\Phi] \) of a right comma morphism \( \Phi : K \to L : X \to * \) sends each \([K]-\)arrow \( h : s \to t \) to the lift of \( K \cdot h \) at \([\Phi] \cdot t \) as indicated in

\[
\begin{array}{ccc}
[\Phi] \cdot s & \xrightarrow{[\Phi] \cdot h = (K \cdot h)(\Phi \cdot t)} & [\Phi] \cdot t \\
K \cdot s & \xrightarrow{K \cdot h} & K \cdot t
\end{array}
\]

- The functor \([\Phi] \) of a left comma morphism \( \Phi : K \to L : * \to A \) sends each \([K]-\)arrow \( h : s \to t \) to the lift of \( K \cdot h \) at \([\Phi] \cdot t \) as indicated in

\[
\begin{array}{ccc}
[\Phi] \cdot s & \xrightarrow{[\Phi] \cdot h = (K \cdot h)(\Phi \cdot s)} & [\Phi] \cdot t \\
K \cdot s & \xrightarrow{K \cdot h} & K \cdot t
\end{array}
\]

**Proof.** By the definition of a comma morphism,

\[ K \cdot h = L \cdot [\Phi] \cdot h \]

; that is, \([\Phi] \cdot h \) is a lift of \( K \cdot h \) along \( L \). The assertion thus follows from the uniqueness of the lift. \( \square \)

**Remark 3.2.6.** A right (resp. left) comma morphism \( \Phi \) is thus determined by the object function of the functor \([\Phi] \).

**Definition 3.2.7.** A \([\text{two-sided}] \) comma \( K \) from a category \( X \) to a category \( A \), written \( K : X \to A \), is given by a category denoted by \([K] \) and a comma fibration \( K : [K] \to X \times A \), i.e. a span

\[
\begin{array}{ccc}
& \text{\( X \)} & \\
\text{\( K \)} & \xleftarrow{K_X} & \text{\( [K] \)} & \xrightarrow{K_A} & \text{\( A \)}
\end{array}
\]

, satisfying the following conditions:

1. for every \( X \)-arrow \( f : x \to K_X \cdot t \) there is a unique \([K]-\)arrow \( f^t : s \to t \), the lift of \( f \) at \( t \), such that \( K_X^t \cdot f^t = f \) and \( K_A^t \cdot f^t = 1_{(K_A^t \cdot t)} \);

2. for every \( A \)-arrow \( f : K_A \cdot s \to a \) there is a unique \([K]-\)arrow \( f^s : s \to t \), the lift of \( f \) at \( s \), such that \( K_A^s \cdot f^s = f \) and \( K_X^s \cdot f^s = 1_{(K_X^s \cdot s)} \);
3. Collages and Commas

3. for every \([K]\)-arrow \(h: s \to t\), the domain of the lift \((K_X \cdot h)^t\) equals the codomain of the lift \((K_A \cdot h)^s\) and the triangle

\[
\begin{array}{ccc}
K_X \cdot t & \xrightarrow{(K_X \cdot h)^t} & K_X \cdot s \\
\downarrow & & \downarrow \\
\uparrow & & \uparrow \\
K_A \cdot t & \xleftarrow{(K_A \cdot h)^s} & K_A \cdot s
\end{array}
\]

commutes.

**Remark 3.2.8.**

1. The fibres of a comma \(K: X \to A\) are defined as follows:
   a) the fibre of \(K\) at \(a \in \|A\|\), written \(\langle K \rangle a\), is the subcategory of \([K]\) consisting of all objects \(t\) with \(K_A \cdot t = a\) and all arrows \(h\) with \(K_A \cdot h = 1_a\).
   b) the fibre of \(K\) at \(x \in \|X\|\), written \(x \langle K \rangle\), is the subcategory of \([K]\) consisting of all objects \(s\) with \(K_X \cdot s = x\) and all arrows \(h\) with \(K_X \cdot h = 1_x\).
   c) the fibre of \(K\) at \((x, a) \in \|X \times A\|\), written \(x \langle K \rangle a\), is the subcategory of \([K]\) consisting of all objects \(k\) with \(K \cdot k = (x, a)\) and all arrows \(h\) with \(K \cdot h = 1_{(x,a)}\).

2. With the notion of fibres, the conditions (1) and (2) in Definition 3.2.7 are restated as follows:
   a) \(K_X\) restricted to \(\langle K \rangle a\) is a right comma fibration for every \(a \in \|A\|\);
   b) \(K_A\) restricted to \(x \langle K \rangle\) is a left comma fibration for every \(x \in \|X\|\).

3. A comma \(K: X \to A\) is called locally small if \(X\) and \(A\) are locally small and the fibre \(x \langle K \rangle a\) is small for every \((x, a) \in \|X \times A\|\).

**Definition 3.2.9.** Given a pair of commas \(K, L: X \to A\), a morphism \(\Phi\) from \(K\) to \(L\), written \(\Phi: K \to L: X \to A\), is defined by a functor \([\Phi]: [K] \to [L]\) such that the triangle

\[
\begin{array}{ccc}
[K] & \xrightarrow{[\Phi]} & [L] \\
\downarrow & & \downarrow \\
X \times A & \xleftarrow{[\Phi]} & L
\end{array}
\]

commutes or, equivalently, the two triangles

\[
\begin{array}{ccc}
X & \xrightarrow{[\Phi]} & A \\
\downarrow & & \downarrow \\
L & \xleftarrow{[\Phi]} & A
\end{array}
\]

commute.
3. Collages and Commas

Remark 3.2.10. The fibre of a comma morphism \( \Phi : K \to L : X \to A \) at \((x, a) \in \|X \times A\|\), written as

\[
x(\Phi) a : x(K) a \to x(L) a
\]

, is the restriction of the functor \([\Phi] : \|K\| \to \|L\|\) to the fibre of \(K\) at \((x, a)\).

Proposition 3.2.11. Given a pair of categories \(X\) and \(A\), all locally small commas \(X \to A\) and morphisms among them define the category \([X \downarrow A]\) with the obvious identities and the composition. Indeed, \([X \downarrow A]\) is fully embedded into the slice category over \(X \times A\).

Proof. Self explanatory.

Remark 3.2.12.

\(\text{\checkmark}\) There is an obvious isomorphism

\[
[X \downarrow] \simeq [X \downarrow \star]
\]

, by which a right comma over \(X\) is identified with a two-sided comma from \(X\) to the terminal category.

\(\text{\checkmark}\) There is an obvious isomorphism

\[
[\downarrow A] \simeq [\star \downarrow A]
\]

, by which a left comma over \(A\) is identified with a two-sided comma from the terminal category to \(A\).

Proposition 3.2.13. The functor \([\Phi] : \|K\| \to \|L\|\) of a comma morphism \(\Phi : K \to L : X \to A\) sends each \([K]\)-arrow \(h : s \to t\) to the composite of the lift of \(K_A : h\) at \([\Phi] : s\) and the lift of \(K_X : h\) at \([\Phi] : t\) as indicated in

\[
\begin{array}{ccc}
[\Phi] : t & \xrightarrow{(\Phi) : h} & [\Phi] : s \\
\downarrow \quad & \quad & \downarrow \quad \\
K_X : t & \xleftarrow{(K_X : h) : \Phi : s} & K_A : t
\end{array}
\]

Proof. By the definition of a comma morphism,

\[
K_X : h = L_X : [\Phi] : h \\
K_A : h = L_A : [\Phi] : h
\]

. The assertion thus follows from the condition (3) in Definition 3.2.7.

Remark 3.2.14. A comma morphism \(\Phi\) is thus determined by the object function of the functor \([\Phi]\).
3. Collages and Commas

**Definition 3.2.15.** Let $K : X \to A$ and $L : Y \to B$ be commas. Given a pair of functors $P : X \to Y$ and $Q : A \to B$, a comma cell $\Phi : P \sim Q : K \to L$ is defined by a functor $[\Phi] : [K] \to [L]$ making the diagram

$$
\begin{array}{ccc}
X & \xymatrix{ \ar[r]^-{K_X} & [K] & \ar[r]^-{K_A} & A } \\
\downarrow & \downarrow_{[\Phi]} & \downarrow & \downarrow \\
Y & \xymatrix{ \ar[r]_-{L_Y} & [L] & \ar[r]_-{L_B} & B }
\end{array}
$$

commute.

**Remark 3.2.16.** The fibre of a comma cell $\Phi : P \sim Q : K \to L$ at $(x, a) \in \|X \times A\|$, written as $\xymatrix{ x(\Phi) a : x(\Phi) a \to (x \cdot P)(L) (Q \cdot a) }$, is the restriction of the functor $[\Phi] : [K] \to [L]$ to the fibre of $K$ at $(x, a)$.

**Proposition 3.2.17.** *All locally small commas and cells among them define the category $\text{COM}$ with the obvious identities and the composition.*

**Proof.** Self explanatory. \qed

**Remark 3.2.18.**
1. There is an obvious forgetful functor $[-] : \text{COM} \to \text{CAT}$, sending each comma $K$ to the category $\|K\|$. 
2. Given a pair of categories $X$ and $A$, there is a canonical embedding $[X \downarrow A] \hookrightarrow \text{COM}$, identical on objects, defined by the arrow function $\Phi \mapsto (\Phi : 1_X \sim 1_A)$. The embedding is not, in general, full.

**Definition 3.2.19.** The comma category $[M^\downarrow]$ of a module $M : X \to A$ is defined in the following way:

1. the objects of $[M^\downarrow]$ are all $M$-arrows $m : x \sim a$, to be precise, all triples $(x, m, a)$ with $x \in \|X\|$, $a \in \|A\|$, and $m \in x(M) a$;
2. an arrow of $[M^\downarrow]$ from $(m : x \sim a)$ to $(n : y \sim b)$ is a pair $(g, f)$ consisting of an $X$-arrow $g : x \to y$ and an $A$-arrow $f : a \to b$ making the quadrangle

$$
\begin{array}{ccc}
x & \xymatrix{ \ar[r]^-m & a } \\
g & \downarrow & \downarrow \ar[r]^-f & b \\
y & \xymatrix{ \ar[r]^-n & b }
\end{array}
$$

commute;
3. the composition law of $[M^\downarrow]$ is that induced by the composition laws of $X$ and $A$.

**Remark 3.2.20.**
3. Collages and Commas

1. In the literature, a comma category is defined for a pair of functors

\[
\begin{array}{ccc}
  X & \xrightarrow{F} & C \\
  \downarrow & & \downarrow \\
  M & \xrightarrow{G} & A
\end{array}
\]

and written \((F \downarrow G)\). Note that

\[
(F \downarrow G) = \left((F(C)G)^{\downarrow}\right)
\]

and, for a module \(M : X \rightarrow A\),

\[
\left[M^1\right] = (M_X \downarrow M_A)
\]

2. The comma category \([M^1]\) is identified with the full subcategory of the functor category \([2,[M]]\) consisting of all sections of the functor \([\Delta_M] : [M] \rightarrow 2\) (see Remark 3.1.2(4)).

**Note.** In the following, \(\rightarrow\) denotes the inclusion given by the identification in Remark 3.2.20(2).

**Definition 3.2.21.**

1. Given a module \(M : X \rightarrow A\), the comma \(M^1 : X \rightarrow A\) is defined by the comma category \([M^1]\) and the pair of functors \(M^1_X : [M^1] \rightarrow X\) and \(M^1_A : [M^1] \rightarrow A\) induced by the pair of evaluations as indicated in the following commutative diagram:

\[
\begin{array}{ccc}
  [M^1] & \xrightarrow{\sim} & [2,[M]] \\
  \downarrow & & \downarrow \\
  M^1_X & \rightarrow & [0,[M]] \\
  \downarrow & & \downarrow \\
  X & \xrightarrow{M_X} & [M] \\
  \downarrow & & \downarrow \\
  M^1_A & \rightarrow & M_A \\
  \downarrow & & \downarrow \\
  A & \rightarrow & A
\end{array}
\]

2. Given a module morphism \(\Phi : M \rightarrow N : X \rightarrow A\), the comma morphism \(\Phi^1 : M^1 \rightarrow N^1 : X \rightarrow A\) is defined by the functor \([\Phi^1] : [M^1] \rightarrow [N^1]\) induced by the postcomposition with the collage functor \([\Phi]\) as indicated in the following commutative diagram:

\[
\begin{array}{ccc}
  [M^1] & \xrightarrow{\sim} & [2,[M]] \\
  \downarrow & & \downarrow \\
  [\Phi^1] & \rightarrow & [2,[\Phi]] \\
  \downarrow & & \downarrow \\
  [N^1] & \xrightarrow{\sim} & [2,[N]]
\end{array}
\]

3. Given a module cell

\[
\begin{array}{ccc}
  X & \xrightarrow{M} & A \\
  \downarrow & \Phi & \downarrow \Phi \\
  Y & \xrightarrow{N} & B
\end{array}
\]

, the comma cell

\[
\begin{array}{ccc}
  X & \xleftarrow{\Phi} & [M^1] \\
  \downarrow & \sim & \downarrow \\
  X & \xrightarrow{M^1_X} & [M^1] \\
  \downarrow & \sim & \downarrow \\
  Y & \xleftarrow{N^1_Y} & [N^1] \\
  \downarrow & \sim & \downarrow \\
  Y & \xrightarrow{N^1_Y} & B
\end{array}
\]

58
3. Collages and Commas

is defined by the functor \([\Phi^i] : [M^i] \rightarrow [N^i]\) induced by the postcomposition with the collage functor \([\Phi]\) as indicated in the following commutative diagram:

\[
\begin{array}{ccc}
[M^i] & \xrightarrow{\phi^i} & [2, [M]] \\
\downarrow & & \downarrow \\
[N^i] & \xrightarrow{\phi^i} & [2, [N]]
\end{array}
\]

Remark 3.2.22. Given an arrow

\[
\begin{array}{ccc}
x & \xrightarrow{m} & a \\
g & \downarrow & f \\
y & \xrightarrow{n} & b
\end{array}
\]

of the comma category \([M^i]\), the decomposition diagram

\[
\begin{array}{ccc}
x & \xrightarrow{m} & a \\
1 & \downarrow & f \\
x & \xrightarrow{m \circ g \circ n} & b \\
g & \downarrow & 1 \\
y & \xrightarrow{n} & b
\end{array}
\]

illustrates the way the comma \(M^i : X \rightarrow A\) satisfies the condition (3) in Definition 3.2.7; note that the upper quadrangle forms the lift of the \(A\)-arrow \(f\) at \(m\) and the lower quadrangle forms the lift of the \(X\)-arrow \(g\) at \(n\).

Proposition 3.2.23. The module-to-comma correspondence given in Definition 3.2.21 is functorial and defines the following functors:

1. \([X : A] \rightarrow [X \downarrow A]\) for categories \(X\) and \(A\);

2. \(MOD \downarrow \rightarrow COM\);

Proof. The functoriality is easily verified. \(\square\)

Remark 3.2.24.

- Given a category \(X\), the functor \([X :] \rightarrow [X \downarrow]\) is defined such that the quadrangle

\[
\begin{array}{ccc}
[X :] & \xrightarrow{\zeta} & [X \downarrow] \\
\downarrow & & \downarrow \\
[X : *] & \xrightarrow{\zeta} & [X \downarrow *]
\end{array}
\]

commutes, where \(\zeta\) denotes the canonical isomorphisms. The functor sends each right module \(M : X \rightarrow *\) to the right comma \(M^i : X \rightarrow *\).
Given a category $A$, the functor $[\cdot : A] \to [\downarrow A]$ is defined such that the quadrangle

$$
\begin{array}{c}
[\cdot : A] \\
\downarrow \cong \\
[\ast : A]
\end{array}
\xrightarrow{i} \begin{array}{c}
[\downarrow A] \\
\downarrow \cong \\
[\ast \downarrow A]
\end{array}
$$

commutes, where $\cong$ denotes the canonical isomorphisms. The functor sends each left module $M : \ast \to A$ to the left comma $M^l : \ast \to A$.

**Definition 3.2.25.**

1. Given a comma $K : X \to A$, the module $K^l : X \to A$ is defined in the following way:
   a) for a pair of objects $x \in \|X\|$ and $a \in \|A\|$, the hom-set $x(K^l) a$ is defined by
      $$
x(K^l) a = \|x(K) a\|
      $$
      , the set of objects of the fibre of $K$ at $(x, a)$.
   b) for an $X$-arrow $g : y \to x$ and an object $a \in \|A\|$, the function $g(K^l) a : x(K^l) a \to y(K^l) a$ is defined such that it maps each $t \in \|x(K) a\|$ to the domain of the lift $g^t$;
   c) for an object $x \in \|X\|$ and an $A$-arrow $f : a \to b$, the function $x(K^l) f : x(K^l) a \to x(K^l) b$ is defined such that it maps each $s \in \|x(K) a\|$ to the codomain of the lift $f^s$.

2. Given a comma morphism $\Phi : K \to L : X \to A$, the module morphism $\Phi^l : K^l \to L^l : X \to A$ is defined by

   $$
   \Phi^l = (\|x(\Phi) a\| : \|x(K) a\| \to \|x(L) a\|)_{(x, a) \in \|X \times A\|}
   $$
   , where $\|x(\Phi) a\|$ is the object function of the fibre of $\Phi$ at $(x, a) \in \|X \times A\|$.

3. Given a comma cell

$$
\begin{array}{c}
X \xrightarrow{X_X} K \\
P \downarrow \cong \\
Y \xrightarrow{L_Y} L
\end{array}
\xrightarrow{\Phi} \begin{array}{c}
A \\
Q \downarrow \cong \\
B
\end{array}
$$

, the module cell

$$
\begin{array}{c}
X \xrightarrow{X_X} K^l \\
P \downarrow \cong \\
Y \xrightarrow{L_Y} L^l
\end{array}
\xrightarrow{\Phi^l} \begin{array}{c}
A \\
Q \downarrow \cong \\
B
\end{array}
$$

is defined by the module morphism $\Phi^l : K^l \to P(L^l) Q : X \to A$ given by

$$
\Phi^l = (\|x(\Phi) a\| : \|x(K) a\| \to \|(x : P) (Q : a)\|)_{(x, a) \in \|X \times A\|}
$$

, where $\|x(\Phi) a\|$ is the object function of the fibre of $\Phi$ at $(x, a) \in \|X \times A\|$.

**Proposition 3.2.26.**

1. $K^l$ defined in Definition 3.2.25(1) is indeed a module.
2. \(\Phi^i\) defined in Definition 3.2.25(2) is indeed a module morphism.

Proof.

1. For each object \(a \in \| A \|\) (resp. \(x \in \| X \|\)), the functoriality of the slice \(\{K^i\} a : X^\circ \to \text{Set}\) (resp. \(x \{K^i\} : A \to \text{Set}\)) follows from the functoriality of \(K_X\) (resp. \(K_A\)). By the bifunctor lemma (see [Ma] p37 Proposition 1), the proof is complete if we show that the quadrangle

\[
x \{K^i\} a \xrightarrow{\gamma \{K^i\} a} y \{K^i\} a \quad \xrightarrow{x \{K^i\} f} \quad x \{K^i\} b \xrightarrow{\gamma \{K^i\} b} y \{K^i\} b
\]

commutes for any \(X\)-arrow \(g : y \to x\) and any \(A\)-arrow \(f : a \to b\). Given an object \(k \in \| x \{K^i\} a \|\), consider the diagram

\[
\begin{array}{ccc}
k & \xrightarrow{g^k} & s \\
\downarrow{f^k} & & \downarrow{f^s} \\
t & \xrightarrow{g^t} & b
\end{array}
\]

\[
x \xrightarrow{g} y
\]

, where \(s\) is domain of the lift \(g^k\) and \(t\) is the codomain of the lift \(f^k\). \(x \{K^i\} f \circ g \{K^i\} b\) maps \(k\) to the domain of \(g^t\), and \(g \{K^i\} a \circ y \{K^i\} f\) maps \(k\) to the codomain of \(f^s\). But they are equal by the condition (3) in Definition 3.2.7.

2. By [Ma] p38 Proposition 2, it suffices to show that \(x \{\Phi^i\} a\) is natural in \(x\) for each \(a \in \| A \|\) and natural in \(a\) for each \(x \in \| X \|\); that is the quadrangles

\[
x \{K^i\} a \xrightarrow{x \{\Phi^i\} a} x \{L^i\} a \quad \xrightarrow{x \{K^i\} f} \quad x \{K^i\} b \xrightarrow{x \{\Phi^i\} b} x \{L^i\} b
\]

commute for any \(X\)-arrow \(g : y \to x\) and any \(A\)-arrow \(f : a \to b\). Let \(k \in \| x \{K\} a \|\). The composite \(g \{K^i\} a \circ y \{\Phi^i\} a\) sends \(k\) to \([\Phi] \circ \text{domain} (g^k)\) and the composite \(x \{\Phi^i\} a \circ g \{L^i\} a\) sends \(k\) to domain \((g^k \circ [\Phi])\); similarly, the composite \(x \{K^i\} f \circ x \{\Phi^i\} b\) sends \(k\) to \([\Phi] \circ \text{codomain} (f^k)\) and the composite \(x \{\Phi^i\} a \circ x \{L^i\} f\) sends \(k\) to \(\text{codomain} (f^k \circ [\Phi])\). But since \([\Phi] : g^k = g^{([\Phi] \circ [\Phi])}\) and \([\Phi] : f^k = f^{([\Phi] \circ [\Phi])}\) by Proposition 3.2.5, we have

\[
\text{domain} (g^{([\Phi] \circ [\Phi])}) = \text{domain} ([\Phi] : g^k) = [\Phi] \circ \text{domain} (g^k)
\]

and

\[
\text{codomain} (f^{([\Phi] \circ [\Phi])}) = \text{codomain} ([\Phi] : f^k) = [\Phi] \circ \text{codomain} (f^k)
\]

. \(\Box\)
**Proposition 3.2.27.** The comma-to-module correspondence given in Definition 3.2.25 is functorial and defines the following functors:

1. \([X \downarrow A] \to [X : A]\) for categories \(X\) and \(A\);
2. \(\text{COM} \to \text{MOD}\).

**Proof.** The functoriality is easily verified. \(\square\)

**Remark 3.2.28.**

- Given a category \(X\), the functor \([X \downarrow] \to [X :]\) is defined such that the quadrangle
  \[
  \begin{array}{ccc}
  [X \downarrow] & \xrightarrow{\dagger} & [X :] \\
  \downarrow & & \downarrow \\
  [X \downarrow *] & \xrightarrow{\dagger} & [X : *]
  \end{array}
  \]
  commutes, where \(\cong\) denotes the canonical isomorphisms. The functor sends each right comma \(K : X \to *\) to the right module \(K! : X \to *\).

- Given a category \(A\), the functor \([\downarrow A] \to [: A]\) is defined such that the quadrangle
  \[
  \begin{array}{ccc}
  [\downarrow A] & \xrightarrow{\dagger} & [: A] \\
  \downarrow & & \downarrow \\
  [* \downarrow A] & \xrightarrow{\dagger} & [* : A]
  \end{array}
  \]
  commutes, where \(\cong\) denotes the canonical isomorphisms. The functor sends each left comma \(K : * \to A\) to the left module \(K! : * \to A\).

**Theorem 3.2.29.**

1. For each module \(M : X \to A\), there is a canonical isomorphism
   \[\epsilon_M : \langle M! \rangle \cong M\]

2. For each comma \(K : X \to A\), there is a canonical isomorphism
   \[\eta_K : K \cong \langle K! \rangle\]

**Proof.** By Definition 3.2.21 and Definition 3.2.25,

1. the module \(\langle M! \rangle\) is obtained from \(M\) by changing each element \(m \in x(M) a\) to the triple \((x, m, a)\).

2. the comma \(\langle K! \rangle\) is obtained from \(K\) by changing each object \(k \in \|x(K) a\|\) to the triple \((x, k, a)\) and changing each \([K]-\text{arrow} h : s \to t\) to the pair \((K_X h, K_A h)\).
Hence \( \epsilon_M \) is defined by the assignment \((x, m, a) \mapsto m\) and \( \eta_K \) is given by the functor \([\eta_K]\) consisting of the object function \( k \mapsto (x, k, a) \) and the arrow function \( h \mapsto (Kx \cdot h, K_A \cdot h) \). The bijectivity of the arrow function of \([\eta_K]\) follows from the condition (3) in Definition 3.2.7. □

**Remark 3.2.30.**

1. In Theorem 9.2.7 and Theorem 9.3.7 we will see that the corresponding functors

\[
\begin{array}{c}
\text{[X ↓ A]} \xrightarrow{\sim} \text{[X : A]} \\
\text{COM} \xrightarrow{\sim} \text{MOD}
\end{array}
\]

in Proposition 3.2.23 and Proposition 3.2.27 form adjoint equivalences with the isomorphisms in Theorem 3.2.29.

2. Given categories \( X \) and \( A \), the functors

\[
\begin{array}{c}
\text{[X ↓ A]} \xrightarrow{\sim} \text{[X ↑ A]} \\
\text{COM} \xrightarrow{\sim} \text{CLG}
\end{array}
\]

are defined so that the diagrams

\[
\begin{array}{c}
\text{[X ↓ A]} \xrightarrow{\sim} \text{[X ↑ A]} \\
\downarrow \downarrow \downarrow \\
\text{[X ↓ A]} \xrightarrow{\sim} \text{[X : A]} \\
\text{COM} \xrightarrow{\sim} \text{CLG}
\end{array}
\]

\[
\begin{array}{c}
\downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow
\end{array}
\]

commute (see Theorem 3.1.15).

**Definition 3.2.31.** Let \( M : X \to A \) be a module.

- The right comma exponential transpose of \( M \) is the functor

\[
[M^\triangledown] : A \to [X \downarrow]
\]

given by the composition

\[
A \xrightarrow{M^\triangledown} [X :] \xrightarrow{\sim} [X \downarrow]
\]

of the right exponential transpose of \( M \) and the functor in Remark 3.2.24.

- The left comma exponential transpose of \( M \) is the functor

\[
[M^\triangledown] : X \to [\downarrow A]^\triangledown
\]

given by the composition

\[
X \xrightarrow{\triangledown M} [: A]^\triangledown \xrightarrow{\sim} [\downarrow A]^\triangledown
\]

of the left exponential transpose of \( M \) and the functor in Remark 3.2.24.

**Remark 3.2.32.**

1.
3. Collages and Commas

- The right comma exponential transpose of a module $\mathcal{M} : X \to A$ sends each object $a \in \| A \|$ to the comma $\langle (\mathcal{M}) a \rangle^\dagger : X \to \ast$ of the right module $(\mathcal{M}) a$. This comma is called the right comma of $\mathcal{M}$ at $a \in \| A \|$ and written as
  $$\langle \mathcal{M} \downarrow a \rangle := \langle (\mathcal{M}) a \rangle^\dagger$$

  Each right comma $\langle \mathcal{M} \downarrow a \rangle : X \to \ast$ of $\mathcal{M}$ consists of the comma category $[\mathcal{M} \downarrow a]$ and the right comma fibration $[\mathcal{M} \downarrow a] : [\mathcal{M} \downarrow a] \to X$.

- The left comma exponential transpose of a module $\mathcal{M} : X \to A$ sends each object $x \in \| X \|$ to the comma $\langle x \langle \mathcal{M} \rangle \rangle^\dagger : \ast \to A$ of the left module $x \langle \mathcal{M} \rangle$. This comma is called the left comma of $\mathcal{M}$ at $x \in \| X \|$ and written as
  $$\langle x \downarrow \mathcal{M} \rangle := \langle x \langle \mathcal{M} \rangle \rangle^\dagger$$

  Each left comma $\langle x \downarrow \mathcal{M} \rangle : \ast \to A$ of $\mathcal{M}$ consists of the comma category $[x \downarrow \mathcal{M}]$ and the left comma fibration $[x \downarrow \mathcal{M}] : [x \downarrow \mathcal{M}] \to A$.

2.

- Given a functor $F : X \to A$, the right comma at $a \in \| A \|$ of the representable module $F \langle A \rangle : X \to A$ is written as
  $$\langle F \downarrow a \rangle := \langle F \langle A \rangle \rangle^\dagger$$

  and called the right comma of $F$ at $a$. Each right comma $\langle F \downarrow a \rangle : X \to \ast$ of $F$ consists of the comma category $[F \downarrow a]$ and the right comma fibration $[F \downarrow a] : [F \downarrow a] \to X$.

- Given a functor $G : A \to X$, the left comma at $x \in \| X \|$ of the corepresentable module $\langle X \rangle G : X \to A$ is written as
  $$\langle x \downarrow G \rangle := \langle x \langle X \rangle G \rangle^\dagger$$

  and called the left comma of $G$ at $x$. Each left comma $\langle x \downarrow G \rangle : \ast \to A$ of $G$ consists of the comma category $[x \downarrow G]$ and the left comma fibration $[x \downarrow G] : [x \downarrow G] \to A$. 

64
4. Frames

4.1. Cylindrical frames

Definition 4.1.1.

1. A [cylindrical] frame \( \alpha \) of an endomodule \( M : E \to E \) is a family of \( M \)-arrows \( \alpha_e : e \to e \), one for each object \( e \in \| E \| \), that is natural in the sense that the quadrangle

\[
\begin{array}{ccc}
e & \xar{\alpha_e} & e' \\
\downarrow h & & \downarrow h \\
e' & \xar{\alpha_{e'}} & e'
\end{array}
\]

commutes for every \( E \)-arrow \( h : e \to e' \). The \( M \)-arrow \( \alpha_e \) is called the component of \( \alpha \) at \( e \). The set of frames of \( M \) is denoted by \( \prod E M \).

Remark 4.1.2.

1. If \( E \) is small, so is \( \prod E M \).

2. If \( E \) is discrete, then \( \prod E M \) is nothing but the cartesian product \( \prod_{e \in \| E \|} e(M) e \).

3. The naturality of a frame \( \alpha \) of an endomodule \( M : E \to E \) is also expressed in the form of “extranaturality” by regarding \( M \) as a right module over \( E \times E^\sim \) or a left module over \( E^\sim \times E \) (see Remark 1.1.6(2)).

- If \( M \) is regarded as a right module \( M : E \times E^\sim \to * \), then the naturality of \( \alpha \) is expressed by the commutativity of the quadrangle

\[
\begin{array}{ccc}
(e, e') & \xar{(e, h)} & (e, e) \\
\downarrow (h, e') & & \downarrow \alpha_e \\
(e', e') & \xar{(e', e')} & *
\end{array}
\]

for each \( E \)-arrow \( h : e \to e' \) (see Remark 1.1.10).

- If \( M \) is regarded as a left module \( M : * \to E^\sim \times E \), then the naturality of \( \alpha \) is expressed by the commutativity of the quadrangle

\[
\begin{array}{ccc}
* & \xar{\alpha_e} & (e, e) \\
\downarrow \alpha_{e'} & & \downarrow (e, h) \\
(e', e') & \xar{(e', e')} & (e, e')
\end{array}
\]

for each \( E \)-arrow \( h : e \to e' \) (see Remark 1.1.10).
4. Frames

4. A frame of an endomodule $\mathcal{M} : \mathcal{E} \to \mathcal{E}$ is the same thing as a frame of the opposite endomodule $\mathcal{M}^* : \mathcal{E}^* \to \mathcal{E}^*$; that is,

$$\prod_{\mathcal{E}} \mathcal{M} = \prod_{\mathcal{E}^*} \mathcal{M}^*$$

Example 4.1.3. Let $\mathcal{E}$ and $\mathcal{C}$ be categories.

1. A natural transformation $\alpha : S \to T : \mathcal{E} \to \mathcal{C}$ is the same thing as a frame $\alpha$ of the composite endomodule $S(\mathcal{C}) T : \mathcal{E} \to \mathcal{E}$.

2. An extranatural transformation $\alpha$ from a bifunctor $\mathcal{K} : \mathcal{E} \times \mathcal{E} \to \mathcal{C}$ to an object $c \in \mathcal{C}$ is the same thing as a cylindrical frame $\alpha$ of the composite right module $\mathcal{K}(\mathcal{C}) c : \mathcal{E} \times \mathcal{E}^* \to \mathcal{E}$.

Theorem 4.1.4. Let $\mathcal{M} : \mathcal{E} \times \mathcal{D} \to \mathcal{E} \times \mathcal{D}$ be an endomodule. A family of $\mathcal{M}$-arrows $\alpha(e,d) : (e,d) \to (e,d)$ indexed by the objects of $\mathcal{E} \times \mathcal{D}$ is a frame of $\mathcal{M}$ if and only if, for each $e \in \mathcal{E}$, $\alpha(e,d) : (e,d) \to (e,d)$ is a frame of the endomodule $[\mathcal{E} \times \mathcal{D}] \mathcal{M} [\mathcal{E} \times \mathcal{D}] : \mathcal{D} \to \mathcal{D}$ (see Example 1.1.18(6)) and, for each $d \in \mathcal{D}$, $\alpha(e,d) : (e,d) \to (e,d)$ is a frame of the endomodule $[\mathcal{E} \times \mathcal{D}] \mathcal{M} [\mathcal{E} \times \mathcal{D}] : \mathcal{E} \to \mathcal{E}$.

Proof. See [Ma] p38 Proposition 2. □

Definition 4.1.5. If $\Phi : \mathcal{M} \to \mathcal{N} : \mathcal{E} \to \mathcal{E}$ is a module morphism and $\alpha$ is a frame of $\mathcal{M}$, then their composite $\alpha \circ \Phi = \Phi \circ \alpha$ is the frame of $\mathcal{N}$ defined by

$$[\alpha \circ \Phi]_e = \alpha_e \cdot (\Phi) e$$

for $e \in \mathcal{E}$. The function

$$\prod_{\mathcal{E}} \Phi : \prod_{\mathcal{E}} \mathcal{M} \to \prod_{\mathcal{E}} \mathcal{N}$$

is defined by

$$\alpha : \prod_{\mathcal{E}} \Phi = \alpha \circ \Phi$$

Remark 4.1.6. The composite $\alpha \circ \Phi$ so defined does form a frame of $\mathcal{N}$. Indeed, any commutative quadrangle

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\alpha_e} & \mathcal{E} \\
| & \searrow & | \\
\mathcal{E}' & \xrightarrow{\alpha_{e'}} & \mathcal{E}'
\end{array}$$

in $\mathcal{M}$ yields the commutative quadrangle

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\alpha_e \circ \Phi} & \mathcal{E} \\
| & \searrow & | \\
\mathcal{E}' & \xrightarrow{\alpha_{e'} \circ \Phi} & \mathcal{E}'
\end{array}$$

in $\mathcal{N}$ by the naturality of $\Phi$.  

66
Proposition 4.1.7. The assignment $\Phi \mapsto \prod E \Phi$ is functorial and defines the left module

$$\prod : * \to [E : E]$$

Proof. The functoriality is easily verified. \qed

Definition 4.1.8. If $F : E \to D$ is a functor and $\alpha$ is a frame of an endomodule $M : D \to D$, then their composite $F \circ \alpha = \alpha \circ F$ is the frame of the endomodule $F \langle M \rangle F : E \to E$ defined by

$$[F \circ \alpha]_e = \alpha (F \cdot e)$$

for $e \in \| E \|$. The function

$$\prod M : \prod D \to \prod E \langle M \rangle F$$

is defined by

$$\alpha : \prod F = F \circ \alpha$$

Remark 4.1.9. The composite $F \circ \alpha$ so defined does form a frame of $F \langle M \rangle F$. Indeed, by the naturality of $\alpha$, the quadrangle

\[
\begin{array}{ccc}
\ e & \ e \cdot F & \ e \\
\ h & \ h \cdot F & \ h \\
\ e' & \ e' \cdot F & \ e'
\end{array}
\]

commutes for every $E$-arrow $h : e \to e'$.

Proposition 4.1.10. Given functors $E \xrightarrow{F} D \xrightarrow{G} C$ and a frame $\alpha$ of an endomodule $M : C \to C$, the associative law

$$[F \circ G] \circ \alpha = F \circ [G \circ \alpha]$$

holds.

Proof. For any $e \in \| E \|$,

$$[[F \circ G] \circ \alpha]_e = \alpha (G \cdot F \cdot e) = \alpha (G \cdot \alpha)_{(F \cdot e)} = [F \circ [G \circ \alpha]]_e$$

Proposition 4.1.11. The function $\prod F M$ is natural in $M$; that is, for every module morphism $\Phi : M \to N : D \to D$, the quadrangle

$$\begin{array}{ccc}
\prod D M & \xrightarrow{\prod E M} & \prod E F \langle M \rangle F \\
\prod D \Phi & \downarrow & \prod E F \langle \Phi \rangle F \\
\prod D N & \xrightarrow{\prod E N} & \prod E F \langle N \rangle F
\end{array}$$

commutes.
4. Frames

Proof. For any frame $\alpha$ of $\mathcal{M}$,

$$\alpha : \prod_D \Phi : \prod_F \mathcal{N} = F \circ [\alpha \circ \Phi]$$

and

$$\alpha : \prod_F \mathcal{M} : \prod_E F(\Phi) F = [F \circ \alpha] \circ F(\Phi) F$$

. Hence we need to verify that

$$F \circ [\alpha \circ \Phi] = [F \circ \alpha] \circ F(\Phi) F$$

. But, for any $e \in \|E\|$,

$$[F \circ [\alpha \circ \Phi]]_e = [\alpha \circ \Phi]_{(F : e)}$$

$$= \alpha_{(F : e)} : (e : F)(\Phi)(F : e)$$

$$= [F \circ \alpha]_e : e(F(\Phi) F) e$$

$$= [[F \circ \alpha] \circ F(\Phi) F]_e$$

. 

Proposition 4.1.12. The family of functions $\Pi_F \mathcal{M} : \Pi_D \mathcal{M} \to \Pi_E F(\mathcal{M}) F$, one for each endomodule $\mathcal{M} : D \to D$, defines a left module cell

$$\begin{array}{ccc}
\ast & \longrightarrow & \prod_D [D : D] \\
1 & \downarrow & \downarrow {\Pi_F} \\
\ast & \longrightarrow & \prod_E [E : E]
\end{array}$$

. Moreover, the assignment $F \mapsto \Pi_F$ is contravariant functorial.

Proof. The first assertion is immediate from Proposition 4.1.11. The functoriality of the assignment is easily verified using Proposition 4.1.10. 

4.2. Conical frames

Definition 4.2.1.

- A [conical] frame $\alpha$ of a left module $\mathcal{M} : \ast \to E$ is a family of $\mathcal{M}$-arrows $\alpha_e : \ast \to e$, one for each object $e \in \|E\|$, that is natural in the sense that the triangle

$$\begin{array}{ccc}
\ast & \longrightarrow & e \\
\downarrow {\alpha_e} & & \downarrow {h} \\
\ast & \longrightarrow & e'
\end{array}$$

commutes for every $E$-arrow $h : e \to e'$. The set of frames of $\mathcal{M}$ is denoted by $\prod_{E^*} \mathcal{M}$. 

68
4. Frames

- A [conical] frame \( \alpha \) of a right module \( \mathcal{M} : E \to * \) is a family of \( \mathcal{M} \)-arrows \( \alpha_e : e \to * \), one for each object \( e \in \|E\| \), that is natural in the sense that the triangle

\[
\begin{array}{c}
\text{e} \\
\downarrow^h \\
\text{e}'
\end{array}
\begin{array}{c}
\alpha_e \\
\alpha_{e'}
\end{array}
\begin{array}{c}
* \\
\downarrow \\
*
\end{array}
\]

commutes for every \( E \)-arrow \( h : e \to e' \). The set of frames of \( \mathcal{M} \) is denoted by \( \prod_{E^*} \mathcal{M} \).

**Remark 4.2.2.**

1. If \( E \) is small, so is \( \prod_{E^*} \mathcal{M} \) (resp. \( \prod_{E^*} \mathcal{M} \)).
2. If \( E \) is discrete, then \( \prod_{E^*} \mathcal{M} \) (resp. \( \prod_{E^*} \mathcal{M} \)) is just the cartesian product \( \prod_{e \in \|E\|} (\mathcal{M}) e \) (resp. \( \prod_{e \in \|E\|} e (\mathcal{M}) \)).
3. A frame of a right module \( \mathcal{M} : E \to * \) is the same thing as a frame of the opposite left module \( \mathcal{M}^- : * \to E^- \); that is,

\[
\prod_{E^*} \mathcal{M} = \prod_{[E^-]^d} \mathcal{M}^-
\]

**Example 4.2.3.** Given a functor \( \mathcal{K} : E \to C \) and an object \( c \in \|C\| \),

- a cone \( \alpha \) from \( c \) to \( \mathcal{K} \) is the same thing as a frame \( \alpha \) of the composite left module \( c(C) \mathcal{K} : * \to E \).
- a cone \( \alpha \) from \( \mathcal{K} \) to \( c \) is the same thing as a frame \( \alpha \) of the composite right module \( \mathcal{K}(C) c : E \to * \).

**Proposition 4.2.4.**

- A frame of a left module \( \mathcal{M} : * \to E \) is the same thing as a frame of the endomodule \( \Delta_E (\mathcal{M}) : E \to E \) (see Example 1.1.18(7)); that is,

\[
\prod_{E^*} \mathcal{M} = \prod_E \Delta_E (\mathcal{M})
\]

- A frame of a right module \( \mathcal{M} : E \to * \) is the same thing as a frame of the endomodule \( \Delta_E (\mathcal{M}) : E \to E \) (see Example 1.1.18(7)); that is,

\[
\prod_{E^*} \mathcal{M} = \prod_E (\mathcal{M}) \Delta_E
\]

**Proof.** Immediate by noting that the triangle

\[
\begin{array}{c}
\text{e} \\
\downarrow^h \\
\text{e}'
\end{array}
\begin{array}{c}
\alpha_e \\
\alpha_{e'}
\end{array}
\begin{array}{c}
* \\
\downarrow \\
*
\end{array}
\]
Definition 4.2.5.

- If $\Phi : M \to N : * \to E$ is a left module morphism and $\alpha$ is a frame of $M$, then their composite $\alpha \circ \Phi = \Phi \circ \alpha$ is the frame of $N$ defined by

$$[\alpha \circ \Phi]_e = \alpha_{e'} (\Phi) e$$

for $e \in [E]$. The function

$$\prod_{E^*} \Phi : \prod_{E^*} M \to \prod_{E^*} N$$

is defined by

$$\alpha : \prod_{E^*} \Phi = \alpha \circ \Phi$$

- If $\Phi : M \to N : E \to *$ is a right module morphism and $\alpha$ is a frame of $M$, then their composite $\alpha \circ \Phi = \Phi \circ \alpha$ is the frame of $N$ defined by

$$[\alpha \circ \Phi]_e = \alpha_{e'} (\Phi) e$$

for $e \in [E]$. The function

$$\prod_{E^*} \Phi : \prod_{E^*} M \to \prod_{E^*} N$$

is defined by

$$\alpha : \prod_{E^*} \Phi = \alpha \circ \Phi$$

Remark 4.2.6. The composite $\alpha \circ \Phi$ so defined does form a frame of $N$. In fact, we have the following.

Proposition 4.2.7.

- The quadrangle

$$\begin{array}{c}
\prod_{E^*} M \\
\downarrow \downarrow \Phi \Downarrow \Delta_E (M) \\
\prod_{E^*} N \\
\downarrow \downarrow \Delta_E (\Phi)
\end{array}$$

commutes; that is,

$$\prod_{E^*} \Phi = \prod_{E} \Delta_E (\Phi)$$
4. Frames

The quadrangle

\[
\begin{array}{cccc}
\prod_{E^*} M & \to & \prod_{E} (\mathcal{M}) \Delta_E \\
\prod_{E^*} \phi & \downarrow & \prod_{E} (\phi) \Delta_E \\
\prod_{E^*} N & \to & \prod_{E} (\mathcal{N}) \Delta_E \\
\end{array}
\]

commutes; that is,

\[\prod_{E^*} \phi = \prod_{E} (\phi) \Delta_E\]

Proof. We need to verify that \(\alpha \circ \phi = \alpha \circ \Delta_E (\phi)\) for any frame \(\alpha\) of \(\mathcal{M}\), i.e. frame \(\alpha\) of \(\Delta_E (\mathcal{M})\). But since \(\mathcal{N} = \phi (\Delta_E (\mathcal{M}))\) for every \(e \in \|E\|\), we have

\([\alpha \circ \phi]_e = (\phi)_e \cdot (\Delta_E (\mathcal{M}))_e = [\alpha \circ \Delta_E (\phi)]_e\]

\[\square\]

Proposition 4.2.8.

- The assignment \(\phi \mapsto \prod_{E^*} \phi\) is functorial and defines the left module

\[\prod_{E^*} : \ast \to [\cdot : E]\]

In fact, \(\prod_{E^*}\) is obtained from the left module \(\prod_E\) in Proposition 4.1.7 by the composition

\[\ast \to \prod_E \xrightarrow{\|\Delta_E : E\|} [E : E] \xrightarrow{\cdot} [\cdot : E]\]

- The assignment \(\phi \mapsto \prod_{E^*} \phi\) is functorial and defines the left module

\[\prod_{E^*} : \ast \to [E : ]\]

In fact, \(\prod_{E^*}\) is obtained from the left module \(\prod_E\) in Proposition 4.1.7 by the composition

\[\ast \to \prod_E \xrightarrow{\|\Delta_E : E\|} [E : E] \xrightarrow{\cdot} [E : ]\]

Proof. The second assertion is immediate from Proposition 4.2.4 and Proposition 4.2.7. The first assertion follows from the second. \[\square\]

Definition 4.2.9.

- If \(F : E \to D\) is a functor and \(\alpha\) is a frame of a left module \(M : \ast \to D\), then their composite \(F \circ \alpha = \alpha \circ F\) is the frame of the left module \(\langle M \rangle F : \ast \to E\) defined by

\[\alpha \circ (F, e)\]

for \(e \in \|E\|\). The function

\[
\prod_F M : \prod_D M \to \prod_{E^*} (\langle M \rangle F)
\]

is defined by

\[\alpha : \prod_F M = F \circ \alpha\]

71
If \( F : E \to D \) is a functor and \( \alpha \) is a frame of a right module \( \mathcal{M} : D \to \ast \), then their composite \( F \circ \alpha = \alpha \circ F \) is the frame of the right module \( F(\mathcal{M}) : E \to \ast \) defined by

\[
[F \circ \alpha]_e = \alpha(e) \circ F(1)
\]

for \( e \in \|E\| \). The function

\[
\prod_{\mathcal{M} : D <\mathcal{M}>} \rightarrow \prod_{E <\mathcal{M}>} F(\mathcal{M})
\]

is defined by

\[
\alpha : \prod_{\mathcal{M} : D} \mathcal{M} = F \circ \alpha
\]

Remark 4.2.10. The composite \( F \circ \alpha \) so defined does form a frame of \( \mathcal{M} \) (resp. \( F(\mathcal{M}) \)). In fact, we have the following.

**Proposition 4.2.11.** Let \( F : E \to D \) be a functor.

- For any left module \( \mathcal{M} : \ast \to D \), the quadrangle

\[
\prod_{D <\mathcal{M}>} \mathcal{M} \xrightarrow{\prod_{\mathcal{M} : D <\mathcal{M}>}} \prod_{D <\mathcal{M}>} \Delta_D(\mathcal{M})
\]

commutes; that is,

\[
\prod_{\mathcal{M} : D} \mathcal{M} = \prod_{\mathcal{M} : D} \Delta_D(\mathcal{M})
\]

- For any right module \( \mathcal{M} : D \to \ast \), the quadrangle

\[
\prod_{D <\mathcal{M}>} \mathcal{M} \xrightarrow{\prod_{\mathcal{M} : D <\mathcal{M}>}} \prod_{D <\mathcal{M}>} (\mathcal{M}) \Delta_D
\]

commutes; that is,

\[
\prod_{\mathcal{M} : D} \mathcal{M} = \prod_{\mathcal{M} : D} (\mathcal{M}) \Delta_D
\]

**Proof.** First note that

\[
F(\Delta_D(\mathcal{M})) F = [F \circ \Delta_D] (\mathcal{M}) F \Delta_E(\mathcal{M}) F
\]

The assertion is now immediate by Definition 4.2.9 and Definition 4.1.8 on noting Proposition 4.2.4.

**Proposition 4.2.12.** Let \( F : E \to D \) be a functor.
4. Frames

- The family of functions $\prod_{F^*} M : \prod_{D^*} M \to \prod_{E^*} F(M) F$, one for each left module $M : * \to D$, defines a left module cell

$$
\begin{array}{c}
* - - \Pi_{D^*} \\
\downarrow 1 \\
* - - \Pi_{E^*} \\
\end{array} 
\xrightarrow{[\vdash D]} 
\begin{array}{c}
[D : D] \\
\end{array} 
\xleftarrow{[\vdash E]} 
\begin{array}{c}
* - - \Pi_{E^*} \\
\downarrow 1 \\
* - - \Pi_{D^*} \\
\end{array}
$$

. In fact, the cell $\prod_{F^*}$ is obtained from the cell $\prod_F$ in Proposition 4.1.12 by the pasting composition

$$
\begin{array}{c}
* - - \Pi_{D^*} \\
\downarrow 1 \\
* - - \Pi_{E^*} \\
\end{array} 
\xrightarrow{[\vdash D \Delta D]} 
\begin{array}{c}
[D : D] \\
\end{array} 
\xleftarrow{[\vdash E \Delta E]} 
\begin{array}{c}
* - - \Pi_{E^*} \\
\downarrow 1 \\
* - - \Pi_{D^*} \\
\end{array}
$$

. Moreover, the assignment $F \mapsto \prod_{F^*}$ is contravariant functorial.

- The family of functions $\prod_{F^*} M : \prod_{D^*} M \to \prod_{E^*} F(M), one for each right module $M : D \to *$, defines a left module cell

$$
\begin{array}{c}
* - - \Pi_{D^*} \\
\downarrow 1 \\
* - - \Pi_{E^*} \\
\end{array} 
\xrightarrow{[\vdash D]} 
\begin{array}{c}
[D : D] \\
\end{array} 
\xleftarrow{[\vdash E]} 
\begin{array}{c}
* - - \Pi_{E^*} \\
\downarrow 1 \\
* - - \Pi_{D^*} \\
\end{array}
$$

. In fact, the cell $\prod_{F^*}$ is obtained from the cell $\prod_F$ in Proposition 4.1.12 by the pasting composition

$$
\begin{array}{c}
* - - \Pi_{D^*} \\
\downarrow 1 \\
* - - \Pi_{E^*} \\
\end{array} 
\xrightarrow{[\vdash D \Delta D]} 
\begin{array}{c}
[D : D] \\
\end{array} 
\xleftarrow{[\vdash E \Delta E]} 
\begin{array}{c}
* - - \Pi_{E^*} \\
\downarrow 1 \\
* - - \Pi_{D^*} \\
\end{array}
$$

. Moreover, the assignment $F \mapsto \prod_{F^*}$ is contravariant functorial.

Proof. The first assertion follows from the second, which is immediate from Proposition 4.2.11. The functoriality of the assignment $F \mapsto \prod_{F^*}$ is now reduced to that of the assignment $F \mapsto \prod_F$ (see Proposition 4.1.12) by Proposition 1.2.32.

4.3. Inner cylinders

Definition 4.3.1. Let $M : X \to A$ be a module.

- Given a functor $G : A \to X$, a right cylinder $\alpha$ from $G$ to $M$, written

$$\alpha : G \rightsquigarrow M$$

or graphically as

$$X \xleftarrow{G} \xrightarrow{M} A$$

, is defined by a frame $\alpha$ of the composite endomodule $G(M) : A \to A$. 

73
4. Frames

Given a functor $F : X \to A$, a left cylinder $\alpha$ from $\mathcal{M}$ to $F$, written
\[ \alpha : \mathcal{M} \rightrightarrows F \]
or graphically as
\[ \begin{array}{c}
X \\
\alpha \\
\mathcal{M} \\
F \\
\alpha \\
A
\end{array} \]
is defined by a frame $\alpha$ of the composite endomodule $(\mathcal{M}) F : X \to X$.

**Remark 4.3.2.**

1. The naturality of a right cylinder $\alpha : G \rightrightarrows \mathcal{M}$ is expressed by the commutativity of the quadrangle
\[ \begin{array}{c}
s : G \\
\alpha_s \\
\mathcal{M} \\
\mathcal{M} \\
\alpha_t \\
t : G
\end{array} \]
for each $A$-arrow $f : s \to t$.

2. The naturality of a left cylinder $\alpha : \mathcal{M} \rightrightarrows F$ is expressed by the commutativity of the quadrangle
\[ \begin{array}{c}
s \sim \alpha_s F : s \\
\mathcal{M} \\
\mathcal{M} \\
F : f \\
\mathcal{M} \\
\alpha_t \\
t \sim F : t
\end{array} \]
for each $X$-arrow $f : s \to t$.

2. By Remark 4.1.2(4) and Remark 1.1.22(4),
\[ \prod \mathcal{X} (\mathcal{M}) F = \prod \mathcal{X} F (\mathcal{M}^\sim) \]
Hence a left cylinder $\alpha : \mathcal{M} \rightrightarrows F$ is the same thing as a right cylinder $\alpha : F \rightrightarrows \mathcal{M}^\sim$.

3. If, in Definition 4.3.1, $A$ is the terminal category, then a right cylinder
\[ \begin{array}{c}
X \\
\alpha \\
\mathcal{M} \\
\mathcal{M} \\
\alpha \\
\star
\end{array} \]
is identified with an arrow of the right module $\mathcal{M} : X \to \star$.

If, in Definition 4.3.1, $X$ is the terminal category, then a left cylinder
\[ \begin{array}{c}
\star \\
\alpha \\
\mathcal{M} \\
\mathcal{M} \\
\alpha \\
A
\end{array} \]
is identified with an arrow of the left module $\mathcal{M} : \star \to A$. 

74
**Definition 4.3.3.** Let $E$ be a category and $M : X \to A$ be a module. Given a pair of functors $S : E \to X$ and $T : E \to A$, a [two-sided] cylinder $\alpha$ from $S$ to $T$ along $M$, written

$$\alpha : S \sim T : E \sim M$$

or graphically as

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & T \\
\downarrow{M} & & \downarrow{M} \\
X & \xrightarrow{\alpha} & A
\end{array}
\]

is defined by a frame $\alpha$ of the composite endomodule $S(M) T : E \to E$.

**Remark 4.3.4.**

1. The naturality of a cylinder $\alpha : S \sim T : E \sim M$ is expressed by the commutativity of the quadrangle

\[
\begin{array}{ccc}
e : S & \xrightarrow{\alpha_e} & T : e \\
\downarrow{h} & & \downarrow{T : h} \\
e' : S & \xrightarrow{\alpha_{e'}} & T : e'
\end{array}
\]

for each $E$-arrow $h : e \to e'$.

2. Since $S(M) T = S ((M) T)$, a two-sided cylinder

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & T \\
\downarrow{M} & & \downarrow{M} \\
X & \xrightarrow{\alpha} & A
\end{array}
\]

from $S$ to $T$ along $M$ is the same thing as a right cylinder

\[
\begin{array}{ccc}
X & \xleftarrow{S} & E \\
\downarrow{(M) T} & & \downarrow{(M) T} \\
X & \xleftarrow{\alpha} & A
\end{array}
\]

from $S$ to the composite module $(M) T$. Conversely, a right cylinder

\[
\begin{array}{ccc}
X & \xleftarrow{G} & A \\
\downarrow{M} & & \downarrow{M} \\
X & \xleftarrow{\alpha} & A
\end{array}
\]

from $G$ to $M$ is the same thing as a two-sided cylinder

\[
\begin{array}{ccc}
A & \xleftarrow{1} & A \\
\downarrow{M} & & \downarrow{M} \\
X & \xleftarrow{\alpha} & A
\end{array}
\]

from $G$ to the identity $1_A$ along $M$. 

75
Since \( S(M) T = (S(M)) T \), a two-sided cylinder

\[
\begin{array}{c}
\text{E} \\
\begin{array}{c}
S \\
\alpha \\
\rightarrow \\
\text{M} \\
\downarrow \\
X \\
\rightarrow \\
\text{T} \\
\downarrow \\
\rightarrow \\
A \\
\end{array}
\end{array}
\]

from \( S \) to \( T \) along \( M \) is the same thing as a left cylinder

\[
\begin{array}{c}
\text{E} \\
\begin{array}{c}
S(M) \\
\alpha \\
\rightarrow \\
\text{T} \\
\downarrow \\
\rightarrow \\
A \\
\end{array}
\end{array}
\]

from the composite module \( S(M) \) to \( T \). Conversely, a left cylinder

\[
\begin{array}{c}
X \\
\begin{array}{c}
\alpha \\
\rightarrow \\
\text{M} \\
\downarrow \\
\rightarrow \\
A \\
\end{array}
\end{array}
\]

from \( M \) to \( F \) is the same thing as a two-sided cylinder

\[
\begin{array}{c}
\text{X} \\
\begin{array}{c}
1 \\
\rightarrow \\
\alpha \\
\rightarrow \\
\text{M} \\
\downarrow \\
\rightarrow \\
\text{F} \\
\rightarrow \\
\rightarrow \\
A \\
\end{array}
\end{array}
\]

from the identity \( 1_X \) to \( F \) along \( M \).

3. By Example 4.1.3(1), a natural transformation \( \alpha : S \rightarrow T : E \rightarrow C \) is the same thing as a cylinder \( \alpha : S \rightarrow T : E \rightarrow (C) \) along the hom of \( C \). Conversely, since \( M = [M_X] ([M]) [MA] \) for any module \( M : X \rightarrow A \) (see Remark 3.1.16(2)), a cylinder

\[
\begin{array}{c}
\text{E} \\
\begin{array}{c}
S \\
\alpha \\
\rightarrow \\
\text{M} \\
\downarrow \\
X \\
\rightarrow \\
\text{T} \\
\downarrow \\
\rightarrow \\
A \\
\end{array}
\end{array}
\]

from \( S \) to \( T \) along \( M \) is the same thing as a natural transformation

\[
\begin{array}{c}
\text{E} \\
\begin{array}{c}
S \\
\alpha \\
\rightarrow \\
\text{M} \\
\downarrow \\
X \\
\rightarrow \\
\text{M}_X [M] [MA] \\
\rightarrow \\
\rightarrow \\
A \\
\end{array}
\end{array}
\]

from \( S \circ M_X \) to \( M_A \circ T \) in the collage category \([M]\).

Example 4.3.5.

1. Since \([S \circ F] (D) T = S (F(D)) T\), a natural transformation

\[
\begin{array}{c}
\text{C} \\
\begin{array}{c}
\alpha \\
\rightarrow \\
\text{F} \\
\downarrow \\
\rightarrow \\
\text{D} \\
\end{array}
\end{array}
\]
4. Frames

from $S \circ F$ to $T$ is the same thing as a cylinder

\[
\begin{array}{c}
\text{from } S \text{ to } T \text{ along the representable module } F \langle D \rangle.
\end{array}
\]

- Since $S \langle D \rangle [F \circ T] = S \langle (D) F \rangle T$, a natural transformation

\[
\begin{array}{c}
\text{from } S \text{ to } F \circ T \text{ is the same thing as a cylinder}
\end{array}
\]

\[
\begin{array}{c}
\text{from } S \text{ to } T \text{ along the corepresentable module } \langle D \rangle F.
\end{array}
\]

2. As a special case of above, consider functors as in:

\[
\begin{array}{c}
\text{since } \langle G \circ F \rangle \langle A \rangle \langle 1_A \rangle = G \langle F \langle A \rangle \rangle, \text{ a natural transformation } \epsilon : G \circ F \rightarrow 1_A : A \rightarrow A \text{ is the same thing as a right cylinder}
\end{array}
\]

\[
\begin{array}{c}
\text{from } G \text{ to the representable module } F \langle A \rangle.
\end{array}
\]

- Since $[1_X \langle X \rangle \langle G \circ F \rangle] = \langle \langle X \rangle G \rangle F$, a natural transformation $\eta : 1_X \rightarrow G \circ F : X \rightarrow X$ is the same thing as a left cylinder

\[
\begin{array}{c}
\text{from the corepresentable module } \langle X \rangle G \text{ to } F.
\end{array}
\]

**Definition 4.3.6.** Given a category $\mathcal{E}$ and a module $\mathcal{M} : X \rightarrow A$, the module of cylinders $\mathcal{E} \Rightarrow \mathcal{M}$,

\[
\langle \mathcal{E}, \mathcal{M} \rangle : [\mathcal{E}, X] \rightarrow [\mathcal{E}, A]
\]

is defined by

\[
\langle S \rangle \langle \mathcal{E}, \mathcal{M} \rangle (T) = \prod_{E} S \langle \mathcal{M} \rangle T
\]

for $S \in [\mathcal{E}, X]$ and $T \in [\mathcal{E}, X]$. 
4. Frames

Remark 4.3.7.

1. For a pair of functors \( S : E \to X \) and \( T : E \to A \), the set \((S,M)/(T)\) consists of all cylinders \( S \times T : E \to M \);

2. If \( \alpha : S \to T \) is a cylinder and \( \sigma : S' \to S \) is a natural transformation as in

\[
\begin{array}{ccc}
S' & \xrightarrow{\sigma} & S \\
\downarrow{\alpha} & & \downarrow{\ } \\
\ M & \xrightarrow{\ } & \ A \\
\end{array}
\]

, then their composite is the cylinder

\[
\begin{array}{ccc}
S' & \xrightarrow{\sigma \circ \alpha} & T \\
\downarrow{\ } & & \downarrow{\ } \\
\ M & \xrightarrow{\ } & \ A \\
\end{array}
\]

defined by

\[
\sigma \circ \alpha = \alpha \circ \sigma (M) T = \alpha : \prod_{E} (M) T
\]

, the image of the frame \( \alpha \in \prod_{E} S(M) T \) under the function

\[
\prod_{E} (M) T : \prod_{E} S(M) T \to \prod_{E} S'(M) T
\]

. By Example 1.1.18(4), each component of the cylinder \( \sigma \circ \alpha \) is given by

\[
\left[ \sigma \circ \alpha \right]_e = \sigma_e \circ \alpha_e
\]

.

- If \( \alpha : S \to T \) is a cylinder and \( \tau : T \to T' \) is a natural transformation as in

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & \ T \\
\downarrow{\tau} & & \downarrow{\ } \\
\ M & \xrightarrow{\ } & \ A \\
\end{array}
\]

, then their composite is the cylinder

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha \circ \tau} & \ T' \\
\downarrow{\ } & & \downarrow{\ } \\
\ M & \xrightarrow{\ } & \ A \\
\end{array}
\]

defined by

\[
\alpha \circ \tau = \alpha \circ S(M) \tau = \alpha : \prod_{E} S(M) \tau
\]

, the image of the frame \( \alpha \in \prod_{E} S(M) T \) under the function

\[
\prod_{E} S(M) \tau : \prod_{E} S(M) T \to \prod_{E} S(M) T'
\]
4. Frames

By Example 1.1.18(4), each component of the cylinder $\alpha \circ \tau$ is given by

$$[\alpha \circ \tau]_e = \alpha_e \circ \tau_e$$

3. If $E$ is small and $M$ is locally small, then the module $\langle E, M \rangle$ is locally small.

**Proposition 4.3.8.** Given a category $E$ and a composite module $P\langle N \rangle Q$ as in

$$
\begin{array}{ccc}
X & \xrightarrow{P\langle N \rangle Q} & A \\
P & \downarrow{1} & \downarrow{Q} \\
Y & \xrightarrow{N} & B
\end{array}
$$

the identity

$$
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, P\langle N \rangle Q)} & [E, A] \\
[E, P] & \downarrow{1} & \downarrow{[E, Q]} \\
[E, Y] & \xrightarrow{(E, N)} & [E, B]
\end{array}
$$

i.e.

$$\langle E, P\langle N \rangle Q \rangle = [E, P][E, N][E, Q]$$

holds.

**Proof.** For any $G \in [E, X]$ and $F \in [E, A],$

$$(G) \langle E, P\langle N \rangle Q \rangle (F) = \prod_{E} G \langle P\langle N \rangle Q \rangle F$$

$$= \prod_{E} [G \circ P \langle N \rangle \langle Q \circ \delta F \rangle]$$

$$= (G \circ P) \langle E, N \rangle (Q \circ \delta F)$$

$$= (G : [E, P]) \langle E, N \rangle ([E, Q] : F)$$

$$= (G) \langle [E, P][E, N][E, Q] \rangle (F)$$

.$$

**Definition 4.3.9.** If $\alpha : E \to M$ is a cylinder and $\Phi : M \to N$ is a module morphism as in

$\begin{array}{ccc}
\alpha & & \Phi \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
X & \xrightarrow{M} & A
\end{array}$

then their composite $\alpha \circ \Phi = \Phi \circ \alpha$ is the cylinder

$\begin{array}{ccc}
\alpha & & \Phi \\
\downarrow{\alpha \circ \Phi} & & \downarrow{\Phi} \\
X & \xrightarrow{N} & A
\end{array}$

79
4. Frames

The image of the frame $\alpha \in \prod_{E} S(M) T$ under the function

$$\prod_{E} S(\Phi) T : \prod_{E} S(M) T \to \prod_{E} S(N) T$$

is denoted by $\alpha \odot \Phi$. By Example 1.1.18(3), each component of the cylinder $\alpha \odot \Phi$ is given by

$$[\alpha \odot \Phi]_e = \alpha_e \odot (e : S)(\Phi)(T : e)$$

for each pair of functors $S : E \to X$ and $T : E \to A$.

**Remark 4.3.10.** The module morphism $(E, \Phi)$ maps each cylinder $\alpha : S \rightsquigarrow T : E \rightsquigarrow M$ to the cylinder $\alpha \odot \Phi : S \rightsquigarrow T : E \rightsquigarrow N$ defined in Definition 4.3.9.

**Definition 4.3.11.** Given a category $E$ and a module morphism $\Phi : M \to N : X \to A$, the module morphism

$$\langle E, \Phi \rangle : \langle E, M \rangle \to \langle E, N \rangle : [E, X] \to [E, A]$$

"postcomposition with $\Phi", is defined by

$$(S) \langle E, \Phi \rangle (T) = \prod_{E} S(\Phi) T$$

for each pair of functors $S : E \to X$ and $T : E \to A$.

**Remark 4.3.12.**

1. The module morphism $(E, \Phi)$ maps each cylinder $\alpha : S \rightsquigarrow T : E \rightsquigarrow M$ to the cylinder $\alpha \odot \Phi : S \rightsquigarrow T : E \rightsquigarrow N$ defined in Definition 4.3.9.

2. The assignment $\Phi \to \langle E, \Phi \rangle$ is functorial; indeed the functor

$$\langle E, - \rangle : [X : A] \to [[E, X] : [E, A]]$$

is defined by

$$(S) \langle E, M \rangle (T) = \prod_{E} S(M) T$$

for $S \in [E, X]$, $T \in [E, A]$, and $M \in [X : A]$.

**Note.** By Remark 1.2.2(2), the following definition is regarded as a special case of Definition 4.3.9 and vice versa.

**Definition 4.3.13.** If $\alpha : E \rightsquigarrow M$ is a cylinder and $\Phi : M \to N$ is a cell as in

```
\begin{array}{ccc}
E & \xrightarrow{\alpha} & T \\
\downarrow S & & \downarrow T \\
X & \xrightarrow{\Phi} & A \\
\downarrow P & & \downarrow Q \\
Y & \xrightarrow{\Phi} & B \\
\end{array}
```

then their composite $\alpha \odot \Phi = \Phi \circ \alpha$ is the cylinder

```
\begin{array}{ccc}
E & \xrightarrow{\alpha \odot \Phi} & Q \circ T \\
\downarrow S \circ P & & \downarrow Q \circ T \\
Y & \xrightarrow{\Phi} & B \\
\end{array}
```
4. Frames

defined by
\[ \alpha \circ \Phi = \alpha \circ S(\Phi) \cdot T = \alpha : \prod E S(\Phi) \cdot T \]

, the image of the frame \( \alpha \in \prod E S(\mathcal{M}) \cdot T \) under the function
\[
\prod E S(\Phi) \cdot T : \prod E S(\mathcal{M}) \cdot T \to \prod E S(P(\mathcal{N}) Q) \cdot T = \prod E [S \circ P] (\mathcal{N}) [Q \circ T]
\].

**Remark 4.3.14.**

1. Each component of the cylinder \( \alpha \circ \Phi \) is given by
\[
[\alpha \circ \Phi]_e = \alpha_e : (e : S) (\Phi) (T \cdot e)
\]
(cf. Remark 4.3.10).

2. If a cell is given by the hom of a functor \( H \) as in

\[
\begin{array}{ccc}
E & \xrightarrow{S} & T \\
\downarrow^\alpha & & \downarrow^H \\
C & \xrightarrow{(C)} & C \\
\downarrow^H & & \downarrow^H \\
B & \xrightarrow{(B)} & B
\end{array}
\]

, then the composite \( \alpha \circ (H) \) is just the usual composite \( \alpha \circ H \) of a natural transformation and a functor.

**Note.** Proposition 4.3.8 allows the following definition.

**Definition 4.3.15.** Given a category \( E \) and a cell

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & A \\
\downarrow^P & \Phi & \downarrow^Q \\
Y & \xrightarrow{\mathcal{N}} & B
\end{array}
\]

, the cell

\[
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, \mathcal{M})} & [E, A] \\
\downarrow^{[E, P]} & \Phi & \downarrow^{[E, Q]} \\
[E, Y] & \xrightarrow{(E, \mathcal{N})} & [E, B]
\end{array}
\]

, “postcomposition with \( \Phi \)”, is defined by the postcomposition module morphism

\[
(E, \mathcal{M}) \xrightarrow{(E, \mathcal{N})} (E, P(\mathcal{N}) Q) = [E, P] (E, \mathcal{N}) [E, Q]
\]

with \( \Phi : \mathcal{M} \to P(\mathcal{N}) Q \).

**Remark 4.3.16.** The cell \( (E, \Phi) \) sends each cylinder \( \alpha : S \sim T : E \sim \mathcal{M} \) to the cylinder \( \alpha \circ \Phi : S \circ P \sim Q \circ T : E \sim \mathcal{N} \) defined in Definition 4.3.13.
Proposition 4.3.17. The assignment $\Phi \mapsto (E, \Phi)$ is functorial.

Proof. Clearly, the assignment $\Phi \mapsto (E, \Phi)$ preserves the identities. To verify that it preserves the composition, let $\Phi$ and $\Psi$ be a composable pair of cells and consider the cells $(E, \Phi)$, $(E, \Psi)$, and $(E, \Phi \circ \Psi)$ depicted in diagram:

\[
\begin{array}{ccc}
X \xrightarrow{M} A & \xrightarrow{\Phi} & [E, X] \xrightarrow{(E, M)} [E, A] \\
Y \xrightarrow{-\Lambda^-} B & \xrightarrow{\Psi} & [E, Y] \xrightarrow{(E, \Psi)} [E, B] \\
Z \xrightarrow{\Lambda^-} C & \xrightarrow{\Phi \circ \Psi} & [E, Z] \xrightarrow{(E, \Phi \circ \Psi)} [E, C]
\end{array}
\]

We need to verify that the composition of the cells $(E, \Phi)$ and $(E, \Psi)$ yields the cell $(E, \Phi \circ \Psi)$. First note that $[E, P \circ P'] = [E, P] \circ [E, P']$ and $[E, Q' \circ Q] = [E, Q'] \circ [E, Q]$ by the functoriality of the operation $[E, -]$. The cell $(E, \Phi) \circ (E, \Psi)$ is defined by the module morphism $(E, \Phi) \circ (E, \Psi)$, and the cell $(E, \Phi \circ \Psi)$ is defined by the module morphism $(E, \Phi \circ P \circ (\Psi) Q)$. But, by Remark 4.3.12(2) and Proposition 4.3.8,

\[
(E, \Phi \circ P \circ (\Psi) Q) = (E, \Phi) \circ (E, P \circ (\Psi) Q) = (E, \Phi) \circ [E, P \circ (\Psi) Q] = [E, A]
\]

Remark 4.3.18. Given a small category $E$, the functor

\[
(E, -) : \text{MOD} \rightarrow \text{MOD}
\]

is defined by the object function $M \mapsto (E, M)$ and the arrow function $\Phi \mapsto (E, \Phi)$, extending the functor $(E, -)$ in Remark 4.3.12(2) as shown in diagram:

\[
\begin{array}{ccc}
[X : A] \xrightarrow{(E, -)} [E, X] : [E, A] \\
\downarrow \hspace{1cm} \downarrow \\
\text{MOD} \xrightarrow{(E, -)} \text{MOD}
\end{array}
\]

where $\rightarrow$ denotes the canonical embedding in Remark 1.2.17(2).

Proposition 4.3.19. If a cell $\Phi$ is fully faithful, so is the cell $(E, \Phi)$.

Proof. Since the operation $(E, -)$ is functorial, it preserves isomorphisms. \qed
Theorem 4.3.20. Let \( E \) be a category.

1. Given a category \( C \), the module of cylinders \( E \to (C) \) along the hom of \( C \) is the same thing as the hom of the functor category \( [E, C] \); that is,
\[
\langle E, (C) \rangle = \langle E, C \rangle
\]
2. Given a functor \( H : C \to B \), the cell
\[
\begin{array}{c}
[E, C] \\
\downarrow \langle E, H \rangle
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
[E, C] \\
\downarrow \langle E, H \rangle
\end{array}
\]
postcomposition with the hom \( (H) \), is the same thing as the hom
\[
\begin{array}{c}
[E, C] \\
\downarrow \langle E, H \rangle
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
[E, C] \\
\downarrow \langle E, H \rangle
\end{array}
\]
of the postcomposition functor \( [E, H] : [E, C] \to [E, B] \); that is,
\[
\langle E, (H) \rangle = \langle E, H \rangle
\]

Proof.

1. If \( M \) is given by the hom \( (C) \), then the composites \( \sigma \circ \alpha \) and \( \alpha \circ \tau \) in Remark 4.3.7(2) are just the vertical composites of two natural transformations.
2. Immediate by Remark 4.3.14(2).

\( \square \)

Remark 4.3.21. In the case \( E \) is small, Theorem 4.3.20 says that the diagram
\[
\begin{array}{c}
\text{CAT} \\
\downarrow \langle - \rangle
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
\text{CAT} \\
\downarrow \langle - \rangle
\end{array}
\]
\[
\begin{array}{c}
\text{MOD} \\
\downarrow \langle E, - \rangle
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
\text{MOD} \\
\downarrow \langle E, - \rangle
\end{array}
\]
commutes.

Definition 4.3.22. If \( F \) is a functor and \( \alpha \) is a cylinder as in
\[
\begin{array}{c}
E \\
\downarrow F
\end{array}
\begin{array}{c}
D \\
\downarrow \alpha \\
\downarrow T
\end{array}
\begin{array}{c}
X \\
\downarrow M \\
\downarrow \ M
\end{array}
\begin{array}{c}
A
\end{array}
\]

83
4. Frames

, then their composite \( F \circ \alpha = \alpha \circ F \) is the cylinder

\[
\begin{array}{c}
\text{E} \\
\downarrow_{F \circ S} \downarrow_{F \circ \alpha} \downarrow_{T \circ F} \\
\text{X} \rightarrow \cdots \rightarrow \text{M} \rightarrow \cdots \rightarrow \text{A}
\end{array}
\]

defined by

\[
F \circ \alpha = \alpha : \prod_F S(\mathcal{M}) \mathcal{T}
\]

, the image of the frame \( \alpha \in \prod_D S(\mathcal{M}) \mathcal{T} \) under the function

\[
\prod_F S(\mathcal{M}) \mathcal{T} : \prod_D S(\mathcal{M}) \mathcal{T} \rightarrow \prod_E F(S(\mathcal{M}) \mathcal{T}) \mathcal{F} = \prod_E [F \circ S](\mathcal{M}) [T \circ F]
\]

Remark 4.3.23.

1. Each component of the cylinder \( F \circ \alpha \) is given by

\[
[F \circ \alpha]_e = \alpha_{(F : e)}
\]

(see Definition 4.1.8).

2. If \( \mathcal{M} \) is given by the hom of a category, then the composite \( F \circ \alpha \) is just the usual composite of a functor and a natural transformation.

Note. By Remark 4.3.4(2), the following definition is a special case of Definition 4.3.22 (and vice versa). It is presented for the sake of reference.

Definition 4.3.24.

- If \( K \) is a functor and \( \alpha \) is a right cylinder as in

\[
\begin{array}{c}
\text{X} \rightarrow \cdots \rightarrow \alpha \rightarrow \text{M} \rightarrow \cdots \rightarrow \text{A} \\
\downarrow_{\alpha} \downarrow_{\mathcal{M}} \\
\text{E}
\end{array}
\]

, then their composite \( K \circ \alpha = \alpha \circ K \) is the two-sided cylinder

\[
\begin{array}{c}
\text{E} \\
\downarrow_{K \circ S} \downarrow_{K \circ \alpha} \downarrow_{K \circ F} \\
\text{X} \rightarrow \cdots \rightarrow \mathcal{M} \rightarrow \cdots \rightarrow \text{A}
\end{array}
\]

defined by

\[
K \circ \alpha = \alpha : \prod_K G(\mathcal{M})
\]

, the image of the frame \( \alpha \in \prod_A G(\mathcal{M}) \) under the function

\[
\prod_K G(\mathcal{M}) : \prod_A G(\mathcal{M}) \rightarrow \prod_E K(G(\mathcal{M})) \mathcal{K} = \prod_E [K \circ G](\mathcal{M}) \mathcal{K}
\]
If \( K \) is a functor and \( \alpha \) is a left cylinder as in
\[
E \xrightarrow{K} X \xrightarrow{\alpha} A
\]
then their composite \( K \circ \alpha = \alpha \circ K \) is the two-sided cylinder
\[
\begin{array}{ccc}
E & \xrightarrow{K \circ \alpha} & F \circ K \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
X & \xrightarrow{K \circ \alpha} & A
\end{array}
\]
defined by
\[
K \circ \alpha = \alpha \cdot \text{product disp } K \text{ product disp } M \text{ product disp } F,
\]
the image of the frame \( \alpha \in \prod_X (\mathcal{M}) F \) under the function
\[
\prod_K (\mathcal{M}) F : \prod_X (\mathcal{M}) F \to \prod_E K (\mathcal{M}) F \to \prod_E K \mathcal{M} [F \circ K]
\]
Remark 4.3.25. Each component of the cylinder \( K \circ \alpha \) is given by
\[
[K \circ \alpha]_e = \alpha_{(K \circ e)}
\]
(see Definition 4.1.8).

**Definition 4.3.26.** Given a functor \( F : E \to D \) and a module \( \mathcal{M} : X \to A \), the cell
\[
\begin{array}{ccc}
[D, X] & \xrightarrow{\mathcal{M}} & [D, A] \\
[F, X] & \downarrow{(F, \mathcal{M})} & \downarrow{(F, A)} \\
[E, X] & \xrightarrow{(E, \mathcal{M})} & [E, A]
\end{array}
\]
"precomposition with \( F \)" is defined by
\[
(S) (F, \mathcal{M}) (T) = \prod_F S (\mathcal{M}) T
\]
for each pair of functors \( S : D \to X \) and \( T : D \to A \).

**Remark 4.3.27.**
1. The cell \( (F, \mathcal{M}) \) sends each cylinder \( \alpha : S \to T : D \to \mathcal{M} \) to the cylinder \( F \circ \alpha : F \circ S \to T \circ F : E \to \mathcal{M} \) defined in Definition 4.3.22.
2. The precomposition cell \( (F, \mathcal{M}) \) is obtained from the left module cell in Proposition 4.1.12 by the pasting composition
\[
\begin{array}{ccc}
\ast & \xrightarrow{\Pi_D} & [D : D] \xleftarrow{-\mathcal{M}-} [D, X] \times [D, A] \\
\downarrow{\Pi_F} & & \downarrow{(F, F)} \\
\ast & \xrightarrow{\Pi_E} & [E : E] \xleftarrow{-\mathcal{M}-} [E, X] \times [E, A]
\end{array}
\]
where \( -\mathcal{M}- \) denotes the functor given by the assignment \( (S, T) \mapsto S \mathcal{M} T \). (See Remark 1.2.4.)
Proposition 4.3.28. Given a module $\mathcal{M}$, the assignment $F \mapsto (F, \mathcal{M})$ defines the contravariant functor

$$\langle -, \mathcal{M} \rangle : \text{Cat}^\text{op} \to \text{MOD}$$

Proof. The functoriality is reduced to that of the assignment $F \mapsto \prod_F$ by Remark 4.3.27(2) and Proposition 1.2.32. □

Example 4.3.29. Let $E$ be a category and $M : X \to A$ be a module. Given an object $e \in |E|$, precomposition with the functor $e : * \to E$ yields the cell

$$\begin{array}{c}
\left[ E, X \right] \xrightarrow{\left( E, M \right)} \left[ E, A \right] \\
\downarrow \left[ e, X \right] \quad \downarrow \left[ (e, M) \right] \\
\left[ X, M \right] \xrightarrow{\left( e, M \right)} \left[ A \right]
\end{array}$$

"evaluation at $e"$, which sends each cylinder $\alpha : S \to T : E \to M$ to the $M$-arrow $\alpha_e : eS \to T e$, the component of $\alpha$ at $e$.

Proposition 4.3.30. Given a functor $F : E \to D$ and a category $C$, the precomposition cell

$$\begin{array}{c}
\left[ D, C \right] \xrightarrow{\left( D, \langle C \rangle \right)} \left[ D, C \right] \\
\downarrow \left[ F, C \right] \quad \downarrow \left( F, \langle C \rangle \right) \\
\left[ E, C \right] \xrightarrow{\left( E, \langle C \rangle \right)} \left[ E, C \right]
\end{array}$$

with respect to the hom of $C$ is the same thing as the hom

$$\begin{array}{c}
\left[ D, C \right] \xrightarrow{\langle D, C \rangle} \left[ D, C \right] \\
\downarrow \left[ F, C \right] \quad \downarrow \left( F, C \right) \\
\left[ E, C \right] \xrightarrow{\langle E, C \rangle} \left[ E, C \right]
\end{array}$$

of the precomposition functor $[F, C]$; that is,

$$\langle F, \langle C \rangle \rangle = \langle F, C \rangle$$

Proof. First note that $\langle D, C \rangle = \langle D, \langle C \rangle \rangle$ and $\langle E, C \rangle = \langle E, \langle C \rangle \rangle$ by Theorem 4.3.20(1). The assertion is then immediate by noting Remark 4.3.23(2). □

Example 4.3.31. Replacing $\mathcal{M}$ in Example 4.3.29 with the hom of a category $C$ yields the cell

$$\begin{array}{c}
\left[ E, C \right] \xrightarrow{\langle E, \langle C \rangle \rangle} \left[ E, C \right] \\
\downarrow \left[ e, C \right] \quad \downarrow \left( e, \langle C \rangle \right) \\
\left[ C, \langle C \rangle \right] \xrightarrow{\langle e, C \rangle} \left[ C, \langle C \rangle \right]
\end{array}$$
4. Frames

, and, by Proposition 4.3.30, this is the same thing as the hom

\[
\begin{array}{ccc}
[E, C] \xrightarrow{(E, C)} [E, C] \\
(e, C) \downarrow & & \downarrow (e, C) \\
C \xrightarrow{(C)} C
\end{array}
\]

of the precomposition functor \([e, C] : [E, C] \to C\); “evaluation at \(e\)” (see Preliminaries(13)).

**Theorem 4.3.32.** There is a functor

\[
\langle -, - \rangle : \text{Cat}^* \times \text{MOD} \to \text{MOD}
\]

such that

1. for each small category \(E\), \((E, -) : \text{MOD} \to \text{MOD}\) coincides with the functor in Remark 4.3.18.

2. for each locally small module \(M\), \((- , M) : \text{Cat}^* \to \text{MOD}\) coincides with the functor in Proposition 4.3.28.

**Proof.** By the bifunctor lemma (see [Ma] p37 Proposition 1), it suffices to show that the quadrangle

\[
\begin{array}{ccc}
(D, M) \xrightarrow{(D, \Phi)} (D, N) \\
(F, M) \downarrow & & \downarrow (F, N) \\
(E, M) \xrightarrow{(E, \Phi)} (E, N)
\end{array}
\]

commutes for any functor \(F : E \to D\) and any cell \(\Phi : M \Rightarrow N\); that is,

\[
\alpha : (D, \Phi) : (F, N) = \alpha : (F, M) : (E, \Phi)
\]

for any cylinder \(\alpha : S \Rightarrow T : D \Rightarrow M\). But by Remark 4.3.16 and Remark 4.3.27(1),

\[
\alpha : (D, \Phi) : (F, N) = F \circ (\alpha \circ \Phi) = (F \circ \alpha) \circ \Phi = \alpha : (F, M) : (E, \Phi)
\]

4.4. Outer cylinders

**Definition 4.4.1.** Let \(E\) be a category and \(M : X \Rightarrow A\) be a module.

- Given an object \(x \in \|X\|\) and a bifunctor \(K : E^* \times E \to A\), an [outer] cylinder \(\alpha\) from \(x\) to \(K\) along \(M\), written

\[
\alpha : x \Rightarrow K : E^* \Rightarrow M
\]

, is defined by a cylindrical frame \(\alpha\) of the composite left module \(x(M) K : * \Rightarrow E^* \times E\) (see Remark 4.1.2(3)).

- Given an object \(a \in \|A\|\) and a bifunctor \(K : E \times E^* \Rightarrow X\), an [outer] cylinder \(\alpha\) from \(K\) to \(a\) along \(M\), written

\[
\alpha : K \Rightarrow a : E^* \Rightarrow M
\]

, is defined by a cylindrical frame \(\alpha\) of the composite right module \(K(M) a : E \times E^* \Rightarrow *\) (see Remark 4.1.2(3)).
4. Frames

Remark 4.4.2.

1. The extranaturality of an outer cylinder \( \alpha : x \rightsquigarrow K : E^+ \rightsquigarrow M \) is expressed by the commutativity of the quadrangle

\[
\begin{array}{ccc}
\alpha & \text{in} & K(e,e) \\
\downarrow & & \downarrow K(e,h) \\
K(e',e') & \xrightarrow{K(h,e')} & K(e,e')
\end{array}
\]

for each \( E \)-arrow \( h : e \to e' \).

- The extranaturality of an outer cylinder \( \alpha : K \rightsquigarrow a : E^+ \rightsquigarrow M \) is expressed by the commutativity of the quadrangle

\[
\begin{array}{ccc}
K(e,e') & \xrightarrow{K(e,h)} & K(e,e) \\
\downarrow K(h,e') & & \downarrow \alpha_e \\
K(e',e') & \xrightarrow{a_{e'}} & a
\end{array}
\]

for each \( E \)-arrow \( h : e \to e' \).

Notation 4.4.3.

- Given a bifunctor \( K : E^+ \times E \to C \) and an object \( c \in \|C\| \), an extranatural transformation \( \alpha \) from \( c \) to \( K \) is denoted by

\[
\alpha : c \rightsquigarrow K : E^+ \to C
\]

- Given a bifunctor \( K : E \times E^+ \to C \) and an object \( c \in \|C\| \), an extranatural transformation \( \alpha \) from \( K \) to \( c \) is denoted by

\[
\alpha : K \rightsquigarrow c : E^+ \to C
\]

Remark 4.4.4. By Example 4.1.3(2), an extranatural transformation in a category \( C \) is the same thing as an outer cylinder along the hom of \( C \). Conversely, an outer cylinder along a module \( M : X \to A \) is the same thing as an extranatural transformation in the collage category \( \|M\| \) (cf. Remark 4.3.4(3)).

Definition 4.4.5. Given a category \( E \) and a module \( M : X \to A \),

- the module of cylinders \( E^+ \rightsquigarrow M \),

\[
\langle E^+, M \rangle : X \to [E^+ \times E, A]
\]

is defined by

\[
(x) \langle E^+, M \rangle (K) = \prod_{E} x \langle M \rangle K
\]

for \( x \in X \) and \( K \in [E^+ \times E, A] \).
4. Frames

the module of cylinders $E^* \rightsquigarrow M$,

$$(E^*, M) : [E \times E^*, X] \to A$$

is defined by

$$(K)(E^*, M)(a) = \prod_E K(M)a$$

for $a \in A$ and $K \in [E \times E^*, X]$.

Remark 4.4.6.

1. For an object $x \in \|X\|$ and a bifunctor $K : E^* \times E \to A$, the set $(x)(E^*, M)(K)$ consists of all cylinders $x \rightsquigarrow K : E^* \rightsquigarrow M$;

2. For an object $a \in \|A\|$ and a bifunctor $K : E \times E^* \to X$, the set $(K)(E^*, M)(a)$ consists of all cylinders $K \rightsquigarrow a : E^* \rightsquigarrow M$;

3. If $\alpha : x \rightsquigarrow K : E^* \rightsquigarrow M$ is a cylinder and $\tau : K \to K' : E^* \times E \to A$ is a natural transformation, then their composite is the cylinder $\alpha \circ \tau : x \rightsquigarrow K' : E^* \rightsquigarrow M$ defined by

$$\alpha \circ \tau = \alpha \circ x(M)\tau = \alpha : \prod_E x(M)\tau$$

the image of the frame $\alpha \in \prod_E x(M)K$ under the function

$$\prod_E x(M)\tau : \prod_E x(M)K \to \prod_E x(M)K'$$

Each component of the cylinder $\alpha \circ \tau$ is given by

$$[\alpha \circ \tau]_e = \alpha_e \circ \tau_{(e,e)}$$

(cf. Remark 4.3.7(2)).

If $\alpha : K \rightsquigarrow a : E^* \rightsquigarrow M$ is a cylinder and $\tau : K' \to K : E \times E^* \to X$ is a natural transformation, then their composite is the cylinder $\tau \circ a : E^* \rightsquigarrow M$ defined by

$$\tau \circ a = \alpha : \prod_E \tau(M)a$$

the image of the frame $\alpha \in \prod_E K(M)a$ under the function

$$\prod_E \tau(M)a : \prod_E K(M)a \to \prod_E K'(M)a$$

Each component of the cylinder $\tau \circ a$ is given by

$$[\tau \circ a]_e = \tau_{(e,e)} \circ \alpha_e$$

(cf. Remark 4.3.7(2)).
4. Frames

- If \( \alpha : x \to K : E^\to \to M \) is a cylinder and \( f : x' \to x \) is an \( X \)-arrow, then their composite is the cylinder \( f \circ \alpha : x' \to M \) defined by

\[
f \circ \alpha = \alpha \circ f \langle M \rangle K = \alpha : \prod_E f \langle M \rangle K
\]

, the image of the frame \( \alpha \in \prod_E x \langle M \rangle K \) under the function

\[
\prod_E f \langle M \rangle K : \prod_E x \langle M \rangle K \to \prod_E x' \langle M \rangle K
\]

. Each component of the cylinder \( f \circ \alpha \) is given by

\[
[f \circ \alpha]_e = f \circ \alpha_e
\]

(cf. Remark 4.3.7(2)).

- If \( \alpha : K \to a : E^\to \to M \) is a cylinder and \( f : a \to a' \) is an \( A \)-arrow, then their composite is the cylinder \( \alpha \circ f : K \to a' : E^\to \to M \) defined by

\[
\alpha \circ f = \alpha \circ K \langle M \rangle f = \alpha : \prod_E K \langle M \rangle f
\]

, the image of the frame \( \alpha \in \prod_E K \langle M \rangle a \) under the function

\[
\prod_E K \langle M \rangle f : \prod_E K \langle M \rangle a \to \prod_E K \langle M \rangle a'
\]

. Each component of the cylinder \( \alpha \circ f \) is given by

\[
[\alpha \circ f]_e = \alpha_e \circ f
\]

(cf. Remark 4.3.7(2)).

4. As a special case where \( M \) is the hom of a category \( C \),

- the module of extranatural transformations \( E^\to \to C \),

\[
(E^\to, C) : C \to [E^\to \times E, C]
\]

, is defined by

\[
(c) (E^\to, C) (K) = \prod_E c \langle C \rangle K
\]

for \( c \in C \) and \( K \in [E^\to \times E, C] \); that is,

\[
\langle E^\to, C \rangle := (E^\to, \langle C \rangle)
\]

.

- the module of extranatural transformations \( E^\to \to C \),

\[
(E^\to, C) : [E \times E^\to, C] \to C
\]

, is defined by

\[
(K) (E^\to, C) (c) = \prod_E K \langle C \rangle c
\]

for \( c \in C \) and \( K \in [E \times E^\to, C] \); that is,

\[
\langle E^\to, C \rangle := (E^\to, \langle C \rangle)
\]

.
4.5. Weighted cylinders

Definition 4.5.1.

- A cylinder

\[
\begin{array}{c}
D \xleftarrow{\alpha} E \\
S \downarrow \alpha \downarrow T \\
X \xleftarrow{\mathcal{M}} A
\end{array}
\]

from \(F \circ S\) to \(T\) along \(\mathcal{M}\) is said to be right weighted by \(F\) (or right \(F\)-weighted).

- A cylinder

\[
\begin{array}{c}
E \xrightarrow{\alpha} D \\
S \downarrow \alpha \downarrow T \\
X \xleftarrow{\mathcal{M}} A
\end{array}
\]

from \(S\) to \(F \circ T\) along \(\mathcal{M}\) is said to be left weighted by \(F\) (or left \(F\)-weighted).

Remark 4.5.2.

1. Since \([F \circ S](\mathcal{M}) \circ T = F(S(\mathcal{M}) \circ T)\), a right \(F\)-weighted cylinder above is the same thing as a right cylinder

\[
\begin{array}{c}
D \xleftarrow{\alpha} E \\
S \downarrow \alpha \downarrow S(\mathcal{M}) \circ T \\
X \xleftarrow{\mathcal{M}} A
\end{array}
\]

. Conversely, a right cylinder is regarded as a special instance of a right weighted cylinder.

- Since \(S(\mathcal{M}) [T \circ F] = (S(\mathcal{M}) \circ T) F\), a left \(F\)-weighted cylinder above is the same thing as a left cylinder

\[
\begin{array}{c}
E \xrightarrow{\alpha} D \\
S \downarrow \alpha \downarrow F \\
X \xleftarrow{\mathcal{M}} A
\end{array}
\]

. Conversely, a left cylinder is regarded as a special instance of a left weighted cylinder.

2. A cylinder

\[
\begin{array}{c}
E \\
S \alpha \downarrow T \\
X \xleftarrow{\mathcal{M}} A
\end{array}
\]

is regarded to be weighted by the identity \(E \xrightarrow{1} E\) and sometimes depicted as

\[
\begin{array}{c}
E \xrightarrow{1} E \\
S \alpha \downarrow T \\
X \xleftarrow{\mathcal{M}} A
\end{array}
\]

\[
\begin{array}{c}
E \xrightarrow{1} E \\
S \alpha \downarrow T \\
X \xleftarrow{\mathcal{M}} A
\end{array}
\]

Definition 4.5.3. Given a functor \(F : E \to D\) and a module \(\mathcal{M} : X \to A\),

91
4. Frames

- the module \( \langle F^\circ, \mathcal{M} \rangle : [D, X] \to [E, A] \)

is defined by the composition

\[
\begin{array}{c}
[D, X] \xrightarrow{[F, X]} [E, X] \xrightarrow{(E, \mathcal{M})} [E, A] \\
\end{array}
\]

- the module \( \langle F^\circ, \mathcal{M} \rangle : [E, X] \to [D, A] \)

is defined by the composition

\[
\begin{array}{c}
[E, X] \xrightarrow{(E, \mathcal{M})} [E, A] \xleftarrow{[F, A]} [D, A] \\
\end{array}
\]

Remark 4.5.4.

1. For any pair of functors \( S : D \to X \) and \( T : E \to A \),

\[
(S) \langle F^\circ, \mathcal{M} \rangle (T) = (S) \langle [F, X] (E, \mathcal{M}) \rangle (T) = (F \circ S) \langle E, \mathcal{M} \rangle (T)
\]

; that is, an \( \langle F^\circ, \mathcal{M} \rangle \)-arrow \( \alpha : S \to T \) is a right \( F \)-weighted cylinder in Definition 4.5.1.

- For any pair of functors \( S : E \to X \) and \( T : D \to A \),

\[
(S) \langle F^\circ, \mathcal{M} \rangle (T) = (S) \langle (E, \mathcal{M}) [F, A] \rangle (T) = (S) \langle E, \mathcal{M} \rangle (T \circ F)
\]

; that is, an \( \langle F^\circ, \mathcal{M} \rangle \)-arrow \( \alpha : S \to T \) is a left \( F \)-weighted cylinder in Definition 4.5.1.

2. As a special case where \( \mathcal{M} \) is the hom of a category \( C \),

- the module \( \langle F^\circ, C \rangle : [E, C] \to [D, C] \)

is defined by the composition

\[
\begin{array}{c}
[D, C] \xrightarrow{[F, C]} [E, C] \xrightarrow{(E, C)} [E, C] \\
\end{array}
\]

; that is,

\( \langle F^\circ, C \rangle := \langle F^\circ, (C) \rangle \)

(recall Theorem 4.3.20(1)). The module \( \langle F^\circ, C \rangle \) is thus nothing but the representable module of the precomposition functor \([F, C]\). An \( \langle F^\circ, (C) \rangle \)-arrow \( \alpha : S \to T \) is a right \( F \)-weighted natural transformation

\[
\begin{array}{c}
D \xleftarrow{F} E \\
\downarrow S \quad \alpha \quad \downarrow T \\
C \xleftarrow{(C)} \to C
\end{array}
\]
4. Frames

- the module
  \[ (F^*, C) : [E, C] \to [D, C] \]
  is defined by the composition
  \[ [E, C] \xrightarrow{(E, C)} [E, C] \xrightarrow{(F, C)} [D, C] \]
; that is,
  \[ (F^*, C) := (F^*, (C)) \]
(recall Theorem 4.3.20(1)). The module \( (F^*, C) \) is thus nothing but the corepresentable module of the precomposition functor \([F, C]\). An \( (F^*, C) \)-arrow \( \alpha : S \to T \) is a left \( F \)-weighted natural transformation

\[
\begin{array}{ccc}
E & \xrightarrow{F} & D \\
\downarrow S & \alpha & \downarrow T \\
C & \xrightarrow{(C)} & C
\end{array}
\]

**Definition 4.5.5.** Given a functor \( F : E \to D \) and a module morphism \( \Phi : M \to N : X \to A \),

- the module morphism
  \[ (F^*, \Phi) : (F^*, M) \to (F^*, N) : [D, X] \to [E, A] \]
  , “postcomposition with \( \Phi \)”, is defined by the composition
  \[ [D, X] \xrightarrow{(F, X)} [E, X] \xrightarrow{(E, M)} [E, A] \]

- the module morphism
  \[ (F^*, \Phi) : (F^*, M) \to (F^*, N) : [E, X] \to [D, A] \]
  , “postcomposition with \( \Phi \)”, is defined by the composition
  \[ [E, X] \xrightarrow{(E, \Phi)} [E, A] \xrightarrow{(E, N)} [D, A] \]

**Remark 4.5.6.**

1. The module morphism \( (F^*, \Phi) \) maps each cylinder \( \alpha : F \circ S \Rightarrow T : E \Rightarrow M \) to the cylinder
  \[ \alpha \circ \Phi : F \circ S \Rightarrow T : E \Rightarrow N \]
  (see Definition 4.3.9).

2. The module morphism \( (F^*, \Phi) \) maps each cylinder \( \alpha : S \Rightarrow T \circ F : E \Rightarrow M \) to the cylinder
  \[ \alpha \circ \Phi : S \Rightarrow T \circ F : E \Rightarrow N \]
  (see Definition 4.3.9).
2. The assignment $\Phi \to (F, \Phi)$ is functorial; indeed the functor

$$(F, -) : [X : A] \to [[D, X] : [E, A]]$$

is defined by

$$\langle F, M \rangle = [F, X] (E, M)$$

- The assignment $\Phi \to (F', \Phi)$ is functorial; indeed the functor

$$(F', -) : [X : A] \to [[E, X] : [D, A]]$$

is defined by

$$\langle F', M \rangle = (E, M) [F, A]$$

**Definition 4.5.7.** Given a functor $F : E \to D$ and a cell

$$\begin{align*}
X &\xrightarrow{M} A \\
P &\downarrow \Phi \\
Y &\xrightarrow{N} B
\end{align*}$$

- the cell

$$\begin{align*}
[D, X] &\xrightarrow{\langle F, M \rangle} [E, A] \\
D, P &\downarrow \Phi \downarrow E, Q \\
[D, Y] &\xrightarrow{\langle F, N \rangle} [E, B]
\end{align*}$$

, “postcomposition with $\Phi$”, is defined by the pasting composition

$$\begin{align*}
[D, X] &\xrightarrow{[F, X]} [E, X] \xrightarrow{\langle E, M \rangle} [E, A] \\
D, P &\downarrow \Phi \downarrow E, Q \\
[D, Y] &\xrightarrow{[F, Y]} [E, Y] \xrightarrow{\langle E, N \rangle} [E, B]
\end{align*}$$

- the cell

$$\begin{align*}
[E, X] &\xrightarrow{\langle F', M \rangle} [D, A] \\
E, P &\downarrow \Phi \downarrow D, Q \\
[E, Y] &\xrightarrow{\langle F', N \rangle} [D, B]
\end{align*}$$

, “postcomposition with $\Phi$”, is defined by the pasting composition

$$\begin{align*}
[E, X] &\xrightarrow{\langle E, M \rangle} [E, A] \xrightarrow{[F, A]} [D, A] \\
E, P &\downarrow \Phi \downarrow E, Q \\
E, Q &\downarrow \Phi \downarrow D, Q \\
[E, Y] &\xrightarrow{\langle E, N \rangle} [E, B] \xrightarrow{[F, B]} [D, B]
\end{align*}$$

94
4. Frames

Remark 4.5.8.

1. The cell \( (F^n, \Phi) \) sends each cylinder \( \alpha : F \circ S \to T : E \to M \) to the cylinder \( \alpha \circ \Phi : F \circ S \circ P \to Q \circ T : E \to N \) given by the composition

\[
\begin{array}{c}
D \xleftarrow{F} E \\
S \downarrow \alpha \downarrow T \\
X \xrightarrow{\Phi} A \\
P \downarrow \Phi \downarrow Q \\
Y \xrightarrow{\sim} B \\
\end{array}
\]

(see Definition 4.3.13).

2. As a special case where \( \Phi \) is given by the hom of a functor \( H : C \to B \),

- the cell

\[
\begin{array}{c}
[D, C] \xrightarrow{(F^n, C)} [E, C] \\
[D, H] \xrightarrow{(F^n, H)} [E, H] \\
[D, B] \xrightarrow{(F^n, B)} [E, B] \\
\end{array}
\]

, "postcomposition with \( H \)"), is defined by the pasting composition

\[
\begin{array}{c}
[D, C] \xrightarrow{(F^n, C)} [E, C] \xrightarrow{(E, C)} [E, C] \\
[D, H] \xrightarrow{(E, H)} [E, H] \xrightarrow{(E, H)} [E, H] \\
[D, B] \xrightarrow{(F^n, B)} [E, B] \xrightarrow{(E, B)} [E, B] \\
\end{array}
\]

; that is,

\( \langle F^n, H \rangle := \langle F^n, (H) \rangle \)

(recall Theorem 4.3.20(2)). The cell \( \langle F^n, H \rangle \) sends each natural transformation \( \alpha : F \circ S \to T : E \to C \) to the natural transformation \( \alpha \circ H : F \circ S \circ H \to H \circ T : E \to B \), the usual composite of a natural transformation and a functor.
4. Frames

- the cell

\[
\begin{array}{c}
\text{[E,C]} \xrightarrow{(F',C)} [D,C] \\
\text{[E,H]} \quad (F',H) \quad [D,H] \\
\text{[E,B]} \xrightarrow{(F',B)} [D,B]
\end{array}
\]

, “postcomposition with \( H \)”, is defined by the pasting composition

\[
\begin{array}{c}
\text{[E,C]} \xrightarrow{(E,C)} [E,C] \xleftarrow{(F,C)} [D,C] \\
\text{[E,H]} \quad (E,H) \quad [E,H] \\
\text{[E,B]} \xrightarrow{(E,B)} [E,B] \xleftarrow{(F,B)} [D,B]
\end{array}
\]

; that is,

\[(F',H) := (F',(H))\]

(recall Theorem 4.3.20(2)). The cell \((F',H)\) sends each natural transformation \( \alpha : S \to T \delta F : E \to C \) to the natural transformation \( \alpha \delta H : S \delta H \to H \delta T \delta F : E \to B \), the usual composite of a natural transformation and a functor.

**Proposition 4.5.9.** The assignment \( \Phi \mapsto (F^\circ, \Phi) \) (resp. \( \Phi \mapsto (F^\circ, \Phi) \)) is functorial.

**Proof.** By Definition 4.5.7 and Proposition 1.2.32, the functoriality of the assignment \( \Phi \mapsto (F^\circ, \Phi) \) is reduced to that of the assignment \( \Phi \mapsto (E, \Phi) \) (see Proposition 4.3.17). \( \square \)

**Remark 4.5.10.** Given a functor \( F : E \to D \) with \( E \) small,

- the functor

\[
\langle F^\circ, - \rangle : \text{MOD} \to \text{MOD}
\]

is defined by the object function \( \mathcal{M} \mapsto (F^\circ, \mathcal{M}) \) and the arrow function \( \Phi \mapsto (F^\circ, \Phi) \), extending the functor \( \langle F^\circ, - \rangle \) in Remark 4.5.6(2) as shown in

\[
\begin{array}{c}
[X : A] \xrightarrow{(F^\circ, -)} [[D,X] : [E,A]] \\
\text{MOD} \xrightarrow{(F^\circ, -)} \text{MOD}
\end{array}
\]

, where \( \hookrightarrow \) denotes the canonical embedding in Remark 1.2.17(2).

- the functor

\[
\langle F^\circ, - \rangle : \text{MOD} \to \text{MOD}
\]

is defined by the object function \( \mathcal{M} \mapsto (F^\circ, \mathcal{M}) \) and the arrow function \( \Phi \mapsto (F^\circ, \Phi) \), extending the functor \( \langle F^\circ, - \rangle \) in Remark 4.5.6(2) as shown in

\[
\begin{array}{c}
[X : A] \xrightarrow{(F^\circ, -)} [[E,X] : [D,A]] \\
\text{MOD} \xrightarrow{(F^\circ, -)} \text{MOD}
\end{array}
\]

, where \( \hookrightarrow \) denotes the canonical embedding in Remark 1.2.17(2).
4. Frames

4.6. Cones

Definition 4.6.1.

- Let \( M : * \to A \) be a left module. Given a functor \( K : E \to A \), a cone \( \alpha \) to \( K \) along \( M \), written
  \[
  \alpha : * \to K : E^a \to M
  \]
  is defined by a frame \( \alpha \) of the composite left module \( (M)K : * \to E \).

- Let \( M : X \to * \) be a right module. Given a functor \( K : E \to X \), a cone \( \alpha \) from \( K \) along \( M \), written
  \[
  \alpha : K \to * : E^b \to M
  \]
  is defined by a frame \( \alpha \) of the composite right module \( K(\mathcal{M}) : E \to * \).

Remark 4.6.2. By Proposition 4.2.4,

- a frame \( \alpha \) of the left module \( (M)K \) is the same thing as a frame of the endomodule \( \Delta_E (M) K : E \to E \). Hence a cone \( \alpha : * \to K : E^a \to M \) is the same thing as a cylinder \( \alpha : \Delta_E \to K : E \to M \) and depicted as

  \[
  \begin{array}{c}
  \Delta \\
  \alpha \\
  \downarrow \\
  K \\
  \hline
  * \\
  \downarrow \\
  M \\
  \end{array}
  \]

- a frame \( \alpha \) of the right module \( K(\mathcal{M}) \) is the same thing as a frame of the endomodule \( K(\mathcal{M}) \Delta_E : E \to E \). Hence a cone \( \alpha : K \to * : E^b \to M \) is the same thing as a cylinder \( \alpha : K \to \Delta_E : E \to M \) and depicted as

  \[
  \begin{array}{c}
  \Delta \\
  \alpha \\
  \downarrow \\
  K \\
  \hline
  X \\
  \downarrow \\
  M \\
  \end{array}
  \]

Definition 4.6.3. Let \( E \) be a category and \( \mathcal{M} : X \to A \) be a module.

- Given an object \( x \in \|X\| \) and a functor \( K : E \to A \), a cone \( \alpha \) from \( x \) to \( K \) along \( \mathcal{M} \), written
  \[
  \alpha : x \to K : E^a \to M
  \]
  is defined by a frame \( \alpha \) of the composite left module \( x(\mathcal{M}) K : * \to E \).

- Given an object \( a \in \|A\| \) and a functor \( K : E \to X \), a cone \( \alpha \) from \( K \) to \( a \) along \( \mathcal{M} \), written
  \[
  \alpha : K \to a : E^b \to M
  \]
  is defined by a frame \( \alpha \) of the composite right module \( K(\mathcal{M}) a : E \to * \).

Remark 4.6.4.
4. Frames

1. By Proposition 4.2.4,
   - a frame $\alpha$ of the left module $x \langle M \rangle K$ is the same thing as a frame of the endomodule $\Delta_x (x \langle M \rangle K) = [ \Delta_x \circ x ] K \langle M \rangle$. Hence a cone $\alpha : x \sim K : E \sim M$ is the same thing as a cylinder $\alpha : \Delta_x \circ x \sim K : E \sim M$ right weighted by $\Delta_x$ and depicted as
     \[
     \begin{array}{c}
     \Delta \\
     \alpha \\
     \downarrow \\
     K
     \end{array}
     \]

     \[
     \begin{array}{c}
     \ast \\
     \downarrow \\
     X - \_ - M - A
     \end{array}
     \]

   - a frame $\alpha$ of the right module $K \langle M \rangle a$ is the same thing as a frame of the endomodule $\langle K \langle M \rangle a \rangle \Delta_x = K \langle M \rangle [ a \circ \Delta_x ]$. Hence a cone $\alpha : K \sim a : E \sim M$ left weighted by $\Delta_x$ and depicted as
     \[
     \begin{array}{c}
     E \\
     \alpha \\
     \downarrow \\
     K
     \end{array}
     \]

     \[
     \begin{array}{c}
     \Delta \\
     \downarrow \\
     a
     \end{array}
     \]

     \[
     \begin{array}{c}
     \ast \\
     \downarrow \\
     X - \_ - M - A
     \end{array}
     \]

2. By Remark 4.2.2(3) and Remark 1.1.22(4),
\[
\prod_{E^e} K \langle M \rangle a = \prod_{[E]^e} a \langle M^\sim \rangle K
\]

Hence a cone $\alpha : K \sim a : E^e \sim M$ is the same thing as a cone $\alpha : a \sim K : [E^\sim]^e \sim M^\sim$.

3. A cone defined in Definition 4.6.1 is a special instance of a cone defined in Definition 4.6.3 where $X$ (resp. $A$) is the terminal category. Conversely,
   - a cone $\alpha : x \sim K$ along a module $M : X \rightarrow A$ is the same thing as a cone $\alpha : * \sim K$ along the left module $x \langle M \rangle : * \rightarrow X$.
   - a cone $\alpha : K \sim a$ along a module $M : X \rightarrow A$ is the same thing as a cone $\alpha : K \sim *$ along the right module $\langle M \rangle a : X \rightarrow *$.

**Definition 4.6.5.** Given a category $E$ and a module $M : X \rightarrow A$,
   - the module of cones $E^e \sim M$,
     \[
     \langle E^e, M \rangle : X \rightarrow [E, A]
     \]
     is defined by
     \[
     (x) \langle E^e, M \rangle (K) = \prod_{E^e} x \langle M \rangle K
     \]
     for $x \in X$ and $K \in [E, A]$.
   - the module of cones $E^p \sim M$,
     \[
     \langle E^p, M \rangle : [E, X] \rightarrow A
     \]
     is defined by
     \[
     (K) \langle E^p, M \rangle (a) = \prod_{E^p} K \langle M \rangle a
     \]
     for $a \in A$ and $K \in [E, X]$.
4. Frames

Remark 4.6.6.

1. For an object $x \in \|X\|$ and a functor $K : E \to A$, the set $(x) (E^o,G) (K)$ consists of all cones $x \Rightarrow K : E^o \Rightarrow M$;

For an object $a \in \|A\|$ and a functor $K : E \to X$, the set $(K) (E^o,G) (a)$ consists of all cones $K \Rightarrow a : E^o \Rightarrow M$;

2. If $\alpha : x \Rightarrow K$ is a cone and $\tau : K \Rightarrow K'$ is a natural transformation as in

\[
\begin{array}{c}
\ast \\
\downarrow \alpha \\
E \\
\downarrow K \\
X \\
\downarrow M \\
A
\end{array}
\]

, then their composite is the cone

\[
\begin{array}{c}
\ast \\
\downarrow \alpha \circ \tau \\
E \\
\downarrow K' \\
X \\
\downarrow M \\
A
\end{array}
\]

defined by

\[\alpha \circ \tau = \alpha \circ x (\mathcal{M}) \tau = \alpha : \prod_{E^o} x (\mathcal{M}) \tau\]

, the image of the frame $\alpha \in \prod_{E^o} x (\mathcal{M}) K$ under the function

\[\prod_{E^o} x (\mathcal{M}) \tau : \prod_{E^o} x (\mathcal{M}) K \to \prod_{E^o} x (\mathcal{M}) K'\]

. If $\alpha : K \Rightarrow a$ is a cone and $\tau : K' \Rightarrow K$ is a natural transformation as in

\[
\begin{array}{c}
E \\
\downarrow \alpha \circ \tau \\
K' \\
\downarrow K \\
X \\
\downarrow M \\
A
\end{array}
\]

, then their composite is the cone

\[
\begin{array}{c}
E \\
\downarrow \tau \circ \alpha \\
K' \\
\downarrow K \\
X \\
\downarrow M \\
A
\end{array}
\]

defined by

\[\tau \circ \alpha = \tau \circ x (\mathcal{M}) a = \alpha : \prod_{E^p} x (\mathcal{M}) a\]

, the image of the frame $\alpha \in \prod_{E^p} K (\mathcal{M}) a$ under the function

\[\prod_{E^p} \tau (\mathcal{M}) a : \prod_{E^p} K (\mathcal{M}) a \to \prod_{E^p} K' (\mathcal{M}) a\]

.
4. Frames

3. If \( \alpha : X \rightarrow K \) is a cone and \( f : X' \rightarrow X \) is an \( X \)-arrow as in

\[
\begin{array}{c}
* & \Delta & \rightarrow & E \\
\downarrow & f & \uparrow & \kappa \\
X & \alpha & \rightarrow & A
\end{array}
\]

, then their composite is the cone

\[
\begin{array}{c}
* & \Delta & \rightarrow & E \\
\downarrow & f \circ \alpha & \uparrow & \kappa \\
X & \rightarrow & A
\end{array}
\]

defined by

\[
f \circ \alpha = \alpha \circ f \colon (M) K = \alpha : \prod_{E^o} f (M) K
\]

, the image of the frame \( \alpha \in \prod_{E^o} x (M) K \) under the function

\[
\prod_{E^o} f (M) K : \prod_{E^o} x (M) K \rightarrow \prod_{E^o} x' (M) K
\]

. If \( \alpha : K \rightarrow a \) is a cone and \( f : a \rightarrow a' \) is an \( A \)-arrow as in

\[
\begin{array}{c}
E & \Delta & \rightarrow & * \\
\downarrow & \kappa & \uparrow & \alpha \circ a \circ f \\
X & \rightarrow & A
\end{array}
\]

, then their composite is the cone

\[
\begin{array}{c}
E & \Delta & \rightarrow & * \\
\downarrow & \kappa & \uparrow & \alpha \circ f \\
X & \rightarrow & A
\end{array}
\]

defined by

\[
\alpha \circ f = \alpha \circ f \colon (M) f = \alpha : \prod_{E^o} K (M) f
\]

, the image of the frame \( \alpha \in \prod_{E^o} K (M) a \) under the function

\[
\prod_{E^o} K (M) f : \prod_{E^o} K (M) a \rightarrow \prod_{E^o} K (M) a'
\]

. If \( E \) is small and \( M \) is locally small, then the module \( (E^o, M) \) (resp. \( (E^o, M) \)) is locally small.
4. Frames

5. By Remark 4.6.4(2), the identity

\[
\begin{array}{ccc}
[E, A]^- & \xrightarrow{\Phi} & X^- \\
\downarrow & \downarrow 1 & \downarrow 1 \\
[E^-, A^-] & \xrightarrow{\Phi^-} & X^- \\
\end{array}
\]

holds, giving a canonical isomorphism

\[
\langle E^-, M \rangle^- \cong \langle [E^-, M]^- \rangle
\]

Definition 4.6.7.

- If \( \alpha : E ^{\circ} \to M \) is a cone and \( \Phi : M \to N \) is a module morphism as in

\[
\begin{array}{ccc}
* & \xleftarrow{\Delta} & E \\
\downarrow x & & \downarrow K \\
X & \xrightarrow{\phi} & A \\
\end{array}
\]

, then their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cone

\[
\begin{array}{ccc}
* & \xleftarrow{\Delta} & E \\
\downarrow x & & \downarrow K \\
X & \xrightarrow{\phi \circ \Phi} & A \\
\end{array}
\]

defined by

\[
\alpha \circ \Phi = \alpha \circ x \cdot (\Phi) K = \alpha \cdot \prod_{E ^{\circ}} x \cdot (\Phi) K
\]

, the image of the frame \( \alpha \in \prod_{E ^{\circ}} x \cdot (M) K \) under the function

\[
\prod_{E ^{\circ}} x \cdot (\Phi) K : \prod_{E ^{\circ}} x \cdot (M) K \to \prod_{E ^{\circ}} x \cdot (N) K
\]

.

- If \( \alpha : E ^{\circ} \to M \) is a cone and \( \Phi : M \to N \) is a module morphism as in

\[
\begin{array}{ccc}
E & \xrightarrow{\Delta} & * \\
\downarrow K & & \downarrow a \\
X & \xrightarrow{\phi} & A \\
\end{array}
\]

, then their composite \( \alpha \circ \Phi = \Phi \circ \alpha \) is the cone

\[
\begin{array}{ccc}
E & \xrightarrow{\Delta} & * \\
\downarrow K & & \downarrow a \\
X & \xrightarrow{\alpha \circ \Phi} & A \\
\end{array}
\]
4. Frames


defined by

\[ \alpha \circ \Phi = \alpha \circ K(\Phi) a = \alpha : \prod_{E^\circ} K(\Phi) a \]

, the image of the frame \( \alpha \in \prod_{E^\circ} K(M) a \) under the function

\[ \prod_{E^\circ} K(\Phi) a : \prod_{E^\circ} K(M) a \to \prod_{E^\circ} K(N) a \]


**Definition 4.6.8.** Given a category \( E \) and a module morphism \( \Phi : M \to N : X \to A \),

- the module morphism

\[ (E^d, \Phi) : (E^d, M) \to (E^d, N) : X \to [E, A] \]

, “postcomposition with \( \Phi \)”, is defined by

\[ (x) (E^d, \Phi) (K) = \prod_{E^\circ} x(\Phi) K \]

for each pair of an object \( x \in \|X\| \) and a functor \( K : E \to A \).

- the module morphism

\[ (E^p, \Phi) : (E^p, M) \to (E^p, N) : [E, X] \to A \]

, “postcomposition with \( \Phi \)”, is defined by

\[ (K) (E^p, \Phi) (a) = \prod_{E^\circ} K(\Phi) a \]

for each pair of an object \( a \in \|A\| \) and a functor \( K : E \to X \).

**Remark 4.6.9.**

1. 

- The module morphism \( (E^d, \Phi) \) maps each cone \( \alpha : x \sim K : E^d \sim M \) to the cone \( \alpha \circ \Phi : x \sim K : E^d \sim N \) defined in Definition 4.6.7.

- The module morphism \( (E^p, \Phi) \) maps each cone \( \alpha : K \sim a : E^p \sim M \) to the cone \( \alpha \circ \Phi : K \sim a : E^p \sim N \) defined in Definition 4.6.7.

2. 

- The assignment \( \Phi \to (E^d, \Phi) \) is functorial; indeed the functor

\[ (E^d, -) : [X : A] \to [X : [E, A]] \]

is defined by

\[ (x) (E^d, M) (K) = \prod_{E^\circ} x(M) K \]

for \( x \in X, K \in [E, A] \), and \( M \in [X : A] \).
The assignment \( \Phi \to (E^\Phi, \Phi) \) is functorial; indeed the functor

\[
(E^\Phi, -) : [X : A] \to [[E, X] : A]
\]

is defined by

\[
(K) (E^\Phi, M) (a) = \prod_{E^\Phi} K (M) a
\]

for \( a \in A, K \in [E, X], \) and \( M \in [X : A] \).

**Note.** We saw in Remark 4.6.4(1) that a cone \( E^\alpha \to M \) may be regarded as a cylinder \( E \to M \) right weighted by the functor \( \Delta_E \) and will see below that the module \( (E^\alpha, M) \) is an instance of the module \( (F^\alpha, M) \) (see Section 4.5) with \( F \) given by \( \Delta_E \).

**Theorem 4.6.10.**

- For any module (resp. module morphism) \( M : X \to A \), the module (resp. module morphism) \( (E^\alpha, M) : X \to [E, A] \) is given by the composition

\[
[X \to (\Delta_E, X)] [E, X] \to (E, X) \to (E^\alpha, M) \to [E, A]
\]

; that is,

\[
(E^\alpha, M) = ([\Delta_E]^\alpha, M)
\]

- For any module (resp. module morphism) \( M : X \to A \), the module (resp. module morphism) \( (E^\circ, M) : [E, X] \to A \) is given by the composition

\[
[E, X] \to (E, X) \to [E, A] \to (\Delta_E, \circ) \to A
\]

; that is,

\[
(E^\circ, M) = ([\Delta_E]^\circ, M)
\]

**Proof.** For any \( x \in X \) and \( K \in [E, A] \),

\[
(x) (E^\alpha, M) (K) = \prod_{E^\alpha} x (M) K
= \prod_{E} \Delta_E (x (M) K)
= \prod_{E} [\Delta_E \circ x] (M) K
= [\Delta_E \circ x] (E, M) (K)
= (x : [\Delta_E, X]) (E, M) (K)
= (x) ([\Delta_E, X] (E, M)) (K)
\]

\[\square\]

**Remark 4.6.11.** By Theorem 4.6.10, the compositions in Remark 4.6.6(2, 3) and Definition 4.6.7 are regarded as the same things as those in Remark 4.3.7(2) and 4.3.9 with the cone \( \alpha \) regarded as the cylinder weighted by \( \Delta_E \).
Note 4.6.12. Theorem 4.6.10 allows the following definition, which is an instance of Definition 4.5.7 with $F$ given by $\Delta_E$.

**Definition 4.6.13.** Given a category $E$ and a cell

\[
\begin{array}{c}
X \rightarrow \Delta_E \rightarrow A \\
P \downarrow \phi \downarrow Q \\
Y \rightarrow \Delta_N \rightarrow B \\
\end{array}
\]

- the cell

\[
\begin{array}{c}
X \rightarrow \Delta_E \rightarrow \Delta_E \rightarrow [E, A] \\
P \downarrow \Delta_{E,P} \downarrow \Delta_{E,\phi} \downarrow \Delta_{E,Q} \\
Y \rightarrow \Delta_E \rightarrow \Delta_E \rightarrow [E, B] \\
\end{array}
\]

, “postcomposition with $\Phi$”, is defined by the pasting composition

\[
\begin{array}{c}
X \\
P \downarrow \Delta_{E,X} \downarrow [E, X] \\
\end{array}
\begin{array}{c}
\rightarrow \Delta_E \rightarrow [E, A] \\
\Delta_{E,P} \downarrow \Delta_{E,\phi} \downarrow \Delta_{E,Q} \\
\end{array}
\begin{array}{c}
Y \\
\end{array}
\begin{array}{c}
\rightarrow \Delta_E \rightarrow [E, B] \\
\end{array}
\]

- the cell

\[
\begin{array}{c}
[E, X] \\
\Delta_{E,P} \downarrow \Delta_{E,\phi} \downarrow \Delta_{E,Q} \\
\end{array}
\begin{array}{c}
\rightarrow \Delta_E \rightarrow [E, A] \\
\Delta_{E,P} \downarrow \Delta_{E,\phi} \downarrow \Delta_{E,Q} \\
\end{array}
\begin{array}{c}
\rightarrow \Delta_E \rightarrow [E, B] \\
\end{array}
\]

, “postcomposition with $\Phi$”, is defined by the pasting composition

\[
\begin{array}{c}
[E, X] \\
\Delta_{E,P} \downarrow \Delta_{E,\phi} \downarrow \Delta_{E,Q} \\
\end{array}
\begin{array}{c}
\rightarrow \Delta_E \rightarrow [E, A] \\
\Delta_{E,P} \downarrow \Delta_{E,\phi} \downarrow \Delta_{E,Q} \\
\end{array}
\begin{array}{c}
\rightarrow \Delta_E \rightarrow [E, B] \\
\end{array}
\]

**Remark 4.6.14.**

- The cell $(E^\phi, \Phi)$ sends each cone $\alpha : x \rightarrow K : E^\phi \rightarrow M$ to the cone $\alpha \circ \Phi : x \circ P \rightarrow Q \circ K : E^\phi \rightarrow N$ given by the composition

\[
\begin{array}{c}
* \rightarrow \Delta \rightarrow E \\
\alpha \downarrow \downarrow \downarrow \Delta \rightarrow M \rightarrow A \\
P \downarrow \phi \downarrow Q \\
Y \rightarrow \Delta_N \rightarrow B \\
\end{array}
\]
4. Frames

- The cell \((E^p, \Phi)\) sends each cone \(\alpha : K \twoheadrightarrow a : E^p \twoheadrightarrow M\) to the cone \(\alpha \circ \Phi : K \circ P \twoheadrightarrow Q \circ a : E^p \twoheadrightarrow N\) given by the composition

\[
\begin{array}{ccc}
E & \overset{\Delta}{\rightarrow} & \ast \\
\downarrow K & \alpha & \downarrow a \\
X & \twoheadrightarrow M & \rightarrow A \\
\downarrow P & \Phi & \downarrow Q \\
Y & \twoheadrightarrow N & \rightarrow B
\end{array}
\]

**Proposition 4.6.15.** The assignment \(\Phi \mapsto (E^s, \Phi)\) (resp. \(\Phi \mapsto (E^p, \Phi)\)) is functorial.

**Proof.** See Proposition 4.5.9. \(\square\)

**Remark 4.6.16.** Given a small category \(E\),

- the functor

\[
(E^s, -) : \text{MOD} \rightarrow \text{MOD}
\]

is defined by the object function \(M \mapsto (E^s, M)\) and the arrow function \(\Phi \mapsto (E^s, \Phi)\), extending the functor \((E^s, -)\) in Remark 4.6.9(2) as shown in

\[
\begin{array}{ccc}
[X : A] & \overset{(E^s, -)}{\rightarrow} & [X : [E, A]] \\
\downarrow \sim & & \downarrow \sim \\
\text{MOD} & \overset{(E^s, -)}{\rightarrow} & \text{MOD}
\end{array}
\]

, where \(\sim\) denotes the canonical embedding in Remark 1.2.17(2).

- the functor

\[
(E^p, -) : \text{MOD} \rightarrow \text{MOD}
\]

is defined by the object function \(M \mapsto (E^p, M)\) and the arrow function \(\Phi \mapsto (E^p, \Phi)\), extending the functor \((E^p, -)\) in Remark 4.6.9(2) as shown in

\[
\begin{array}{ccc}
\downarrow \sim & & \downarrow \sim \\
\text{MOD} & \overset{(E^p, -)}{\rightarrow} & \text{MOD}
\end{array}
\]

, where \(\sim\) denotes the canonical embedding in Remark 1.2.17(2).

**Definition 4.6.17.**

- If \(F\) is a functor and \(\alpha\) is a cone as in

\[
\begin{array}{ccc}
\ast & \overset{\Delta}{\leftarrow} & D \\
\downarrow x & \alpha & \downarrow K \\
X & \twoheadrightarrow M & \rightarrow A
\end{array}
\]
4. Frames

, then their composite \( F \circ \alpha = \alpha \circ F \) is the cone

\[
\begin{array}{c}
\ast \\
\downarrow \Delta \\
E \\
\downarrow x \\
F \circ \alpha \\
\downarrow K \circ F \\
X - \rightarrow M - A
\end{array}
\]
defined by

\[ F \circ \alpha = \alpha : \prod_{F^\circ} x\langle \mathcal{M} \rangle K \]

, the image of the frame \( \alpha \in \prod_{D^\circ} x\langle \mathcal{M} \rangle K \) under the function

\[
\prod_{F^\circ} x\langle \mathcal{M} \rangle K : \prod_{D^\circ} x\langle \mathcal{M} \rangle K \to \prod_{E^\circ} x\langle \mathcal{M} \rangle K \]

\[ F = \prod_{E^\circ} x\langle \mathcal{M} \rangle [K \circ F] \]

. If \( F \) is a functor and \( \alpha \) is a cone as in

\[
\begin{array}{c}
E \\
\downarrow F \\
D \\
\downarrow \Delta \\
\ast \\
\downarrow a \\
X - \rightarrow M - A
\end{array}
\]

, then their composite \( F \circ \alpha = \alpha \circ F \) is the cone

\[
\begin{array}{c}
\ast \\
\downarrow \Delta \\
E \\
\downarrow F \circ K \\
\downarrow F \circ \alpha \\
\downarrow a \\
X - \rightarrow M - A
\end{array}
\]
defined by

\[ F \circ \alpha = \alpha : \prod_{F^\circ} K\langle \mathcal{M} \rangle a \]

, the image of the frame \( \alpha \in \prod_{D^\circ} K\langle \mathcal{M} \rangle a \) under the function

\[
\prod_{F^\circ} K\langle \mathcal{M} \rangle a : \prod_{D^\circ} K\langle \mathcal{M} \rangle a \to \prod_{E^\circ} F\langle K\langle \mathcal{M} \rangle a \rangle = \prod_{E^\circ} [F \circ K] \langle \mathcal{M} \rangle a
\]

.

Remark 4.6.18. Each component of the cone \( F \circ \alpha \) is given by

\[ [F \circ \alpha]_e = \alpha_{(F \circ e)} \]

(see Definition 4.2.9).

Definition 4.6.19. Given a functor \( F : E \to D \) and a module \( M : X \to A \),

\[
\begin{array}{c}
X \\
\downarrow 1 \\
\downarrow (F^\circ, M) \\
\downarrow [F, A] \\
[\langle F^\circ, M \rangle]_e [D, A]
\end{array}
\]

, “precomposition with \( F \)”, is defined by

\[ (x) \langle F^\circ, M \rangle (K) = \prod_{F^\circ} x\langle \mathcal{M} \rangle K \]

for \( x \in [X] \) and \( K \) a functor \( D \to A \).
4. Frames

- the cell

\[
\begin{array}{c}
[D,X] \xrightarrow{(D^p,M)} A \\
[F,A] \xrightarrow{(F^p,M)} 1 \\
[E,X] \xrightarrow{(E^p,M)} A
\end{array}
\]

, “precomposition with \( F \), is defined by

\[(K)\langle F^p,M \rangle (a) = \prod_{F^p} K(M) a\]

for \( a \in \|A\| \) and \( K \) a functor \( D \to X \).

Remark 4.6.20.

- The cell \( \langle F^q,M \rangle \) sends each cone \( \alpha: x \to K: D^q \to M \) to the cone \( F \circ \alpha: x \to K \circ F: E^q \to M \) defined in Definition 4.6.17.

- The cell \( \langle F^p,M \rangle \) sends each cone \( \alpha: K \to a: D^p \to M \) to the cone \( F \circ \alpha: F \circ K \to a: E^p \to M \) defined in Definition 4.6.17.

Proposition 4.6.21.

- The cell \( \langle F^q,M \rangle \) is obtained from the cell \( \langle F,M \rangle \) in Definition 4.3.26 by the pasting composition

\[
\begin{array}{c}
X \xleftarrow{[\Delta D,X]} [D,X] \xrightarrow{(D,M)} [D,A] \\
| 1 \downarrow \quad \downarrow [F,X] \quad \downarrow (F,M) \quad \downarrow [F,A] \\
X \xrightarrow{[\Delta E,X]} [E,X] \xrightarrow{(E,M)} [E,A]
\end{array}
\]

- The cell \( \langle F^p,M \rangle \) is obtained from the cell \( \langle F,M \rangle \) in Definition 4.3.26 by the pasting composition

\[
\begin{array}{c}
[D,X] \xrightarrow{(D,M)} [D,A] \xrightarrow{[\Delta D,A]} A \\
| [F,X] \downarrow (F,M) \downarrow [F,A] \downarrow 1 \\
[E,X] \xrightarrow{(E,M)} [E,A] \xrightarrow{[\Delta E,A]} A
\end{array}
\]

Proof. We need to verify that \( \langle F^q,M \rangle = [\Delta D,X] \langle F,M \rangle \). But, by Proposition 4.2.11, for any object \( x \in \|X\| \) and any functor \( K: D \to A \),

\[
(x)\langle F^q,M \rangle (K) = \prod_{F^q} x(M) K
= \prod_F \Delta D (x(M) K)
= \prod_F [\Delta D \circ x] (M) K
= [\Delta D \circ x] \langle F,M \rangle (K)
= (x\cdot [\Delta D,X]) \langle F,M \rangle (K)
= (x) ([\Delta D,X] \langle F,M \rangle) (K)
\]

\[\square\]
Proposition 4.6.22. If $\mathcal{M}$ is a locally small module, then

1. the assignment $F \mapsto \{F^\circ, \mathcal{M}\}$ defines the contravariant functor
   $$\{[-]^\circ, \mathcal{M}\} : \text{Cat}^\circ \to \text{MOD}$$

2. the assignment $F \mapsto \{F^\circ, \mathcal{M}\}$ defines the contravariant functor
   $$\{[-]^\circ, \mathcal{M}\} : \text{Cat}^\circ \to \text{MOD}$$

Proof. The functoriality of $\{[-]^\circ, \mathcal{M}\}$ is reduced to that of the functor $\{-, \mathcal{M}\} : \text{Cat}^\circ \to \text{MOD}$ (see Proposition 4.3.28) by Proposition 4.6.21 and Proposition 1.2.32.

Theorem 4.6.23.

1. There is a functor
   $$\{[-]^\circ, -\} : \text{Cat}^\circ \times \text{MOD} \to \text{MOD}$$
   such that
   1. for each small category $E$, $\{E^\circ, -\} : \text{MOD} \to \text{MOD}$ coincides with the functor in Remark 4.6.16.
   2. for each locally small module $\mathcal{M}$, $\{[-]^\circ, \mathcal{M}\} : \text{Cat}^\circ \to \text{MOD}$ coincides with the functor in Proposition 4.6.22.

2. There is a functor
   $$\{[-]^\circ, -\} : \text{Cat}^\circ \times \text{MOD} \to \text{MOD}$$
   such that
   1. for each small category $E$, $\{E^\circ, -\} : \text{MOD} \to \text{MOD}$ coincides with the functor in Remark 4.6.16.
   2. for each locally small module $\mathcal{M}$, $\{[-]^\circ, \mathcal{M}\} : \text{Cat}^\circ \to \text{MOD}$ coincides with the functor in Proposition 4.6.22.

Proof. Similar to the proof of Theorem 4.3.32.

4.7. Bicylinders and wedges

Definition 4.7.1. A two-sided cylinder from a product category is called a bicylinder.

Remark 4.7.2. A bicylinder along the hom of a category is just a natural transformation between bifunctors.

Definition 4.7.3. The right and left exponential transposes of a bicylinder
4. Frames

are the cylinders

\[
\begin{array}{c}
\text{D} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[E, X]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{E} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[D, X]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{D} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[E, A]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{E} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[D, A]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\]

defined by

\[
[\alpha \downarrow]_e = \alpha(e, d) = [\alpha \downarrow]_d
\]

for \( e \in \|E\| \) and \( d \in \|D\| \).

\textit{Remark 4.7.4.}

1. The right slice of a bicylinder \( \alpha : S \to T : E \times D \sim M \) at \( d \in \|D\| \) is the cylinder

\[
\begin{array}{c}
\text{E} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[E, X]} \quad \rightarrow \\
\text{M} \\
\end{array}
\quad
\begin{array}{c}
\text{D} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[D, A]} \quad \rightarrow \\
\text{M} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[D, X]} \quad \rightarrow \\
\text{M} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[E, A]} \quad \rightarrow \\
\text{M} \\
\end{array}
\]

given by the component at \( d \) of the right exponential transpose of \( \alpha \), and the left slice of \( \alpha \) at \( e \in \|E\| \) is the cylinder

\[
\begin{array}{c}
\text{D} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[E, X]} \quad \rightarrow \\
\text{M} \\
\end{array}
\quad
\begin{array}{c}
\text{E} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[D, A]} \quad \rightarrow \\
\text{M} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[D, X]} \quad \rightarrow \\
\text{M} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \quad \text{T}^\ast \\
\text{S}^\ast \quad \rightarrow \\
\text{[E, A]} \quad \rightarrow \\
\text{M} \\
\end{array}
\]

given by the component at \( e \) of the left exponential transpose of \( \alpha \).

2. The right and left exponential transpositions do yield cylinders by Theorem 4.1.4, and form the iso cells

\[
\begin{array}{c}
\text{E} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[E, X]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{D} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[D, X]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{E} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[E, A]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{D} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[D, A]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\]

, natural in \( E, D, \) and \( M \).

3. The transpose of a cylinder

\[
\begin{array}{c}
\text{E} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[E, X]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{D} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[D, X]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[E, A]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[D, A]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\]

is the cylinder

\[
\begin{array}{c}
\text{D} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[E, X]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{E} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[D, A]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[E, A]} \quad \rightarrow \\
\text{(E, M)} \\
\end{array}
\quad
\begin{array}{c}
\text{X} \quad \alpha^\ast \\
\text{S} \quad \rightarrow \\
\text{[D, A]} \quad \rightarrow \\
\text{(D, M)} \\
\end{array}
\]

109
defined by 
\[ [\alpha_d]_e = [\alpha_e]_d \]
for \( e \in \|E\| \) and \( d \in \|D\| \). The transposition \( \alpha \mapsto \alpha^\top \) forms the iso cell
\[
\begin{array}{c}
\vdash [E, [D, X]] \xrightarrow{(E, \langle D, \mathcal{M} \rangle)} [E, [D, A]] \\
\alpha^\top \end{array}
\begin{array}{c}
\vdash [D, [E, X]] \xrightarrow{(D, \langle E, \mathcal{M} \rangle)} [D, [E, A]] \\
\alpha^\top \end{array}
\]
, natural in \( E, D \) and \( M \), making the diagram
\[
\langle E \times D, \mathcal{M} \rangle
\]
commute.

4. The diagram
\[
\langle D, \langle E, \mathcal{M} \rangle \rangle \xrightarrow{\langle D, \langle e, \mathcal{M} \rangle \rangle} \langle E, \langle D, \mathcal{M} \rangle \rangle
\]
commutes for every \( e \in \|E\| \), where \( \langle e, \mathcal{M} \rangle \) and \( \langle e, \langle D, \mathcal{M} \rangle \rangle \) are evaluations at \( e \) (see Example 4.3.29); that is, for any cylinder
\[
\begin{array}{c}
S \xrightarrow{\alpha} \vdash [D, X] \xrightarrow{(D, \langle E, \mathcal{M} \rangle)} [D, A] \\
\alpha^\top \end{array}
\]
, its component at \( e \) is given by the composition
\[
\begin{array}{c}
S^\top \xrightarrow{\alpha^\top} \vdash [E, X] \xrightarrow{(E, \langle D, \mathcal{M} \rangle)} [E, A] \\
\vdash \langle e, X \rangle \xrightarrow{\langle e, \mathcal{M} \rangle} \langle e, A \rangle \\
\end{array}
\]
(cf. Preliminaries(13)).

Note. A bicylinder weighted by a projection is called a wedge. Defined below are representative instances of wedges.

**Definition 4.7.5.** Let \( E \) and \( D \) be categories and \( \mathcal{M} : X \to A \) be a module.
Given a functor \( S : E \to X \) and a bifunctor \( K : E \times D \to A \), a wedge \( \alpha \) from \( S \) to \( K \) along \( \mathcal{M} \), written
\[
\alpha : S \leadsto K : E \times D^q \leadsto \mathcal{M}
\]
, is a cylinder
\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & D \\
\downarrow & & \downarrow K \\
X & \xrightarrow{\mathcal{M}} & A
\end{array}
\]
right weighted by the projection \( E \times \Delta_D \).

Given a functor \( T : E \to A \) and a bifunctor \( K : E \times D \to X \), a wedge \( \alpha \) from \( K \) to \( T \) along \( \mathcal{M} \), written
\[
\alpha : K \leadsto T : E \times D^p \leadsto \mathcal{M}
\]
, is a cylinder
\[
\begin{array}{ccc}
E \times D & \xrightarrow{\alpha} & E \\
\downarrow & & \downarrow T \\
X & \xrightarrow{\mathcal{M}} & A
\end{array}
\]
left weighted by the projection \( E \times \Delta_D \).

**Remark 4.7.6.** A cone is a special instance of a wedge. What cones are to cylinders is what wedges are to bicylinders.

**Definition 4.7.7.** Let \( E \) and \( D \) be categories and \( \mathcal{M} : X \to A \) be a module.

\(\triangleright\) The module of wedges \( E \times D^q \leadsto \mathcal{M} \),
\[
\langle E \times D^q, \mathcal{M} \rangle : [E, X] \to [E \times D, A]
\]
, is defined by
\[
\langle E \times D^q, \mathcal{M} \rangle = ([E \times \Delta_D]^+, \mathcal{M})
\]
; that is, by the composition
\[
[E, X] \xrightarrow{[E \times \Delta_D, X]} [E \times D, X] \xrightarrow{\langle E \times D^q, \mathcal{M} \rangle} [E \times D, A]
\]
(see Definition 4.5.3).

\(\triangleright\) The module of wedges \( E \times D^p \leadsto \mathcal{M} \),
\[
\langle E \times D^p, \mathcal{M} \rangle : [E \times D, X] \to [E, A]
\]
, is defined by
\[
\langle E \times D^p, \mathcal{M} \rangle = ([E \times \Delta_D]^-, \mathcal{M})
\]
; that is, by the composition
\[
[E \times D, X] \xrightarrow{\langle E \times D^p, \mathcal{M} \rangle} [E \times D, A] \xrightarrow{[E \times \Delta_D, A]} [E, A]
\]
(see Definition 4.5.3).
Remark 4.7.8.

- For a functor $S : E \rightarrow X$ and a bifunctor $K : E \times D \rightarrow A$, the set $(S)(E \times D^\ast, M)(K)$ consists of all wedges $S \triangleright K : E \times D^\ast \rightarrow M$.
- For a functor $T : E \rightarrow A$ and a bifunctor $K : E \times D \rightarrow X$, the set $(K)(E \times D^\ast, M)(T)$ consists of all wedges $K \triangleright T : E \times D^\ast \rightarrow M$.

Note. The following definition is analogous to Definition 4.7.3.

Definition 4.7.9.

- The right and left exponential transposes of a wedge

$$
\begin{array}{c}
E^\leftarrow \xrightarrow{E \times \Delta_D} E \times D \\
\downarrow s \hspace{1cm} \alpha \hspace{1cm} \downarrow K \\
X \hspace{1cm} \rightarrow \hspace{1cm} M \hspace{1cm} \rightarrow \hspace{1cm} A
\end{array}
$$

are the cone and the cylinder

$$
\begin{array}{c}
* \leftarrow \xrightarrow{\Delta_D} D \\
\downarrow s \hspace{1cm} \alpha^{-} \hspace{1cm} \downarrow \bigl(K^\ast}\bigr) \\
[E, X] \xrightarrow{(E,M)} [E, A] \\
\downarrow [E, X] \xrightarrow{(D^\ast, M)} [D, A]
\end{array}
$$

defined by

$$
[\alpha_{d}^{-}]_e = \alpha_{(e,d)} = [\alpha_{e}^{-}]_d
$$

for $e \in \|E\|$ and $d \in \|D\|$.

- The right and left exponential transposes of a wedge

$$
\begin{array}{c}
E \times D \xrightarrow{E \times \Delta_D} E \\
\downarrow K \hspace{1cm} \alpha \hspace{1cm} \downarrow T \\
X \hspace{1cm} \rightarrow \hspace{1cm} M \hspace{1cm} \rightarrow \hspace{1cm} A
\end{array}
$$

are the cone and the cylinder

$$
\begin{array}{c}
D \xrightarrow{\Delta_D} * \leftarrow \\
\downarrow K^{-} \hspace{1cm} \alpha^{-} \hspace{1cm} \downarrow T \\
[E, X] \xrightarrow{(E, M)} [E, A] \\
\downarrow D, X \xrightarrow{(D^\ast, M)} [D, A]
\end{array}
$$

defined by

$$
[\alpha_{d}^{-}]_e = \alpha_{(e,d)} = [\alpha_{e}^{-}]_d
$$

for $e \in \|E\|$ and $d \in \|D\|$.

Remark 4.7.10.

1.
4. Frames

- The right slice of a wedge $\alpha : S \rightarrow K : E \times D^\ast \rightarrow \mathcal{M}$ at $d \in \|D\|$ is the cylinder

\[
\begin{array}{c}
\text{E} \leftarrow \text{E} \\
\downarrow \quad [\alpha^\ast]_d \\
\text{X} \rightarrow \mathcal{M} \rightarrow \text{A}
\end{array}
\]

given by the component at $d$ of the right exponential transpose of $\alpha$, and the left slice of $\alpha$ at $e \in \|E\|$ is the cone

\[
\begin{array}{c}
* \leftarrow \Delta_D \\
\downarrow \quad [\alpha^\ast]_e \\
\text{X} \rightarrow \mathcal{M} \rightarrow \text{A}
\end{array}
\]

given by the component at $e$ of the left exponential transpose of $\alpha$.

- The right slice of a wedge $\alpha : K \rightarrow T : E \times D^\ast \rightarrow \mathcal{M}$ at $d \in \|D\|$ is the cylinder

\[
\begin{array}{c}
\text{E} \leftarrow \text{E} \\
\downarrow \quad [\alpha^\ast]_d \\
\text{X} \rightarrow \mathcal{M} \rightarrow \text{A}
\end{array}
\]

given by the component at $d$ of the right exponential transpose of $\alpha$, and the left slice of $\alpha$ at $e \in \|E\|$ is the cone

\[
\begin{array}{c}
D \leftarrow * \\
\downarrow \quad [\alpha^\ast]_e \\
\text{X} \rightarrow \mathcal{M} \rightarrow \text{A}
\end{array}
\]

given by the component at $e$ of the left exponential transpose of $\alpha$.

2. The right and left exponential transpositions of wedges $E \times D^\ast \rightarrow M$ form the iso cells

\[
\begin{array}{ccc}
[E, X] & \rightarrow & \rightarrow [E \times D, A] \\
\downarrow \quad 1 & \quad \cdots & \quad \cdots \\
\rightarrow & \rightarrow & \rightarrow
\end{array}
\]

, natural in $E$, $D$, and $M$. In fact, these iso cells are obtained from the iso cells in Remark 4.7.4(2) by pasting a commutative diagram of diagonal functors (see Preliminaries(16)) as shown in

\[
\begin{array}{ccc}
[E, X] & \rightarrow & \rightarrow [E \times D, A] \\
\downarrow \quad 1 & \quad \cdots & \quad \cdots \\
\rightarrow & \rightarrow & \rightarrow
\end{array}
\]

113
4. Frames

\[
\begin{align*}
[E, X] @> [E \times \Delta_D, X] >> [E \times D, X] @> (E \times D, M) >> [E \times D, A] \\
1 \downarrow & \downarrow 1 & \downarrow 1 \\
[E, X] @> [E, [D, X]] >> [E, [D, A]] @> (E, [D, M]) >> [E, [D, A]]
\end{align*}
\]

- The right and left exponential transpositions of wedges \(E \times D^\triangleright \Rightarrow \mathcal{M}\) form the iso cells

\[
\begin{align*}
[E \times D, X] @> (E \times D^\triangleright, M) >> [E, A] & \quad [E \times D, X] @> (E \times D^\triangleright, M) >> [E, A] \\
\downarrow 1 & \downarrow 1 & \downarrow 1 \\
[D, [E, X]] @> (D^\triangleright, [E, M]) >> [D, [E, A]] & \quad [D, [E, A]] @> (D^\triangleright, [E, M]) >> [E, A] \\
[E \times D, X] @> (E \times D, M) >> [E \times D, A] @> [E \times \Delta_D, A] >> [E, A] \\
\downarrow 1 & \downarrow 1 & \downarrow 1 \\
[E, [D, X]] @> (E, [D, M]) >> [E, [D, A]] @> [E, [\Delta_D, A]] >> [E, [D, A]]
\end{align*}
\]

3.

- The transpose of a cone

\[
\begin{align*}
* & \Rightarrow \Delta_D \\
S & \downarrow \alpha \downarrow \kappa \\
[E, X] @> \Rightarrow [E, A]
\end{align*}
\]

is the cylinder

\[
\begin{align*}
E & \Rightarrow E \\
S & \downarrow \alpha \downarrow \kappa \uparrow \\
X @> \Rightarrow [D, A]
\end{align*}
\]

defined by

\[
[\alpha^*]_e = [\alpha]_d
\]

for \(e \in |E|\) and \(d \in |D|\); conversely, the transpose of a cylinder

\[
\begin{align*}
E & \Rightarrow E \\
S & \downarrow \alpha \downarrow \kappa \\
X @> \Rightarrow [D, A]
\end{align*}
\]
4. Frames

is the cone

\[
\begin{array}{c}
* \xleftarrow{\Delta_D} D \\
\downarrow \alpha \downarrow \Delta_K \downarrow \\
[E, X] \xrightarrow{(E, M)} [E, A]
\end{array}
\]

defined by

\[
\alpha^*_d = [\alpha_e]_d
\]

for \( e \in |E| \) and \( d \in |D| \). These transpositions form the iso cell

\[
\begin{array}{c}
[E, X] \xrightarrow{(D^q, (E, M))} [D, [E, A]] \\
\downarrow \tau \downarrow \\
[E, X] \xrightarrow{(E, (D^q, M))} [E, [D, A]]
\end{array}
\]

, natural in \( E, D, \) and \( M \), making the diagram

\[
\begin{array}{c}
(E \times D^q, M) \\
\downarrow \tau \downarrow \\
(D^q, (E, M)) \xrightarrow{\tau} (E, (D^q, M))
\end{array}
\]

commute.

- The transpose of a cone

\[
\begin{array}{c}
D \xrightarrow{\Delta_D} * \\
\downarrow \alpha \downarrow \Delta_K \downarrow \\
[E, X] \xrightarrow{(E, M)} [E, A]
\end{array}
\]

is the cylinder

\[
\begin{array}{c}
E \xrightarrow{E} E \\
\downarrow \alpha \downarrow \Delta_K \downarrow \\
[D, X] \xrightarrow{(D^q, M)} A
\end{array}
\]

defined by

\[
[\alpha^*_d]_e = [\alpha_e]_d
\]

for \( e \in |E| \) and \( d \in |D| \); conversely, the transpose of a cylinder

\[
\begin{array}{c}
E \xrightarrow{E} E \\
\downarrow \alpha \downarrow \Delta_K \downarrow \\
[D, X] \xrightarrow{(D^q, M)} A
\end{array}
\]

is the cone

\[
\begin{array}{c}
D \xrightarrow{\Delta_D} * \\
\downarrow \alpha \downarrow \Delta_K \downarrow \\
[E, X] \xrightarrow{(E, M)} [E, A]
\end{array}
\]
4. Frames

defined by

\[ [\alpha_d]_e = [\alpha_e]_d \]

for \( e \in \|E\| \) and \( d \in \|D\| \). These transpositions form the iso cell

\[
\begin{align*}
[D, [E, X]] & \xrightarrow{\alpha_{(E, \mathcal{M})}} [E, A] \\
[\tau] & \\
[E, [D, X]] & \xrightarrow{\alpha_{(D, \mathcal{M})}} [E, A]
\end{align*}
\]

, natural in \( E, D \), and \( M \), making the diagram

\[
\begin{align*}
(E \times D^\alpha, \mathcal{M}) & \\
\xrightarrow{\alpha_{(E, \mathcal{M})}} (D^\alpha, (E, \mathcal{M})) & \xrightarrow{\alpha_{(D^\alpha, \mathcal{M})}} (E, (D^\alpha, \mathcal{M}))
\end{align*}
\]

commute.

4.

- The diagram

\[
\begin{align*}
(D^\alpha, (E, \mathcal{M})) & \xrightarrow{\alpha_{(E, \mathcal{M})}} (E, (D^\alpha, \mathcal{M})) \\
\xrightarrow{\alpha_{(E^\alpha, \mathcal{M})}} (E^\alpha, (D^\alpha, \mathcal{M})) & \xrightarrow{\alpha_{(E^\alpha, \mathcal{M})}} (D^\alpha, (E^\alpha, \mathcal{M}))
\end{align*}
\]

commutes for every \( e \in \|E\| \), where \( (e, \mathcal{M}) \) and \( (e, (D^\alpha, \mathcal{M})) \) are evaluations at \( e \) (see Example 4.3.29); that is, for any cylinder

\[
\begin{align*}
E & \xrightarrow{\alpha} E \\
S & \\
X & \xrightarrow{(D^\alpha, \mathcal{M})} [D, A]
\end{align*}
\]

, its component at \( e \) is given by the composition

\[
\begin{align*}
* & \xleftarrow{\Delta_D} D \\
S & \\
[e, X] & \xrightarrow{\alpha^{-1}_{(E, \mathcal{M})}} [E, A] \\
\xrightarrow{(e, \mathcal{M})} & \xrightarrow{(e, \mathcal{M})} A
\end{align*}
\]

- The diagram

\[
\begin{align*}
(D^\alpha, (E, \mathcal{M})) & \xrightarrow{\alpha_{(E, \mathcal{M})}} (E, (D^\alpha, \mathcal{M})) \\
\xrightarrow{\alpha_{(D^\alpha, \mathcal{M})}} (D^\alpha, (E, \mathcal{M})) & \xrightarrow{\alpha_{(D^\alpha, \mathcal{M})}} (E, (D^\alpha, \mathcal{M}))
\end{align*}
\]
commutes for every $e \in \|E\|$, where $\langle e, M \rangle$ and $\langle e, \langle D^o, M \rangle \rangle$ are evaluations at $e$ (see Example 4.3.29); that is, for any cylinder

$$
\begin{array}{c}
E \xrightarrow{\alpha} E \\
\downarrow \quad \quad \quad \downarrow T \\
[D, X] \xrightarrow{\langle(D^o, M) \rangle} A
\end{array}
$$

, its component at $e$ is given by the composition

$$
\begin{array}{c}
D \xrightarrow{\Delta_D} \ast \\
\downarrow \quad \quad \quad \downarrow T \\
[E, X] \xrightarrow{\langle(E, M) \rangle} [E, A] \\
\downarrow \quad \quad \quad \downarrow [e, A] \\
X \xrightarrow{\alpha} A
\end{array}
$$

4.8. Cones and wedges in a category

**Notation 4.8.1.** Given a functor $K : E \to C$ and an object $c \in \|C\|$, there are cones:

- a cone $\alpha$ from $c$ to $K$ is denoted by
  $$
  \alpha : c \Rightarrow K : E^d \to C
  $$

- a cone $\alpha$ from $K$ to $c$ is denoted by
  $$
  \alpha : K \Rightarrow c : E^b \to C
  $$

**Remark 4.8.2.**

1. By Example 4.2.3, a cone in a category $C$ is just a special instance of a cone in Definition 4.6.3 where $M$ is the hom of $C$. Conversely, a cone along a module $M$ is the same thing as a cone in the collage category $[\mathcal{M}]$ (cf. Remark 4.3.4(3)).

2. By Remark 4.6.4(1),

- A cone $\alpha : c \Rightarrow K : E^d \Rightarrow C$ is the same thing as a natural transformation

$$
\begin{array}{c}
\ast \xrightarrow{\Delta} E \\
\downarrow \quad \quad \quad \downarrow K \\
c \xrightarrow{\alpha} C
\end{array}
$$

right weighted by $\Delta_E$. 

117
4. Frames

- A cone \( \alpha : K \Rightarrow c : E^p \Rightarrow C \) is the same thing as a natural transformation

\[
\begin{array}{ccc}
E & \xrightarrow{\Delta} & \ast \\
\downarrow{\alpha} & & \downarrow{c} \\
C & \xrightarrow{\cdot} & C
\end{array}
\]

left weighted by \( \Delta_E \).

Note. The following definition is a special case of Definition 4.6.5 where \( M \) is given by the hom of a category.

**Definition 4.8.3.** Given categories \( E \) and \( C \),

- the module of cones \( E^s \Rightarrow C \),

\[
\langle E^s, C \rangle : C \rightarrow [E, C]
\]

is defined by

\[
(c) \langle E^s, C \rangle (K) = \prod_{E^s} c(\sigma) K
\]

for \( c \in C \) and \( K \in [E, C] \); that is,

\[
\langle E^s, C \rangle := \langle E^s, (C) \rangle
\]

- the module of cones \( E^p \Rightarrow C \),

\[
\langle E^p, C \rangle : [E, C] \rightarrow C
\]

is defined by

\[
(K) \langle E^p, C \rangle (c) = \prod_{E^p} K(\sigma) a
\]

for \( c \in C \) and \( K \in [E, C] \); that is,

\[
\langle E^p, C \rangle := \langle E^p, (C) \rangle
\]

**Remark 4.8.4.**

1. For an object \( c \in \parallel C \parallel \) and a functor \( K : E \Rightarrow C \),

- the set \( (c) \langle E^s, C \rangle (K) \) consists of all cones \( c \Rightarrow K : E^s \Rightarrow C \).
- the set \( (K) \langle E^p, C \rangle (c) \) consists of all cones \( K \Rightarrow c : E^p \Rightarrow C \).

2. By Theorem 4.6.10

\[
\langle E^s, C \rangle = \langle [\Delta_E]^s, C \rangle \quad \langle E^p, C \rangle = \langle [\Delta_E]^p, C \rangle
\]

; hence, by Remark 4.5.4(2),

- the module \( \langle E^s, C \rangle : C \Rightarrow [E, C] \) is given by the composition

\[
C \xrightarrow{[\Delta_E, C]} [E, C] \xrightarrow{(E, C)} [E, C]
\]

; that is, \( \langle E^s, C \rangle \) is the representable module of the diagonal functor \( [\Delta_E, C] \).
4. Frames

- the module $\langle E^a, C \rangle : [E, C] \to C$ is given by the composition

$$[E, C] \xrightarrow{\langle E, C \rangle} [E, C] \xrightarrow{\Delta_{E, C}} C$$

; that is, $\langle E^a, C \rangle$ is the corepresentable module of the diagonal functor $[\Delta_{E, C}]$.

*Note.* The following definition is a special case of Definition 4.6.13 where $\Phi$ is given by the hom of a functor (cf. Remark 4.5.8(2)).

**Definition 4.8.5.** Given a category $E$ and a functor $H : C \to B$,

- the cell

$$C \xrightarrow{H} \xrightarrow{\langle E^a, C \rangle} [E, C] \xrightarrow{\langle E, H \rangle} [E, H] \xrightarrow{\langle E, H \rangle} [E, B]$$

, “postcomposition with $H$”, is defined by the pasting composition

$$\xrightarrow{\Delta_{E, B}} [E, B] \xrightarrow{\langle E, B \rangle} \xrightarrow{\langle E, B \rangle} [E, B]$$

; that is,

$$\langle E^a, H \rangle := \langle E^a, \langle H \rangle \rangle$$

- the cell

$$[E, C] \xrightarrow{\langle E^a, C \rangle} \xrightarrow{\langle E, H \rangle} \xrightarrow{\langle E, H \rangle} [E, B] \xrightarrow{\langle E, B \rangle} \xrightarrow{\langle E, B \rangle} [E, B]$$

, “postcomposition with $H$”, is defined by the pasting composition

$$\xrightarrow{\Delta_{E, B}} [E, B] \xrightarrow{\langle E^a, B \rangle} \xrightarrow{\langle E^a, B \rangle} [E, B]$$

; that is,

$$\langle E^a, H \rangle := \langle E^a, \langle H \rangle \rangle$$

**Remark 4.8.6.**

- The cell $\langle E^a, H \rangle$ sends each cone $\alpha : c \to K : E^a \to C$ to the cone $\alpha \circ H : c \circ H \to H \circ K : E^a \to B$, the usual composite of a cone and a functor.
4. Frames

The cell $(E^p, H)$ sends each cone $\alpha : K \to c : E^p \to C$ to the cone $\alpha \circ H : K \circ H \to H \circ c : E^p \to B$, the usual composite of a cone and a functor.

**Note.** The following definition is a special case of Definition 4.6.19 where $M$ is given by the hom of a category.

**Definition 4.8.7.** Given a functor $F : E \to D$ and a category $C$,

- the cell

$$
\begin{array}{ccc}
C & \xrightarrow{\text{precomposition with } F} & [D, C] \\
1 & \downarrow & \downarrow \\
C & \xrightarrow{\text{precomposition with } F} & [E, C]
\end{array}
$$

, “precomposition with $F$”, is defined by

$$(c) \langle F^p, C \rangle (K) = \prod_{F^p} c(C) K$$

for $c \in \| C \|$ and $K$ a functor $D \to C$; that is,

$$\langle F^p, C \rangle := \langle F^p, \{C\} \rangle$$

- the cell

$$
\begin{array}{ccc}
[D, C] & \xrightarrow{\text{precomposition with } F} & C \\
[F, C] & \downarrow & \downarrow \\
[E, C] & \xrightarrow{\text{precomposition with } F} & C
\end{array}
$$

, “precomposition with $F$”, is defined by

$$(K) \langle F^p, C \rangle (c) = \prod_{F^p} K(C) c$$

for $c \in \| C \|$ and $K$ a functor $D \to C$; that is,

$$\langle F^p, C \rangle := \langle F^p, \{C\} \rangle$$

**Remark 4.8.8.**

1. The cell sends each cone $\alpha : x \simeq K : D^e \to C$ to the cone $F \circ \alpha : x \simeq K \circ F : E^e \to C$, the usual composite of a functor and a cone.

2. By Proposition 4.6.21 and noting Proposition 4.3.30,
4. Frames

- The cell \( (F^\circ, C) \) is obtained from the hom of the precomposition functor \([F, C]\) by the pasting composition

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta D, C} & [D, C] \\
1 & \downarrow & \downarrow \\
C & \xrightarrow{\Delta E, C} & [E, C]
\end{array}
\]

- The cell \( (F^\triangleright, C) \) is obtained from the hom of the precomposition functor \([F, C]\) by the pasting composition

\[
\begin{array}{ccc}
[D, C] & \xrightarrow{\Delta D, C} & [D, C] \\
[F, C] & \downarrow & \downarrow \\
[E, C] & \xrightarrow{\Delta E, C} & [E, C]
\end{array}
\]

Example 4.8.9.

1. Let \( C \) and \( E \) be categories and \( I \) be a subcategory of \( E \).

- The precomposition with the inclusion \( 1_I : I \to E \) yields the cell

\[
\begin{array}{ccc}
C & \xrightarrow{(E^\circ, C)} & [E, C] \\
1 & \downarrow & \downarrow \\
C & \xrightarrow{(E, C)} & [I, C]
\end{array}
\]

, "restriction to \( \Gamma \)”, which sends each cone \( \alpha : c \leadsto K : E^\circ \to C \) to the cone \( I \circ \alpha : c \leadsto K \circ I : I^\circ \to C \), \( \alpha \) restricted to \( I \).

- The precomposition with the inclusion \( 1_I : I \to E \) yields the cell

\[
\begin{array}{ccc}
[E, C] & \xrightarrow{(E^\circ, C)} & [E, C] \\
[I, C] & \downarrow & \downarrow \\
[I, C] & \xrightarrow{(E, C)} & [C]
\end{array}
\]

, "restriction to \( \Gamma \)”, which sends each cone \( \alpha : K \leadsto c : E^\circ \to C \) to the cone \( I \circ \alpha : I \circ K \leadsto c : I^\circ \to C \), \( \alpha \) restricted to \( I \).

2. Let \( C \) and \( E \) be categories and \( e \) be an object of \( E \).

- The precomposition with the functor \( e : * \to E \) yields the cell

\[
\begin{array}{ccc}
C & \xrightarrow{(E^\circ, C)} & [E, C] \\
1 & \downarrow & \downarrow \\
C & \xrightarrow{(e^\circ, C)} & [e, C]
\end{array}
\]

, "evaluation at \( e \)”, which sends each cone \( \alpha : c \leadsto K : E^\circ \to C \) to the \( C \)-arrow \( \alpha_e : c \to K e \), the component of \( \alpha \) at \( e \).
4. Frames

- The precomposition with the functor $e : * \to E$ yields the cell

$$
\begin{array}{ccc}
\{ E, C \} & \xrightarrow{(E^\circ, C)} & C \\
\{ e, C \} & \xrightarrow{(e^\circ, C)} & 1 \\
C & \xrightarrow{(C)} & C
\end{array}
$$

, "evaluation at $e$", which sends each cone $\alpha : K \to c : E^\circ \to C$ to the $C$-arrow $\alpha_e : e^\circ K \to c$, the component of $\alpha$ at $e$.

Note. The following definition is a special case of Definition 4.7.5 where $M$ is given by the hom of a category.

**Definition 4.8.10.** Let $E$, $D$, and $C$ be categories.

- Given a functor $S : E \to C$ and a bifunctor $K : E \times D \to C$, a wedge $\alpha$ from $S$ to $K$, written

$$
\alpha : S \to K : E \times D^\circ \to C
$$

, is a natural transformation

$$
\begin{array}{ccc}
E & \xleftarrow{E \times \Delta_D} & E \times D \\
\downarrow S & \alpha & \downarrow K \\
C & \xrightarrow{(C)} & C
\end{array}
$$

right weighted by the projection $E \times \Delta_D$.

- Given a functor $T : E \to C$ and a bifunctor $K : E \times D \to C$, a wedge $\alpha$ from $K$ to $T$, written

$$
\alpha : K \to T : E \times D^\circ \to C
$$

, is a natural transformation

$$
\begin{array}{ccc}
E \times D & \xrightarrow{E \times \Delta_D} & E \\
\downarrow K & \alpha & \downarrow T \\
C & \xrightarrow{(C)} & C
\end{array}
$$

left weighted by the projection $E \times \Delta_D$.

**Remark 4.8.11.** A wedge in a category $C$ defined above is just a special instance of a wedge in Definition 4.7.5 where $M$ is the hom of $C$. Conversely, a wedge along a module $M$ is the same thing as a wedge in the collage category $[M]$ (cf. Remark 4.3.4(3)).

Note. The following definition is a special case of Definition 4.7.7 where $M$ is given by the hom of a category.

**Definition 4.8.12.** Let $E$, $D$, and $C$ be categories.

- The module of wedges $E \times D^\circ \to C$,

$$
\langle E \times D^\circ, C \rangle : [E, C] \to [E \times D, C]
$$

, is defined by

$$
\langle E \times D^\circ, C \rangle = \langle [E \times \Delta_D]^\circ, C \rangle
$$

; that is,

$$
\langle E \times D^\circ, C \rangle := \langle E \times D^\circ, (C) \rangle
$$
4. Frames

- The module of wedges \( E \times D^o \to C \),
  \[ \langle E \times D^o, C \rangle : [E \times D, C] \to [E, C] \]
  is defined by
  \[ \langle E \times D^o, C \rangle = \langle [E \times \Delta_D]^+, C \rangle \]
  ; that is,
  \[ \langle E \times D^o, C \rangle := \langle E \times D^o, \{C\} \rangle \]

**Remark 4.8.13.**

1. For a functor \( S : E \to C \) and a bifunctor \( K : E \times D \to C \), the set \( (S) \langle E \times D^o, C \rangle (K) \) consists of all wedges \( S \Rightarrow K : E \times D^o \Rightarrow C \).

- For a functor \( T : E \to C \) and a bifunctor \( K : E \times D \to C \), the set \( (K) \langle E \times D^o, C \rangle (T) \) consists of all wedges \( K \Rightarrow T : E \times D^o \Rightarrow C \).

2. By Remark 4.5.4(2),
   - The module \( \langle E \times D^o, C \rangle : [E, C] \to [E \times D, C] \) is given by the composition
     \[ [E, C] \xrightarrow{\langle E \times D^o, C \rangle} [E \times D, C] \xrightarrow{\langle E \times D^o, C \rangle} [E \times D, C] \]
     ; that is, \( \langle E \times D^o, C \rangle \) is the representable module of the diagonal functor \( [E \times \Delta_D, C] \).

- The module \( \langle E \times D^p, C \rangle : [E \times D, C] \to [E, C] \) is given by the composition
  \[ [E \times D, C] \xrightarrow{\langle E \times D^p, C \rangle} [E \times D, C] \xrightarrow{\langle E \times D, C \rangle} [E, C] \]
  ; that is, \( \langle E \times D^p, C \rangle \) is the corepresentable module of the diagonal functor \( [E \times \Delta_D, C] \).

**Note.** The following definition is a special case of Definition 4.7.9 where \( M \) is given by the hom of a category.

**Definition 4.8.14.**

- The right and left exponential transposes of a wedge
  \[
  \begin{array}{ccc}
  E & \xleftarrow{E \times \Delta_D} & E \times D \\
  \downarrow \alpha & & \downarrow \kappa \\
  C & \xrightarrow{(C)} & C
  \end{array}
  \]
  are the cone and the cylinder
  \[
  \begin{array}{ccc}
  * & \xleftarrow{\Delta_D} & D \\
  \downarrow \alpha^{-1} & & \downarrow \kappa^{-1} \\
  [E, C] & \xrightarrow{(E,C)} & [E, C]
  \end{array}
  \quad
  \begin{array}{ccc}
  E & \xleftarrow{E} & E \\
  \downarrow \alpha^{-1} & & \downarrow \kappa^{-1} \\
  C & \xrightarrow{(D^e,C)} & [D, C]
  \end{array}
  \]
  defined by
  \[ [\alpha^{-1}_d]_e = \alpha_{(e,d)} = [\alpha^{-1}_e]_d \]
  for \( e \in \|E\| \) and \( d \in \|D\| \).
4. Frames

- The right and left exponential transposes of a wedge

\[
\begin{array}{ccc}
E \times D & \xrightarrow{E \times \Delta D} & E \\
& \downarrow \alpha & \downarrow T \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

are the cone and the cylinder

\[
\begin{array}{ccc}
D & \xrightarrow{\Delta D} & \ast \\
& \downarrow \alpha' & \downarrow T \\
\langle E, C \rangle & \xrightarrow{\langle \alpha' \rangle} & \langle E, C \rangle
\end{array}
\quad
\begin{array}{ccc}
E & \xrightarrow{E} & E \\
& \downarrow \alpha' & \downarrow T \\
\langle D, C \rangle & \xrightarrow{\langle \alpha' \rangle} & \langle D, C \rangle
\end{array}
\]

defined by

\[
[\alpha_d]_e = \alpha_{(e,d)} = [\alpha_e]_d
\]

for \(e \in \|E\|\) and \(d \in \|D\|\).

**Remark 4.8.15.**

1. The right slice of a wedge \(\alpha : S \to K : E \times D^\leq \to M\) at \(d \in \|D\|\) is the natural transformation

\[
\begin{array}{ccc}
E & \xleftarrow{E} & E \\
& \downarrow [\alpha']_d & \downarrow \alpha'_{d} \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

given by the component at \(d\) of the right exponential transpose of \(\alpha\), and the left slice of \(\alpha\) at \(e \in \|E\|\) is the cone

\[
\begin{array}{ccc}
\ast & \xleftarrow{\Delta D} & D \\
& \downarrow [\alpha']_e & \downarrow \alpha'_{e} \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

given by the component at \(e\) of the left exponential transpose of \(\alpha\).

- The right slice of a wedge \(\alpha : K \to T : E \times D^\geq \to C\) at \(d \in \|D\|\) is the natural transformation

\[
\begin{array}{ccc}
E & \xrightarrow{E} & E \\
& \downarrow [\alpha']_d & \downarrow T \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

given by the component at \(d\) of the right exponential transpose of \(\alpha\), and the left slice of \(\alpha\) at \(e \in \|E\|\) is the cone

\[
\begin{array}{ccc}
D & \xrightarrow{\Delta D} & \ast \\
& \downarrow [\alpha']_e & \downarrow T \cdot e \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

given by the component at \(e\) of the left exponential transpose of \(\alpha\).
4. Frames

2. The right and left exponential transpositions of wedges \( E \times D^\circ \to C \) form the iso cells

\[
\begin{align*}
[E, C] & \xrightarrow{(E \times D^\circ, C)} [E \times D, C] \\
& \xrightarrow{1} \xrightarrow{-} [E \times D, C] \\
& \xrightarrow{1} \xrightarrow{-} [E \times D, C]
\end{align*}
\]


\[
\begin{align*}
[D, [E, C]] & \xrightarrow{(E, D, C)} [E, [D, C]] \\
& \xrightarrow{1} \xrightarrow{-} [E, [D, C]] \\
& \xrightarrow{1} \xrightarrow{-} [E, [D, C]]
\end{align*}
\]

, natural in \( E, D, \) and \( C \). In fact, these iso cells are obtained from the homs of the functors \( [E \times D, C] \xrightarrow{\sim} [D, [E, C]] \) and \( [E \times D, C] \xrightarrow{\sim} [E, [D, C]] \) by pasting a commutative diagram of diagonal functors (see Preliminaries(16)) as shown in

\[
\begin{align*}
[E, C] & \xrightarrow{E \times D, C} [E \times D, C] \\
& \xrightarrow{1} \xrightarrow{-} [E \times D, C] \\
& \xrightarrow{1} \xrightarrow{-} [E \times D, C]
\end{align*}
\]

3. The right and left exponential transpositions of wedges \( E \times D^\circ \to C \) form the iso cells

\[
\begin{align*}
[E \times D, C] & \xrightarrow{(E \times D^\circ, C)} [E, C] \\
& \xrightarrow{1} \xrightarrow{-} [E, C] \\
& \xrightarrow{1} \xrightarrow{-} [E, C]
\end{align*}
\]

\[
\begin{align*}
[D, [E, C]] & \xrightarrow{(E, D, C)} [E, [D, C]] \\
& \xrightarrow{1} \xrightarrow{-} [E, [D, C]] \\
& \xrightarrow{1} \xrightarrow{-} [E, [D, C]]
\end{align*}
\]

, natural in \( E, D, \) and \( C \). In fact, these iso cells are obtained from the homs of the functors \( [E \times D, C] \xrightarrow{\sim} [D, [E, C]] \) and \( [E \times D, C] \xrightarrow{\sim} [E, [D, C]] \) by pasting a commutative diagram of diagonal functors (see Preliminaries(16)) as shown in

\[
\begin{align*}
[E \times D, C] & \xrightarrow{(E \times D^\circ, C)} [E \times D, C] \\
& \xrightarrow{1} \xrightarrow{-} [E \times D, C] \\
& \xrightarrow{1} \xrightarrow{-} [E \times D, C]
\end{align*}
\]

\[
\begin{align*}
[D, [E, C]] & \xrightarrow{(E, D, C)} [E, [D, C]] \\
& \xrightarrow{1} \xrightarrow{-} [E, [D, C]] \\
& \xrightarrow{1} \xrightarrow{-} [E, [D, C]]
\end{align*}
\]
4. Frames

- The transpose of a cone

\[
\begin{array}{c}
\ast \leftarrow \Delta_D \\
\downarrow S \quad \downarrow \alpha \quad \downarrow \kappa \\
[E, C] - (E, C)^\ast = [E, C]
\end{array}
\]

is the cylinder

\[
\begin{array}{c}
E \rightarrow E \rightarrow E \\
\downarrow S \quad \downarrow \alpha^\ast \quad \downarrow \kappa^\ast \\
C - (D, C)^\ast = [D, C]
\end{array}
\]

defined by

\[
[a_e^\ast]_d = [a_d]_e
\]

for \( e \in \|E\| \) and \( d \in \|D\| \); conversely, the transpose of a cylinder

\[
\begin{array}{c}
E \rightarrow E \rightarrow E \\
\downarrow S \quad \downarrow \alpha \quad \downarrow \kappa \\
C - (D, C)^\ast = [D, C]
\end{array}
\]

is the cone

\[
\begin{array}{c}
\ast \leftarrow \Delta_D \\
\downarrow S \quad \downarrow \alpha^\ast \quad \downarrow \kappa^\ast \\
[E, C] - (E, C)^\ast = [E, C]
\end{array}
\]

defined by

\[
[a_d^\ast]_e = [a_e]_d
\]

for \( e \in \|E\| \) and \( d \in \|D\| \). These transpositions form the iso cell

\[
\begin{array}{c}
[E, C] - (\mathcal{D}^e, (E, C)) \rightarrow [D, [E, C]] \\
\downarrow \tau \quad \downarrow \tau \\
[E, C] - (E, (D^e, C)) \rightarrow [E, [D, C]]
\end{array}
\]

, natural in \( E, D \), and \( C \), making the diagram

\[
\begin{array}{c}
\langle E \times D^e, C \rangle \\
\downarrow \tau \\
\langle D^e, \langle E, C \rangle \rangle \\
\downarrow \tau \\
\langle E, \langle D^e, C \rangle \rangle
\end{array}
\]

commute.

- The transpose of a cone

\[
\begin{array}{c}
D \rightarrow \Delta_D \\
\downarrow \kappa \quad \downarrow \alpha \quad \downarrow \tau \\
[E, C] - (E, C)^\ast = [E, C]
\end{array}
\]

126
4. Frames

is the cylinder

\[
\begin{array}{c}
E \xrightarrow{K^}\ E \\
\downarrow \quad \downarrow \quad \downarrow \\
[D, C] \xrightarrow{(D^e, C)} \rightarrow C
\end{array}
\]

defined by

\[
[\alpha^\dagger]_d = [\alpha_d]_e
\]

for \( e \in \|E\| \) and \( d \in \|D\| \); conversely, the transpose of a cylinder

\[
\begin{array}{c}
E \xrightarrow{K^}\ E \\
\downarrow \quad \downarrow \quad \downarrow \\
[D, C] \xrightarrow{(D^e, C)} \rightarrow C
\end{array}
\]

is the cone

\[
\begin{array}{c}
D \xrightarrow{\Delta_D} \ast \\
\downarrow \quad \downarrow \quad \downarrow \\
[E, C] \xrightarrow{(E, C)} \rightarrow [E, C]
\end{array}
\]

defined by

\[
[\alpha^\dagger]_e = [\alpha]_d
\]

for \( e \in \|E\| \) and \( d \in \|D\| \). These transpositions form the iso cell

\[
\begin{array}{c}
[D, E, C] \xrightarrow{(D^e, (E, C))} \rightarrow [E, C] \\
\downarrow \quad \downarrow \quad \downarrow \\
[E, D, C] \xrightarrow{(E, (D^e, C))} \rightarrow [E, C]
\end{array}
\]

, natural in \( E, D, \) and \( C \), making the diagram

\[
\begin{array}{c}
(E \times D^e, C) \\
\downarrow \\
(D^e, (E, C)) \xrightarrow{\tau} (E, (D^e, C)) \rightarrow (E, (D^e, C))
\end{array}
\]

commute.

4. 

- The diagram

\[
\begin{array}{c}
(D^e, (E, C)) \xrightarrow{\tau} (E, (D^e, C)) \\
\downarrow \\
(D^e, (E, C)) \xrightarrow{\alpha} (E, (D^e, C)) \rightarrow (D^e, C)
\end{array}
\]
4. Frames

commutes for every \( e \in \|E\| \), where \( \langle e, C \rangle \) and \( \langle e, \langle D^p, C \rangle \rangle \) are evaluations at \( e \) (see Example 4.3.31); that is, for any cylinder

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & E \\
\downarrow S & & \downarrow K \\
C & \xrightarrow{\langle D^p, C \rangle} & [D, C]
\end{array}
\]

, its component at \( e \) is given by the composition

\[
\begin{array}{ccc}
\ast & \xleftarrow{\Delta D} & D \\
\downarrow S & & \downarrow K^t \\
[e, C] & \xrightarrow{\langle e, C \rangle} & [e, C] \\
\downarrow (e, C) & & \downarrow (e, C) \\
[C] & \xrightarrow{} & C
\end{array}
\]

- The diagram

\[
\begin{array}{ccc}
\langle D^p, \langle E, C \rangle \rangle & \xleftarrow{} & \langle E, \langle D^p, C \rangle \rangle \\
\downarrow \langle D^p, \langle e, C \rangle \rangle & & \downarrow \langle e, \langle D^p, C \rangle \rangle \\
\langle D^p, \langle e, C \rangle \rangle & \xleftarrow{} & \langle e, \langle D^p, C \rangle \rangle
\end{array}
\]

commutes for every \( e \in \|E\| \), where \( \langle e, C \rangle \) and \( \langle e, \langle D^p, C \rangle \rangle \) are evaluations at \( e \) (see Example 4.3.31); that is, for any cylinder

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & E \\
\downarrow K & & \downarrow T \\
[D, C] & \xrightarrow{\langle D^p, C \rangle} & C
\end{array}
\]

, its component at \( e \) is given by the composition

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\Delta D} & \ast \\
\downarrow K^t & & \downarrow T \\
[e, C] & \xrightarrow{\langle e, C \rangle} & [e, C] \\
\downarrow (e, C) & & \downarrow (e, C) \\
[C] & \xrightarrow{} & C
\end{array}
\]

4.9. Cones and wedges in \textit{Set}

\textit{Notation} 4.9.1.

1.
4. Frames

- A cone $\alpha$ from a set $S$ to a left module $M : \ast \to E$ (i.e. a functor $M : E \to \text{Set}$) is denoted by
  \[ \alpha : S \to M : \ast \to E^* \]
rather than $\alpha : S \to M : E^d \to \text{Set}$, and the component at $e \in |E|$ is written as $\langle \alpha \rangle e : S \to \langle M \rangle e$. The module of cones $E^d \to \text{Set}$ is denoted by
  \[ \langle E^* \rangle : \text{Set} \to [\ast : E] \]
rather than $(E^d, \text{Set}) : \text{Set} \to [E, \text{Set}]$.

- A cone $\alpha$ from a set $S$ to a right module $M : E \to \ast$ (i.e. a functor $M : E^- \to \text{Set}$) is denoted by
  \[ \alpha : S \to M : E^* \to \ast \]
rather than $\alpha : S \to M : [E^-]^d \to \text{Set}$, and the component at $e \in |E|$ is written as $e \langle \alpha \rangle : S \to e \langle M \rangle$. The module of cones $[E^-]^d \to \text{Set}$ is denoted by
  \[ \langle E^* \rangle : \text{Set} \to [E : E] \]
rather than $([E^-]^d, \text{Set}) : \text{Set} \to [E^-, \text{Set}]$.

2.

- A wedge $\alpha$ from a right module $L : X \to \ast$ (i.e. a functor $L : X^- \to \text{Set}$) to a module $M : X \to E$ (i.e. a functor $M : X^- \times E \to \text{Set}$) is denoted by
  \[ \alpha : L \to M : X \to E^* \]
rather than $\alpha : L \to M : X^- \times E^d \to \text{Set}$, and the component at $(x, e) \in |X \times E|$ is written as $x \langle \alpha \rangle e : x \langle L \rangle \to x \langle M \rangle e$. The module of wedges $X^- \times E^d \to \text{Set}$ is denoted by
  \[ \langle X : E^* \rangle : [X : E] \to [X : E] \]
rather than $(X^- \times E^d, \text{Set}) : [X^-, \text{Set}] \to [X^-, \text{Set}]$.

- A wedge $\alpha$ from a left module $L : \ast \to A$ (i.e. a functor $L : A \to \text{Set}$) to a module $M : E \to A$ (i.e. a functor $M : E^- \times A \to \text{Set}$) is denoted by
  \[ \alpha : L \to M : E^* \to A \]
rather than $\alpha : L \to M : [E^-]^d \times A \to \text{Set}$, and the component at $(e, a) \in |E \times A|$ is written as $e \langle \alpha \rangle a : \langle L \rangle a \to e \langle M \rangle a$. The module of wedges $[E^-]^d \times A \to \text{Set}$ is denoted by
  \[ \langle E^* : A \rangle : [A] \to [E : A] \]
rather than $([E^-]^d \times A, \text{Set}) : [A, \text{Set}] \to [E^- \times A, \text{Set}]$.

Remark 4.9.2.

1. By Remark 4.8.4(2),

- $\langle E^* \rangle : \text{Set} \to [\ast : E]$ is the representable module of the diagonal functor $[\ast : \Delta_E] : \text{Set} \to [\ast : E]$ (see Example 1.1.18(7)); that is, $\langle E^* \rangle$ is given by the composition
  \[ \text{Set} \xleftarrow{[\Delta_E]} [\ast : E] \xrightarrow{(E)} [E : E] \]
  , and a cone $\alpha : S \to M : \ast \to E^*$ is the same thing as a left module morphism $\alpha : \langle S \rangle \Delta_E \to M : \ast \to E$.  

129
4. Frames

- \((E^*: \text{Set}) : \text{Set} \to [E:]\) is the representable module of the diagonal functor \([\Delta_E:] : \text{Set} \to [E:]\) (see Example 1.1.18(7)); that is, \((E^*: \text{Set})\) is given by the composition

\[
\text{Set} \xrightarrow{[\Delta_E]} [E:] \xrightarrow{(E^*)} \text{Set}
\]

, and a cone \(\alpha : S \to M : E^* \to \ast\) is the same thing as a right module morphism \(\alpha : \Delta_E (S) \to M : E \to \ast\).

2. By Remark 4.8.13(2),

- \((X : E^*) : [X : E] \to [X : E]\) is the representable module of the diagonal functor \([X : \Delta_E] : [X :] \to [X : E]\) (see Example 1.1.18(7)); that is, \((X : E^*)\) is given by the composition

\[
[X :] \xrightarrow{[X : \Delta_E]} [X : E] \xrightarrow{(X^E)} [X : E]
\]

, and a wedge \(L \to M : X \to E^*\) is the same thing as a module morphism \(\alpha : (L) \Delta_E \to M : E \to E\).

- \((E^*: A) : [:A] \to [E : A]\) is the representable module of the diagonal functor \([\Delta_E : A] : [:A] \to [E : A]\) (see Example 1.1.18(7)); that is, \((E^*: A)\) is given by the composition

\[
[:A] \xrightarrow{[\Delta_E : A]} [E : A] \xrightarrow{(E^A)} [E : A]
\]

, and a wedge \(L \to M : E^* \to A\) is the same thing as a module morphism \(\alpha : \Delta_E (L) \to M : E \to A\).

Notation 4.9.3.

- The right and left exponential transpositions of wedges \(X^* \times E^d \to \text{Set}\) are denoted by

\[
[X :] \xrightarrow{\langle X^*,E^d \rangle} \xrightarrow{X^* \times E} [X : E]
\]

rather than

\[
[X^*, \text{Set}] \xrightarrow{\langle X^* \times E^d, \text{Set} \rangle} [X^* \times E, \text{Set}]
\]

- The right and left exponential transpositions of wedges \([E^*]^d \times A \to \text{Set}\) are denoted by

\[
[:A] \xrightarrow{\langle E^*,A \rangle} [E : A]
\]

rather than

\[
[:A] \xrightarrow{\langle E^*,A \rangle} [E : A]
\]
rather than

\[
\begin{array}{ccc}
A, \text{Set} & \to & [E^\sim \times A, \text{Set}] \\
\downarrow 1 & & \downarrow 1 \\
A, \text{Set} & \to & [A, [E^\sim, \text{Set}]]
\end{array}
\]

\[
\begin{array}{ccc}
A, \text{Set} & \to & [E^\sim \times A, \text{Set}] \\
\downarrow 1 & & \downarrow 1 \\
A, \text{Set} & \to & [E^\sim, [A, \text{Set}]]
\end{array}
\]

. The cell \( (E^*: A) \sim (\[E^\sim\], \langle \cdot\rangle) \) often appears in the opposite form

\[
\begin{array}{ccc}
[E: A]^\sim & \to & [\langle \cdot\rangle] \\
\downarrow \wedge & & \downarrow 1 \\
[E, \langle \cdot\rangle]^\sim & \to & [\langle \cdot\rangle]^\sim
\end{array}
\]

(note that \( (\[E^\sim\], \langle \cdot\rangle) \sim (E^*: \langle \cdot\rangle)^\sim \) by Remark 4.6.6(5)).

**Remark 4.9.4.** By Remark 4.8.15(2),

- the iso cell \( \langle X: E^* \rangle \to \langle E^*: \{X: \_\} \rangle \) is obtained from the hom of the functor \( \langle X: E \rangle \to [E, \{X: \_\}] \) by pasting a commutative diagram of diagonal functors (see Example 2.1.3) as shown in

\[
\begin{array}{ccc}
[X: \_] & \to & [X: E] \\
\downarrow 1 & & \downarrow \rho \\
[X: \_] & \to & [E, \{X: \_\}]
\end{array}
\]

. Likewise, the iso cell \( \langle X: E^* \rangle \to \langle X^\sim, \langle : E^* \rangle \rangle \) is obtained from the hom of the functor \( \langle X: E \rangle \to [X^\sim, \langle : E \rangle] \) by the pasting composition

\[
\begin{array}{ccc}
[X: \_] & \to & [X: E] \\
\downarrow 1 & & \downarrow \rho \\
[X: \_] & \to & [X^\sim, \langle : E \rangle]
\end{array}
\]

- the iso cell \( (E^*: A) \sim \langle A, (E^* : \_\) \rangle \) is obtained from the hom of the functor \( \langle E: A \rangle \to [A, \{E: \_\}] \) by pasting a commutative diagram of diagonal functors (see Example 2.1.3) as shown in

\[
\begin{array}{ccc}
[\cdot: A] & \to & [E: A] \\
\downarrow 1 & & \downarrow \rho \\
[\cdot: A] & \to & [A, \{E: \_\}]
\end{array}
\]
Frames

Likewise, the iso cell \( \langle E^o : A \rangle \to \langle E^p, \vdash A \rangle \) is obtained from the hom of the functor \([E : A] \to [E, \vdash A] \) by the pasting composition

\[
\begin{array}{ccc}
\left[ [E : A] \right] \to \left[ \Delta_{E^p A} \right] & \to & \left[ [E, \vdash A] \right] \\
\downarrow & & \downarrow \\
\left[ \left[ E, \vdash A \right] \right] \to \left[ [E, \vdash A] \right] & \to & \left[ [E, \vdash A] \right]
\end{array}
\]

\[
\begin{array}{ccc}
\left[ [E, \vdash A] \right] \to \left[ [E, \vdash A] \right] & \to & \left[ [E, \vdash A] \right] \\
\downarrow & & \downarrow \\
\left[ [E, \vdash A] \right] \to \left[ [E, \vdash A] \right] & \to & \left[ [E, \vdash A] \right]
\end{array}
\]

**Definition 4.9.5.** Let \( E \) be a small category.

- The “universal” cone of a left module \( M : * \to E \) is the cone
  \[
  E^\text{\small{\#}}_M : \prod_{E^p} M \leadsto M : * \to E^o
  \]
  defined by
  \[
  \alpha : \langle E^\text{\small{\#}}_M \rangle e = \alpha_e
  \]
  for \( e \in \| E \| \) and \( \alpha \) a frame of \( M \).

- The “universal” cone of a right module \( M : E \to * \) is the cone
  \[
  E^\text{\small{\#}}_M : \prod_{E^o} M \leadsto M : * \to E^p
  \]
  defined by
  \[
  \alpha : e \langle E^\text{\small{\#}}_M \rangle = \alpha_e
  \]
  for \( e \in \| E \| \) and \( \alpha \) a frame of \( M \).

**Remark 4.9.6.**

1. The smallness of \( E \) guarantees the smallness of \( \prod_{E^p} M \) (resp. \( \prod_{E^o} M \)).

2. The component
  \[
  \langle E^\text{\small{\#}}_M \rangle e : \prod_{E^p} M \to \langle M \rangle e
  \]
  of the cone \( E^\text{\small{\#}}_M \) at \( e \in \| E \| \) maps each frame \( \alpha \) of \( M \) to its component at \( e \).

3. We will see that the cone \( E^\text{\small{\#}}_M \) (resp. \( E^\text{\small{\#}}_M \)) is indeed universal in Theorem 7.4.1.

**Proposition 4.9.7.**

- The family of cones \( E^\text{\small{\#}}_M : \prod_{E^p} M \leadsto M : * \to E^o \), one for each left module \( M : * \to E \), defines the right cylinder
  \[
  \text{Set} \quad \prod_{E^p} E^\text{\small{\#}}_M \to \langle E^o \rangle \leadsto [E]
  \]
The family of cones $\mathbf{E}_\mathcal{M}^p : \prod_{\mathbf{E}^p} \mathcal{M} \sim \mathcal{M} : \mathbf{E}^p \to \ast$, one for each right module $\mathcal{M} : \mathbf{E} \to \ast$, defines the right cylinder

\[ \text{Set} \xrightarrow{\prod_{\mathbf{E}^p}} \mathbf{E}_\mathcal{M}^p \xrightarrow{\mathcal{M}} \mathcal{E} \]

Proof. For any left module morphism $\Phi : \mathcal{M} \to \mathcal{N}$, we need to verify that the quadrangle

\[
\begin{array}{ccc}
\prod_{\mathbf{E}^p} \mathcal{M} & \xrightarrow{\mathcal{E}^p_{\mathcal{M}} \cdot \mathbf{E}^p} & \mathcal{N} \\
\prod_{\mathbf{E}^p} \mathcal{M} & \xrightarrow{\Phi} & \prod_{\mathbf{E}^p} \mathcal{N} \\
\end{array}
\]

commutes, i.e. that

\[ \alpha : (\mathbf{E}_\mathcal{M}^p \cdot \Phi) e = \alpha : \left( \prod_{\mathbf{E}^p} \Phi \cdot \mathbf{E}_{\mathcal{N}}^p \right) e \]

for any frame $\alpha \in \prod_{\mathbf{E}^p} \mathcal{M}$ and any object $e \in [\mathcal{E}]$. But,

\[ \alpha : (\mathbf{E}_\mathcal{M}^p \cdot \Phi) e = \alpha : (\mathbf{E}_\mathcal{M}^p)^e : (\Phi) e = \alpha_e : (\Phi) e \]

and

\[ \alpha : \left( \prod_{\mathbf{E}^p} \Phi \cdot \mathbf{E}_{\mathcal{N}}^p \right) e = \alpha : \prod_{\mathbf{E}^p} \Phi : (\mathbf{E}_{\mathcal{N}}^p)^e = [\alpha \cdot (\Phi) ] : (\mathbf{E}_{\mathcal{N}}^p) e = [\alpha \cdot (\Phi) ] e = \alpha_e : (\Phi) e \]

Definition 4.9.8.

Given a pair of categories $\mathbf{E}$ and $\mathbf{X}$ with $\mathbf{E}$ small, the functor $\prod_{[\mathbf{X} : \mathbf{E}]} : [\mathbf{X} : \mathbf{E}] \to [\mathbf{X} : ]$ is defined by

\[ x \left( \prod_{[\mathbf{X} : \mathbf{E}]} M \right) = \prod_{\mathbf{E}^p} x (M) \]

for $x \in \mathbf{X}$ and $M \in [\mathbf{X} : \mathbf{E}]$, and the right cylinder

\[ [\mathbf{X} : ] \xrightarrow{\prod_{[\mathbf{X} : \mathbf{E}]}^p} [\mathbf{X} : \mathbf{E}] \]

is defined by

\[ x \left( [\mathbf{X} : \mathbf{E}]^p \right)_{\mathcal{M}} = \mathbf{E}_{x(\mathcal{M})}^p \]

for $\mathcal{M}$ a module $\mathbf{X} \to \mathbf{E}$ and $x \in [\mathbf{X}]$.

Given a pair of categories $\mathbf{E}$ and $\mathbf{A}$ with $\mathbf{E}$ small, the functor $\prod_{[\mathbf{E} : \mathbf{A}]} : [\mathbf{E} : \mathbf{A}] \to [\mathbf{A}]$ is defined by

\[ \left( \prod_{[\mathbf{E} : \mathbf{A}]} M \right) a = \prod_{\mathbf{E}^p} (M) a \]

for $a \in \mathbf{A}$ and $M \in [\mathbf{E} : \mathbf{A}]$, and the right cylinder

\[ [\mathbf{A}] \xrightarrow{\prod_{[\mathbf{E} : \mathbf{A}]}^p} [\mathbf{E} : \mathbf{A}] \]
is defined by
\[
\langle [E : A]_M \rangle a = E_{(M)a}^E
\]
for \( M \) a module \( E \rightarrow A \) and \( a \in \| A \| \).

Remark 4.9.9. By Remark 4.3.2(2), the right cylinder
\[
\begin{array}{ccc}
\Pi_{[E : A]} & \text{[E : A]} & \text{[E : A]} \\
\end{array}
\]
is the same thing as the left cylinder
\[
\begin{array}{ccc}
\Pi_{[E : A]} & \text{[E : A]} & \text{[E : A]} \\
\end{array}
\]

Proposition 4.9.10. Given a small category \( E \) and a module \( M : X \rightarrow A \),

- the triangle

\[
\begin{array}{ccc}
( E^a, M ) & [E, A] & M \rightarrow E \\
[ X : ] & \Pi_{[X : E]} & [ X : E ]
\end{array}
\]

commutes, where \( ( E^a, M ) \) is the right exponential transpose of the module \( ( E^a, M ) : X \rightarrow [E, A] \) and \( M \times E \) is the right action of \( M \) on \( [E, A] \).

- the triangle

\[
\begin{array}{ccc}
E \times M & [E, X] & \times ( E^a, M ) \\
[ E : A ] & \Pi_{[E : A]} & [ : A ]
\end{array}
\]

commutes, where \( \times ( E^a, M ) \) is the left exponential transpose of the module \( ( E^a, M ) : [E, X] \rightarrow A \) and \( E \times M \) is the left action of \( M \) on \( [E, X] \).

Proof. Indeed, for any \( K \in [E, A] \) and \( x \in X \),

\[
( x ) ( E^a, M ) ( K ) = \prod_{E^a} x ( M ) K = x \left( \prod_{[X : E]^a} ( M ) K \right)
\]

\[\square\]

Note. Proposition 4.9.10 allows the following definition.

Definition 4.9.11. Given a small category \( E \) and a module \( M : X \rightarrow A \),
4. Frames

- the cylinder

\[
\begin{array}{c}
\xymatrix{ & [E, A] \ar[rd]^{M \cdot E} \ar[ld]_{\langle E^s, M \rangle} & \\
[X :] \ar[r]_{(X:E^s)} & [X : E] \ar[r]_{M \cdot E} & [E, A]}
\end{array}
\]

is defined by the composition

\[
[X :] \xrightarrow{\Pi_{[X:E]^g}} [X : E] \xrightarrow{M \cdot E} [E, A]
\]

(see Definition 4.3.24), where \([X : E]^g\) is the right cylinder defined in Definition 4.9.8 and \(M \cdot E\) is the right action of \(M\) on \([E, A]\).

- the cylinder

\[
\begin{array}{c}
\xymatrix{ & [E, X] \ar[rd]^{E \cdot M} \ar[ld]_{\langle E^s, M \rangle} & \\
[E^s, M] \ar[r]_{\langle E^s, M \rangle} & [E : A] \ar[r]_{E \cdot M} & [E : A]^s}
\end{array}
\]

is defined by the composition

\[
[E, X] \xrightarrow{E \cdot M} [E : A]^s \xrightarrow{(E^s : A)^g} [E : A]^s \xrightarrow{\Pi_{[E : A]^g}} : [A]^s
\]

(see Definition 4.3.24), where \([E : A]^g\) is the left cylinder in Remark 4.9.9 and \(E \cdot M\) is the left action of \(M\) on \([E, X]\).

Remark 4.9.12. By Remark 4.3.25,

- the component of \([M \cdot E]^g\) at a functor \(K : E \to A\) is given by the component of \([X : E]^g\) at \((M)K\); that is, for each \(x \in \|X\|\),

\[
x([M \cdot E]^g) = x([X : E]^g)_{(M)K} = E^s_{x(M)K}
\]

- the component of \([E \cdot M]^g\) at a functor \(K : E \to X\) is given by the component of \([E : A]^g\) at \(K(M)\); that is, for each \(a \in \|A\|\),

\[
([E \cdot M]^g) a = ([E : A]^g)_{K(M)} a = E^g_{K(M)a}
\]
5. Yoneda Lemma

5.1. Yoneda modules

Definition 5.1.1.

- The right Yoneda module for $X$ is the module
  $$(X^r) : X \to [X:]$$
given by the evaluation
  $$(x, M) \mapsto x(M) : X^r \times [X:] \to \text{Set}$$
  ; that is,
  $$(x)(X^r)(M) := x(M)$$
  for $x \in X$ and $M \in [X:]$.

- The left Yoneda module for $A$ is the module
  $$(^rA) : [A]^r \to A$$
given by the evaluation
  $$(M, a) \mapsto (M)a : [A] \times A \to \text{Set}$$
  ; that is,
  $$(M)(^rA)(a) := (M)a$$
  for $a \in A$ and $M \in [:A]$.

Remark 5.1.2.

1. The module $X^r$ (resp. $^rA$) is called the Yoneda module just because it is represented (resp. corepresented) by the Yoneda functor $X^r$ (resp. $^rA$) (see Theorem 5.2.15).

2. 
   - For an object $x \in |X|$ and a right module $M : X \to \ast$, the set $(x)(X^r)(M)$ consists of all $M$-arrows $x \to \ast$.
   - For an object $a \in |A|$ and a left module $M : \ast \to A$, the set $(M)(^rA)(a)$ consists of all $M$-arrows $\ast \to a$.

Proposition 5.1.3.

- the right exponential transpose of the right Yoneda module $(X^r) : X \to [X:]$ yields the identity $[X:] \to [X:]$; that is,
  $$[X:] \xrightarrow{([X]^r)^*} [X:]$$
  ; the right slice of $X^r$ at a right module $M : X \to \ast$ is $M$ itself:
  $$(X^r)(M) = M$$
5. Yoneda Lemma

- the left exponential transpose of the left Yoneda module \( \langle \set{A} \rangle : [\set{A}] \rightarrow \set{A} \) yields the identity \([\set{A}] \rightarrow [\set{A}]\); that is,

\[
\set{A} \xrightarrow{[\set{A}] \rightarrow [\set{A}]} [\set{A}]
\]

; the left slice of \( \set{A} \) at a left module \( \set{M} : \set{A} \rightarrow \set{A} \) is \( \set{M} \) itself:

\[
(\set{M}) \langle \set{A} \rangle = \set{M}
\]

Proof. Immediate from the definition.

\[\square\]

Proposition 5.1.4. Given a module (resp. module morphism) \( \set{M} : \set{X} \rightarrow \set{A} \),

- the identity

\[
\set{M} = (\set{X}, \set{A}) \left[ \set{M}, \set{A} \right]
\]

holds; that is, \( \set{M} \) is recovered from its right exponential transpose by the composition

\[
\set{X} \xrightarrow{\set{X}, \set{A}} [\set{M}, \set{A}] \xrightarrow{M, \set{A}} \set{A}
\]

. Hence the right action of the right Yoneda module \( \set{X}, \set{A} \) on the functor category \([\set{A}, [\set{X}]])\) yields the inverse of the right exponential transposition of \([\set{X}: \set{A}]\); that is,

\[
[\set{X}: \set{A}] \xleftarrow{[\set{X}, \set{A}] = [\set{A}, [\set{X}]])}\]

(see Definition 2.1.1).

- the identity

\[
\set{M} = [\set{X}, \set{A}] (\set{A}, \set{A})
\]

holds; that is, \( \set{M} \) is recovered from its left exponential transpose by the composition

\[
\set{X} \xrightarrow{\set{X}, \set{A}} [\set{A}] \xrightarrow{\set{A}, \set{A}} \set{A}
\]

. Hence the left action of the left Yoneda module \( \set{X}, \set{A} \) on the functor category \([\set{X}, [\set{A}]])\) yields the inverse of the left exponential transposition of \([\set{X}: \set{A}]\); that is,

\[
[\set{X}: \set{A}] \xleftarrow{[\set{X}, \set{A}] = [\set{A}, [\set{X}]])}\]

(see Definition 2.1.1).

Proof. Since the exponential transposition is bijective, it suffices to show that

\[
[M, \set{A}] = [([X, \set{A}] [M, \set{A}]), \set{A}]
\]

. But, by Proposition 2.1.5 and Proposition 5.1.3,

\[
([X, \set{A}] [M, \set{A}]) \set{A} = [(X, \set{A}) \set{A}] \circ [M, \set{A}] = [M, \set{A}]
\]

\[\square\]
5. Yoneda Lemma

Remark 5.1.5. For a module $M : X \to A$, the identities in Proposition 5.1.4 are expressed by

\[
\begin{array}{ccc}
X & \xrightarrow{\sim} & M \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\sim} & 1
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\sim} & M \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\sim} & 1
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\sim} & A \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\sim} & 1
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\sim} & A \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\sim} & 1
\end{array}
\]

These fully faithful cells identify each $M$-arrow $m : x \sim a$ with the $(M)$-arrow $m : x \sim *$ and with the $x(M)$-arrow $m : * \sim a$ respectively.

Definition 5.1.6. Given categories $X$ and $A$,

- the right general Yoneda module for $[A, X]$,

\[
\langle X^* A \rangle : [A, X] \to [X : A]
\]

is defined by

\[
\langle G \rangle \langle X^* A \rangle (M) := \prod_A G(M)
\]

for $G \in [A, X]$ and $M \in [X : A]$.

- the left general Yoneda module for $[X, A]$,

\[
\langle X^* A \rangle : [X : A] \to [X, A]
\]

is defined by

\[
\langle M \rangle \langle X^* A \rangle (F) := \prod_X \langle M \rangle F
\]

for $M \in [X : A]$ and $F \in [X, A]$.

Remark 5.1.7.

1. The module $X^* A$ (resp. $X^* A$) is called the general Yoneda module just because it is represented (resp. corepresented) by the general Yoneda functor $X^* A$ (resp. $X^* A$) (see Theorem 5.3.15).

2.

- For a functor $G : A \to X$ and a module $M : X \to A$, the set $(G) \langle X^* A \rangle (M)$ consists of all right cylinders $G \sim M$.

- For a functor $F : X \to A$ and a module $M : X \to A$, the set $(M) \langle X^* A \rangle (F)$ consists of all left cylinders $M \sim F$.

3. The identification in Remark 4.3.2(3) yields canonical isomorphisms

\[
\langle X^* \rangle \cong \langle X^* \rangle
\]

and

\[
\langle *^* \rangle \cong \langle *^* \rangle
\]

. Yoneda modules are thus special instances of general Yoneda modules.
Theorem 5.1.8. Given a category $E$ and a module $M : X \to A$, the identities
\[
\begin{align*}
[E, X] & \to [E, [E, A]] \\
\downarrow & \downarrow \\
[X, E] & \to [E, A]
\end{align*}
\]
hold.

Proof. Immediate from Definition 4.3.6 and Definition 5.1.6; indeed, for any $S \in [E, X]$ and any $T \in [E, A]$, 
\[
S (E, M) T = \prod_{E} S (M) T
\]
\[
= \prod_{E} S ([M] T)
\]
\[
= S ([X, E]) ([M] T)
\]
\[
= S (X, E) ([M, E] : T)
\]
\[
= S ([X, E] [M, E]) T
\]
hold. \qed

Remark 5.1.9.
1. This is the identity stated in Remark 4.3.4(2); the fully faithful cells in Theorem 5.1.8 identify each two-sided cylinder 
\[
\begin{array}{c}
S \quad E \\
\alpha \quad T \\
X \quad M \quad A
\end{array}
\]
with the right cylinder 
\[
\begin{array}{c}
X \\
\alpha \quad T \\
E
\end{array}
\]
and with the left cylinder 
\[
\begin{array}{c}
E \\
\alpha \quad T \\
A
\end{array}
\]
respectively.

2. If we replace $E$ with the terminal category, then we obtain the cells in Remark 5.1.5.

Corollary 5.1.10. Given categories $X$ and $A$, the identities
\[
\begin{align*}
[A, X] & \to [X, A] \\
\downarrow & \downarrow \\
[A, X] & \to [A, [X, A]]
\end{align*}
\]
hold.
5. Yoneda Lemma

Proof. Replacing $\mathcal{M}$ in Theorem 5.1.8 with the right and left Yoneda modules, we have

\[
\begin{array}{ccc}
[A, X] \xrightarrow{(A, X^*)} [A, [X :]] & \xrightarrow{[X, [\cdot, A]^\top]} [X, A] \\
\downarrow 1 & \downarrow 1 & \downarrow 1 \\
[A, X] \xrightarrow{\otimes} [X : A] & [X : A]^\top \xrightarrow{\otimes} [X, A]
\end{array}
\]

Since $[(X^*) \otimes A] = \vee$ and $[X \otimes (\cdot, A)] = \wedge$ by Proposition 5.1.4, the assertion results by taking the inverses.

Remark 5.1.11.

- The iso cell $(X^* A) \rightarrow (A, (X^*) A)$ identifies each right cylinder

\[
\begin{array}{ccc}
\xrightarrow{G}
\end{array}
\]

with the two-sided cylinder

\[
\begin{array}{ccc}
A & \xleftarrow{\alpha} & M^* \\
\downarrow \alpha & \downarrow \xrightarrow{M^*} & \downarrow \alpha \\
X & \xleftarrow{\otimes} & [X :]
\end{array}
\]

- The iso cell $(X \otimes A) \rightarrow (X, (\cdot, A))$ identifies each left cylinder

\[
\begin{array}{ccc}
\xleftarrow{M} \xrightarrow{F}
\end{array}
\]

with the two-sided cylinder

\[
\begin{array}{ccc}
\xleftarrow{F}
\end{array}
\]

5.2. Yoneda morphisms

Definition 5.2.1. Let $\mathcal{M} : X \rightarrow A$ be a module.

1. The right hom of $\mathcal{M}$ is the module

\[
(X|\mathcal{M}) : X \rightarrow [\mathcal{M}]
\]

, from $X$ to the collage category of $\mathcal{M}$, given by the composition

\[
\begin{array}{ccc}
X & \xrightarrow{M_X} & [\mathcal{M}] & \xrightarrow{([\mathcal{M}])} & [\mathcal{M}]
\end{array}
\]
5. Yoneda Lemma

; that is, \( (X|M) \) is the representable module of the inclusion \( M_X : X \to [M] \); in short,
\[
(X|M) := M_X ([M])
\]

- The left hom of \( M \) is the module
\[
(M|A) : [M] \to A
\]
, from the collage category of \( M \) to \( A \), given by the composition
\[
[M] \xrightarrow{([M])} [M] \xrightarrow{M_A} A
\]
; that is, \( (M|A) \) is the corepresentable module of the inclusion \( M_A : A \to [M] \); in short,
\[
(M|A) := ([M])M_A
\]

Remark 5.2.2.
- The composition
\[
X \xrightarrow{(X|M)} [M] \xrightarrow{M_X} X
\]
, i.e.
\[
X \xrightarrow{M_X} [M] \xrightarrow{([M])} [M] \xrightarrow{M_X} X
\]
, yields \( (X) \), the hom of \( X \), and the composition
\[
X \xrightarrow{(X|M)} [M] \xrightarrow{M_A} A
\]
, i.e.
\[
X \xrightarrow{M_X} [M] \xrightarrow{([M])} [M] \xrightarrow{M_A} A
\]
, yields \( M \) (see Remark 3.1.16(2)).
- The composition
\[
A \xrightarrow{M_A} [M] \xrightarrow{(M|A)} A
\]
, i.e.
\[
A \xrightarrow{M_A} [M] \xrightarrow{([M])} [M] \xrightarrow{M_A} A
\]
, yields \( (A) \), the hom of \( A \), and the composition
\[
X \xrightarrow{M_X} [M] \xrightarrow{(M|A)} A
\]
, i.e.
\[
X \xrightarrow{M_X} [M] \xrightarrow{([M])} [M] \xrightarrow{M_A} A
\]
, yields \( M \) (see Remark 3.1.16(2)).

Definition 5.2.3. Let \( M : X \to A \) be a module.
5. Yoneda Lemma

- The right exponential transpose

\[ \{(X|M) \twoheadrightarrow \}: [M] \to [X:] \]

of the right hom of \( M \) is called the right Yoneda functor for \( M \).

- The left exponential transpose

\[ \land (M|A) [M] \to [A:]^\sim \]

of the left hom of \( M \) is called the left Yoneda functor for \( M \).

**Remark 5.2.4.**

- By Remark 5.2.2, the right slices of \( (X|M) \) at \( s \in \|X\| \) and \( t \in \|A\| \) are given by

\[ (X|M) s = (X|M) (M_X \downarrow s) = (X|M)(M_X) s = (X)s \]

and

\[ (X|M) t = (X|M) (M_A \downarrow t) = (X|M)(M_A) t = (X)t \]

; hence the right Yoneda functor \( (X|M) \twoheadrightarrow \) sends each \( M \)-arrow \( m: s \to t \) to the right module morphism

\[ (X|M) m : (X) s \to (M) t : X \to \star \]

which maps each \( X \)-arrow \( h: x \to s \) to the \( M \)-arrow \( h \circ m : x \to t \) as indicated in

\[
\begin{array}{c}
\xymatrix{
X \\
\ar_{h}^{(X|M)m}
\quad s \ar_{m}^{t} \\
}
\end{array}
\]

(cf. Remark 2.1.2(2)). When \( M \) is understood, \( (X|M) \) is also written as

\[ X|m : (X)s \to (M)t : X \to \star \]

and called the right module morphism generated by \( X \) direct along \( m \).

- By Remark 5.2.2, the left slices of \( (M|A) \) at \( s \in \|X\| \) and \( t \in \|A\| \) are given by

\[ s(M|A) = (s : M_X)(M|A) = s(M_X(M|A)) = s(M) \]

and

\[ t(M|A) = (t : M_A)(M|A) = t(M_A(M|A)) = t(A) \]

; hence the left Yoneda functor \( \land (M|A) \) sends each \( M \)-arrow \( m: s \to t \) to the left module morphism

\[ m(M|A) : t(A) \to s(M) : \star \to \star A \]

which maps each \( A \)-arrow \( h: t \to a \) to the \( M \)-arrow \( m \circ h : s \to a \) as indicated in

\[
\begin{array}{c}
\xymatrix{
X \\
\ar_{h}^{m(M|A)m}
\quad t \ar_{h}^{a} \\
}
\end{array}
\]

(cf. Remark 2.1.2(2)). When \( M \) is understood, \( m(M|A) \) is also written as

\[ m\uparrow A : t(A) \to s(M) : \star \to \star A \]

and called the left module morphism generated by \( A \) inverse along \( m \).
5. Yoneda Lemma

**Definition 5.2.5.** Let \( \mathcal{M} : X \to A \) be a module.

- The right Yoneda morphism for \( \mathcal{M} \) is the cell

\[
\begin{array}{ccc}
X & \rightarrow & A \\
\downarrow_{X} & \Downarrow & \downarrow_{\mathcal{M}} \\
[X :] & \rightarrow & [X :]
\end{array}
\]

sending each \( \mathcal{M} \)-arrow \( m : s \to t \) to the right module morphism

\[
X \uparrow m = (X|\mathcal{M}) m : (X) s \to (\mathcal{M}) t : X \to *
\]

- The left Yoneda morphism for \( \mathcal{M} \) is the cell

\[
\begin{array}{ccc}
\downarrow_{\mathcal{M}} & \Downarrow & \downarrow_{\mathcal{A}} \\
\left(\mathcal{M}\right)^{-} & \left(\mathcal{A}\right)^{-} & \left(\mathcal{A}\right)^{-}
\end{array}
\]

sending each \( \mathcal{M} \)-arrow \( m : s \to t \) to the left module morphism

\[
m\downarrow A = m (\mathcal{M}|A) : t (A) \to s (\mathcal{M}) : * \to A
\]

**Remark 5.2.6.**

1. The right Yoneda morphism for \( \mathcal{M} \) is formally given by the cell adjunct (see Theorem 3.1.17) to the right Yoneda functor for \( \mathcal{M} \), i.e. by the composition

\[
\begin{array}{ccc}
X & \rightarrow & A \\
\downarrow_{\mathcal{M}} & \Downarrow & \downarrow_{\mathcal{M}_A} \\
[M] & \rightarrow & [M] \\
(X|\mathcal{M}) & \rightarrow & (X|\mathcal{M}) \\
\left(\mathcal{M}\right)^{-} & \left(\mathcal{A}\right)^{-} & \left(\mathcal{A}\right)^{-}
\end{array}
\]

of the unit cell \( \mathcal{I}_M \) and the hom of the right Yoneda functor for \( \mathcal{M} \).

- The left Yoneda morphism for \( \mathcal{M} \) is formally given by the cell adjunct (see Theorem 3.1.17) to the left Yoneda functor for \( \mathcal{M} \), i.e. by the composition

\[
\begin{array}{ccc}
X & \rightarrow & A \\
\downarrow_{\mathcal{M}} & \Downarrow & \downarrow_{\mathcal{M}_A} \\
[M] & \rightarrow & [M] \\
\left(\mathcal{M}\right)^{-} & \left(\mathcal{A}\right)^{-} & \left(\mathcal{A}\right)^{-}
\end{array}
\]

of the unit cell \( \mathcal{I}_M \) and the hom of the left Yoneda functor for \( \mathcal{M} \).
The Yoneda morphism

\[
\begin{array}{c}
\mathbf{X} \\
\downarrow^X
\end{array}
\xrightarrow{\mathbf{X} \rightarrow \mathbf{M}}
\begin{array}{c}
\mathbf{M} \\
\downarrow^M
\end{array}
\xrightarrow{\mathbf{M} \rightarrow \mathbf{A}}
\begin{array}{c}
\mathbf{A} \\
\downarrow^A
\end{array}
\]

for a right module \( \mathbf{M} : \mathbf{X} \rightarrow \mathbf{A} \) is defined as a special case of Definition 5.2.5 where \( \mathbf{A} \) is the terminal category under the identification \([\mathbf{X}] \cong [\mathbf{X} : \mathbf{A}] \). This conical cell sends each \( \mathbf{M} \)-arrow \( m : r \rightarrow s \) to the right module morphism

\[
\mathbf{X} \vdash m : (\mathbf{X}) r \rightarrow \mathbf{M} : \mathbf{X} \rightarrow \mathbf{A}
\]

which maps each \( \mathbf{X} \)-arrow \( h : x \rightarrow r \) to the \( \mathbf{M} \)-arrow \( h \circ m : x \rightarrow * \) as indicated in

\[
\begin{array}{c}
x \\
\downarrow^h
\end{array}
\xrightarrow{h : (\mathbf{X}) m}
\begin{array}{c}
h \circ m \\
\downarrow^r
\end{array}
\xrightarrow{m \circ h : \mathbf{X} \rightarrow *}
\]

Conversely, given an arrow \( m : s \rightarrow t \) of a two-sided module \( \mathbf{M} : \mathbf{X} \rightarrow \mathbf{A} \), the right module morphism

\[
\mathbf{X} \vdash m : (\mathbf{X}) s \rightarrow (\mathbf{M}) t : \mathbf{X} \rightarrow \mathbf{A}
\]

coincides with that generated by \( \mathbf{X} \) direct along the arrow \( m : s \rightarrow * \) of the right module \( (\mathbf{M}) t : \mathbf{X} \rightarrow \mathbf{A} \).

The Yoneda morphism

\[
\begin{array}{c}
* \\
\downarrow^\mathbf{M}
\end{array}
\xrightarrow{\mathbf{M} \rightarrow \mathbf{A}}
\begin{array}{c}
\mathbf{A} \\
\downarrow^A
\end{array}
\xrightarrow{\mathbf{A} \rightarrow \mathbf{M}}
\begin{array}{c}
[\mathbf{A}]^{-} \\
\downarrow^{\mathbf{A}^{-}}
\end{array}
\xrightarrow{[\mathbf{A}]^{-} \rightarrow [\mathbf{A}]^{-}}
\begin{array}{c}
[*]^{-} \\
\downarrow^{[*]}^{-}
\end{array}
\xrightarrow{[*]^{-} \rightarrow [*]^{-}}
\begin{array}{c}
[\mathbf{A}]^{-} \\
\downarrow^A
\end{array}
\xrightarrow{[\mathbf{A}]^{-} \rightarrow [\mathbf{A}]^{-}}
\begin{array}{c}
[\mathbf{A}]^{-} \\
\downarrow^\mathbf{M}
\end{array}
\xrightarrow{\mathbf{M} \rightarrow \mathbf{A}}
\begin{array}{c}
* \\
\downarrow^\mathbf{M}
\end{array}
\xrightarrow{\mathbf{M} \rightarrow \mathbf{A}}
\]

for a left module \( \mathbf{M} : \mathbf{X} \rightarrow \mathbf{A} \) is defined as a special case of Definition 5.2.5 where \( \mathbf{X} \) is the terminal category under the identification \([\mathbf{X}] \cong [\mathbf{X} : \mathbf{A}] \). This conical cell sends each \( \mathbf{M} \)-arrow \( m : * \rightarrow r \) to the left module morphism

\[
m \upharpoonright \mathbf{A} : r (\mathbf{A}) \rightarrow \mathbf{M} : * \rightarrow \mathbf{A}
\]

which maps each \( \mathbf{A} \)-arrow \( h : r \rightarrow a \) to the \( \mathbf{M} \)-arrow \( m \circ h : * \rightarrow a \) as indicated in

\[
\begin{array}{c}
* \\
\downarrow^m
\end{array}
\xrightarrow{m \circ h : \mathbf{X} \rightarrow *}
\begin{array}{c}
r \\
\downarrow^h
\end{array}
\xrightarrow{h : (\mathbf{M}) t}
\begin{array}{c}
a \\
\downarrow^a
\end{array}
\xrightarrow{m \circ h : [\mathbf{A}]^{-} \rightarrow [\mathbf{A}]^{-}}
\]

Conversely, given an arrow \( m : s \rightarrow t \) of a two-sided module \( \mathbf{M} : \mathbf{X} \rightarrow \mathbf{A} \), the left module morphism

\[
m \upharpoonright \mathbf{A} : t (\mathbf{A}) \rightarrow s (\mathbf{M}) : * \rightarrow \mathbf{A}
\]

coincides with that generated by \( \mathbf{A} \) inverse along the arrow \( m : * \rightarrow t \) of the left module \( s (\mathbf{M}) : * \rightarrow \mathbf{A} \).
5. Yoneda Lemma

**Proposition 5.2.7.** Let \( \mathcal{M} : \mathbf{X} \rightarrow \mathbf{A} \) be a module.

- **The right slice** (see Definition 2.1.7)

\[
\begin{array}{ccc}
\mathbf{X} & \rightarrow & \mathbf{X}^t \mathcal{M} \\
\downarrow & & \downarrow \mathcal{M} \\
\mathbf{X} : \rightarrow & \mathbf{X}^t \mathcal{M} \\
\end{array}
\]

at \( t \) of the right Yoneda morphism for \( \mathcal{M} \) is given by the Yoneda morphism

\[
\begin{array}{ccc}
\mathbf{X} & \rightarrow & \mathbf{X}^t \mathcal{M} \\
\downarrow & & \downarrow \mathcal{M} \\
\mathbf{X}^t \mathcal{M} & \rightarrow & \mathbf{X} \\
\end{array}
\]

for the right module \( \mathcal{M} \mathbf{t} \), the right slice of \( \mathcal{M} \) at \( t \).

- **The left slice** (see Definition 2.1.7)

\[
\begin{array}{ccc}
\mathbf{X} & \rightarrow & \mathbf{X} \mathcal{M}^s \\
\downarrow & & \downarrow \mathcal{M} \\
\mathbf{X} : \rightarrow & \mathbf{X} \mathcal{M}^s \\
\end{array}
\]

at \( s \) of the left Yoneda morphism for \( \mathcal{M} \) is given by the Yoneda morphism

\[
\begin{array}{ccc}
\mathbf{X} & \rightarrow & \mathbf{X} \mathcal{M}^s \\
\downarrow & & \downarrow \mathcal{M} \\
\mathbf{X} \mathcal{M}^s & \rightarrow & \mathbf{X} \\
\end{array}
\]

for the left module \( \mathbf{s} \mathcal{M} \), the left slice of \( \mathcal{M} \) at \( s \).

**Proof.** Immediate from Remark 5.2.6(2). \( \square \)

**Proposition 5.2.8.**

- **Given a category \( \mathbf{X} \), the right Yoneda morphism**

\[
\begin{array}{ccc}
\mathbf{X} & \rightarrow & \mathbf{X} \mathcal{X} \\
\downarrow & & \downarrow \mathcal{X} \\
\mathbf{X} & \rightarrow & \mathbf{X} \mathcal{X} \\
\end{array}
\]

for the hom of \( \mathbf{X} \) is the same thing as the hom

\[
\begin{array}{ccc}
\mathbf{X} & \rightarrow & \mathbf{X} \mathcal{X} \\
\downarrow & & \downarrow \mathcal{X} \\
\mathbf{X} & \rightarrow & \mathbf{X} \mathcal{X} \\
\end{array}
\]

145
of the right Yoneda functor for $X$; that is, for any $X$-arrow $f : s \to t$, the right module morphism

$$X \vdash f = (X \vdash (X)) m : (X) s \to (X) t : X \to *$$

coincides with

$$(X) f : (X) s \to (X) t : X \to *$$

Given a category $A$, the left Yoneda morphism

$$A \vdash A \vdash A$$

for the hom of $A$ is the same thing as the hom

$$A \vdash A \vdash A$$

of the left Yoneda functor for $A$; that is, for any $A$-arrow $f : s \to t$, the left module morphism

$$f \vdash A = m \langle (A) | A \rangle : t \langle A \rangle \to s \langle A \rangle : * \to A$$

coincides with

$$f \langle A \rangle : t \langle A \rangle \to s \langle A \rangle : * \to A$$

Proof. Both $X \vdash f$ and $(X) f$ map each $X$-arrow $h : x \to s$ to the $X$-arrow $h \circ f : x \to t$ (see Remark 2.3.2).

Example 5.2.9.

1. Let $K$ be a functor and $M$ be a module as in

$$E \xrightarrow{K} X \xrightarrow{M} A$$

do. The right Yoneda morphism for the composite module $K \langle M \rangle : E \to A$ sends each $K \langle M \rangle$-arrow $m : s \sim t$ to the right module morphism

$$E \vdash m : (E) s \to K \langle M \rangle t : E \to *$$

which maps each $E$-arrow $h : e \to s$ to the $K \langle M \rangle$-arrow $h \circ m : e \sim t$ as indicated in

$$
\begin{array}{ccc}
  e & \xrightarrow{h \circ m} & t \\
  h & \downarrow & \\
  s & \xrightarrow{m} & t
\end{array}
$$
5. Yoneda Lemma

This commutative diagram shows that \( E|m \) is given by the composition

\[
\begin{array}{c}
\langle E \rangle s \\
\downarrow \langle K \rangle s \\
\langle K \langle X \rangle K \rangle s
\end{array}
\quad
\begin{array}{c}
\xrightarrow{E|m} \\
\xrightarrow{K(X|m)} \\
\xrightarrow{K\langle X \rangle (K \cdot s)}
\end{array}
\quad
\begin{array}{c}
K \langle M \rangle t \\
K \langle X \rangle (K \cdot s)
\end{array}
\]

Let \( K \) be a functor and \( M \) be a module as in

\[
X \xrightarrow{M} A \xleftarrow{K} E
\]

The left Yoneda morphism for the composite module \( \langle M \rangle K : X \to E \) sends each \( \langle M \rangle K \)-arrow \( m : s \to t \) to the left module morphism

\[
m|E : t \langle E \rangle \to s \langle M \rangle K : * \to E
\]

which maps each \( E \)-arrow \( h : t \to e \) to the \( \langle M \rangle K \)-arrow \( m \circ h : s \to e \) as indicated in

\[
\begin{array}{c}
s \xrightarrow{m} K \cdot t \\
\downarrow \langle m|E \rangle \circ h \\
K \cdot e
\end{array}
\quad
\begin{array}{c}
t \\
\downarrow h \\
e
\end{array}
\quad
\begin{array}{c}
s \xrightarrow{m} K \cdot t \\
\downarrow \langle m|E \rangle \circ h \\
K \cdot e
\end{array}
\quad
\begin{array}{c}
t \\
\downarrow h \\
e
\end{array}
\]

This commutative diagram shows that \( m|E \) is given by the composition

\[
\begin{array}{c}
t \langle E \rangle \\
\downarrow t \langle K \rangle \\
t \langle K \langle A \rangle K \rangle
\end{array}
\quad
\begin{array}{c}
\xrightarrow{m|E} \\
\xrightarrow{(m|A)K} \\
\xrightarrow{(t|K) \langle A \rangle K}
\end{array}
\quad
\begin{array}{c}
s \langle M \rangle K \\
\downarrow \langle m|A \rangle K \\
(t \langle K \rangle) \langle A \rangle K
\end{array}
\]

2. As a special case of (1) above,

- consider a representable module \( F \langle D \rangle : E \to D \) given by

\[
E \xrightarrow{F} D \xleftarrow{(D)} D
\]

The right Yoneda morphism for \( F \langle D \rangle \) sends each \( F \langle D \rangle \)-arrow \( f : s \to t \) to the right module morphism

\[
E|f : \langle E \rangle s \to F \langle D \rangle t : E \to *
\]

which maps each \( E \)-arrow \( h : e \to s \) to the \( F \langle D \rangle \)-arrow \( h \circ f : e \to t \) as indicated in

\[
\begin{array}{c}
e \xrightarrow{h} e \xrightarrow{h \circ f} t \\
\downarrow f \\
s \xrightarrow{h \circ f} t
\end{array}
\]

This commutative diagram shows that \( E|f \) is given by the composition

\[
\begin{array}{c}
\langle E \rangle s \\
\downarrow \langle F \rangle s \\
\langle F \langle D \rangle F \rangle s
\end{array}
\quad
\begin{array}{c}
\xrightarrow{E|f} \\
\xrightarrow{F \langle D \rangle f} \\
\xrightarrow{F \langle D \rangle (F \cdot s)}
\end{array}
\quad
\begin{array}{c}
F \langle D \rangle t \\
F \langle D \rangle (F \cdot s)
\end{array}
\]
consider a corepresentable module \((D)\) \(F: D \to E\) given by
\[
D \xrightarrow{(D)} \to D \xleftarrow{F} E
\]

The left Yoneda morphism for \((D)\) \(F\) sends each \((D)\) \(F\)-arrow \(f: s \to t\) to the left module morphism
\[
f|E: t(E) \to s(D)F : s \to t
\]
which maps each \(E\)-arrow \(h: t \to e\) to the \((D)\) \(F\)-arrow \(f \circ h: s \to e\) as indicated in

\[
\begin{array}{ccc}
s & \xrightarrow{f} & t \\
\downarrow{(f|E)} & & \downarrow{f}\Phi \\
E & \xrightarrow{h} & E \\
\end{array}
\]

This commutative diagram shows that \(f|E\) is given by the composition
\[
\begin{array}{ccc}
t(E) & \xrightarrow{f|E} & s(D)F \\
\downarrow{t(F)} & & \downarrow{t(F)} \\
t(F(D)F) & \xrightarrow{(t:F)(D)F} & (t:F)(D)F
\end{array}
\]

**Theorem 5.2.10.** (Yoneda Lemma : Part one).

- Given a right module \(M: X \to *\) and an object \(r \in \|X\|\), the assignment \(m \mapsto X|m\) yields a bijection
  \[
  r(M) \cong ((X)r)(X)(M)
  \]
  from the set of \(M\)-arrows \(r \to *\) to the set of right module morphisms \((X)r \to M: X \to *\), whose inverse sends each right module morphism \(\Phi: (X)r \to M\) to the \(M\)-arrow \(1r: \Phi: r \to *\), the image of the identity \(r \to r\) under the function \(r(\Phi): r(X)r \to r(M)\).

- Given a left module \(M: * \to A\) and an object \(r \in \|A\|\), the assignment \(m \mapsto m|A\) yields a bijection
  \[
  (M)r \cong (r(A))(A)(M)
  \]
  from the set of \(M\)-arrows \(* \to r\) to the set of left module morphisms \(r(A) \to M: * \to A\), whose inverse sends each left module morphism \(\Phi: r(A) \to M\) to the \(M\)-arrow \(\Phi 1r: * \to r\), the image of the identity \(r \to r\) under the function \((\Phi)r: r(A)r \to (M)r\).

**Proof.** Let \(m: r \to *\) be an \(M\)-arrow and \(\Phi: (X)r \to M\) be a right module morphism. We need to show that \(m = 1r: (X|m)\) and \(\Phi = X|(1r: \Phi)\). Replacing \(h: x \to r\) with \(1r: r \to r\) in the triangle in Remark 5.2.6(2), we have

\[
\begin{array}{ccc}
r & \xrightarrow{1r} & (X|m) \\
\downarrow{1r} & & \downarrow{1r} \\
r & \xrightarrow{m} & *
\end{array}
\]

, i.e.
\[
m = 1r: (X|m)
\]
5. Yoneda Lemma

For any object \( x \in \|X\| \) and any arrow \( h : x \to r \), the commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{h} & r \\
\downarrow & & \downarrow \\
r & \xrightarrow{1_r} & r
\end{array}
\]

yields the commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{h \cdot \Phi} & r \\
\downarrow & & \downarrow \\
r & \xrightarrow{1_r \cdot \Phi} & r
\end{array}
\]

by the naturality of \( \Phi \). Comparing this triangle with that in Remark 5.2.6(2), we have

\[ \Phi = X| (1_r \cdot \Phi) \]

\[ \square \]

**Theorem 5.2.11.** Let \( \mathcal{M} : X \to A \) be a module.

- The right Yoneda morphism

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & A \\
\downarrow & & \downarrow \\
[X:] & \xrightarrow{\mathcal{M}} & [X:]
\end{array}
\]

for \( \mathcal{M} \) is fully faithful. Specifically, for each pair of objects \( s \in \|X\| \) and \( t \in \|A\| \), the assignment \( m \mapsto X| m \) yields a bijection

\[ s(\mathcal{M}) t \cong ((X) s)(X:)((\mathcal{M}) t) \]

from the set of \( \mathcal{M} \)-arrows \( s \sim t \) to the set of right module morphisms \( (X) s \to (\mathcal{M}) t : X \to * \), whose inverse sends each right module morphism \( \Phi : (X) s \to (\mathcal{M}) t \) to the \( \mathcal{M} \)-arrow \( 1_s \cdot \Phi : s \sim t \), the image of the identity \( s \to s \) under the function \( s(\Phi) : s(X)s \to s(\mathcal{M})t \).

- The left Yoneda morphism

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & A \\
\downarrow \mathcal{M} & & \downarrow \mathcal{A} \\
[\cdot : A]^- & \xrightarrow{\mathcal{M}(\mathcal{A})} & [\cdot : A]^-
\end{array}
\]

for \( \mathcal{M} \) is fully faithful. Specifically, for each pair of objects \( s \in \|X\| \) and \( t \in \|A\| \), the assignment \( m \mapsto m \cap A \) yields a bijection

\[ s(\mathcal{M}) t \cong (t(\mathcal{A}))(\cdot : A)(s(\mathcal{M})) \]

from the set of \( \mathcal{M} \)-arrows \( s \sim t \) to the set of left module morphisms \( t(\mathcal{A}) \to s(\mathcal{M}) : * \to A \), whose inverse sends each left module morphism \( \Phi : t(\mathcal{A}) \to s(\mathcal{M}) \) to the \( \mathcal{M} \)-arrow \( 1_t \cdot \Phi : s \sim t \), the image of the identity \( t \to t \) under the function \( (\Phi) t : t(\mathcal{A})t \to s(\mathcal{M})t \).

**Proof.** See Remark 5.2.12. \( \square \)
5. Yoneda Lemma

Remark 5.2.12. Theorem 5.2.11 follows from Theorem 5.2.10 (Yoneda lemma) on noting Proposition 5.2.7 and Proposition 2.1.8. Conversely, the Yoneda lemma is a special case of Theorem 5.2.11 where $A$ (resp. $X$) is the terminal category.

Corollary 5.2.13. (Yoneda Embedding).

- For any category $X$, the right Yoneda functor $[X^*] : X \to [X:]$ is fully faithful. Specifically, for each pair of objects $s, t \in \|X\|$, the assignment $f \mapsto (X)f$ yields a bijection
  $$s(X) t \cong ((X)s)(X:)((X)t)$$
  from the set of $X$-arrows $s \to t$ to the set of right module morphisms $(X)s \to (X)t : X \to \ast$, whose inverse sends each module morphism $\Phi : (X)s \to (X)t$ to the $X$-arrow $1_s : \Phi : s \to t$, the image of the identity $s \to s$ under the function $s(\Phi) : s(X)s \to s(X)t$.

- For any category $A$, the left Yoneda functor $[\land A] : A \to [\land A]^{-}$ is fully faithful. Specifically, for each pair of objects $s, t \in |A|$, the assignment $f \mapsto f(A)$ yields a bijection
  $$s(A)t \cong (t(A))(A)(s(A))$$
  from the set of $A$-arrows $s \to t$ to the set of left module morphisms $t(A) \to s(A) : \ast \to A$, whose inverse sends each module morphism $\Phi : t(A) \to s(A)$ to the $A$-arrow $\Phi : 1_t : s \to t$, the image of the identity $t \to t$ under the function $(\Phi)t : t(A)t \to s(A)t$.

Proof. See Remark 5.2.14. □

Remark 5.2.14. By Proposition 5.2.8, Corollary 5.2.13 (Yoneda embedding) is a special case of Theorem 5.2.11; to put it the other way round, Theorem 5.2.11 generalizes the Yoneda embedding for general modules.

Theorem 5.2.15.

- Given a category $X$, the right Yoneda morphism
  $$X \to X^* \cong [X:]$$
  for the right Yoneda module $X^*$ yields a representation
  $$\langle X^* \rangle \cong [X^*],[X:] : X \to [X:]$$
  of the right Yoneda module $X^*$ by the right Yoneda functor $X^*$. For a right module $M : X \to \ast$ and an object $r \in \|X\|$, the representation sends each $(X^*)$-arrow $m : r \to M$ (i.e. $M$-arrow $m : r \to \ast$) to the right module morphism $X|m : (X)r \to M$.

- Given a category $A$, the left Yoneda morphism
  $$[\land A]^{-} \to \land A \cong [\land A]^{-}$$
  for the left Yoneda module $[\land A]$ yields a representation
  $$\langle \land A \rangle \cong [\land A],[\land A]^{-} : [\land A]^{-} \to [\land A]^{-}$$
  of the left Yoneda module $[\land A]$ by the left Yoneda functor $\land A$. For a left module $M : \ast \to X$ and an object $r \in \|X\|$, the representation sends each $(\land A)$-arrow $m : r \to M$ (i.e. $M$-arrow $m : r \to \ast$) to the left module morphism $\land A|m : (\land A)r \to M$. 

150
for the left Yoneda module $\ast, A$ yields a corepresentation
\[(\ast, A) \cong (A)^{-} \cdot (\ast, A) : (A)^{-} \rightarrow A\]
of the left Yoneda module $\ast, A$ by the left Yoneda functor $\ast, A$. For a left module $M : \ast \rightarrow A$ and an object $r \in \|A\|$, the representation sends each $\ast$-arrow $m : M \rightarrow r$ (i.e. $M$-arrow $m : \ast \rightarrow r$) to the left module morphism $m \upharpoonright A : r(A) \rightarrow M$.

Proof. The first assertion follows from Theorem 5.2.11 on noting Proposition 5.1.3. We claim that the right slice of the representation cell at $M$ is given by the Yoneda morphism for $M$. Indeed, by Proposition 5.2.7 and Proposition 5.1.3,
\[(\langle X | (X, r) \rangle, M) = \langle X | ((X, r) (M)) \rangle = (X | M, r)\]
The second assertion now follows from the claim.

Remark 5.2.16. The representation (resp. corepresentation) in Theorem 5.2.15 is called the Yoneda representation (resp. corepresentation) and denoted by $X \upharpoonright (\ast, A)$ as in:

\[
\begin{array}{ccc}
X & \xrightarrow{r} & X \\
\downarrow & & \downarrow \\
[X] & \xrightarrow{r} & [X] \\
\end{array}
\quad
\begin{array}{ccc}
[A] & \xrightarrow{r} & A \\
\downarrow & & \downarrow \\
[(A)] & \xrightarrow{r} & [(A)] \\
\end{array}
\]

Corollary 5.2.17. (Yoneda Lemma : Part two). The bijection in Theorem 5.2.10 is natural in $r$ and $M$.

Proof. This is a restatement of Theorem 5.2.15.

Remark 5.2.18. Using the notation introduced in Remark 5.2.16, the bijections of the Yoneda lemma are written as
\[(r) (X \upharpoonright) (M) \cong (X) (r) (M) \cong ((X) r) (M) \cong (r (X) (A)) (M)\]
and
\[(M) (\upharpoonright A) (r) (M) \cong (r) (M) (\ast, A) \cong (r (A)) (\ast, A) (M)\]

5.3. Yoneda morphisms for cylinders

Definition 5.3.1. Let $E$ be a category and $M : X \rightarrow A$ be a module.

- The right action
\[
\langle X | M \rangle, \ast \rightarrow E : [E, [M]] \rightarrow [X : E]
\]
of the right hom of $M$ on the functor category $[E, [M]]$ is called the right general Yoneda functor for $\langle E, M \rangle$.

- The left action
\[
E \backslash \langle M | A \rangle : [E, [M]] \rightarrow [E : A]^\ast
\]
of the left hom of $M$ on the functor category $[E, [M]]$ is called the left general Yoneda functor for $\langle E, M \rangle$. 

151
Remark 5.3.2. Consider a cylinder

\[
\begin{array}{c}
E \\
S \alpha \\
X \rightarrow \underset{\sim}{M} \rightarrow A \\
T
\end{array}
\]

i.e. a natural transformation \( \alpha : S \circ M_X \rightarrow M_A \circ T : E \rightarrow |M| \) (see Remark 4.3.4(3)).

- By Remark 5.2.2, the module \((X|M)\) acts on \(M_X \circ S\) and \(M_A \circ T\) and yields the modules

\[
(X|M)[M_X \circ S] = \langle (X|M)M_X \rangle S = \langle X \rangle S
\]

and

\[
(X|M)[M_A \circ T] = \langle (X|M)M_A \rangle T = \langle M \rangle T
\]

hence \((X|M)\) acts on \(\alpha\) and yields the module morphism

\[
(X|M)\alpha : \langle X \rangle S \rightarrow \langle M \rangle T : X \rightarrow E
\]

which maps each \((X)S\)-arrow \(h : x \sim e\) to the \((M)T\)-arrow \(h \circ \alpha_e : x \sim e\) as indicated in

\[
\begin{array}{c}
x \\
h \downarrow \\
\overset{\alpha_e}{\sim} \rightarrow \\
e : S \sim \rightarrow \underset{T}{e}
\end{array}
\]

(cf. Remark 2.2.2(2)). When \(M\) is understood, \((X|M)\alpha\) is also written as

\[
X|\alpha : \langle X \rangle S \rightarrow \langle M \rangle T : X \rightarrow E
\]

and called the module morphism generated by \(X\) direct along \(\alpha\).

- By Remark 5.2.2, the module \((M|A)\) acts on \(S \circ M_X\) and \(T \circ M_A\) and yields the modules

\[
[S \circ M_X](M|A) = S(M_X(M|A)) = S\langle M \rangle
\]

and

\[
[T \circ M_A](M|A) = T(M_A(M|A)) = T\langle A \rangle
\]

hence \((M|A)\) acts on \(\alpha\) and yields the module morphism

\[
\alpha(M|A) : T\langle A \rangle \rightarrow S\langle M \rangle : E \rightarrow A
\]

which maps each \(T\langle A\rangle\)-arrow \(h : e \sim a\) to the \(S\langle M\rangle\)-arrow \(\alpha_e \circ h : e \sim a\) as indicated in

\[
\begin{array}{c}
e : S \sim \rightarrow \underset{T}{e} \\
\overset{\alpha_e}{\sim} \rightarrow \\
\alpha(M|A) \downarrow \\
\overset{h}{\sim} \rightarrow \\
a
\end{array}
\]

(cf. Remark 2.2.2(2)). When \(M\) is understood, \(\alpha(M|A)\) is also written as

\[
\alpha \uparrow A : T\langle A \rangle \rightarrow S\langle M \rangle : E \rightarrow A
\]

and called the module morphism generated by \(A\) inverse along \(\alpha\).
5. Yoneda Lemma

**Definition 5.3.3.** Let $E$ be a category and $M : X \to A$ be a module.

- The right general Yoneda morphism for $(E, M)$ (see Definition 4.3.6) is the cell

$$
\begin{array}{c}
[ E, X ] - \rightarrow \rightarrow \rightarrow \rightarrow [ E, A ] \\
\downarrow (E|M)_E \downarrow \downarrow (M \rightarrow E) \downarrow \downarrow \downarrow \\
[ X : E ] - \rightarrow \rightarrow \rightarrow \rightarrow [ X : E ]
\end{array}
$$

sending each cylinder $\alpha : S \rightarrow T : E \rightarrow M$ to the module morphism

$$
X \uparrow \alpha = (X|M) \alpha : (X) S \rightarrow (M) T : X \rightarrow E
$$

- The left general Yoneda morphism for $(E, M)$ (see Definition 4.3.6) is the cell

$$
\begin{array}{c}
[ E, X ] - \rightarrow \rightarrow \rightarrow \rightarrow [ E, A ] \\
\downarrow E \cdot (M|A) \downarrow \downarrow E \cdot A \\
[ E : A ]^- - \rightarrow \rightarrow \rightarrow \rightarrow [ E : A ]^-
\end{array}
$$

sending each cylinder $\alpha : S \rightarrow T : E \rightarrow M$ to the module morphism

$$
\alpha \uparrow A = \alpha (M|A) : T (A) \rightarrow S (M) : E \rightarrow A
$$

**Remark 5.3.4.**

1. The right general Yoneda morphism for $(E, M)$ is formally given by the composition

$$
\begin{array}{c}
[ E, X ] - \rightarrow \rightarrow \rightarrow \rightarrow [ E, A ] \\
\downarrow (E,M)_X \downarrow \downarrow (E,I_M) \downarrow \downarrow (E,M_A) \\
\downarrow (X|M)_E \downarrow \downarrow (X|M)_E \\
[ X : E ] - \rightarrow \rightarrow \rightarrow \rightarrow [ X : E ]
\end{array}
$$

, where $(E,I_M)$ is the postcomposition with the unit cell $I_M : M \rightarrow ([M])$ (see Remark 3.1.16(2)) and $(X|M) \rightarrow E$ is the hom of the right general Yoneda functor.

- The left general Yoneda morphism for $(E, M)$ is formally given by the composition

$$
\begin{array}{c}
[ E, X ] - \rightarrow \rightarrow \rightarrow \rightarrow [ E, A ] \\
\downarrow (E,M)_X \downarrow \downarrow (E,I_M) \downarrow \downarrow (E,M_A) \\
\downarrow E \cdot (M|A) \downarrow \downarrow E \cdot (M|A) \\
[ E : A ]^- - \rightarrow \rightarrow \rightarrow \rightarrow [ E : A ]^-
\end{array}
$$
5. Yoneda Lemma

, where \( (E, I_M) \) is the postcomposition with the unit cell \( I_M : \mathcal{M} \to \langle [\mathcal{M}] \rangle \) (see Remark 3.1.16(2)) and \( (E \cdot (\mathcal{M} \mid A)) \) is the hom of the left general Yoneda functor.

2. The Yoneda morphism for \( \mathcal{M} \) is identified with the special instance of the general Yoneda morphism for \( (E, \mathcal{M}) \) where \( E \) is the terminal category.

Example 5.3.5.

1. Given a right cylinder

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow{\alpha} & & \\
M & \xrightarrow{} & M
\end{array}
\]

, i.e. a two-sided cylinder

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow{\alpha} & & \\
M & \xrightarrow{} & M
\end{array}
\]

(see Remark 4.3.4(2)), the category \( X \) acts on \( \alpha \) and generates the module morphism

\[
X \uparrow \alpha = \langle X \mid \mathcal{M} \rangle \alpha : \langle X \rangle G \to \mathcal{M} : X \to A
\]

direct along \( \alpha \), mapping each \( \langle X \rangle \) G-arrow \( h : x \rightsquigarrow a \) to the \( \mathcal{M} \)-arrow \( h \circ \alpha_a : x \rightsquigarrow a \) as indicated in

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow{\alpha} & & \\
M & \xrightarrow{} & M
\end{array}
\]

, and the category \( A \) acts on \( \alpha \) and generates the module morphism

\[
\alpha \uparrow A = \alpha \langle \mathcal{M} \mid A \rangle : \langle A \rangle \to G \langle \mathcal{M} \rangle : A \to A
\]

inverse along \( \alpha \), mapping each \( A \)-arrow \( h : a \to b \) to the \( G \langle \mathcal{M} \rangle \)-arrow \( \alpha_a \circ h : a \rightsquigarrow b \) as indicated in

\[
\begin{array}{ccc}
a & \xrightarrow{\alpha_a} & a \\
\downarrow{\alpha} & & \\
X & \xrightarrow{\alpha} & A
\end{array}
\]

. The naturality of \( \alpha \) gives the commutative diagram

\[
\begin{array}{ccc}
a & \xrightarrow{\alpha_a} & a \\
\downarrow{\alpha} & & \\
b & \xrightarrow{\alpha_b} & b
\end{array}
\]

for each \( A \)-arrow \( h : a \to b \), from which we can read off the equation

\[
h : G \cdot (X \uparrow \alpha) = (\alpha \uparrow A) \cdot h
\]
and the commutativity of the triangle

\[
\begin{array}{ccc}
G(X) & \xrightarrow{(G)} & A \\
\downarrow^{G(X|\alpha)} & & \downarrow^{\alpha|A} \\
G(M) & & \\
\end{array}
\]

- Given a left cylinder

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow^{F} & & \downarrow^{F} \\
\end{array}
\]

, i.e. a two-sided cylinder

\[
\begin{array}{ccc}
X & \xrightarrow{1} & A \\
\downarrow^{\alpha} & \xrightarrow{\alpha} & \downarrow^{\alpha} \\
\end{array}
\]

(see Remark 4.3.4(2)), the category \( A \) acts on \( \alpha \) and generates the module morphism

\[
\alpha|A = \alpha(\langle M|A \rangle : F(A) \rightarrow \langle M : X \rightarrow A \rangle
\]

inverse along \( \alpha \), mapping each \( F(A) \)-arrow \( h : x \rightarrow a \) to the \( M \)-arrow \( \alpha_x \circ h : x \rightarrow a \) as indicated in

\[
\begin{array}{ccc}
x & \xrightarrow{\alpha_x} & F : x \\
\downarrow^{(\alpha|A) \circ h} & & \downarrow^{h} \\
a & & \\
\end{array}
\]

, and the category \( X \) acts on \( \alpha \) and generates the module morphism

\[
X|\alpha = (X|\langle M \rangle \alpha : (X) \rightarrow (\langle M \rangle) F : X \rightarrow X
\]

direct along \( \alpha \), mapping each \( X \)-arrow \( h : y \rightarrow x \) to the \( \langle M \rangle F \)-arrow \( h \circ \alpha_x : y \rightarrow x \) as indicated in

\[
\begin{array}{ccc}
y & \xrightarrow{h} & F : x \\
\downarrow^{h \circ (X|\alpha)} & & \downarrow^{F \circ h} \\
x & \xrightarrow{\alpha_x} & F : x \\
\end{array}
\]

. The naturality of \( \alpha \) gives the commutative diagram

\[
\begin{array}{ccc}
y & \xrightarrow{\alpha_y} & F \circ y \\
\downarrow^{h} & & \downarrow^{F \circ h} \\
x & \xrightarrow{\alpha_x} & F \circ x \\
\end{array}
\]

for each \( X \)-arrow \( h : y \rightarrow x \), from which we can read off the equation

\[
h : \langle X | \alpha \rangle = \langle \alpha | A \rangle : F \circ h
\]

and the commutativity of the triangle

\[
\begin{array}{ccc}
\langle X \rangle & \xrightarrow{(F)} & F \langle A \rangle F \\
\downarrow^{X|\alpha} & & \downarrow^{(\alpha|A)F} \\
\langle M \rangle F & & \\
\end{array}
\]
5. Yoneda Lemma

2. Consider a pair of functors as in:

\[
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
F & \downarrow & \downarrow \\
\end{array}
\]

- Given a natural transformation \( \epsilon : G \circ F \rightarrow 1_A : A \rightarrow A \), i.e. a right cylinder

\[
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
F & \downarrow & \downarrow \\
\end{array}
\]

(see Example 4.3.5(2)), the category \( X \) acts on \( \epsilon \) and generates the module morphism

\[
X|\epsilon = (\langle X | (F \langle A \rangle) \rangle \epsilon : \langle X | G \rangle \rightarrow F \langle A \rangle : X \rightarrow A
\]

direct along \( \epsilon \), mapping each \( \langle X \rangle G \)-arrow \( h : x \rightsquigarrow a \) to the \( F \langle A \rangle \)-arrow \( h \circ \epsilon_a : x \rightsquigarrow a \) as indicated in

\[
\begin{array}{ccc}
x & \xrightarrow{X|\epsilon} & x : F \\
h & \downarrow & \downarrow \\
G : a & \rightsquigarrow & a : G : F \xrightarrow{\epsilon_a} a
\end{array}
\]

- Given a natural transformation \( \eta : 1_X \rightarrow G \circ F : X \rightarrow X \), i.e. a left cylinder

\[
\begin{array}{ccc}
X & \xleftarrow{\eta} & A \\
F & \downarrow & \downarrow \\
\end{array}
\]

(see Example 4.3.5(2)), the category \( A \) acts on \( \eta \) and generates the module morphism

\[
\eta|A = \eta \langle \langle X | G \rangle | A \rangle : F \langle A \rangle \rightarrow \langle X | G \rangle : X \rightarrow A
\]

inverse along \( \eta \), mapping each \( F \langle A \rangle \)-arrow \( h : x \rightsquigarrow a \) to the \( \langle X \rangle G \)-arrow \( \eta_x \circ h : x \rightsquigarrow a \) as indicated in

\[
\begin{array}{ccc}
x & \xleftarrow{\eta_x} & G : F \circ x \\
\langle \eta|A \rangle : h & \downarrow & \downarrow \\
G : a & \rightsquigarrow & a \xrightarrow{\eta_a}
\end{array}
\]

3.

- Given a right \( F \)-weighted cylinder

\[
\begin{array}{ccc}
D & \xleftarrow{F} & E \\
S & \downarrow \alpha & \downarrow \alpha \\
X & \xleftarrow{\alpha} & A \\
\end{array}
\]

, i.e. a right cylinder

\[
\begin{array}{ccc}
D & \xleftarrow{F} & E \\
S(\alpha) & \downarrow & \downarrow \\
S & \xleftarrow{\alpha} & E
\end{array}
\]
5. Yoneda Lemma

(see Remark 4.5.2(1)), the category \( \text{D} \) acts on \( \alpha \) and generates the module morphism

\[
\text{D}|\alpha = (\text{D}|(\text{S} \{\text{M}\} \text{\ T})) \epsilon : (\text{D})\text{F} \to \text{S} \{\text{M}\} \text{\ T} : \text{D} \to \text{E}
\]

, i.e. a cell

\[
\begin{array}{ccc}
\text{D} & \xrightarrow{\text{(D)F}} & \text{E} \\
\text{S} & \downarrow \text{D}|\alpha & \text{T} \\
\text{X} & \downarrow \text{M} & \text{A}
\end{array}
\]

direct along \( \alpha \), mapping each \( \text{D} \)-arrow \( h : d \to e \) to the \( \text{M} \)-arrow \( (h : S) \circ \alpha_e : d : S \to T : e \) as indicated in

\[
\begin{array}{ccc}
d & d : S \\
\downarrow h & \downarrow (h : (\text{D}|\alpha)) \\
\text{F} : e & \text{e} : F : S \xrightarrow{\alpha_e} T : e
\end{array}
\]

- Given a left \( F \)-weighted cylinder

\[
\begin{array}{ccc}
\text{E} & \xrightarrow{\text{F}} & \text{D} \\
\text{S} & \downarrow \text{\alpha} & \text{T} \\
\text{X} & \downarrow \text{M} & \text{A}
\end{array}
\]

, i.e. a left cylinder

\[
\begin{array}{ccc}
\text{E} & \xrightarrow{\text{S(M)T}} & \text{D} \\
\downarrow \text{F} & \downarrow \text{\alpha} & \downarrow \text{\alpha} \\
\end{array}
\]

(see Remark 4.5.2(1)), the category \( \text{D} \) acts on \( \alpha \) and generates the module morphism

\[
\alpha|\text{D} = \alpha \langle (\text{S} \{\text{M}\} \text{\ T})|\text{D} \rangle : \text{F} \{\text{D}\} \to \text{S} \{\text{M}\} \text{\ T} : \text{E} \to \text{D}
\]

, i.e. a cell

\[
\begin{array}{ccc}
\text{E} & \xrightarrow{\text{F} \{\text{D}\}} & \text{D} \\
\text{S} & \downarrow \alpha|\text{D} & \text{T} \\
\text{X} & \downarrow \text{M} & \text{A}
\end{array}
\]

inverse along \( \alpha \), mapping each \( \text{F} \{\text{D}\} \)-arrow \( h : e \to d \) to the \( \text{M} \)-arrow \( \alpha_e \circ (T : h) : e : S \to T : d \) as indicated in

\[
\begin{array}{ccc}
e : S & \xrightarrow{\alpha_e} T : F : e \\
\downarrow (\alpha|\text{D} : h) & \downarrow T : h \\
\text{T} : d & \downarrow h \\
\text{d}
\end{array}
\]

4.

- Given a cone \( \alpha : r \to K : E^a \to \text{M} \), i.e. a cylinder

\[
\begin{array}{ccc}
\ast & \xrightarrow{\Delta} & \text{E} \\
r & \downarrow \alpha & \downarrow K \\
\text{X} & \downarrow \text{M} & \text{A}
\end{array}
\]
5. Yoneda Lemma

, the category $\mathbf{X}$ acts on $\alpha$ and generates the wedge

$$\mathbf{X} \downarrow \alpha = \langle \mathbf{X}, \mathcal{M} \rangle \alpha : \langle \mathbf{X} \rangle \mathbf{r} \sim \langle \mathcal{M} \rangle \mathbf{K} : \mathbf{X} \rightarrow \mathbf{E}^*$$

direct along $\alpha$, mapping each $\mathbf{X}$-arrow $h : x \rightarrow r$ to the $\langle \mathcal{M} \rangle \mathbf{K}$-arrow $h \circ \alpha_e : x \sim e$ as indicated in

$$\begin{array}{c}
\mathbf{X} \\
\downarrow h \\
r \\
\alpha_e \sim e \rightarrow \mathbf{K} \cdot e \\
\end{array}$$

for each $e \in \|\mathbf{E}\|$.

- Given a cone $\alpha : \mathbf{K} \sim \mathbf{r} : \mathbf{E}^* \sim \mathcal{M}$, i.e. a cylinder

$$\begin{array}{c}
\mathbf{E} \\
\downarrow \Delta \\
\mathbf{K} \\
\end{array} \rightarrow \leftarrow \begin{array}{c}
\mathbf{r} \\
\mathbf{A} \\
\mathbf{X} \downarrow \mathcal{M} \\
\end{array}$$

, the category $\mathbf{A}$ acts on $\alpha$ and generates the wedge

$$\alpha \uparrow \mathbf{A} = \alpha \langle \mathcal{M} \rangle \mathbf{A} : \mathbf{r} \langle \mathbf{A} \rangle \sim \mathbf{K} \langle \mathcal{M} \rangle : \mathbf{E}^* \rightarrow \mathbf{A}$$

inverse along $\alpha$, mapping each $\mathbf{A}$-arrow $r \rightarrow a$ to the $\langle \mathcal{M} \rangle \mathbf{K}$-arrow $\alpha_e \circ h : e \sim a$ as indicated in

$$\begin{array}{c}
e : \mathbf{K} \\
\downarrow \alpha_e \sim \mathbf{r} \\
\langle \mathbf{A} \rangle \mathbf{h} \\
\end{array} \rightarrow \leftarrow \begin{array}{c}
a \\
\mathbf{A} \\
\mathbf{E} \\
\end{array}$$

for each $e \in \|\mathbf{E}\|$.

Note. A two-sided cylinder

$$\begin{array}{c}
\mathbf{E} \\
\downarrow \alpha \\
\mathbf{X} \downarrow \mathcal{M} \\
\end{array} \rightarrow \leftarrow \begin{array}{c}
\mathbf{S} \\
\mathbf{T} \\
\mathbf{A} \\
\end{array}$$

is also depicted as a right cylinder

$$\begin{array}{c}
\mathbf{X} \downarrow \mathcal{M} \\
\downarrow \alpha \\
\leftarrow \begin{array}{c}
\mathbf{S} \\
\mathbf{E} \\
\end{array}$$

or a left cylinder

$$\begin{array}{c}
\mathbf{E} \downarrow \mathcal{M} \\
\leftarrow \begin{array}{c}
\alpha \\
\mathbf{A} \downarrow \mathbf{T} \\
\end{array}$$

(see Remark 4.3.4(2)). Proposition 5.3.6 says that $\mathbf{X}$ (resp. $\mathbf{A}$) generates the same module morphism direct (resp. inverse) along $\alpha$ irrespective of the way $\alpha$ is depicted.

**Proposition 5.3.6.** Consider a cylinder as in Note above. Then
5. Yoneda Lemma

- the module morphism
  \[ X \mid \alpha = (X|\mathcal{M}) \alpha : (X) S \to (\mathcal{M}) T : X \to E \]
  coincides with
  \[ X \mid \alpha = (X|\mathcal{M}) T \alpha : (X) S \to (\mathcal{M}) T : X \to E \]

- the module morphism
  \[ \alpha \mid A = \alpha (\mathcal{M}|A) : T(A) \to S(\mathcal{M}) : E \to A \]
  coincides with
  \[ \alpha \mid A = \alpha (S(\mathcal{M})|A) : T(A) \to S(\mathcal{M}) : E \to A \]

Proof. Both map each \((X) S\)-arrow \(h : x \sim e\) to the \((\mathcal{M}) T\)-arrow \(h \circ \alpha_e : x \sim e\) (see Example 5.3.5(1)). \(\square\)

Note. Given a pair of categories \(X\) and \(A\), we saw in Theorem 4.3.20(1) that the hom of the functor category \([A, X]\) is the same thing as the module \((A, (X))\) of cylinders \(A \rightsquigarrow (X)\), and will see in Proposition 5.3.7 that the hom of the right general Yoneda functor for \([A, X]\) is the same thing as the right general Yoneda morphism for \((A, (X))\).

**Proposition 5.3.7.**

- Given a pair of categories \(X\) and \(A\), the right general Yoneda morphism
  
  \[
  \begin{array}{ccc}
  [A, X] & \xrightarrow{(A, (X))} & [A, X] \\
  \xrightarrow{x \cdot A} & (X|\mathcal{X}) \cdot A & \xrightarrow{X \cdot A} \\
  [X : A] & \xrightarrow{(X \cdot A)} & [X : A]
  \end{array}
  \]

  for \((A, (X))\) is the same thing as the hom
  
  \[
  \begin{array}{ccc}
  [A, X] & \xrightarrow{(A, X)} & [A, X] \\
  \xrightarrow{x \cdot A} & (X \cdot A) & \xrightarrow{X \cdot A} \\
  [X : A] & \xrightarrow{(X \cdot A)} & [X : A]
  \end{array}
  \]

  of the right general Yoneda functor for \([A, X]\); that is, for any natural transformation \(\tau : S \to T : A \to X\), the module morphism
  
  \[ X \mid \tau = (X|\mathcal{X}) \tau : (X) S \to (X) T : X \to A \]
  coincides with
  
  \[ (X) \tau : (X) S \to (X) T : X \to A \]
5. Yoneda Lemma

- Given a pair of categories $X$ and $A$, the left general Yoneda morphism

$$
\begin{array}{c}
[X, A] - \xrightarrow{(X, (A))} - [X, A] \\
\downarrow \xrightarrow{X \cdot A} \downarrow \xrightarrow{(X \cdot A) \cdot A} \downarrow \xrightarrow{X \cdot A}
\end{array}
$$

for $(X, (A))$ is the same thing as the hom

$$
\begin{array}{c}
[X, A] - \xrightarrow{(X, A)} - [X, A] \\
\downarrow \xrightarrow{(X, A)} \downarrow \xrightarrow{(X, A)} \downarrow \xrightarrow{X \cdot A}
\end{array}
$$

of the left general Yoneda functor for $[X, A]$; that is, for any natural transformation $\tau : S \rightarrow T : X \rightarrow A$, the module morphism

$$
\tau \cdot A = \tau (\langle A | A \rangle) : T (A) \rightarrow S (A) : X \rightarrow A
$$

coincides with

$$
\tau (A) : T (A) \rightarrow S (A) : X \rightarrow A
$$

Proof.

1. Both $X \cdot \tau$ and $(X) \tau$ map each $(X)$ $S$-arrow $h : x \rightsquigarrow a$ to the $(X)$ $T$-arrow $h \circ \tau_a : x \rightsquigarrow a$ (see Remark 2.3.6(1)).

\[\square\]

**Theorem 5.3.8.** Given a category $E$ and a module $\mathcal{M} : X \rightarrow A$,

- the triangle

$$
\begin{array}{c}
\{E, \mathcal{M}\} \\
\downarrow \xrightarrow{(X|\mathcal{M}) \cdot E} \downarrow \xrightarrow{(E, \langle X|\mathcal{M} \rangle)} \\
\langle X : E \rangle \rightarrow - \rightarrow - \rightarrow - \rightarrow \langle E, \langle X : \rangle \rangle
\end{array}
$$

commutes; that is, the composition

$$
\begin{array}{c}
[X, E] - \xrightarrow{(E, \mathcal{M})} - [E, A] \\
\downarrow \xrightarrow{X \cdot E} \downarrow \xrightarrow{(X|\mathcal{M}) \cdot E} \downarrow \xrightarrow{\mathcal{M} \cdot E}
\end{array}
$$

$$
\begin{array}{c}
[X : E] - \xrightarrow{(X : E)} - [X : E] \\
\downarrow \xrightarrow{\langle \cdot \rangle} \\
\downarrow \xrightarrow{\langle \cdot \rangle}
\end{array}
$$

$$
\begin{array}{c}
[E, [X : ]] - \xrightarrow{(E, [X : ])} - [E, [X : ]] \\
\downarrow \xrightarrow{\langle E, [X : ] \rangle}
\end{array}
$$
of the right general Yoneda morphism for \((E, M)\) and the right exponential transposition yields the cell

\[
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, M)} & [E, A] \\
\downarrow & \downarrow & \downarrow \\
[E, (X : M)] & \xrightarrow{(E, (X : M) \cdot \cdot)} & [E, (X : M)]
\end{array}
\]

postcomposition with the right Yoneda morphism for \(M\).

\[
\begin{array}{ccc}
(E, M) & \xrightarrow{\cdot \cdot \cdot \cdot \cdot \cdot \cdot} & (E, A) \\
\downarrow & \downarrow & \downarrow \\
(E, (A : M)) & \xrightarrow{(E, (A : M))} & (E, A)
\end{array}
\]

commutes; that is, the composition

\[
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, M)} & [E, A] \\
\downarrow & \downarrow & \downarrow \\
[E, (\cdot : A)] & \xrightarrow{(E, (\cdot : A))} & [E, (\cdot : A)]
\end{array}
\]

of the left general Yoneda morphism for \((E, M)\) and the left exponential transposition yields the cell

\[
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, M)} & [E, A] \\
\downarrow & \downarrow & \downarrow \\
[E, (\cdot : A)] & \xrightarrow{(E, (\cdot : A))} & [E, (\cdot : A)]
\end{array}
\]

postcomposition with the left Yoneda morphism for \(M\).

**Proof.** By Theorem 2.2.3, the diagram

\[
\begin{array}{ccc}
[X : E] & \xrightarrow{(E, M)} & [E, X] \\
\downarrow & \downarrow & \downarrow \\
[X : E] & \xrightarrow{(E, (X : M))} & [E, (X : M)]
\end{array}
\]

commutes. The hom operation and the composition with \((E, I_M)\) as shown in

\[
\begin{array}{ccc}
(E, M) & \xrightarrow{\cdot \cdot} & (E, M) \\
\downarrow & \downarrow & \downarrow \\
(E, (I_M)) & \xrightarrow{\cdot \cdot \cdot \cdot} & (E, (I_M))
\end{array}
\]

then yield the desired commutative triangle by Remark 5.3.4(1) and Remark 5.2.6(1).
**Corollary 5.3.9.** For any cylinder

\[
\begin{array}{c}
\xymatrix{
E 
\ar[rd]_{\alpha} 
\ar[rr]_{T} 
\ar[rrd]_{X} \ar[ld]^{S} \ar[lldd]_{M} & & A \\
X
& &
M
& &
A
}
\end{array}
\]

- the equality
  \[
  \langle (X|M) \alpha \rangle \tau = \langle (X|M) \tau \rangle \alpha
  \]
holds; that is, the right exponential transpose of the module

\[
(X|M) \alpha : (X) S \rightarrow (M) T : X \rightarrow E
\]
is the natural transformation given by the composition

\[
\begin{array}{c}
\xymatrix{
E 
\ar[rd]_{\alpha} 
\ar[rr]_{T} 
\ar[rrd]_{X} \ar[ld]^{S} \ar[lldd]_{M} & & A \\
X
& &
M
& &
A
}
\end{array}
\]

of \( \alpha \) and the right Yoneda morphism for \( M \).

- the equality
  \[
  \kappa \langle \alpha (M|A) \rangle = \alpha \circ \langle \kappa (M|A) \rangle
  \]
holds; that is, the left exponential transpose of the module

\[
\begin{array}{c}
\xymatrix{
E 
\ar[rd]_{\alpha} 
\ar[rr]_{T} 
\ar[rrd]_{X} \ar[ld]^{S} \ar[lldd]_{M} & & A \\
X
& &
M
& &
A
}
\end{array}
\]

is the natural transformation given by the composition

\[
\begin{array}{c}
\xymatrix{
E 
\ar[rd]_{\alpha} 
\ar[rr]_{T} 
\ar[rrd]_{X} \ar[ld]^{S} \ar[lldd]_{M} & & A \\
X
& &
M
& &
A
}
\end{array}
\]

of \( \alpha \) and the left Yoneda morphism for \( M \).

**Proof.** Immediate from Theorem 5.3.8. \( \square \)

**Corollary 5.3.10.** For any cylinder

\[
\begin{array}{c}
\xymatrix{
E 
\ar[rd]_{\alpha} 
\ar[rr]_{T} 
\ar[rrd]_{X} \ar[ld]^{S} \ar[lldd]_{M} & & A \\
X
& &
M
& &
A
}
\end{array}
\]

and any object \( e \in [E] \).
the equality
\[ \langle X \uparrow \alpha \rangle e = X \uparrow \alpha_e \]
holds; that is, the right slice
\[ \langle X \uparrow \alpha \rangle e : \langle \langle X \rangle S \rangle e \to \langle \langle M \rangle T \rangle e : X \to * \]
at \( e \) of the module generated by \( X \) direct along \( \alpha \) is given by the right module
\[ X \uparrow \alpha_e : \langle X \rangle (S \cdot e) \to \langle M \rangle (T \cdot e) : X \to * \]
generated by \( X \) direct along the component of \( \alpha \) at \( e \).

\[ \begin{aligned}
\text{Proof.} \quad & \text{By Corollary 5.3.9,} \\
& \langle X \uparrow \alpha \rangle e = \langle \langle X \rangle | M \rangle \alpha \rangle e = \langle \langle X \rangle | M \rangle \alpha \rangle : \alpha_e = X \uparrow \alpha_e \\
\end{aligned} \]

\[ \square \]

**Theorem 5.3.11.** Let \( E \) be a category and \( M : X \to A \) be a module.

\[ \begin{aligned}
& \text{The right general Yoneda morphism} \\
& [E, X] \xrightarrow{\langle E, M \rangle} [E, A] \\
& [X : E] \xrightarrow{\langle X \rangle | M \rangle} [X : A] \\
\end{aligned} \]

for \( \langle E, M \rangle \) is fully faithful. Specifically, for each pair of functors \( S : E \to X \) and \( T : E \to A \), the assignment \( \alpha \mapsto X \uparrow \alpha \) yields a bijection

\[ (S | E, M) (T) \cong (\langle X \rangle S | X, E) (\langle M \rangle T) \]

from the set of cylinders \( S \sim T : E \to M \) to the set of module morphisms \( \langle X \rangle S \to (M) T : X \to E \), whose inverse sends each module morphism \( \Phi : \langle X \rangle S \to (M) T \) to the cylinder \( [\Phi] : S \sim T : E \to M \) defined by

\[ [\Phi]_e = 1_{(e : S)} : \Phi \]

for \( e \in [E] \), where \( 1_{(e : S)} \cdot \Phi \) is the image of the identity \( X \)-arrow \( e : S \to S \cdot e \) (i.e. \( \langle X \rangle S \)-arrow \( e : S \sim e \)) under the function

\[ (e : S) (X) (S \cdot e) = (e : S) (\langle X \rangle S) e \xrightarrow{(e : S) (\Phi)e} (e : S) (\langle M \rangle T) e = (e : S) (M) (T \cdot e) \]
5. Yoneda Lemma

- The left general Yoneda morphism

\[
\begin{array}{c}
\text{[E, X]} - \xrightarrow{(E,M)} - \text{[E, A]}
\end{array}
\]

\[
\begin{array}{c}
\text{E\times A}
\end{array}
\]

\[
\begin{array}{c}
\text{[E : A]} - \xrightarrow{(E,A)} - \text{[E : A]}
\end{array}
\]

for \(\langle E, M \rangle\) is fully faithful. Specifically, for each pair of functors \(S : E \to X\) and \(T : E \to A\), the assignment \(\alpha \mapsto \alpha \mid A\) yields a bijection

\[(S \langle E, M \rangle) (T) \cong (T \langle A \rangle) (E : A) (S \langle M \rangle)\]

from the set of cylinders \(S \to T : E \to M\) to the set of module morphisms \(T \langle A \rangle \to S \langle M \rangle : E \to A\), whose inverse sends each module morphism \(\Phi : T \langle A \rangle \to S \langle M \rangle\) to the cylinder \([\Phi] : S \to T : E \to M\) defined by

\[\text{[\Phi]}_e = \Phi \cdot 1_{(T \cdot e)}\]

for \(e \in [E]\), where \(\Phi \cdot 1_{(T \cdot e)}\) is the image of the identity \(A\)-arrow \(e : T \to T \cdot e\) (i.e. \(T \langle A \rangle\)-arrow \(e \mapsto T \cdot e\)) under the function

\[(e : T) \langle A \rangle (T \cdot e) = e (T \langle A \rangle) (T \cdot e) \xrightarrow{e(\Phi)(T \cdot e)} e (S \langle M \rangle) (T \cdot e) = (e : S) \langle X \rangle (T \cdot e)\]

Proof. By Theorem 5.3.8, \(\langle X \mid M \rangle \Rightarrow E\) is fully faithful iff so is \(\langle E, \langle X \mid M \rangle \rangle\). But \(\langle E, \langle X \mid M \rangle \rangle\) is fully faithful by Theorem 5.2.11 and Proposition 4.3.19. For the second assertion, it suffices to show that a natural transformation \(\alpha : S \to T : E \to M\) is recovered from the module morphism \(\langle X \rangle \alpha\) by \(\alpha_e = 1_{(e : S)} \cdot (e : S) \langle X \rangle \alpha e\). But, by Theorem 5.2.11 and Corollary 5.3.10,

\[\alpha_e = 1_{(e : S)} \cdot (e : S) \langle X \rangle \alpha \]

Remark 5.3.12. Theorem 5.2.11 is regarded as a special case of Theorem 5.3.11 where \(E\) is the terminal category.

Corollary 5.3.13. (General Yoneda embedding). Let \(X\) and \(A\) be categories.

- The right general Yoneda functor \([X \times A] : [A, X] \to [X : A]\) is fully faithful. Specifically, for each pair of functors \(S, T : A \to X\), the assignment \(\tau \mapsto \langle X \rangle \tau\) yields a bijection

\[(S \langle A, X \rangle) (T) \cong (\langle X \rangle S) \langle A \rangle ((X) T)\]

from the set of natural transformations \(S \to T : A \to X\) to the set of module morphisms \((X) S \to (X) T : X \to A\), whose inverse sends each module morphism \(\Phi : (X) S \to (X) T\) to the natural transformation \([\Phi] : S \to T\) defined by

\[\text{[\Phi]}_a = 1_{(a : S)} \cdot \Phi\]

for \(a \in [A]\), where \(1_{(a : S)} \cdot \Phi\) is the image of the identity \(X\)-arrow \(a^S \to S^a\) (i.e. \((X) S\)-arrow \(a^S \to a\)) under the function

\[(a : S) \langle X \rangle (S^a) = (a : S) \langle (X) S \rangle a \xrightarrow{(a : S)(\Phi)a} (a : S) \langle (X) T \rangle a = (a : S) \langle X \rangle (T^a)\]
Theorem 5.3.15. Let \( \text{X} \) and \( \text{A} \) be categories.

- The left general Yoneda morphism for \( \langle \text{A}, \langle \text{X}, \text{A} \rangle \rangle \) composed with the iso cell in Corollary 5.1.10 as shown in

\[
\begin{array}{c}
\text{[X:X]} \quad \downarrow \quad \text{[X:A]} \\
\text{1} \quad \text{1} \quad \text{1} \\
\text{[X:X]} \quad \downarrow \quad \text{[X:A]} \\
\text{X \times A} \quad \text{(X \times A)} \cdot \text{A} \quad \text{(X \times A)} \cdot \text{A} = \text{X} \\
\text{[X:A]} \quad \downarrow \quad \text{[X:A]} \\
\end{array}
\]

yields a representation

\[
\langle \text{X}, \text{A} \rangle \equiv \text{[X \times A]} \cdot \text{[X:A]} : \text{[X:A]} \rightarrow \text{[X:A]}
\]

of the right general Yoneda module \( \text{X, A} \) by the right general Yoneda functor \( \text{X, A} \). For a functor \( \text{G} : \text{A} \rightarrow \text{X} \) and a module \( \text{M} : \text{X} \rightarrow \text{A} \), the representation sends each right cylinder \( \alpha : \text{G} \rightarrow \text{M} \) to the module morphism \( \text{X} \cdot \alpha : \langle \text{X} \rangle \text{G} \rightarrow \text{M} \) in Example 5.3.5(1).

- The left general Yoneda morphism for \( \langle \text{X}, \langle \text{X}, \text{A} \rangle \rangle \) composed with the iso cell in Corollary 5.1.10 as shown in

\[
\begin{array}{c}
\text{[X:A]} \quad \downarrow \quad \text{[X:A]} \\
\text{1} \quad \text{1} \quad \text{1} \\
\text{[X:A]} \quad \downarrow \quad \text{[X:A]} \\
\text{X \times A} \quad \text{(X \times A)} \cdot \text{A} \quad \text{(X \times A)} \cdot \text{A} = \text{X} \\
\text{[X:A]} \quad \downarrow \quad \text{[X:A]} \\
\end{array}
\]

Proof. This is a special case of Theorem 5.3.11 on noting Proposition 5.3.7. □

Remark 5.3.14. Corollary 5.2.13 (Yoneda embedding) is regarded as a special case of Corollary 5.3.13 where \( \text{A} \) (resp. \( \text{X} \)) is the terminal category.
yields a corepresentation
\[
\langle X^\ast, A \rangle \cong \langle X : A \rangle \ast [X : A] : [X : A] \to [X, A]
\]
of the left general Yoneda module \(X^\ast, A\) by the left general Yoneda functor \(X\ast, A\). For a functor \(F : X \to A\) and a module \(M : X \to A\), the representation sends each left cylinder \(\alpha : M \sim F\) to the module morphism \(\alpha \ast A : F(A) \to M\) in Example 5.3.5(1).

Proof. First recall Proposition 5.1.4. The first assertion now follows from Theorem 5.3.11. The second assertion follows from Remark 5.1.11 and Proposition 5.3.6. □

Remark 5.3.16. The representation (resp. corepresentation) in Theorem 5.3.15 is called the general Yoneda representation (resp. corepresentation) and denoted by \(X\ast A\) (resp. \(X\ast X\)) as
\[
\begin{array}{c|c|c}
[A, X] & \xrightarrow{(X, A)} & [X : A] \\
X^\ast A & \xrightarrow{1} & X^\ast A \\
[X : A] & \xrightarrow{(X, A)} & [X : A]
\end{array}
\]
. The Yoneda representation (resp. corepresentation) in Remark 5.2.16 is identified with a special case of the general Yoneda representation (resp. corepresentation) where \(A\) (resp. \(X\)) is the terminal category (see Remark 5.3.18(2)).

Note. The following is a componentwise description of Theorem 5.3.15.

Corollary 5.3.17. (General Yoneda Lemma).

- Given a module \(M : X \to A\) and a functor \(G : A \to X\), the assignment \(\alpha \mapsto X\ast \alpha\) (see Example 5.3.5(1)) yields a bijection
\[
(G)(X^\ast A)(M) \cong \langle (X) G \rangle \langle X : A \rangle (M)
\]
from the set of right cylinders \(G \sim M\) to the set of module morphisms \(\langle X \rangle G \to M\), whose inverse sends each module morphism \(\Phi : \langle X \rangle G \to M\) to the right cylinder \([\Phi] : G \sim M\) defined by
\[
[\Phi]_a = 1_{(a : G)} : \Phi
\]
for \(a \in [A]\), where \(1_{(a : G)} : \Phi\) is the image of the identity \(X\)-arrow \(a : G \to G \ast a\) (i.e. \(\langle X \rangle G\)-arrow \(a : G \sim a\)) under the function
\[
(a : G)(X)(G : a) = (a : G)(\langle X \rangle G) a \xrightarrow{(a : G)(\Phi)a} (a : G)(M) a
\]
. Moreover, the bijection is natural in \(G\) and \(M\).

- Given a module \(M : X \to A\) and a functor \(F : X \to A\), the assignment \(\alpha \mapsto \alpha \ast A\) (see Example 5.3.5(1)) yields a bijection
\[
(M)(X^\ast A)(F) \cong \langle F(A) \rangle \langle X : A \rangle (M)
\]
from the set of left cylinders \(M \sim F\) to the set of module morphisms \(F(A) \to M\), whose inverse sends each module morphism \(\Phi : F(A) \to M\) to the left cylinder \([\Phi] : M \sim F\) defined by
\[
[\Phi]_x = \Phi : 1_{(F : x)}
5. Yoneda Lemma

for \( x \in \|X\| \), where \( \Phi \cdot 1_{(F \cdot x)} \) is the image of the identity \( A \)-arrow \( x : F \to F \cdot x \) (i.e. \( F(A) \)-arrow \( x \to F \cdot x \)) under the function

\[
x(F(A)) (F \cdot x) \xlongrightarrow{x(\Phi \cdot F \cdot x)} x(\mathcal{M}) (F \cdot x)
\]

Moreover, the bijection is natural in \( F \) and \( \mathcal{M} \).

Proof. The bijective correspondence follows from Theorem 5.3.11 by identifying right cylinders \( G \to \mathcal{M} \) with two-sided cylinders \( G \to 1_A : A \to \mathcal{M} \). The naturality of the bijection in \( G \) and \( \mathcal{M} \) is just a restatement of Theorem 5.3.15.

Remark 5.3.18.

1. Using the notation introduced in Remark 5.3.16, the bijections of the general Yoneda lemma are written as

\[
(G \cdot x \cdot A) (\mathcal{M}) : (G \cdot x \cdot A) (\mathcal{M}) \cong ((X \cdot G) \cdot (X : A) (\mathcal{M})
\]

and

\[
(\mathcal{M} \cdot x \cdot A) (F) : (\mathcal{M} \cdot x \cdot A) (F) \cong (F (A)) \cdot (X : A) (\mathcal{M})
\]

2. The Yoneda lemma (Theorem 5.2.10 and Corollary 5.2.17) is a special case of the general Yoneda lemma where \( A \) (resp. \( X \)) is the terminal category; namely, the bijections in Remark 5.2.18 are identified with

\[
(r \cdot x \cdot *) (\mathcal{M}) \cong ((X \cdot r) \cdot (X : *) (\mathcal{M})
\]

and

\[
(\mathcal{M} \cdot x \cdot A) (r) \cong (r (A)) \cdot (X : A) (\mathcal{M})
\]

(see Remark 5.1.7(3)) respectively, special cases of the bijections in Corollary 5.3.17.

Corollary 5.3.19. Let \( \Theta : \mathcal{M} \to N : X \to A \) be a module morphism and consider the bijection in Corollary 5.3.17.

For any right cylinder \( \alpha : G \to \mathcal{M} \),

\[
X \cdot [\alpha \cdot \Theta] = (X \cdot \alpha) \cdot \Theta
\]

, and for any module morphism \( \Phi : (X \cdot G) \to \mathcal{M} : X \to A \),

\[
[\Phi \cdot \Theta] = [\Phi] \cdot \Theta
\]

For any left cylinder \( \alpha : \mathcal{M} \to F \),

\[
[\alpha \cdot \Theta] \cdot A = (\alpha \cdot A) \cdot \Theta
\]

, and for any module morphism \( \Phi : F (A) \to \mathcal{M} : X \to A \),

\[
[\Phi \cdot \Theta] = [\Phi] \cdot \Theta
\]

Proof. Immediate by the naturality of the bijection. \( \square \)
5. Yoneda Lemma

5.4. Yoneda morphisms for cones

**Definition 5.4.1.** Let $E$ be a category and $M: X \to A$ be a module.

- The right general Yoneda morphism for $(E^a, M)$ (see Definition 4.6.5) is the cell

  $\begin{array}{c}
  X \xrightarrow{\{E^a, M\}} \{E, A\} \\
  X \times \xrightarrow{\times E} \{X|M| \times E^a \} \\
  \{X : - \to X : E\} \xrightarrow{\to M \times E} \{X : E\}
  \end{array}$

  sending each cone $\alpha : r \rightsquigarrow K : E^a \rightsquigarrow M$ to the wedge $\langle X|M| \alpha : \langle X| r \rightsquigarrow \langle M| K : X \to E^*$ (see Example 5.3.5(4)).

- The left general Yoneda morphism for $(E^p, M)$ (see Definition 4.6.5) is the cell

  $\begin{array}{c}
  \{E, X\} \xrightarrow{\{E^p, M\}} A \\
  \{E^p, (M|A) \} \xrightarrow{\times A} \{E : A\} \xrightarrow{\to [E : E]} A
  \end{array}$

  sending each cone $\alpha : K \rightsquigarrow r : E^p \rightsquigarrow M$ to the wedge $\alpha \langle M|A : \langle r| A \rightsquigarrow \langle K|E^* \to A$ (see Example 5.3.5(4)).

**Remark 5.4.2.** By Theorem 4.6.10 and Remark 4.9.2(2),

- the right general Yoneda morphism for $(E^a, M)$ is given by the pasting composition

  $\begin{array}{c}
  X \xrightarrow{\{\Delta_X, X\}} \{E, X\} \xrightarrow{\{E^a, M\}} \{E, A\} \\
  X \times \xrightarrow{\times E} \{X|M| \times E^a \} \\
  \{X : - \to X : E\} \xrightarrow{\to M \times E} \{X : E\}
  \end{array}$

  of the right general Yoneda morphism for $(E, M)$ and the commutative quadrangle in Proposition 2.3.7.

- the left general Yoneda morphism for $(E^p, M)$ is given by the pasting composition

  $\begin{array}{c}
  \{E, X\} \xrightarrow{\{E^p, M\}} \{E, A\} \xleftarrow{\{\Delta_X, A\}} A \\
  \{E^p, (M|A) \} \xrightarrow{\times A} \{E : A\} \xrightarrow{\to [E : E]} A
  \end{array}$

  of the left general Yoneda morphism for $(E, M)$ and the commutative quadrangle in Proposition 2.3.7.

**Proposition 5.4.3.** The right (resp. left) general Yoneda morphism for $(E^a, M)$ (resp. $(E^p, M)$) is fully faithful.
**5. Yoneda Lemma**

*Proof.* The assertion follows from Theorem 5.3.11 on noting Remark 5.4.2 and Proposition 1.2.31.

**Definition 5.4.4.** Let $\mathcal{E}$ and $\mathcal{C}$ be categories.

- The right general Yoneda morphism for $(\mathcal{E}, \mathcal{C})$ (see Definition 4.8.3) is the cell

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathbf{(E^\alpha, C)}} & \mathbf{[E, C]} \\
\xrightarrow{\mathcal{C} \to \mathcal{E}^\alpha} & & \xrightarrow{\mathcal{C} \to \mathcal{E}} \\
\mathbf{[C :]} & \xrightarrow{- \mathbf{(C^E)}} & \mathbf{[C : \mathcal{E}]} \\
\end{array}
$$

sending each cone $\alpha : r \to \mathcal{E}^\alpha \to \mathcal{C}$ to the wedge $(C) \alpha : \mathcal{C} \to \langle C \rangle \mathcal{E}^\alpha \to \mathcal{E}^\alpha$.

- The left general Yoneda morphism for $(\mathcal{E}, \mathcal{C})$ (see Definition 4.8.3) is the cell

$$
\begin{array}{ccc}
\mathcal{E} \times \mathcal{C} & \xrightarrow{\mathbf{(E^\alpha, C)}} & \mathbf{C} \\
\xrightarrow{\mathcal{E} \times \mathcal{C}} & & \xrightarrow{\mathcal{C}} \\
\mathbf{[E : C]} & \xrightarrow{- \mathbf{(E^\alpha C)}} & \mathbf{[C :]} \\
\end{array}
$$

sending each cone $\alpha : \mathcal{K} \to \mathcal{E}^\alpha \to \mathcal{C}$ to the wedge $\alpha (C) : \mathcal{C} \to \langle C \rangle \mathcal{E}^\alpha \to \mathcal{C}$.

**Remark 5.4.5.**

- The right general Yoneda morphism for $(\mathcal{E}, \mathcal{C})$ is just a special instance of the right general Yoneda morphism for $(\mathcal{E}, \mathcal{M})$ where $\mathcal{M}$ is the hom of $\mathcal{C}$; that is,

$$
\mathcal{C} \to \mathcal{E}^\alpha = \langle \mathcal{C} \rangle \mathcal{M} \to \mathcal{E}^\alpha
$$

(cf. Proposition 5.3.7), and obtained from the hom of the right general Yoneda functor for $[\mathcal{E}, \mathcal{C}]$ by the pasting composition

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathbf{[C^E, C]}} & \mathbf{[E, C]} \\
\xrightarrow{\mathcal{C} \to \mathcal{E}} & & \xrightarrow{\mathcal{C} \to \mathcal{E}} \\
\mathbf{[C :]} & \xrightarrow{- \mathbf{[C^E]}} & \mathbf{[C : \mathcal{E}]} \\
\end{array}
$$

- The left general Yoneda morphism for $(\mathcal{E}, \mathcal{C})$ is just a special instance of the left general Yoneda morphism for $(\mathcal{E}, \mathcal{M})$ where $\mathcal{M}$ is the hom of $\mathcal{C}$; that is,

$$
\mathcal{E}^\alpha \times \mathcal{C} = \mathcal{E}^\alpha \times \langle \langle \mathcal{C} \rangle \mathcal{C} \rangle
$$

(cf. Proposition 5.3.7), and obtained from the hom of the left general Yoneda functor for $[\mathcal{E}, \mathcal{C}]$ by the pasting composition

$$
\begin{array}{ccc}
\mathcal{E} \times \mathcal{C} & \xrightarrow{\mathbf{(E^\alpha, C)}} & \mathbf{[E, C]} \\
\xrightarrow{\mathcal{E} \times \mathcal{C}} & & \xrightarrow{\mathcal{E} \times \mathcal{C}} \\
\mathbf{[E : C]} & \xrightarrow{- \mathbf{(E^\alpha C)}} & \mathbf{[C :]} \\
\end{array}
$$
Proposition 5.4.6. Given a category $E$ and a module $M : X \to A$, the triangle 

$$
\begin{array}{ccc}
\langle E^a, M \rangle & \to & \langle E^a, (X : M) \rangle \\
\langle X : E^a \rangle & \to & \langle X : M \rangle
\end{array}
$$

commutes; that is, the composition 

$$
\begin{array}{ccc}
X & \to & \langle E^a, M \rangle \\
\downarrow & & \downarrow \\
\langle X : E^a \rangle & \to & \langle X : M \rangle
\end{array}
$$

of the right general Yoneda morphism for $(E^a, M)$ and the right exponential transposition of wedges $X \to E^a$ (see Notation 4.9.3) yields the cell 

$$
\begin{array}{ccc}
\langle E^a, M \rangle & \to & \langle E^a, M \rangle \\
\langle X : E^a \rangle & \to & \langle X : M \rangle
\end{array}
$$

postcomposition with the right Yoneda morphism for $M$. 

- the triangle 

$$
\begin{array}{ccc}
\langle E^a, M \rangle & \to & \langle E^a, (X : M) \rangle \\
\langle E^a, (X : M) \rangle & \to & \langle E^a, (X : M) \rangle
\end{array}
$$

commutes; that is, the composition 

$$
\begin{array}{ccc}
[\langle E^a, M \rangle] & \to & [\langle E^a, M \rangle] \\
\langle E^a, (X : M) \rangle & \to & \langle E^a, (X : M) \rangle
\end{array}
$$

of the left general Yoneda morphism for $(E^a, M)$ and the left exponential transposition of wedges $E^a \to A$ (see Notation 4.9.3) yields the cell 

$$
\begin{array}{ccc}
[\langle E^a, M \rangle] & \to & [\langle E^a, M \rangle] \\
\langle E^a, (X : M) \rangle & \to & \langle E^a, (X : M) \rangle
\end{array}
$$

170
5. Yoneda Lemma

, postcomposition with the left Yoneda morphism for $\mathcal{M}$.

Proof. Consider the cells and commutative diagrams as in

\[
\begin{array}{ccc}
X & \overset{[\Delta E, X]}{\longrightarrow} & [E, X] \\
\downarrow X^\times & & \downarrow (E, \times) \\
[X:] & \longrightarrow (X \cdot X^\times) & \longrightarrow [X : E] \\
\end{array}
\]

by Definition 4.6.13. The assertion now follows from Proposition 1.2.32.

Note. If $E$ is a small category and $\mathcal{M} : X \to A$ is a locally small category, then the module $(E, \mathcal{M})$ is locally small and we have two Yoneda morphisms for $(E, \mathcal{M})$ depicted in

\[
\begin{array}{ccc}
X & \overset{[\Delta E, X]}{\longrightarrow} & [E, X] \\
\downarrow X^\times & & \downarrow (E, \times) \\
[X:] & \longrightarrow (X \cdot X^\times) & \longrightarrow [X : E] \\
\end{array}
\]

by Remark 5.4.2 and Remark 4.9.4, and the vertical composition yields

\[
\begin{array}{ccc}
X & \overset{[\Delta E, X]}{\longrightarrow} & [E, X] \\
\downarrow X^\times & & \downarrow (E, \times) \\
[X:] & \longrightarrow (X \cdot X^\times) & \longrightarrow [X : E] \\
\end{array}
\]

by Theorem 5.3.8, which then produces

\[
\begin{array}{ccc}
X & \overset{[\Delta E, X]}{\longrightarrow} & [E, X] \\
\downarrow X^\times & & \downarrow (E, \times) \\
[X:] & \longrightarrow (X \cdot X^\times) & \longrightarrow [X : E] \\
\end{array}
\]

by Definition 4.6.13. The assertion now follows from Proposition 1.2.32. \qed

Note. If $E$ is a small category and $\mathcal{M} : X \to A$ is a locally small category, then the module $(E, \mathcal{M})$ is locally small and we have two Yoneda morphisms for $(E, \mathcal{M})$ depicted in

\[
\begin{array}{ccc}
X & \overset{[\Delta E, X]}{\longrightarrow} & [E, X] \\
\downarrow X^\times & & \downarrow (E, \times) \\
[X:] & \longrightarrow (X \cdot X^\times) & \longrightarrow [X : E] \\
\end{array}
\]

, one defined in Definition 5.2.5 and the other in Definition 5.4.1. The following reconciles these two Yoneda morphisms.
5. Yoneda Lemma

**Theorem 5.4.7.** Let $E$ be a small category and $M : X \to A$ be a locally small module.

- The right general Yoneda morphism for $(E^\circ, M)$ is obtained by “pasting” the cylinder $[M \cdot E]^\circ$ defined in Definition 4.9.11 to the right Yoneda morphism for $(E^\circ, M)$ as shown in

  $$
  \begin{array}{ccc}
  X \ar@{}[rr]_{(E^\circ, M)} & \ar@{-->}[rr] & \ar@{}[rr]_{[E, A]} \\
  \ar@{}[rr]_{(X | (E^\circ, M))} \ar@{-->}[rr] & & \ar@{}[rr]_{[M \cdot E]} \\
  [X] \ar@{}[rr]_{(E^\circ, M)} & \ar@{-->}[rr] & \ar@{}[rr]_{[X : E]} \\
  \end{array}
  $$

  ; that is, given a cone $\alpha : r \sim K : E^\circ \sim M$, the wedge

  $(X | M) \alpha : (X) r \sim (M) K : X \to E^*$

  is obtained by the composition of the right module morphism

  $(X | (E^\circ, M)) \alpha : (X) r \to (E^\circ, M) (K) : X \to *$

  and the wedge

  $[M \cdot E]^\circ : (E^\circ, M) (K) \sim (M) K : X \to E^*$

- The left general Yoneda morphism for $(E^\circ, M)$ is obtained by “pasting” the cylinder $[E \cdot M]^\circ$ defined in Definition 4.9.11 to the left Yoneda morphism for $(E^\circ, M)$ as shown in

  $$
  \begin{array}{ccc}
  [E, A] \ar@{}[rr]_{(E^\circ, M)} & \ar@{-->}[rr] & \ar@{}[rr]_{A} \\
  \ar@{}[rr]_{(E \cdot M)} \ar@{-->}[rr] & & \ar@{}[rr]_{A} \\
  [E : A]^- \ar@{}[rr]_{(E^\circ, M)} & \ar@{-->}[rr] & \ar@{}[rr]_{A^-} \\
  \end{array}
  $$

  ; that is, given a cone $\alpha : K \sim r : E^\circ \sim M$, the wedge

  $\alpha (M | A) : r (A) \sim K (M) : E^* \to A$

  is obtained by the composition of the left module morphism

  $\alpha ((E^\circ, M) | A) : r (A) \to (K) (E^\circ, M) : * \to A$

  and the wedge

  $[E \cdot M]^\circ : (K) (E^\circ, M) \sim K (M) : E^* \to A$

**Proof.** We need to show that

$$
\alpha ((X | M) \alpha) e = x ((X | (E^\circ, M)) \alpha) : x ([M \cdot E]^\circ) e
$$

for $x \in \|X\|$ and $e \in \|E\|$. But, for an $X$-arrow $h : x \to r$,

$$
h : x ((X | M) \alpha) e = h \circ \alpha_e
$$

and, by Remark 4.9.12,

$$
h : x ((X | (E^\circ, M)) \alpha) : x ([M \cdot E]^\circ) e = (h \circ \alpha) : (E^\circ_{x(M)} \circ) e = (h \circ \alpha)_e = h \circ \alpha_e
$$
5. Yoneda Lemma

5.5. Correspondences between frames and cells

Note. The following is a special case of the general Yoneda lemma (Corollary 5.3.17) where \( \mathcal{M} \) is a representable module.

**Theorem 5.5.1.** Given a pair of functors

\[
\begin{array}{c}
X \xrightarrow{G} A \\
\downarrow F
\end{array}
\]

- the assignment \( \epsilon \mapsto X \rhd \epsilon \) (see Example 5.3.5(2)) yields a bijection

\[
(G \circ F)(A, A)(1_A) \cong (\langle X \rangle G \langle X : A \rangle (F(A))
\]

from the set of natural transformations \( G \circ F \to 1_A : A \to A \) to the set of module morphisms \( (X) G \to F(A) : X \to A \), whose inverse sends each module morphism \( \Phi : (X) G \to F(A) \) to the natural transformation \([\Phi] : G \circ F \to 1_A \) defined by

\[
[\Phi]_a = 1_{(a \cdot G)} \cdot \Phi
\]

for \( a \in [A] \), where \( 1_{(a \cdot G)} \cdot \Phi \) is the image of the identity \( X \text{-arrow} a : G \to G : a \) (i.e. \( (X) G \text{-arrow} a : G \to a \)) under the function

\[
(a : G) \langle X \rangle (a : G) = (a : G) (\langle X \rangle G a) (a : G) (F(A)) a = (a : G) (F(A)) a
\]

Moreover, the bijection is natural in \( G \) and \( F \).

- the assignment \( \eta \mapsto \eta \gambar A \) (see Example 5.3.5(2)) yields a bijection

\[
(1_X) \langle X, X \rangle (G \circ F) \cong (F(A)) \langle X : A \rangle (\langle X \rangle G)
\]

from the set of natural transformations \( 1_X \to G \circ F : X \to X \) to the set of module morphisms \( F(A) \to (X) G : X \to A \), whose inverse sends each module morphism \( \Phi : F(A) \to (X) G \) to the natural transformation \([\Phi] : 1_X \to G \circ F \) defined by

\[
[\Phi]_a = \Phi \cdot 1_{(G : a)}
\]

for \( a \in [X] \), where \( \Phi \cdot 1_{(G : a)} \) is the image of the identity \( A \text{-arrow} x : F \to F : x \) (i.e. \( F(A) \text{-arrow} x \Rightarrow F : x \)) under the function

\[
(x : F) (A) (F : x) = x (F(A)) (F : x) \xrightarrow{x \cdot \Phi (F : x)} x (\langle X \rangle G (F : x)) = x (\langle X \rangle G : F : x)
\]

Moreover, the bijection is natural in \( G \) and \( F \).

**Proof.** Replacing \( \mathcal{M} \) with \( F(A) \) in Corollary 5.3.17, we have a bijection

\[
(G) \langle X \cdot A \rangle (F(A)) \cong (\langle X \rangle G \langle X : A \rangle (F(A))
\]

, natural in \( G \) and \( F \). But by definition

\[
(G) \langle X \cdot A \rangle (F(A)) = \prod_A G(F(A)) = \prod_A [G \circ F(A)] [1_A] = (G \circ F(A), A)(1_A)
\]

(see Example 4.3.5(2)). \( \square \)
Note. The following is a special case of the general Yoneda lemma (Corollary 5.3.17) where $\mathcal{M}$ is given by the composite module $S(\mathcal{M}) T$.

**Theorem 5.5.2.** Given a functor $F : E \to D$ and a module $\mathcal{M} : X \to A$,

- there is a canonical module isomorphism

$$\psi_{\mathcal{M}}^F : (F^\circ, \mathcal{M}) \cong ([D, F, \mathcal{M}] : [D, X] \to [E, A])$$

, natural in $F$ and $\mathcal{M}$, giving for each pair of functors $S : D \to X$ and $T : E \to A$ a bijection

$$(S) \psi_{\mathcal{M}}^F (T) : (S)(F^\circ, \mathcal{M})(T) \cong (S)([D, F, \mathcal{M}](T))$$

from the set of cylinders $F \circ S \to T : E \to \mathcal{M}$ to the set of cells $S \to T : (D)F \to \mathcal{M}$. Specifically, the bijection sends each cylinder

$$\begin{array}{c}
D \xrightarrow{F} E \\
S \downarrow \alpha \downarrow T \\
X \xrightarrow{\mathcal{M}} A
\end{array}$$

to the cell

$$\begin{array}{c}
D \xrightarrow{(D)F} E \\
S \downarrow \downarrow \downarrow T \\
X \xrightarrow{\mathcal{M}} A
\end{array}$$

in Example 5.3.5(3); the inverse sends each cell

$$\begin{array}{c}
D \xleftarrow{F} E \\
S \downarrow \Theta \downarrow T \\
X \xrightarrow{\mathcal{M}} A
\end{array}$$

to the cylinder

$$\begin{array}{c}
D \xleftarrow{F} E \\
S \downarrow [\Theta] \downarrow T \\
X \xrightarrow{\mathcal{M}} A
\end{array}$$

defined by

$$[\Theta]_e = 1_{(e : F)^\circ} \cdot \Theta$$

for $e \in [E]$, where $1_{(e : F)^\circ} \cdot \Theta$ is the image of the identity $D$-arrow $e : F \to F \cdot e$ (i.e. $(D)F$-arrow $e : F \to e$) under the function

$$(e : F)(D)(F \cdot e) = (e : F)([D](M) \cdot T) e = (e : F)(S(\mathcal{M}) T) e = (e : F \cdot S)(\mathcal{M})(T \cdot e)$$

Moreover, $\psi_{\mathcal{M}}^F$ is natural in $\mathcal{M}$ with $\mathcal{M}$ varying in $\text{MOD}$; that is, the quadrangle

$$\begin{array}{c}
(F^\circ, \mathcal{M}) \xrightarrow{\psi_{\mathcal{N}}^F} ([D, F, \mathcal{M}] : [D, X] \to [E, A]) \\
\downarrow \downarrow \downarrow \downarrow \\
(F^\circ, \mathcal{N}) \xrightarrow{\psi_{\mathcal{N}}^F} ([D, F, \mathcal{N}] : [D, X] \to [E, A])
\end{array}$$

commutes for every cell $\Phi : \mathcal{M} \to \mathcal{N}$.
5. Yoneda Lemma

There is a canonical module isomorphism

\[ \Psi^F_M : (F^r, M) \cong (F(D), M) : [E, X] \to [D, A] \]

natural in \( F \) and \( M \), giving for each pair of functors \( S : D \to X \) and \( T : E \to A \) a bijection

\[ (S)(\Psi^F_M)(T) : (S)(F^r, M)(T) \cong (S)(F(D), M)(T) \]

from the set of cylinders \( S \sim T \circ F : E \to M \) to the set of cells \( S \sim T : F(D) \to M \). Specifically, the bijection sends each cylinder

\[
\begin{array}{c}
E \xrightarrow{F} D \\
S \downarrow \alpha \downarrow T \\
X \xrightarrow{M} A
\end{array}
\]

to the cell

\[
\begin{array}{c}
E \xrightarrow{F(D)} D \\
S \downarrow \alpha \Downarrow \downarrow T \\
X \xrightarrow{M} A
\end{array}
\]

in Example 5.3.5(3); the inverse sends each cell

\[
\begin{array}{c}
E \xrightarrow{F(D)} D \\
S \downarrow \Theta \downarrow T \\
X \xrightarrow{M} A
\end{array}
\]

to the cylinder

\[
\begin{array}{c}
E \xrightarrow{F} D \\
S \downarrow [\Theta] \downarrow T \\
X \xrightarrow{M} A
\end{array}
\]

defined by

\[ [\Theta]_e = \Theta \cdot 1_{(F \cdot e)} \]

for \( e \in \|E\| \), where \( \Theta \cdot 1_{(F \cdot e)} \) is the image of the identity \( D \)-arrow \( e \cdot F \to F \cdot e \) (i.e. \( F(D) \)-arrow \( e \sim F \cdot e \)) under the function

\[ (e : F)(D)(F \cdot e) = e(F(D))(F \cdot e) \xrightarrow{e(\Theta)(F \cdot e)} e(S(M)T)(F \cdot e) = (e \cdot S)(M)(T \cdot F \cdot e) \]

Moreover, \( \Psi^F_M \) is natural in \( M \) with \( M \) varying in \( MOD \); that is, the quadrangle

\[
\begin{array}{c}
(F^r, M) \xrightarrow{\Psi^F_M} (F(D), M) \\
(F^r, \Phi) \downarrow \downarrow (F(D), \Phi) \\
(F^r, N) \xrightarrow{\Psi^F_N} (F(D), N)
\end{array}
\]

commutes for every cell \( \Phi : M \to N \).
5. Yoneda Lemma

Proof. Replacing $\mathcal{M}$ with $S(\mathcal{M}) T$ in Corollary 5.3.17, we have a bijection

$$(F) (D, E) (S(\mathcal{M}) T) \cong ((D) F) (D : E) (S(\mathcal{M}) T)$$

, natural in $F, S, T,$ and $\mathcal{M}$. But

$$(F) (D, E) (S(\mathcal{M}) T) = \prod_{E} F (S(\mathcal{M}) T) = (F \circ S) (E, \mathcal{M}) (T) = (S) (F^*, \mathcal{M}) (T)$$

and

$$((D) F) (D : E) (S(\mathcal{M}) T) = (S) ((D) F, \mathcal{M}) (T)$$

by the definitions. Hence an isomorphism $\Psi_{M}^{F} : (F^*, \mathcal{M}) \cong ((D) F, \mathcal{M})$ is given by

$$(S) (\Psi_{M}^{F}) (T) = (F) (D : E) (S(\mathcal{M}) T)$$

(see Remark 5.3.18(1) for $D : E$). The last assertion holds, since, for any cell

$$\begin{array}{c}
X - M - A \\
| P | \phi | Q \\
Y - N - B
\end{array}$$

, the quadrangle

$$\begin{array}{ccc}
(S) (F^*, \mathcal{M}) (T) & \overset{(S) (\Psi_{M}^{F}) (T)}{\longrightarrow} & (S) ((D) F, \mathcal{M}) (T) \\
\downarrow (S) (F^*, \phi) (T) & & \downarrow (S) ((D) F, \phi) (T) \\
(S \circ P) (F^*, \mathcal{N}) (Q \circ T) & \overset{(S \circ P) (\Psi_{M}^{F}) (T)}{\longrightarrow} & (S \circ P) ((D) F, \mathcal{N}) (Q \circ T)
\end{array}$$

, i.e.

$$\begin{array}{ccc}
(F) (D, E) (S(\mathcal{M}) T) & \overset{(F) (D : E) (S(\mathcal{M}) T)}{\longrightarrow} & ((D) F) (D : E) (S(\mathcal{M}) T) \\
\downarrow (F) (D, E) (S(\phi) T) & & \downarrow ((D) F) (D : E) (S(\phi) T) \\
(F) (D, E) (S(P, \mathcal{N}) Q) T & \overset{(F) (D : E) (S(P, \mathcal{N}) Q) T}{\longrightarrow} & ((D) F) (D : E) (S(P, \mathcal{N}) Q) T
\end{array}$$

commutes by the naturality of $D : E$. $\square$

Remark 5.5.3.

1. Although derived as a special case of the general Yoneda lemma, Theorem 5.5.2 is more versatile; Corollary 5.3.17 turns to be a special case of the bijection in Theorem 5.5.2 where $S$ and $T$ are the identities.

2. Theorem 5.3.11 is viewed as a special case of Theorem 5.5.2 by depicting a cylinder

$$\begin{array}{c}
E \\
\alpha \\
S \quad \quad T \\
X - M - A
\end{array}$$

176
5. Yoneda Lemma

as

\[
\begin{array}{c}
X \xrightarrow{S} E \\
\downarrow \alpha \downarrow T \\
\xrightarrow{\mathcal{M}} A
\end{array}
\quad
\begin{array}{c}
E \xrightarrow{T} A \\
\downarrow S \downarrow \alpha \downarrow 1 \\
\xrightarrow{\mathcal{M}} A
\end{array}
\]

- The category \( X \) acts on \( \alpha \) and generates the cell

\[
\begin{array}{c}
X \xrightarrow{(X)S} E \\
\downarrow X{\mid}\alpha \downarrow T \\
\xrightarrow{\mathcal{M}} A
\end{array}
\]

direct along \( \alpha \), and the assignment \( \alpha \mapsto X{\mid}\alpha \) yields a bijection from the set of cylinders \( S \sim T : E \sim \mathcal{M} \) to the set of cells \( 1_X \sim T : (X) S \to \mathcal{M} \), i.e. the set of module morphisms \( (X) S \to (\mathcal{M}) T : X \to E \).

- The category \( A \) acts on \( \alpha \) and generates the cell

\[
\begin{array}{c}
E \xrightarrow{T(A)} A \\
\downarrow S \downarrow \alpha{\mid}A \downarrow 1 \\
\xrightarrow{\mathcal{M}} A
\end{array}
\]

inverse along \( \alpha \), and the assignment \( \alpha \mapsto \alpha{\mid}A \) yields a bijection from the set of cylinders \( S \sim T : E \sim \mathcal{M} \) to the set of cells \( S \sim 1_A : (A) \to \mathcal{M} \), i.e. the set of module morphisms \( T(A) \to S(\mathcal{M}) : E \to A \).

3. Theorem 5.5.1 is also be viewed as a special case of Theorem 5.5.2 by depicting natural transformations \( \epsilon : G \circ F \to 1_A : A \to A \) and \( \eta : 1_X \to G \circ F : X \to X \) as

\[
\begin{array}{c}
X \xleftarrow{G} A \\
\downarrow F \downarrow \epsilon \downarrow 1 \\
A \xrightarrow{(A)} A
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{F} A \\
\downarrow 1 \downarrow \eta \downarrow G \\
X \xrightarrow{(X)} X
\end{array}
\]

- The category \( X \) acts on \( \epsilon \) and generates the cell

\[
\begin{array}{c}
X \xrightarrow{(X)G} A \\
\downarrow F \downarrow X{\mid}\epsilon \downarrow 1 \\
\xrightarrow{\mathcal{M}} A
\end{array}
\]

direct along \( \epsilon \), and the assignment \( \epsilon \mapsto X{\mid}\epsilon \) yields a bijection from the set of natural transformations \( G \circ F \to 1_A : A \to A \) to the set of cells \( F \sim 1_A : (X) G \to (A) \), i.e. the set of module morphisms \( (X) G \to F(A) : X \to A \).

- The category \( A \) acts on \( \eta \) and generates the cell

\[
\begin{array}{c}
X \xrightarrow{F(A)} A \\
\downarrow 1 \downarrow \eta{\mid}A \downarrow G \\
X \xrightarrow{(X)} X
\end{array}
\]

177
inverse along $\eta$, and the assignment $\eta \mapsto \eta \cdot A$ yields a bijection from the set of natural transformations $1_X \to G \circ F : X \to X$ to the set of cells $1_X \mapsto (X) \to (X)$, i.e. the set of module morphisms $F(A) \to (X) G : (X) \to A$.

**Corollary 5.5.4.** Given a category $E$ and a module $M : X \to A$, there is a canonical module isomorphism

$$\psi^E_M : (E, M) \rightarrow ([E, M] : [E, A])$$

, giving for each pair of functors $S : E \to X$ and $T : E \to A$ a bijection

$$(S)(\psi^E_M)(T) : (S)(E, M)(T) \cong (S)((E), (M))(T)$$

from the set of cylinders $S \sim T : E \sim M$ to the set of cells $S \sim T : (E) \sim M$. Specifically, the bijection sends each cylinder

```
X -------- M -------- A
|         |         |
S|         |         | T
|         |         |
E -------- (E)
```

which maps each $E$-arrow $h : e \to e'$ to the $M$-arrow

$$(h : S) \circ \alpha_{e'} = \alpha_e \circ (T \cdot h) : e : S \sim T \cdot e'$$

as indicated in

```
e -------- e : S -------- T : e -------- e
|         |         |         |
h|         |         |         | h
|         |         |         |
e' -------- e' : S -------- T : e' -------- e'
```

; the inverse sends each cell

```
X -------- M -------- A
|         |         |
S|         |         | T
|         |         |
E -------- (E)
```

to the cylinder

```
X -------- M -------- A
|         |         |
S| [\Theta]|         | T
|         |         |
E -------- [\Theta]
```

defined by

$$[\Theta]_e = 1_e \circ \Theta$$

for $e \in |E|$, where $1_e \circ \Theta$ is the image of the identity $e \to e$ under the function

$$e \cdot (E) e \xrightarrow{e(\Theta)e} e \cdot (S(M) T) e = (e : S)(M)(T \cdot e)$$
Moreover, $\Psi^E_M$ is natural in $M$ with $M$ varying in $\text{MOD}$; that is, the quadrangle

$$
\begin{array}{ccc}
\langle E, M \rangle & \xrightarrow{\Psi^E_M} & \langle \langle E \rangle, M \rangle \\
\langle E, \Phi \rangle & \xrightarrow{\Phi} & \langle \langle E \rangle, \Phi \rangle \\
\langle E, N \rangle & \xrightarrow{\Psi^E_N} & \langle \langle E \rangle, N \rangle
\end{array}
$$

commutes for every cell $\Phi : M \to N$.

**Proof.** Since the hom $\langle E \rangle$ is represented and corepresented by the identity $E \to E$, the assertion follows from Theorem 5.5.2 by depicting $\alpha$ as

$$
\begin{array}{ccc}
E & \xrightarrow{1} & E \\
S & \xrightarrow{\alpha} & T \\
X & \xrightarrow{\exists M} & A
\end{array}
\quad
\begin{array}{ccc}
E & \xrightarrow{1} & E \\
S & \xrightarrow{\alpha} & T \\
X & \xrightarrow{\exists M} & A
\end{array}
$$

(cf. Remark 5.5.3(2)). The cell $\{ \alpha \}$ is given by $E \| \alpha = \alpha \| E$. The module isomorphism $\Psi^E_M$ is defined by

$$
\Psi^E_M = \Psi_{\exists M}^{[1E]} = \Psi_{\exists M}^{[1E]^*}
$$

. \qed

**Corollary 5.5.5.** Given a category $E$ and a module $M : X \to A$,

- there is a canonical module isomorphism

$$
\Psi^E_M : \langle E^\circ, M \rangle \cong \langle \ast E, M \rangle : X \to [E, A]
$$

(see Example 1.1.18(8) for $\ast E$), giving for each pair of an object $x \in \| X \|$ and a functor $K : E \to A$ a bijection

$$
(x) \langle \Psi^E_M \rangle (K) : (x) \langle E^\circ, M \rangle (K) \cong (x) \langle \ast E, M \rangle (K)
$$

from the set of cones $x \searrow K : E^\circ \searrow M$ to the set of conical cells $x \searrow K : \ast E \to M$. Specifically, the bijection sends each cone

$$
\begin{array}{ccc}
\ast & \xrightarrow{\Delta} & E \\
\downarrow x & \xrightarrow{\alpha} & \downarrow K \\
X & \xrightarrow{\exists M} & A
\end{array}
$$

to the conical cell

$$
\begin{array}{ccc}
\ast & \xrightarrow{\ast E} & E \\
\downarrow x & \xrightarrow{\ast \alpha} & \downarrow K \\
X & \xrightarrow{\exists M} & A
\end{array}
$$

which maps each $\ast E$-arrow $1 : \ast \to e$ to the component of $\alpha$ at $e$; the inverse sends each conical cell

$$
\begin{array}{ccc}
\ast & \xrightarrow{\ast E} & E \\
\downarrow x & \xrightarrow{\Theta} & \downarrow K \\
X & \xrightarrow{\exists M} & A
\end{array}
$$
5. Yoneda Lemma

to the cone

\[
\begin{array}{c}
\ast \leftarrow \Delta \\
\times & [\Theta] & \kappa \\
X & \rightarrow \mathcal{M} & \rightarrow A
\end{array}
\]

defined by

\[\left[[\Theta]\right]_e = 1_\ast : \Theta\]

for \(e \in [E]\), where \(1_\ast : \Theta\) is the image of the \((\ast E)\)-arrow \(1_\ast : \ast \rightarrow e\) under the function

\[(\ast E) e \xrightarrow{\Theta e} (x (M) K) e = x (M) (K : e)\]

Moreover, \(\Psi^E_{\mathcal{M}}\) is natural in \(\mathcal{M}\) with \(\mathcal{M}\) varying in \(\text{MOD}\); that is, the quadrangle

\[
\begin{array}{c}
(E^\circ, M) \xrightarrow{\Psi^E_{\mathcal{M}}} (\ast E, M) \\
(E^\circ, \Phi) \downarrow & \downarrow (\ast E, \Phi) \\
(E^\circ, N) \xrightarrow{\Psi^E_{\mathcal{N}}} (\ast E, N)
\end{array}
\]

commutes for every cell \(\Phi : M \rightarrow N\).

There is a canonical module isomorphism

\[
\Psi^E_{\mathcal{M}} : (E^\circ, M) \cong (E^\ast, M) : [E, X] \rightarrow A
\]

(see Example 1.1.18(8) for \(E^\ast\)), giving for each pair of an object \(a \in \parallel A\) and a functor \(K : E \rightarrow X\) a bijection

\[
(K) (\Psi^E_{\mathcal{M}}) (a) : (K) (E^\circ, M) (a) \cong (K) (E^\ast, M) (a)
\]

from the set of cones \(K \rhd a : E^\circ \rhd M\) to the set of conial cells \(K \rhd a : E^\ast \rhd M\). Specifically, the bijection sends each cone

\[
\begin{array}{c}
E \xrightarrow{\Delta} \ast \\
\kappa \downarrow & \alpha \downarrow a \\
X & \rightarrow \mathcal{M} & \rightarrow A
\end{array}
\]

to the conical cell

\[
\begin{array}{c}
E \xrightarrow{E^\ast} \ast \\
\kappa \downarrow & \alpha \downarrow a \\
X & \rightarrow \mathcal{M} & \rightarrow A
\end{array}
\]

which maps each \(E^\ast\)-arrow \(1_\ast : e \rightarrow \ast\) to the component of \(\alpha\) at \(e\); the inverse sends each conical cell

\[
\begin{array}{c}
E \xrightarrow{E^\ast} \ast \\
\kappa \downarrow & \Theta \downarrow a \\
X & \rightarrow \mathcal{M} & \rightarrow A
\end{array}
\]
5. Yoneda Lemma

to the cone

\[
\begin{array}{ccc}
E & \xrightarrow{\Delta} & * \\
\kappa \downarrow & [\Theta] & \downarrow a \\
X & \xrightarrow{\Delta_M} & A
\end{array}
\]

defined by

\[
[\Theta]_e = \Theta \cdot 1_*,
\]

for \( e \in [E] \), where \( \Theta \cdot 1_* \) is the image of the \((E*)\)-arrow \( 1_* : e \to * \) under the function

\[
e \langle E* \rangle \xrightarrow{e(\Theta)} \langle K(M) a \rangle = \langle e : K \rangle (M) a
\]

Moreover, \( \Psi^E_M \) is natural in \( M \) with \( M \) varying in \( \text{MOD} \); that is, the quadrangle

\[
\begin{array}{ccc}
\langle E^*,M \rangle & \xrightarrow{\Psi^E_M} & \langle E^*,M \rangle \\
\langle E^*,N \rangle & \xrightarrow{\Psi^E_N} & \langle E^*,N \rangle
\end{array}
\]

commutes for every cell \( \Phi : M \to N \).

Proof. Since \( *E \) is the corepresentable of \( \Delta_E \), the assertion follows from Theorem 5.5.2 with the module isomorphism \( \Psi^E_M \) defined by

\[
\Psi^E_M = \psi_{M}^{[\Delta_E]^*}
\]

\[\Box\]

Theorem 5.5.6. Given an endomodule \( M : E \to E \), there is a canonical bijection

\[\Phi^E_M : \prod_E M \cong ([E]) (E : E)(M)\]

from the set of frames of \( M \) to the set of module morphisms \( (E) \to M : E \to E \). The bijection sends each frame \( \alpha \) of \( M \) to the module morphism \( \langle \alpha \rangle : (E) \to M \) which maps each \( E \)-arrow \( h : e \to e' \) to the \( M \)-arrow \( h \circ \alpha_e = \alpha_{e'} \circ h : e \to e' \) as indicated in

\[
e \sim_{\alpha_e} \sim \xrightarrow{\alpha_e} e \\
\h \downarrow \quad \h \cdot (\alpha) \quad \downarrow h \\
e' \sim_{\alpha_{e'}} \sim \xrightarrow{\alpha_{e'}} e'
\]

the inverse sends each module morphism \( \Phi : (E) \to M \) to the frame \( [\Phi] \) of \( M \) defined by

\[
[\Phi]_e = 1_e : \Phi
\]

for \( e \in [E] \), where \( 1_e : \Phi \) is the image of the identity \( e \to e \) under the function

\[
e \langle \Phi \rangle e : e(E) e \to e (M) e
\]

Moreover, the bijection is natural in \( M \).
5. **Yoneda Lemma**

**Proof.** The assertion follows from Corollary 5.5.4 by setting $X = A = E$ and $S = T = 1_E$ and noting that $(1_E)(E, M)(1_E) = \prod_E M$ and $(1_E)((E, M)(1_E) = ((E))(E)(M)$ by the definitions. \hfill \Box

**Theorem 5.5.7.**

- **Given an endomodule $M : \ast \to E^- \times E$, there is a canonical bijection**

  \[ \Phi^E_M : \prod_E M \cong ((E)) \langle E^- \times E \rangle (M) \]

  from the set of cylindrical frames of $M$ to the set of module morphisms $(E) \to M : \ast \to E^- \times E$. The bijection sends each cylindrical frame $\alpha$ of $M$ to the module morphism $(\alpha) : (E) \to M$ which maps each $E$-arrow $h : e \to e'$ to the $M$-arrow $\alpha_e \circ (h, e') = \alpha_e \circ (e, h) : \ast \to (e, e')$ as indicated in

  \[ \begin{array}{ccc}
  \ast & \xrightarrow{\alpha_e} & (e, e) \\
  \downarrow{\alpha_e} & \searrow{h \cdot (\alpha)} & (e, h) \\
  (e', e') & \xleftarrow{h \cdot (\alpha)} & (e, e') \\
  \end{array} \]

  . The inverse sends each module morphism $\Phi : (E) \to M$ to the cylindrical frame $[\Phi]$ of $M$ defined by

  \[ [\Phi]_e = \Phi \cdot 1_e \]

  for $e \in [E]$, where $\Phi \cdot 1_e$ is the image of the identity $e \to e$ under the function

  \[ \langle \Phi \rangle (e, e) : (E)(e, e) \to (M)(e, e) \]

  . Moreover, the bijection is natural in $M$.

- **Given an endomodule $M : E \times E^- \to \ast$, there is a canonical bijection**

  \[ \Phi^E_M : \prod_E M \cong ((E)) \langle E \times E^- \rangle (M) \]

  from the set of cylindrical frames of $M$ to the set of module morphisms $(E) \to M : E \times E^- \to \ast$. The bijection sends each cylindrical frame $\alpha$ of $M$ to the module morphism $(\alpha) : (E) \to M$ which maps each $E$-arrow $h : e \to e'$ to the $M$-arrow $(h, e') \circ \alpha_e = (e, h) \circ \alpha_e : (e, e') \to \ast$ as indicated in

  \[ \begin{array}{ccc}
  (e, e') & \xleftarrow{(h, e')} & (e, e) \\
  \downarrow{h \cdot (\alpha)} & \swarrow{\alpha_e} & (e, h) \\
  (e', e') & \xrightarrow{\alpha_e} & (e, e') \\
  \end{array} \]

  . The inverse sends each module morphism $\Phi : (E) \to M$ to the cylindrical frame $[\Phi]$ of $M$ defined by

  \[ [\Phi]_e = 1_e \cdot \Phi \]

  for $e \in [E]$, where $1_e \cdot \Phi$ is the image of the identity $e \to e$ under the function

  \[ (e, e) \langle \Phi \rangle : (e, e)(E) \to (e, e)(M) \]

  . Moreover, the bijection is natural in $M$. 

182
5. Yoneda Lemma

Proof. This is a restatement of Theorem 5.5.6 with $\mathcal{M} : E \to E$ regarded as a right module $\mathcal{M} : E \times E^* \to \ast$. □

**Theorem 5.5.8.** Given a category $E$ and a module $\mathcal{M} : X \to A$,

- there is a canonical module isomorphism

$$
\Psi_{\mathcal{M}}^{E^+} : (E^+, \mathcal{M}) \cong ((E), \mathcal{M}) : X \to [E^* \times E, A]
$$

, natural in $\mathcal{M}$, giving for each pair of an object $x \in \|X\|$ and a bifunctor $K : E^* \times E \to X$ a bijection

$$(x) \left(\Psi_{\mathcal{M}}^{E^+}\right)(K) : (x)(E^*, \mathcal{M})(K) \cong (x)((E), \mathcal{M})(K)$$

from the set of cylinders $x \leadsto K : E^* \leadsto \mathcal{M}$ to the set of cells $x \leadsto K : (E) \leadsto \mathcal{M}$. Specifically, the bijection sends each cylinder $\alpha : x \leadsto K : E^* \leadsto \mathcal{M}$ to the cell

$$
\begin{array}{ccc}
\ast & \to & E^* \times E \\
\hole & \downarrow & \downarrow K \\
X & \overline{\mathcal{M}} & \to A
\end{array}
$$

which maps each $E$-arrow $h : e \to e'$ to the $\mathcal{M}$-arrow

$$
\alpha_{e'} \circ K(h, e') = \alpha_e \circ K(e, h) : x \leadsto K(e, e')
$$

as indicated in

$$
\begin{array}{ccc}
x & \overline{\alpha_e} & \to K(e, e) \\
\alpha_{e'} & \downarrow & \downarrow K(e, h) \\
K(e', e') & \leftarrow & K(e, e')
\end{array}
$$

; the inverse sends each cell

$$
\begin{array}{ccc}
\ast & \to & E^* \times E \\
\hole & \downarrow & \downarrow K \\
X & \overline{\mathcal{M}} & \to A
\end{array}
$$

to the cylinder $[\Theta] : x \leadsto K : E^* \leadsto \mathcal{M}$ defined by

$$
[\Theta]_e = \Theta \cdot 1_e
$$

for $e \in \|E\|$, where $\Theta \cdot 1_e$ is the image of the identity $e \to e$ under the function

$$
\langle E \rangle (e, e) \xrightarrow{\langle E \rangle (e, e)} \langle (E \cdot \mathcal{M})(e, e) \rangle = \langle (E \cdot \mathcal{M})(K(e, e)) \rangle
$$

.$$
5. Yoneda Lemma

from the set of cylinders $K \sim a : E^\rightarrow \sim M$ to the set of cells $K \sim a : (E) \rightarrow M$. Specifically, the bijection sends each cylinder $\alpha : K \sim a : E^\rightarrow \sim M$ to the cell

$$
\begin{array}{c}
E \times E^- \xrightarrow{(E)} \\
\alpha \downarrow \\
\bigcirc_{M} \\
\end{array}
$$

which maps each $E$-arrow $h : e \rightarrow e'$ to the $M$-arrow

$$
K(h, e') \circ \alpha_{e'} = K(e, h) \circ \alpha_e : K(e, e') \sim a
$$
as indicated in

$$
\begin{array}{c}
K(e, e') \xrightarrow{K(e, h)} K(e, e) \\
\alpha_e \\
\end{array}
$$

; the inverse sends each cell

$$
\begin{array}{c}
E \times E^- \xrightarrow{(E)} \\
\Theta \\
\bigcirc_{M} \\
\end{array}
$$
to the cylinder $[\Theta] : K \sim a : E^\rightarrow \sim M$ defined by

$$
[[\Theta]]_e = 1_e \cdot \Theta
$$

for $e \in [E]$, where $1_e \cdot \Theta$ is the image of the identity $e \rightarrow e$ under the function

$$
(e, e) \langle E \rangle \xrightarrow{(e, e)\langle \Theta \rangle} (e, e) \langle K(M) a \rangle = (K(e, e)) \langle M a \rangle
$$

Proof. Replacing $M$ with $x \langle M \rangle K$ in Theorem 5.5.7, we have a bijection

$$
\Phi_{x \langle M \rangle K}^{E^\rightarrow} : \prod_{E} x \langle M \rangle K \cong (\langle E \rangle) \langle : E^\rightarrow \times E \rangle (x \langle M \rangle K)
$$

, natural in $x$ and $K$. But

$$
\prod_{E} x \langle M \rangle K = (x \langle E^\rightarrow, M \rangle K)
$$

and

$$
(\langle E \rangle) \langle : E^\rightarrow \times E \rangle (x \langle M \rangle K) = (x \langle (E) , M \rangle K)
$$

by the definitions. Hence an isomorphism $\Psi_{x \langle M \rangle K}^{E^\rightarrow} : (E^\rightarrow, M) \cong (\langle E \rangle , M)$ is given by

$$
(x \langle \Psi_{x \langle M \rangle K}^{E^\rightarrow} \rangle K) = \Phi_{x \langle M \rangle K}^{E^\rightarrow}
$$

. □

Theorem 5.5.9.
5. Yoneda Lemma

- Given a pair of natural transformations \( \beta : P \to F \circ Q : X \to D \) and \( \alpha : F \circ S \to T \) as in

\[
\begin{array}{c}
X \\
| \\
\beta \\
| \\
P \\
\downarrow \\
| \\
D \\
\downarrow \\
| \\
S \\
\downarrow \\
\alpha \\
| \\
T \\
\downarrow \\
| \\
E \\
\downarrow \\
| \\
A
\end{array}
\]

their pasting composite \([\beta \circ S] \circ [Q \circ \alpha] : P \circ S \to T \circ Q\) is given by the composition

\[
\begin{array}{c}
X \\
| \\
\beta \\
| \\
P \\
\downarrow \\
| \\
D \\
\downarrow \\
| \\
S \\
\downarrow \\
\alpha \\
| \\
E \\
\downarrow \\
| \\
F \\
\downarrow \\
\alpha \circ D \\
\downarrow \\
\alpha \circ (A) \\
\downarrow \\
A
\end{array}
\]

of the cylinder \(\beta\) (see Example 4.3.5(1)) and the cell \(D \circ \alpha\) (see Theorem 5.5.2).

- Given a pair of natural transformations \(\beta : P \to F \circ Q : X \to D\) and \(\alpha : S \to T \circ F\) as in

\[
\begin{array}{c}
X \\
| \\
\beta \\
| \\
P \\
\downarrow \\
| \\
E \\
\downarrow \\
| \\
S \\
\downarrow \\
\alpha \circ D \\
\downarrow \\
\alpha \circ (A) \\
\downarrow \\
A
\end{array}
\]

their pasting composite \([P \circ \alpha] \circ [\beta \circ T] : P \circ S \to T \circ Q\) is given by the composition

\[
\begin{array}{c}
X \\
| \\
\beta \\
| \\
P \\
\downarrow \\
| \\
E \\
\downarrow \\
| \\
F \circ (D) \\
\downarrow \\
\alpha \circ D \\
\downarrow \\
\alpha \circ (A) \\
\downarrow \\
A
\end{array}
\]

of the cylinder \(\beta\) (see Example 4.3.5(1)) and the cell \(\alpha \circ \bullet\) (see Theorem 5.5.2).

**Proof.** For any \(x \in \|X\|\),

\[
[[\beta \circ S] \circ [Q \circ \alpha]]_x = [\beta \circ S]_x \circ [Q \circ \alpha]_x = (\beta_x \circ S) \circ \alpha_{(x \circ Q)}
\]

and

\[
[\beta \circ (D \circ \alpha)]_x = \beta_x \circ (D \circ \alpha)
\]

We thus need to show that \(\beta_x : (D \circ \alpha) = (\beta_x \circ S) \circ \alpha_{(x \circ Q)}\). But, by replacing \(h : d \to F \circ e\) with \(x : T \circ F \circ e \to T \circ e\) in the commutative diagram of Example 5.3.5(3), we have

\[
\begin{array}{c}
x : P \\
\beta_x \\
x : P \circ S \\
\beta_x \circ S \\
F \circ x \\
x : Q \circ F \circ S \\
\alpha_{(x \circ Q)} \\
T \circ Q \circ x
\end{array}
\]

as required. \(\square\)
Corollary 5.5.10.

Given a pair of natural transformations \( \eta : 1_X \to G \circ F : X \to X \) and \( \epsilon : G \circ H \to 1_A \) as in

\[
\begin{array}{ccc}
X & \xrightarrow{1} & X \\
\downarrow{\eta} & & \downarrow{\epsilon} \\
G & \xrightarrow{F} & A \\
\downarrow{H} & & \downarrow{1} \\
A & \xrightarrow{1} & A
\end{array}
\]

, the triangle

\[
\begin{array}{ccc}
F(A) & \xrightarrow{[\eta \circ H \circ F \circ \epsilon]} & H(A) \\
\downarrow{\eta \uparrow A} & & \downarrow{X \uparrow \epsilon} \\
\langle X \rangle G & & \langle X \rangle H \\
\end{array}
\]

commutes, where \([\eta \circ H] \circ [F \circ \epsilon] : H \to F\) is the pasting composite of \(\eta\) and \(\epsilon\).

Given a pair of natural transformations \( \epsilon : G \circ F \to 1_A : A \to A \) and \( \eta : 1_X \to H \circ F \) as in

\[
\begin{array}{ccc}
X & \xrightarrow{1} & X \\
\downarrow{\eta} & & \downarrow{1} \\
G & \xrightarrow{F} & A \\
\downarrow{H} & & \downarrow{\eta \uparrow A} \\
A & \xrightarrow{1} & A
\end{array}
\]

, the triangle

\[
\begin{array}{ccc}
\langle X \rangle G & \xrightarrow{(X) \circ [G \circ \eta \circ F \circ \epsilon]} & \langle X \rangle H \\
\downarrow{X \uparrow \epsilon} & & \downarrow{\eta \uparrow A} \\
F(A) & & \langle X \rangle H \\
\end{array}
\]

commutes, where \([G \circ \eta] \circ [\epsilon \circ H] : G \to H\) is the pasting composite of \(\epsilon\) and \(\eta\).

Proof. Indeed,

\[
[[\eta \circ H] \circ [F \circ \epsilon]](A) = [\eta \circ (X \uparrow \epsilon)](A) = [\eta \circ (X \uparrow \epsilon)](A) = (\eta \uparrow A) \circ (X \uparrow \epsilon)
\]

\((*^1 \text{ by Theorem } 5.5.9; *^2 \text{ by Proposition } 5.3.7; *^3 \text{ by Corollary } 5.3.19). \square\)
6. Universals

6.1. Units of one-sided modules

Definition 6.1.1.

- An arrow $u : r \to \ast$ of a right module $M : X \to \ast$ is called a unit if the right module morphism $X \downarrow u : (X) r \to M : X \to \ast$ is iso.

- An arrow $u : \ast \to r$ of a left module $M : \ast \to A$ is called a unit if the left module morphism $u \uparrow A : r \langle A \rangle \to M : \ast \to A$ is iso.

Remark 6.1.2.

1. By the Yoneda lemma (Theorem 5.2.10),

   - the representations and units of a right module correspond one-to-one; if an $M$-arrow $u : r \to \ast$ is a unit of $M$, then the object $r$ and the right module isomorphism $(X \downarrow u)^{-1} : M \cong (X) r$ form a representation of $M$; conversely, if an object $r \in \|X\|$ and a right module isomorphism $\Upsilon : M \cong (X) r$ form a representation of $M$, then the $M$-arrow $1_r : \Upsilon^{-1} : r \to \ast$ is a unit of $M$.

   - the representations and units of a left module correspond one-to-one; if an $M$-arrow $u : \ast \to r$ is a unit of $M$, then the object $r$ and the left module isomorphism $(u \uparrow A)^{-1} : M \cong r \langle A \rangle$ form a representation of $M$; conversely, if an object $r \in \|A\|$ and a left module isomorphism $\Upsilon : M \cong r \langle A \rangle$ form a representation of $M$, then the $M$-arrow $\Upsilon^{-1} : 1_r : \ast \to r$ is a unit of $M$.

2. Let $u : r \to \ast$ be a unit of a right module $M : X \to \ast$. Given an $M$-arrow $m : x \to \ast$, its inverse image under $X \downarrow u$ is called the adjunct of $m$ along $u$ and written $m/u$; that is,

   $\frac{m}{u} := m \circ (X \downarrow u)^{-1}$

   this is the unique $X$-arrow $x \to r$ making the triangle

   \[
   \begin{array}{ccc}
   X & \rightarrow & r \\
   m/u & \swarrow & m \\
   & r & \\
   \end{array}
   \]

   commute. An $M$-arrow $u : r \to \ast$ is a unit if and only if to every $M$-arrow $m : x \to \ast$ there is a unique $X$-arrow $m/u : x \to r$ as above.

\[^{1}\text{A unit is called a universal element in the literature.}\]
6. Universals

- Let \( u : \ast \rightarrow r \) be a unit of a left module \( M : \ast \rightarrow A \). Given an \( M \)-arrow \( m : \ast \rightarrow a \), its inverse image under \( u \uparrow A \) is called the adjunct of \( m \) along \( u \) and written \( u \backslash m \); that is,

\[
u \backslash m := (u \uparrow A)^{-1} : m\]

; this is the unique \( A \)-arrow \( r \rightarrow a \) making the triangle

\[
\begin{array}{ccc}
\ast & \xrightarrow{u} & \ast \\
\downarrow{m} & & \downarrow{u \backslash m} \\
r & \rightarrow & a
\end{array}
\]

commute. An \( M \)-arrow \( u : \ast \rightarrow r \) is a unit if and only if to every \( M \)-arrow \( m : \ast \rightarrow a \) there is a unique \( A \)-arrow \( u \backslash m : r \rightarrow a \) as above.

**Proposition 6.1.3.**

- an arrow of a right module \( M \) is a unit if and only if it is a terminal object of the comma category \( [M^1] \).
- an arrow of a left module \( M \) is a unit if and only if it is an initial object of the comma category \( [M^1] \).

**Proof.** Immediate by the last sentence of Remark 6.1.2(2).

**Proposition 6.1.4.**

- A right module \( M \) is representable if and only if the comma category \( [M^1] \) has a terminal object.
- A left module \( M \) is representable if and only if the comma category \( [M^1] \) has an initial object.

**Proof.** Immediate from Proposition 6.1.3 on noting Remark 6.1.2(1).

**Proposition 6.1.5.** For any arrow \( f : r \rightarrow s \) of a category \( C \), the following conditions are equivalent:

1. \( C \)-arrow \( f : r \rightarrow s \) is invertible;
2. \( (C) \)-arrow \( f : r \rightarrow \ast \) is a unit;
3. \( r (C) \)-arrow \( f : \ast \rightarrow s \) is a unit,

where \( (C) s \) and \( r (C) \) are the right and left slices of the hom of \( C \) at \( s \) and \( r \).

**Proof.** By definition, \( f : r \rightarrow \ast \) is a unit of the right module \( (C) s : C \rightarrow \ast \) iff the right module morphism \( C \uparrow f : (C) r \rightarrow (C) s \) is iso. By Proposition 5.2.8 \( C \uparrow f = (C) f \), and, by the fully faithfulness of the Yoneda functor, \( (C) f \) is iso iff \( f \) is invertible. The conditions (1) and (2) are thus equivalent. The equivalence of the conditions (1) and (3) is proved dually.

**Theorem 6.1.6.**
6. Universals

- Let $\mathcal{M} : X \to \ast$ be a right module. If $\mathcal{M}$ is representable, a representing object is unique up to isomorphism. In fact, if $u : r \to \ast$ and $v : s \to \ast$ are two units, then the adjunct $v/u : s \to r$ of $v$ along $u$ and the adjunct $u/v : r \to s$ of $u$ along $v$, as indicated in

\[
\begin{array}{c}
\ast \\
v/u \\
\downarrow \\
u \\
\ast \\
\end{array}
\quad
\begin{array}{c}
r \\
u \\
\downarrow \\
v/u \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
v \\
\downarrow \\
v/u \\
\ast \\
\end{array}
\quad
\begin{array}{c}
r \\
u \\
\downarrow \\
v/u \\
\ast \\
\end{array}
\]

are the inverse of each other.

- Let $\mathcal{M} : \ast \to A$ be a left module. If $\mathcal{M}$ is representable, a representing object is unique up to isomorphism. In fact, if $u : \ast \to r$ and $v : \ast \to s$ are two units, then the adjunct $u/v : r \to s$ of $v$ along $u$ and the adjunct $v/u : s \to r$ of $u$ along $v$, as indicated in

\[
\begin{array}{c}
r \\
u \\
\downarrow \\
v \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
v/u \\
\downarrow \\
\ast \\
r \\
\end{array}
\quad
\begin{array}{c}
\ast \\
v \\
\downarrow \\
v/u \\
\ast \\
\end{array}
\quad
\begin{array}{c}
r \\
u \\
\downarrow \\
v \\
\ast \\
\end{array}
\]

are the inverse of each other.

Proof. Since

\[u/v \circ v/u = u, \quad u/v \circ u/v = 1_s\]

and $u$ is a unit, $u/v \circ v/u = 1_r$ by the uniqueness of the factorization. Symmetrically, $v/u \circ u/v = 1_s$.

\[\square\]

Definition 6.1.7.

- A right module cell

\[
\begin{array}{c}
X \\
P \\
\downarrow \Phi \\
Y \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\]

is said to

1. preserve units if every unit $u : r \to \ast$ of $\mathcal{M}$ yields a unit $u : \Phi : r : P \to \ast$ of $\mathcal{N}$;
2. reflect units if an $\mathcal{M}$-arrow $u : r \to \ast$ is a unit whenever the $\mathcal{N}$-arrow $u : \Phi : r : P \to \ast$ is a unit;
3. create units if for every unit $v : s \to \ast$ of $\mathcal{N}$ there is exactly one $\mathcal{M}$-arrow $u : r \to \ast$ with $u : \Phi = v$, and if this $u$ is a unit.

- A left module cell

\[
\begin{array}{c}
\ast \\
1 \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\quad
\begin{array}{c}
\ast \\
\ast \\
\downarrow \Phi \\
\ast \\
\ast \\
\end{array}
\]

is said to

1. preserve units if every unit $u : \ast \to r$ of $\mathcal{M}$ yields a unit $u : \Phi : \ast \to Q : r$ of $\mathcal{N}$;
6. Universals

2. reflect units if an $M$-arrow $u : * \to r$ is a unit whenever the $N$-arrow $u : \Phi : * \to Q \cdot r$ is a unit;

3. create units if for every unit $v : * \to s$ of $N$ there is exactly one $M$-arrow $u : * \to r$ with $u : \Phi = v$, and if this $u$ is a unit.

**Proposition 6.1.8.** If a right (resp. left) module cell is iso, then it preserves, reflects and creates units.

**Proof.** Evident. □

**Theorem 6.1.9.**

- Suppose that a right module $M : X \to *$ has a representation $(s, \Upsilon)$. Then an $M$-arrow $u : r \to *$ is a unit if and only if its image under $\Upsilon : M \to (X) s$ is an invertible $X$-arrow.

- Suppose that a left module $M : * \to A$ has a representation $(s, \Upsilon)$. Then an $M$-arrow $u : * \to r$ is a unit if and only if its image under $\Upsilon : M \to s(A)$ is an invertible $A$-arrow.

**Proof.** Since $\Upsilon : M \to (X)s$ is iso, it preserves and reflects units. The assertion thus follows from Proposition 6.1.5. □

**Theorem 6.1.10.**

- Suppose that a right module $M : X \to *$ has a unit $u : r \sim *$. Then an $X$-arrow $f : s \to r$ is invertible if and only if the composite $f \circ u : s \to *$ is a unit of $M$; to put it the other way round, an $M$-arrow $v : s \sim *$ is a unit if and only if its adjunct $v/u : s \to r$ along $u$ is invertible.

- Suppose that a left module $M : * \to A$ has a unit $u : * \sim r$. Then an $A$-arrow $f : r \to s$ is invertible if and only if the composite $u \circ f : * \sim s$ is a unit of $M$; to put it the other way round, an $M$-arrow $v : * \sim s$ is a unit if and only if its adjunct $u/v : r \to s$ along $u$ is invertible.

**Proof.** By Remark 6.1.2(1) and (2), this is just a restatement of Theorem 6.1.9. □

### 6.2. Universal arrows

**Definition 6.2.1.** Let $M : X \to A$ be a module.

- An $M$-arrow $u : r \sim a$ is inverse universal if the right module morphism $X \upharpoonright u : (X) r \to (M) a : X \to *$ is iso. Given an object $a \in ||A||$, an inverse universal $M$-arrow $u : r \sim a$ or the pair $(r, u)$, or the object $r$ itself, is called a universal of a inverse along $M$.

- An $M$-arrow $u : x \sim r$ is direct universal if the left module morphism $u \upharpoonright A : r(A) \to x(M) : * \to A$ is iso. Given an object $x \in ||X||$, a direct universal $M$-arrow $u : x \sim r$ or the pair $(r, u)$, or the object $r$ itself, is called a universal of $x$ direct along $M$.

**Remark 6.2.2.**

1. An $M$-arrow $u : x \sim r$ is direct universal if and only if the $M$-arrow $u : r \sim x$ is inverse universal.

2. By Remark 5.2.6(2),
6. Universals

- an $M$-arrow $u : r \rightarrow a$ is inverse universal if and only if it is a unit of the right module $(M) a : X \rightarrow \ast$, and a unit $u : r \rightarrow \ast$ of a right module $M : X \rightarrow \ast$ is the same thing as an inverse universal arrow of $M$ regarded as a two-sided module from $X$ to the terminal category.
- an $M$-arrow $u : x \rightarrow r$ is direct universal if and only if it is a unit of the left module $x (M) : \ast \rightarrow A$, and a unit $u : \ast \rightarrow r$ of a left module $M : \ast \rightarrow A$ is the same thing as a direct universal arrow of $M$ regarded as a two-sided module from the terminal category to $A$.

3. Remark 6.1.2(2) is repeated below in terms of universal arrows.

- Let $u : r \rightarrow a$ be an inverse universal $M$-arrow. Given an $M$-arrow $m : x \rightarrow a$, its inverse image under $X \upharpoonright u$ is called the adjunct of $m$ along $u$ and written $m / u$; that is,
  $$m / u := m \cdot (X \upharpoonright u)^{-1}$$
  ; this is the unique $X$-arrow $x \rightarrow r$ making the triangle

$$\begin{array}{ccc}
x & \rightarrow & u \\
\downarrow & & \downarrow m \\
r & \rightarrow & a
\end{array}$$

commute. An $M$-arrow $u : r \rightarrow a$ is inverse universal if and only if to every $M$-arrow $m : x \rightarrow a$ there is a unique $X$-arrow $m / u : x \rightarrow r$ as above.

- Let $u : x \rightarrow r$ be a direct universal $M$-arrow. Given an $M$-arrow $m : x \rightarrow a$, its inverse image under $u \upharpoonright A$ is called the adjunct of $m$ along $u$ and written $u \backslash m$; that is,
  $$u \backslash m := m \cdot (u \upharpoonright A)^{-1}$$
  ; this is the unique $A$-arrow $r \rightarrow a$ making the triangle

$$\begin{array}{ccc}
x & \rightarrow & u \\
\downarrow & & \downarrow m \\
r & \rightarrow & a
\end{array}$$

commute. An $M$-arrow $u : x \rightarrow r$ is direct universal if and only if to every $M$-arrow $m : x \rightarrow a$ there is a unique $A$-arrow $u \backslash m : r \rightarrow a$ as above.

4. An $M$-arrow $u : r \rightarrow s$ is called two-way universal if it is both inverse and direct universal.

**Example 6.2.3.** Let $F : E \rightarrow D$ be a functor.

- An inverse universal of an object $d \in \mathcal{D}$ along the representable module $F (D) : E \rightarrow D$ is an $F (D)$-arrow $u : r \rightarrow d$ (i.e. $D$-arrow $u : r : F \rightarrow d$) such that to every $F (D)$-arrow $f : e \rightarrow d$ (i.e. $D$-arrow $f : e : F \rightarrow d$), there is a unique $E$-arrow $f / u : e \rightarrow r$ such that the triangle

$$\begin{array}{ccc}
e & \rightarrow & F \\
\downarrow f / u & & \downarrow (f / u) : F \\
r & \rightarrow & u \\
d \rightarrow & \rightarrow & d
\end{array}$$

commutes. A $D$-arrow $u : r : F \rightarrow d$ is said to be universal from $F$ to $d$ if the $F (D)$-arrow $u : r \rightarrow d$ is inverse universal.
6. Universals

A direct universal of an object \( d \in \| D \| \) along the corepresentable module \( (D) F : D \to E \) is a \( (D) \) \( F \)-arrow \( u : d \to r \) (i.e. \( D \)-arrow \( u : d \to F \cdot r \)) such that to every \( (D) \) \( F \)-arrow \( f : d \to e \) (i.e. \( D \)-arrow \( f : d \to F \cdot e \)), there is a unique \( E \)-arrow \( u \cdot f : r \to e \) such that the triangle

\[
\begin{array}{ccc}
  d & \xrightarrow{u} & F \cdot r \\
  \downarrow{f} \quad \text{and} \quad \downarrow{u \cdot f} & \quad & \downarrow{\text{triangle}} \\
  F \cdot e & \xrightarrow{u \cdot f} & e
\end{array}
\]

commutes. A \( D \)-arrow \( u : d \to F \cdot r \) is said to be universal from \( d \) to \( F \) if the \( (D) \) \( F \)-arrow \( u : d \to r \) is direct universal.

Remark 6.2.4. The module \( F (D) : E \to D \) and \( (D) F : D \to E \) abstract the commutative diagrams above into

\[
\begin{array}{ccc}
  e & \xrightarrow{f / u} & d \\
  \downarrow{f} \quad \text{and} \quad \downarrow{u} & \quad & \downarrow{u / f} \\
  r & \xrightarrow{u / f} & e
\end{array}
\]

and present a simpler and more conceptual view of universals.

Note. The following is a restatement of Proposition 6.1.5 in terms of universal arrows.

Proposition 6.2.5. For any arrow \( f : r \to s \) of a category \( C \), the following conditions are equivalent:

1. \( C \)-arrow \( f : r \to s \) is invertible;
2. \( (C) \)-arrow \( f : r \to s \) is inverse universal;
3. \( (C) \)-arrow \( f : r \to s \) is direct universal;
4. \( (C) \)-arrow \( f : r \to s \) is two-way universal,

where \( (C) \) is the hom of \( C \).

Proof. The equivalence of the conditions (1), (2), and (3) follows from Proposition 6.1.5 on noting Remark 6.2.2(2). The equivalence of the conditions (1) and (4) then follows. \( \square \)

Note. The following is a restatement of Theorem 6.1.6 in terms of universal arrows.

Theorem 6.2.6. Let \( M : X \to A \) be a module.

- A universal of an object \( a \in \| A \| \) inverse along \( M \), if exists, is unique up to isomorphism. In fact, if \( u : r \to a \) and \( v : s \to a \) are two inverse universal \( M \)-arrows, then the adjunct \( v / u : s \to r \) of \( v \) along \( u \) and the adjunct \( u / v : r \to s \) of \( u \) along \( v \), as indicated in

\[
\begin{array}{cc}
  s & \xrightarrow{v} \quad \text{and} \quad \xrightarrow{u / v} \quad \text{up} \\
  \downarrow{v / u} \quad \text{and} \quad \downarrow{u / v} & \quad \downarrow{\text{triangle}} \\
  r & \xrightarrow{u / v} \quad \text{up} \\
  \downarrow{u / v} \quad \text{and} \quad \downarrow{u / v} & \quad \downarrow{\text{triangle}} \\
  a & \xrightarrow{u / v}
\end{array}
\]

, are the inverse of each other.
6. Universals

- A universal of an object \( x \in \mathbb{X} \) direct along \( M \), if exists, is unique up to isomorphism. In fact, if \( u : x \to r \) and \( v : x \to s \) are two direct universal \( M \)-arrows, then the adjunct \( u \backslash v : r \to s \) of \( v \) along \( u \) and the adjunct \( v \backslash u : r \to s \) of \( u \) along \( v \), as indicated in

\[
\begin{array}{ccc}
  r & \xrightarrow{\nu} & s \\
\downarrow{u} & & \downarrow{v} \\
  x & \xrightarrow{} & \text{ (diagram)}
\end{array}
\]

, are the inverse of each other.

Proof. The assertion follows from Theorem 6.1.6 on noting Remark 6.2.2(2). \( \square \)

**Corollary 6.2.7.** Let \( F : E \to D \) be a functor and \( d \) be an object of \( D \).

- If there is an invertible \( D \)-arrow \( u : r \to F \) universal from \( F \) to \( d \), then every \( D \)-arrow \( v : s \to d \) universal from \( F \) to \( d \) is invertible.

- If there is an invertible \( D \)-arrow \( u : d \to F \) universal from \( d \) to \( F \), then every \( D \)-arrow \( v : d \to F \) universal from \( d \) to \( F \) is invertible.

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
  s & \xrightarrow{(v/u)} & F \\
\downarrow{v} & & \downarrow{u} \\
  r & \xrightarrow{v} & F \\
\end{array}
\]

. We need to show that if \( u \) is invertible, so is \( v \). For this, it suffices to show that \( (v/u) : F \) is invertible. But this the case since \( v/u \) is invertible by Theorem 6.2.6 and any functor preserves invertibility. \( \square \)

**Note.** The following is a restatement of Theorem 6.1.10 in terms of universal arrows.

**Theorem 6.2.8.** Let \( M : X \to A \) be a module.

- Suppose that an object \( a \in \mathbb{A} \) has a universal \( u : r \to a \) inverse along \( M \). Then an \( X \)-arrow \( f : s \to r \) is invertible if and only if the composite \( f \circ u : s \to a \) is an inverse universal \( M \)-arrow; to put it the other way round, an \( M \)-arrow \( v : s \to a \) is inverse universal if and only if its adjunct \( v/u \) along \( u \) is invertible.

- Suppose that an object \( x \in \mathbb{X} \) has a universal \( u : x \to r \) direct along \( M \). Then an \( A \)-arrow \( f : r \to s \) is invertible if and only if the composite \( u \circ f : x \to s \) is a direct universal \( M \)-arrow; to put it the other way round, an \( M \)-arrow \( v : x \to s \) is direct universal if and only if its adjunct \( u \backslash v \) along \( u \) is invertible.

Proof. The assertion follows from Theorem 6.1.10 on noting Remark 6.2.2(2). \( \square \)

**Theorem 6.2.9.**

- Consider a composable pair of a module and a functor

\[
\begin{array}{ccc}
  X & \xrightarrow{M} & A \\
\downarrow{K} & & \downarrow{K} \\
  E & \xrightarrow{K} & E
\end{array}
\]

. An \( M \)-arrow \( u : r \to K : e \) is inverse universal if and only if so is the \( (M)K \)-arrow \( u : r \to e \).

193
6. Universals

- Consider a composable pair of a module and a functor

\[
E \xrightarrow{K} X \xrightarrow{M} A
\]

. An \( M \)-arrow \( u : e \cdot K \rightharpoonup r \) is direct universal if and only if so is the \( K \langle M \rangle \)-arrow \( u : e \rightharpoonup r \).

**Proof.** \( u : r \rightharpoonup e \) is an inverse universal \( \langle M \rangle K \)-arrow iff \( u : r \rightharpoonup \ast \) is a unit of \( \langle (M) K \rangle e \), and \( u : r \rightharpoonup K \cdot e \) is an inverse universal \( M \)-arrow iff \( u : r \rightharpoonup \ast \) is a unit of \( \langle M \rangle (K \cdot e) \). But \( \langle (M) K \rangle e = \langle M \rangle (K \cdot e) \).

**Corollary 6.2.10.** Let \( F : E \to D \) be a functor.

- A \( D \)-arrow \( u : r \to F \cdot e \) is invertible if and only if the \( \langle D \rangle F \)-arrow \( u : r \rightharpoonup e \) is inverse universal.

- A \( D \)-arrow \( u : e \cdot F \to r \) is invertible if and only if the \( F \langle D \rangle \)-arrow \( u : e \rightharpoonup r \) is direct universal.

**Proof.** By Proposition 6.2.5, \( u : r \to F \cdot e \) is invertible iff it is an inverse universal \( \langle D \rangle \)-arrow. The assertion thus follows from Theorem 6.2.9.

**Theorem 6.2.11.**

- Consider a composable pair of a functor and a module

\[
E \xrightarrow{K} X \xrightarrow{M} A
\]

with \( K \) fully faithful. If an \( M \)-arrow \( u : r \cdot K \rightharpoonup a \) is inverse universal, so is the \( K \langle M \rangle \)-arrow \( u : r \rightharpoonup a \).

- Consider a composable pair of a functor and a module

\[
X \xrightarrow{M} A \xrightarrow{K} E
\]

with \( K \) fully faithful. If an \( M \)-arrow \( u : x \rightharpoonup K \cdot r \) is direct universal, so is the \( \langle M \rangle K \)-arrow \( u : x \rightharpoonup r \).

**Proof.** By Example 5.2.9(1), \( E \upharpoonright u \) is given by the composite \( \langle K \rangle r \circ K \langle X \upharpoonright u \rangle \). Since \( K \) is fully faithful, \( \langle K \rangle r \) is iso; since \( u : r \cdot K \rightharpoonup a \) is an inverse universal \( M \)-arrow, \( X \upharpoonright u \) is iso, and hence so is \( K \langle X \upharpoonright u \rangle \). \( E \upharpoonright u \) is thus iso.

**Corollary 6.2.12.** Let \( F : E \to D \) be a fully faithful functor.

- If a \( D \)-arrow \( u : r \cdot F \to d \) is invertible, then it is universal from \( F \) to \( d \), i.e. the \( \langle D \rangle F \)-arrow \( u : r \rightharpoonup d \) is inverse universal.

- If a \( D \)-arrow \( u : d \to F \cdot r \) is invertible, then it is universal from \( d \) to \( F \), i.e. the \( \langle D \rangle F \)-arrow \( u : d \rightharpoonup r \) is direct universal.

**Proof.** By Proposition 6.2.5, \( u : r \cdot F \rightharpoonup d \) is invertible iff it is an inverse universal \( \langle D \rangle \)-arrow. By Theorem 6.2.11, \( u : r \cdot F \rightharpoonup d \) is an inverse universal \( \langle D \rangle \)-arrow iff \( u : r \rightharpoonup d \) is an inverse universal \( F \langle D \rangle \)-arrow.
Theorem 6.2.13. Consider a pair of functors

\[
\begin{array}{c}
E \xrightarrow{F} C \rightarrow (C) \leftarrow C \xrightarrow{G} D
\end{array}
\]

with both \( F \) and \( G \) fully faithful. If a \( C \)-arrow \( u : r \cdot F \to G \cdot s \) is invertible, then the \( F(C) \) \( G \)-arrow \( u : r \to s \) is two-way universal.

Proof. By Corollary 6.2.12, the \( F(C) \)-arrow \( u : r \to G \cdot s \) (resp. \( (C) \) \( G \)-arrow \( u : r \cdot F \to s \)) is inverse (resp. direct) universal; hence, by Theorem 6.2.9, the \( F(C) \) \( G \)-arrow \( u : r \to s \) is inverse (resp. direct) universal. □

Definition 6.2.14. A cell

\[
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow \quad \Phi \\
Y \xrightarrow{N} B
\end{array}
\]

is said to

1. preserve
   - inverse universals if every inverse universal \( M \)-arrow \( u : r \to a \) yields an inverse universal \( N \)-arrow \( \Phi : r : P \to Q : a \).
   - direct universals if every direct universal \( M \)-arrow \( u : x \to r \) yields a direct universal \( N \)-arrow \( \Phi : x : P \to Q : r \).

2. reflect
   - inverse universals if an \( M \)-arrow \( u : r \to a \) is inverse universal whenever the \( N \)-arrow \( u : \Phi : r : P \to Q : a \) is inverse universal.
   - direct universals if an \( M \)-arrow \( u : x \to r \) is direct universal whenever the \( N \)-arrow \( u : \Phi : x : P \to Q : r \) is direct universal.

3. create
   - inverse universals if for every object \( a \in \| A \| \) and for every inverse universal \( N \)-arrow \( v : s \to Q : a \) there is exactly one \( M \)-arrow \( u : r \to a \) with \( u : \Phi = v \), and if this \( u \) is inverse universal.
   - direct universals if for every object \( x \in \| X \| \) and for every direct universal \( N \)-arrow \( v : x : P \to s \) there is exactly one \( M \)-arrow \( u : x \to r \) with \( u : \Phi = v \), and if this \( u \) is direct universal.

Remark 6.2.15. By Remark 6.2.2(2), a cell \( \Phi \)

- preserves (resp. reflects, creates) inverse universals if and only if its right slice

\[
\begin{array}{c}
X \xrightarrow{(M)a} * \\
\downarrow \quad \Phi \\
Y \xrightarrow{(N)a} B
\end{array}
\]

(see Definition 2.1.7) at each \( a \in \| A \| \) preserves (resp. reflects, creates) units.
6. Universals

- preserves (resp. reflects, creates) direct universals if and only if its left slice

\[
\begin{array}{ccc}
\ast & \rightarrow & \text{A} \\
\xrightarrow{x} \cdot P & \xrightarrow{x(\Phi)} Q \\
Y & \rightarrow & \text{B}
\end{array}
\]

(see Definition 2.1.7) at each \( x \in \| X \| \) preserves (resp. reflects, creates) units.

**Proposition 6.2.16.** Consider a cell \( \Phi \) as in Definition 6.2.14.

- If \( \Phi \) is fully faithful and \( P \) is iso, then \( \Phi \) preserves, reflects and creates inverse universals.
- If \( \Phi \) is fully faithful and \( Q \) is iso, then \( \Phi \) preserves, reflects and creates direct universals.

**Proof.** The assertion follows from Proposition 6.1.8 on noting Remark 6.2.15 and Proposition 2.1.8.

**Theorem 6.2.17.** A right (resp. left) Yoneda morphism preserves and reflects inverse (resp. direct) universal.

**Proof.** Given a module \( M : X \rightarrow A \), the right Yoneda morphism for \( M \) (see Definition 5.2.5) sends each \( M \)-arrow \( u : r \rightarrow a \) to the right module \( X|u : (X) r \rightarrow (M) a : X \rightarrow \ast \). But, by definition, \( u : r \sim a \) is inverse universal iff \( X|u \) is iso, and, by Proposition 6.2.5, \( X|u \) is iso iff \( X|u \) is inverse universal.

**Definition 6.2.18.** Let \( M : X \rightarrow A \) be a module.

- Given a pair of inverse universal \( M \)-arrows \( u : r \sim a \) and \( v : s \sim b \), the conjugate of an \( A \)-arrow \( f : a \rightarrow b \) inverse along \((u,v)\) is the \( X \)-arrow given by \((u \circ f)/v\), the adjunct of the composite \( u \circ f \) along \( v \), i.e. the unique \( X \)-arrow \( g : r \rightarrow s \) making the quadrangle

\[
\begin{array}{ccc}
r & \sim & a \\
\downarrow g & & \downarrow f \\
\sim & \rightarrow & \sim \\
\downarrow & & \downarrow \\
s & \sim & b
\end{array}
\]

commute. This commutative quadrangle, or the assignment \( f \mapsto g \), is called an inverse conjugation by \((u,v)\).

- Given a pair of direct universal \( M \)-arrows \( u : x \rightarrow r \) and \( v : y \rightarrow s \), the conjugate of an \( X \)-arrow \( g : x \rightarrow y \) direct along \((u,v)\) is the \( A \)-arrow given by \( u \setminus (g \circ v) \), the adjunct of the composite \( g \circ v \) along \( u \), i.e. the unique \( A \)-arrow \( f : r \rightarrow s \) making the quadrangle

\[
\begin{array}{ccc}
x & \rightarrow & r \\
\downarrow g & & \downarrow f \\
\sim & \rightarrow & \sim \\
\downarrow & & \downarrow \\
y & \rightarrow & s
\end{array}
\]

commute. This commutative quadrangle, or the assignment \( g \mapsto f \), is called a direct conjugation by \((u,v)\).

**Proposition 6.2.19.** Let \( M : X \rightarrow A \) be a module.
6. Universals

1. Inverse conjugation is functorial in the following sense:

   a) for any inverse (resp. direct) universal M-arrow u, the identities form an inverse conjugation

   \[
   \begin{array}{c}
   r \sim u \sim s \\
   1 \downarrow \\
   r \sim u \sim s
   \end{array}
   \]

   b) if, in the diagram

   \[
   \begin{array}{c}
   r \sim u \sim s \\
   g \downarrow \\
   r' \sim u' \sim s' \\
   g' \downarrow \\
   r'' \sim u'' \sim s''
   \end{array}
   \]

   , each of the two inner quadrangle is an inverse (resp. direct) conjugation, so is the outer quadrangle.

2. Conjugation is universal in the following sense.

   • Given an inverse conjugation

   \[
   \begin{array}{c}
   r \sim u \sim a \\
   g \downarrow \\
   s \sim v \sim b
   \end{array}
   \]

   and any commutative quadrangle

   \[
   \begin{array}{c}
   x \sim m \sim a \\
   h \downarrow \\
   y \sim n \sim b
   \end{array}
   \]

   , the adjunct of m along u and the adjunct of n along v yield a unique pair of X-arrows making the diagram

   \[
   \begin{array}{c}
   x \sim m \sim a \\
   h \downarrow \\
   y \sim n \sim b \\
   \end{array}
   \]

   commute.

   • Given a direct conjugation

   \[
   \begin{array}{c}
   x \sim u \sim r \\
   g \downarrow \\
   y \sim v \sim s
   \end{array}
   \]
and any commutative quadrangle

\[
\begin{array}{ccc}
\ast & \sim & \ast \\
\downarrow & & \downarrow \\
\ast & \sim & \ast
\end{array}
\]

, the adjunct of m along u and the adjunct of n along v yield a unique pair of A-arrows making the diagram

\[
\begin{array}{ccc}
x & \sim & r \\
g & \downarrow & f \\
\ast & \sim & \ast \\
v & \downarrow & h \\
y & \sim & s \\
\end{array}
\]

commute.

Proof.

1. Evident.

2. The uniqueness of such a pair follows from the uniqueness of the adjunct. The proof is thus complete if we show that the quadrangle

\[
\begin{array}{ccc}
x & \sim & r \\
h & \downarrow & g \\
y & \sim & s \\
\end{array}
\]

commutes. But since
\[
h \circ n \circ v = h \circ n = m \circ f = m \circ u \circ f = m \circ u \circ g \circ v
\]

and v is inverse universal, we have
\[
h \circ n \circ v = m \circ u \circ g
\]

by the uniqueness of the factorization.

\[\square\]

6.3. Units of two-sided modules

**Definition 6.3.1.** Let \( M : X \to A \) be a module.

- A right cylinder

\[
\begin{array}{ccc}
X & \sim & R \\
\mu & \downarrow & \lambda \\
M & \sim & A
\end{array}
\]

is called a counit of M if the module morphism \( X \uparrow \mu : (X) R \to M : X \to A \) is iso.
6. Universals

- A left cylinder

\[ X \xrightarrow{\mu} \frac{M}{R} \xrightarrow{\rho} A \]

is called a unit of \( \mathcal{M} \) if the module morphism \( \mu \upharpoonright A : R(A) \to \mathcal{M} : X \to A \) is iso.

**Remark 6.3.2.** By the general Yoneda lemma (Corollary 5.3.17),

- the corepresentations and counits of a module correspond one-to-one; if a right cylinder \( \mu : R \to \mathcal{M} \) is a counit of a module \( \mathcal{M} : X \to A \), then the functor \( R \) and the module isomorphism \( (X \upharpoonright \mu)^{-1} : \mathcal{M} \cong (X) R \) form a corepresentation of \( \mathcal{M} \); conversely, if a functor \( R : A \to X \) and a module isomorphism \( \Upsilon : \mathcal{M} \cong (X) R \) form a corepresentation of \( \mathcal{M} \), then the right cylinder \( [\Upsilon^{-1}] : R \to \mathcal{M} \) is a counit of \( \mathcal{M} \).

- the representations and units of a module correspond one-to-one; if a left cylinder \( \mu : \mathcal{M} \to R \) is a unit of a module \( \mathcal{M} : X \to A \), then the functor \( R \) and the module isomorphism \( (\mu \upharpoonright A)^{-1} : \mathcal{M} \cong (X) \) form a representation of \( \mathcal{M} \); conversely, if a functor \( R : X \to A \) and a module isomorphism \( \Upsilon : \mathcal{M} \cong (X) \) form a representation of \( \mathcal{M} \), then the left cylinder \( [\Upsilon^{-1}] : \mathcal{M} \to R \) is a unit of \( \mathcal{M} \).

**Proposition 6.3.3.** Let \( \mathcal{M} : X \to A \) be a module.

- A right cylinder \( \mu : R \to \mathcal{M} \) is a counit of \( \mathcal{M} \) if and only if its each component \( \mu_a : a : R \to a \) is an inverse universal \( \mathcal{M} \)-arrow.

- A left cylinder \( \mu : \mathcal{M} \to R \) is a unit of \( \mathcal{M} \) if and only if its each component \( \mu_x : x \to R \cdot x \) is a direct universal \( \mathcal{M} \)-arrow.

**Proof.** By Proposition 2.1.4, \( X \upharpoonright \mu \) is iso iff its each slice \( (X \upharpoonright \mu) a \) is iso. Since \( (X \upharpoonright \mu) a = X \upharpoonright \mu_a \) (see Corollary 5.3.10), \( (X \upharpoonright \mu) a \) is iso iff \( \mu_a \) is inverse universal.

**Remark 6.3.4.**

1. Proposition 6.3.3 gives an alternative definition of units of a module.

2. By Proposition 6.3.3 and Remark 6.2.2(2), under the identification in Remark 4.3.2(3),

- a unit of a right module \( \mathcal{M} : X \to * \) is the same thing as a counit of \( \mathcal{M} \) regarded as the two-sided module from \( X \) to the terminal category.

- a unit of a left module \( \mathcal{M} : * \to A \) is the same thing as a unit of \( \mathcal{M} \) regarded as the two-sided module from the terminal category to \( A \).

**Proposition 6.3.5.** Let \( \mathcal{M} : X \to A \) be a module.

- A counit of \( \mathcal{M} \) is a universal of \( \mathcal{M} \) inverse along the right general Yoneda module \( X^\ast A \), i.e. a unit (in the sense of Definition 6.1.1) of the right module \( (X^\ast A)(\mathcal{M}) \).

- A unit of \( \mathcal{M} \) is a universal of \( \mathcal{M} \) direct along the left general Yoneda module \( X^\ast A \), i.e. a unit (in the sense of Definition 6.1.1) of the left module \( (\mathcal{M})(X^\ast A) \).
6. Universals

Proof. Let $\mu : R \sim M$ be a counit and $\alpha : G \sim M$ be a right cylinder. We need to show that there is a unique natural transformation $\alpha / \mu : G \to R : A \to X$ making the triangle

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha / \mu} & M \\
\downarrow\alpha & & \downarrow\alpha \\
R & \xleftarrow{\mu} & M \\
\end{array}
\]

commute. For any $A$-arrow $f : a \to b$, consider the commutative quadrangles

\[
\begin{array}{ccc}
\alpha_a & \xrightarrow{\alpha_a / \mu_a} & a \\
\downarrow f & & \downarrow f \\
b & \xleftarrow{\mu_a} & b \\
\end{array} \quad \quad \begin{array}{ccc}
\alpha_b & \xrightarrow{\alpha_b / \mu_b} & b \\
\downarrow f & & \downarrow f \\
a & \xleftarrow{\mu_b} & a \\
\end{array}
\]

Since $\mu_a$ and $\mu_b$ are inverse universal by Proposition 6.3.3, the right quadrangle forms an inverse conjugation. Hence, by Proposition 6.2.19(2), the adjunct of $\alpha_a$ along $\mu_a$ and the adjunct of $\alpha_b$ along $\mu_b$ yield a unique pair of $X$-arrows making the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha_a / \mu_a} & a \\
\downarrow f & & \downarrow f \\
b & \xleftarrow{\mu_b} & b \\
\end{array}
\]

commute. The family of $X$-arrows $\alpha_a / \mu_a$, one for each $a \in |A|$, thus forms a unique natural transformation $\alpha / \mu : G \to R$ such that $\alpha / \mu \circ \mu = \alpha$.

Remark 6.3.6. The converse does not hold. See Example 6.3.7.

Example 6.3.7. Let $2$ denote the discrete category consisting of objects $\{0, 1\}$, $\mathbf{2}$ denote the interval category, and let $\mathcal{M} : 2 \to \mathbf{2}$ be a module which looks like

\[
\begin{array}{ccc}
0 & \xrightarrow{\Delta 0} & 0 \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\mu} & 1 \\
\end{array}
\]

Then $\mathcal{M}$ admits only one right cylinder

\[
\begin{array}{ccc}
2 & \xrightarrow{\mu} & 2 \\
\end{array}
\]

along it. $\mu$ is a universal of $\mathcal{M}$ inverse along the module $2 \to \mathbf{2}$; however, $\mu$ is not a unit of $\mathcal{M}$ since $\mu_1 : 0 \sim 1$ is not an inverse universal $\mathcal{M}$-arrow.

Theorem 6.3.8. Let $\mathcal{M}$ be a module.

- If $\mu : R \sim \mathcal{M}$ and $\nu : S \sim \mathcal{M}$ are two counits of $\mathcal{M}$, then $R$ and $S$ are isomorphic.
6. Universals

- If $\mu : M \rightarrow R$ and $\nu : M \rightarrow S$ are two units of $M$, then $R$ and $S$ are isomorphic.

**Proof.** By Proposition 6.3.5, the assertion is reduced to Theorem 6.2.6. \qed

**Corollary 6.3.9.** A representing functor of a module, if exists, is unique up to isomorphism.

**Proof.** By Remark 6.3.2, this is just a restatement of Theorem 6.3.8. \qed

**Theorem 6.3.10.** Let $M : X \rightarrow A$ be a module.

- Suppose that every object $a \in \|A\|$ has a universal inverse along $M$ and an inverse universal $M$-arrow $\mu_a : r_a \sim a$ is chosen. Then there is a unique functor $R : A \rightarrow X$ with $a^* R = r_a$ such that the family of $M$-arrows $\mu_a$, one for each $a \in \|A\|$, forms a right cylinder $\mu : R \sim M$; moreover, $\mu$ is a counit of $M$.

- Suppose that every object $x \in \|X\|$ has a universal direct along $M$ and a direct universal $M$-arrow $\mu_x : x \rightarrow r_x$ is chosen. Then there is a unique functor $R : X \rightarrow A$ with $x^* R = r_x$ such that the family of $M$-arrows $\mu_x$, one for each $x \in \|X\|$, forms a left cylinder $\mu : M \sim R$; moreover, $\mu$ is a unit of $M$.

**Proof.** The arrow function of $R$ is given by the inverse conjugation

\[
\begin{array}{ccc}
r_a & \sim & a \\
R : f & \sim & f \\
r_b & \sim & \mu_b
\end{array}
\]

(see Definition 6.2.18) for each $A$-arrow $f : a \rightarrow b$. $R$ is functorial by Proposition 6.2.19(1) and the uniqueness $R$ follows from the uniqueness of a conjugate. Since each $\mu_a$ is inverse universal, $\mu$ forms a counit of $M$ by Proposition 6.3.3. \qed

**Note.** The axiom of choice is used in the proof of the following.

**Corollary 6.3.11.** Given a module $M : X \rightarrow A$,

- the following conditions are equivalent:
  1. $M$ is corepresentable;
  2. $M$ has a counit;
  3. the right module $\langle M \rangle a : X \rightarrow *$ is representable for every object $a \in \|A\|$;
  4. the right module $\langle M \rangle a : X \rightarrow *$ has a unit for every object $a \in \|A\|$; that is, every $a \in \|A\|$ has a universal inverse along $M$.

- the following conditions are equivalent:
  1. $M$ is representable;
  2. $M$ has a unit;
  3. the left module $x(M) : * \rightarrow A$ is representable for every object $x \in \|X\|$;
  4. the left module $x(M) : * \rightarrow A$ has a unit for every object $x \in \|X\|$; that is, every $x \in \|X\|$ has a universal direct along $M$.

**Proof.**
6. Universals

(1) \iff (2) See Remark 6.3.2.

(3) \iff (4) See Remark 6.1.2(1).

(2) \implies (4) Immediate from Proposition 6.3.3.

(4) \implies (2) A family of inverse universal $\mathcal{M}$-arrows $\mu_a : r_a \rightsquigarrow a$, one chosen for each $a \in |A|$, yields a counit of $\mathcal{M}$ by Theorem 6.3.10.

\begin{proof}
\end{proof}

**Theorem 6.3.12.** Let $\mathcal{M} : X \to A$ be a module.

- For a counit

\[
\begin{array}{c}
X \xrightarrow{\mu_R} \mathcal{M} \xrightarrow{\mu} A
\end{array}
\]

of $\mathcal{M}$, the following conditions are equivalent:

1. the functor $R$ is fully faithful;
2. each component of $\mu$ is not only inverse universal but direct universal, i.e. two-way universal.

- For a unit

\[
\begin{array}{c}
X \xleftarrow{\mu_R} \mathcal{M} \xleftarrow{\mu} A
\end{array}
\]

of $\mathcal{M}$, the following conditions are equivalent:

1. the functor $R$ is fully faithful;
2. each component of $\mu$ is not only direct universal but inverse universal, i.e. two-way universal.

**Proof.** Let $a$ and $b$ be objects of $A$. By the naturality of $\mu$, the quadrangle

\[
\begin{array}{c}
a : R \xrightarrow{\mu_a} a \\
f : R \downarrow \downarrow f \\
b : R \xrightarrow{\mu_b} b
\end{array}
\]

commutes for every $A$-arrow $f : a \to b$, yielding the commutative triangle

\[
\begin{array}{c}
(a : R) \xrightarrow{\mu} (a : R) \langle M \rangle b
\end{array}
\]

. Since $\mu_b$ is inverse universal, the assignment $h \mapsto h \circ \mu_b$ is bijective. Hence the assignment $f \mapsto f : R$ is bijective (i.e. $R$ is fully faithful) iff the assignment $f \mapsto \mu_a \circ f$ is bijective (i.e. $\mu_a$ is direct universal).

\end{proof}
6.4. Universal cylinders

**Definition 6.4.1.**

- A cylinder

\[ \begin{array}{ccc}
  E & \xrightarrow{\mu} & K \\
  \downarrow{\Rightarrow} & \downarrow{\Rightarrow} \\
  X & \xrightarrow{\Rightarrow} & A
\end{array} \]

is called

1. inverse universal if it is an inverse universal \( (E, M) \)-arrow (see Definition 4.3.6);
2. pointwise inverse universal if each component \( \mu_e : e \sim R \sim K \sim e \) is an inverse universal \( M \)-arrow.

Given a functor \( K : E \to A \), an inverse universal (resp. pointwise inverse universal) cylinder \( \mu : R \sim K : E \sim M \) or the pair \((R, \mu)\), or the functor \( R \) itself, is called a lift (resp. pointwise lift) of \( K \) inverse along \( M \).

- A cylinder

\[ \begin{array}{ccc}
  E & \xrightarrow{\mu} & K \\
  \downarrow{\Rightarrow} & \downarrow{\Rightarrow} \\
  X & \xrightarrow{\Rightarrow} & A
\end{array} \]

is called

1. direct universal if it is a direct universal \( (E, M) \)-arrow (see Definition 4.3.6);
2. pointwise direct universal if each component \( \mu_e : e \sim K \sim R \sim e \) is a direct universal \( M \)-arrow.

Given a functor \( K : E \to X \), a direct universal (resp. pointwise direct universal) cylinder \( \mu : K \sim R : E \sim M \) or the pair \((R, \mu)\), or the functor \( R \) itself, is called a lift (resp. pointwise lift) of \( K \) direct along \( M \).

**Remark 6.4.2.**

1. ▶ A cylinder \( \alpha : R \sim K : E \sim M \) is inverse universal if and only if to every cylinder \( \alpha : S \sim K : E \sim M \) there is a unique natural transformation \( \alpha/\mu : S \to R \) such that \( \alpha = \alpha/\mu \circ \mu \).

- ▶ A cylinder \( \alpha : K \sim R : E \sim M \) is direct universal if and only if to every cylinder \( \alpha : K \sim T : E \sim M \), there is a unique natural transformation \( \mu/\alpha : R \to T \) such that \( \alpha = \mu \circ \mu/\alpha \).

2. By Proposition 6.3.3 and Theorem 6.2.9,

- ▶ a pointwise lift

\[ \begin{array}{ccc}
  E & \xrightarrow{\mu} & K \\
  \downarrow{\Rightarrow} & \downarrow{\Rightarrow} \\
  X & \xrightarrow{\Rightarrow} & A
\end{array} \]
of $K$ inverse along $\mathcal{M}$ is the same thing as a counit

$$
\begin{array}{c}
\xymatrix{ X \ar@{=}[r]^\mu_{(M)K} \\
| K |\ar@{=}[u] & E \ar@{=}[l]_R \\
}
\end{array}
$$

of the composite module $(M)K$; conversely, a counit

$$
\begin{array}{c}
\xymatrix{ X \ar@{=}[r]^\mu_{\mathcal{M}} \\
| A |\ar@{=}[u] & A \ar@{=}[l]_R \\
}
\end{array}
$$

of a module $\mathcal{M}$ is the same thing as a pointwise lift

$$
\begin{array}{c}
\xymatrix{ X \ar[r]_{\mathcal{M}} & A \\
A \ar@{=}[u] & A \ar@{=}[l]_R \\
}
\end{array}
$$
of the identity $A \to A$ inverse along $\mathcal{M}$.

- a pointwise lift

$$
\begin{array}{c}
\xymatrix{ X \ar[r]_{\mathcal{M}} & A \\
E \ar@{=}[u] & A \ar@{=}[l]_R \\
}
\end{array}
$$
of $K$ direct along $\mathcal{M}$ is the same thing as a unit

$$
\begin{array}{c}
\xymatrix{ X \ar[r]^\mu_{\mathcal{M}} \\
A \ar@{=}[u] & A \ar@{=}[l]_R \\
}
\end{array}
$$
of the composite module $K(\mathcal{M})$; conversely a unit

$$
\begin{array}{c}
\xymatrix{ X \ar[r]_{\mathcal{M}} & A \\
A \ar@{=}[u] & A \ar@{=}[l]_R \\
}
\end{array}
$$
of a module $\mathcal{M}$ is the same thing as a pointwise lift

$$
\begin{array}{c}
\xymatrix{ X \ar[r]_{\mathcal{M}} & A \\
X \ar@{=}[u] & A \ar@{=}[l]_R \\
}
\end{array}
$$
of the identity $X \to X$ direct along $\mathcal{M}$.

3. A cylinder $\mu : R \to S : E \to \mathcal{M}$ is called two-way universal (resp. pointwise two-way universal) if it is both inverse and direct universal (resp. pointwise inverse and direct universal).

**Proposition 6.4.3.**

- A cylinder $\mu : R \to K : E \to \mathcal{M}$ is pointwise inverse universal if and only if the module morphism $X|\mu : (X)R \to (\mathcal{M})K : X \to E$ is iso.
6. Universals

- A cylinder $\mu : K \to R \colon E \to M$ is pointwise direct universal if and only if the module morphism $\mu \upharpoonright A \colon R\langle A \rangle \to K\langle M \rangle \colon E \to A$ is iso.

**Proof.** By Proposition 2.1.4, $X \upharpoonright \mu$ is iso iff its each slice $(X \upharpoonright \mu)$ is iso. Since $(X \upharpoonright \mu) e = X \upharpoonright \mu_e$ (see Corollary 5.3.10), $(X \upharpoonright \mu)$ is iso iff $\mu_e$ is inverse universal.

**Remark 6.4.4.** Proposition 6.4.3 gives an alternative definition of the pointwise universality of a cylinder.

**Proposition 6.4.5.** A pointwise inverse (resp. direct) universal cylinder is inverse (resp. direct) universal.

**Proof.** Let $\mu : R \to K \colon E \to M$ be a pointwise inverse universal cylinder. By Remark 6.4.2(2), $\mu$ is a unit of $(M)K$. Hence, by Proposition 6.3.5, $\mu$ is a universal of $(M)K$ along $X \cdot A$. Hence, by the identity in Theorem 5.1.8, $\mu$ is a universal of $K$ inverse along $(E, M)$.

**Proposition 6.4.6.** Let $F : E \to D$ be a functor.

- If $\mu : R \to K \colon D \to M$ is a pointwise inverse universal cylinder, so is the cylinder $F \circ \mu : F \circ R \to K \circ F : E \to M$.

- If $\mu : K \to R \colon D \to M$ is a pointwise direct universal cylinder, so is the cylinder $F \circ \mu : F \circ K \to R \circ F : E \to M$.

**Proof.** Obvious since $[F \circ \mu]_e = \mu(F_{\cdot e})$ for each $e \in |E|$.

**Note.** By Remark 6.4.2(2), Theorem 6.4.7 and Theorem 6.3.10 are special cases of each other.

**Theorem 6.4.7.** Let $E$ be a category and $M : X \to A$ be a module.

- Given a functor $K : E \to A$, suppose that, for every object $e \in |E|$, $K : e$ has a universal inverse along $M$ and an inverse universal $M$-arrow $\mu_e : r_e \to K : e$ is chosen. Then there is a unique functor $R : E \to X$ with $e : R = r_e$ such that the family of $M$-arrows $\mu_e$, one for each $e \in |E|$, forms a cylinder $\mu : R \to K : E \to M$; moreover, $\mu$ is pointwise inverse universal.

- Given a functor $K : E \to X$, suppose that, for every object $e \in |E|$, $e : K$ has a universal direct along $M$ and a direct universal $M$-arrow $\mu_e : e : K \to r_e$ is chosen. Then there is a unique functor $R : E \to A$ with $e : R = r_e$ such that the family of $M$-arrows $\mu_e$, one for each $e \in |E|$, forms a cylinder $\mu : K \to R : E \to M$; moreover, $\mu$ is pointwise direct universal.

**Proof.** See Note above.

**Note.** The axiom of choice is used in the proof of the following.

**Corollary 6.4.8.** Let $M : X \to A$ be a module.

- Given a functor $K : E \to A$, the following conditions are equivalent:
  1. $K$ has a pointwise lift inverse along $M$;
  2. for every $e \in |E|$, $K : e$ has a universal inverse along $M$.

- Given a functor $K : E \to X$, the following conditions are equivalent:
  1. $K$ has a pointwise lift direct along $M$;
6. Universals

2. for every $e \in |E|$, $e : K$ has a universal direct along $M$.

Proof.

$(1) \Rightarrow (2)$ Immediate by the definition of a pointwise lift.

$(2) \Rightarrow (1)$ A family of inverse universal $M$-arrows $\mu_e : r_e \Rightarrow K \cdot e$, one chosen for each $e \in |E|$, yields a pointwise lift inverse along $M$ by Theorem 6.4.7.

\[ \square \]

Theorem 6.4.9. Let $E$ be a category and $M : X \to A$ be a module.

- If a cylinder $\mu : R \Rightarrow K : E \Rightarrow M$ is inverse universal (resp. pointwise inverse universal), then a natural transformation $\tau : S \Rightarrow R : E \Rightarrow X$ is iso if and only if the cylinder $\tau \circ \mu : S \Rightarrow K : E \Rightarrow M$ is inverse universal (resp. pointwise inverse universal).

- If a cylinder $\mu : K \Rightarrow R : E \Rightarrow M$ is direct universal (resp. pointwise direct universal), then a natural transformation $\tau : R \Rightarrow S : E \Rightarrow A$ is iso if and only if the cylinder $\mu \circ \tau : K \Rightarrow S : E \Rightarrow M$ is direct universal (resp. pointwise direct universal).

Proof. Suppose that $\mu$ is inverse universal, i.e. an inverse universal $(E, M)$-arrow. Then the assertion is just an instance of Theorem 6.2.8 where $M$ is given by $(E, M)$. Now suppose that $\mu$ is pointwise inverse universal. Since $[\tau \circ \mu]_e = \tau_e \circ \mu_e$ for each $e \in |E|$, the assertion follows from Theorem 6.2.8 on noting that a natural transformation is iso iff each its component is invertible.

Corollary 6.4.10. Let $E$ be a category and $M : X \to A$ be a module.

- If a functor $K : E \to A$ has a pointwise lift inverse along $M$, then every lift of $K$ inverse along $M$ is pointwise.

- If a functor $K : E \to X$ has a pointwise lift direct along $M$, then every lift of $K$ direct along $M$ is pointwise.

Proof. Let $\mu : R \Rightarrow K : E \Rightarrow M$ and $\nu : S \Rightarrow K : E \Rightarrow M$ be two lifts of $K$ inverse along $M$ and suppose that $\mu$ is a pointwise lift. By Proposition 6.4.5, $\mu$ is inverse universal; hence, by Theorem 6.2.6, there is a natural isomorphism $\nu : S \Rightarrow R$ such that $\nu = \nu \circ \mu$. $\nu$ is thus pointwise inverse universal by Theorem 6.4.9.

\[ \square \]

6.5. Kan lifts

Note. Example 4.3.5(1) allows the following definition.

Definition 6.5.1. Given a pair of functors

\[ E \xrightarrow{F} D \xleftarrow{K} C \]

- a natural transformation

\[ E \xrightarrow{R} C \xleftarrow{K} D \]

Diagram 6.5.1.
from $R \circ F$ to $K$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a right Kan lift (resp. pointwise right Kan lift) of $K$ along $F$ if the cylinder

$$
\begin{array}{ccc}
C & \xrightarrow{\mu} & K \\
\downarrow & & \downarrow \\
E & \xrightarrow{\mu} & D
\end{array}
$$

is inverse universal (resp. pointwise inverse universal).

- a natural transformation

$$
\begin{array}{ccc}
K & \xrightarrow{\mu} & R \\
\downarrow & & \downarrow \\
D & \xrightarrow{\mu} & E
\end{array}
$$

from $K$ to $F \circ R$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a left Kan lift (resp. pointwise left Kan lift) of $K$ along $F$ if the cylinder

$$
\begin{array}{ccc}
C & \xrightarrow{\mu} & R \\
\downarrow & & \downarrow \\
D & \xrightarrow{\mu} & E
\end{array}
$$

is direct universal (resp. pointwise direct universal).

**Remark 6.5.2.**

1. A Kan lift (resp. pointwise Kan lift) is thus a special instance of a lift (resp. pointwise lift) defined in Definition 6.4.1 where $M$ is representable.

2. A natural transformation $\mu : R \circ F \to K$ forms a pointwise right Kan lift of $K$ along $F$ if and only if each component $\mu_c : c \circ R \circ F \to K \circ c$ is universal from $F$ to $K \circ c$ (see Example 6.2.3).

- A natural transformation $\mu : K \to F \circ R$ forms a pointwise left Kan lift of $K$ along $F$ if and only if each component $\mu_c : c \circ K \to F \circ R \circ c$ is universal from $c \circ K$ to $F$ (see Example 6.2.3).

3. See Example 6.5.5(1) for an example of a non-pointwise Kan lift.

**Theorem 6.5.3.**

- Assume that a natural isomorphism

$$
\begin{array}{ccc}
C & \xrightarrow{\mu} & K \\
\downarrow & & \downarrow \\
E & \xrightarrow{\mu} & D
\end{array}
$$

from $R \circ F$ to $K$ is given. In this situation, if $F$ is fully faithful, then $\mu$ forms a pointwise right Kan lift of $K$ along $F$. The converse holds if we assume in addition that $R$ is surjective.
Assume that a natural isomorphism

\[
\begin{array}{c}
\text{C} \\
\downarrow \mu \\
\text{D} & \leftarrow & \text{E}
\end{array}
\]

from \( K \) to \( F \circ R \) is given. In this situation, if \( F \) is fully faithful, then \( \mu \) forms a pointwise left Kan lift of \( K \) along \( F \). The converse holds if we assume in addition that \( R \) is surjective.

**Proof.** By Example 5.2.9(2), the diagram

\[
\begin{array}{c}
\langle E \rangle (R \cdot c) \\
\downarrow (F)(R \cdot c) \\
\langle F \langle D \rangle \rangle (R \cdot c) \\
\downarrow \langle F \langle D \rangle \rangle (R \cdot R \cdot c)
\end{array}
\]

\[
\begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
F \langle D \rangle (K \cdot c) \\
\downarrow F(D) \mu_c \\
F \langle D \rangle (F \cdot R \cdot c)
\end{array}
\]

commutes for every \( c \in |C| \). Since \( \mu_c \) is an invertible \( D \)-arrow by the assumption, \( F \langle D \rangle \mu_c \) is iso. Hence \( E \downarrow \mu_c \) is iso iff \( (F)(R \cdot c) \) is iso. The assertion now follows on noting that \( E \downarrow \mu_c \) is iso iff \( \mu_c \) is an inverse universal \( F \langle D \rangle \)-arrow, and that \( F \) is fully faithful iff \( (F)e \) is iso for each \( e \in |E| \).

**Remark 6.5.4.** If \( F \) is not fully faithful in Theorem 6.5.3, then a natural isomorphism \( \mu \), even an identity, need not form a Kan lift. See Example 6.5.5(2) for an example.

**Example 6.5.5.**

1. Example 6.3.7 is repeated below in terms of Kan lifts. Let \( 2 \) and \( \Delta \) be as in Example 6.3.7 and let \( F : \Delta \to 2 \) be the inclusion functor. Then the representable module of \( F \) looks like this:

\[
\begin{array}{c}
0 \\
\Rightarrow \\
\downarrow \\
1
\end{array}
\]

The identity \( F : \Delta \to 2 \) has only one natural transformation

\[
\begin{array}{c}
2 \\
\mu \\
\downarrow \\
2
\end{array}
\]

along \( F \). The constant functor \( \Delta \) and \( \mu \) form a right Kan lift of the identity \( 2 \to 2 \) along \( F \); however, the lift is not pointwise since \( \mu_1 : 0 : F \to 1 \) is not universal from \( F \) to 1.

2. Consider functors as in

\[
\begin{array}{c}
E \\
\downarrow \Delta \\
* 
\end{array}
\]
6. Universals

, where $*$ is the terminal category and $E$ is any category. Given $e \in \|E\|$, the functor $e : * \to E$ and the identity natural transformation

form a right Kan lift of the identity $* \to *$ along $\Delta$ only when $e$ is a terminal object of $E$. 
7. Limits

7.1. Universal cones

Definition 7.1.1.

- A cone
  \[
  \begin{array}{c}
  \ast \\
  r \\
  X \\
  \end{array} \begin{array}{c}
  \xrightarrow{\Delta} \\
  \mu \\
  M \rightarrow \\
  \end{array} \begin{array}{c}
  E \\
  \xrightarrow{K} \\
  A \\
  \end{array}
  \]

  is called universal if it is an inverse universal \((E^\ast, M)\)-arrow (see Definition 4.6.5). Given a functor \(K : E \to A\), a universal cone \(\mu : r \rightsquigarrow K : E^\ast \rightsquigarrow M\) or the pair \((r, \mu)\), or the object \(r\) itself, is called a limit of \(K\) inverse along \(M\).

- A cone
  \[
  \begin{array}{c}
  E \\
  \xrightarrow{\Delta} \\
  \end{array} \begin{array}{c}
  \ast \\
  r \\
  X \\
  \xrightarrow{\mu} \\
  \xrightarrow{M} \\
  \end{array} \begin{array}{c}
  \xrightarrow{\kappa} \\
  A \\
  \end{array}
  \]

  is called universal if it is a direct universal \((E^\ast, M)\)-arrow (see Definition 4.6.5). Given a functor \(K : E \to X\), a universal cone \(\mu : K \rightsquigarrow r : E^\ast \rightsquigarrow M\) or the pair \((r, \mu)\), or the object \(r\) itself, is called a limit of \(K\) direct along \(M\).

Remark 7.1.2.

- A cone \(\mu : r \rightsquigarrow K : E^\ast \rightsquigarrow M\) is universal if and only if to every cone \(\alpha : x \rightsquigarrow K : E^\ast \rightsquigarrow M\) there is a unique \(X\)-arrow \(\alpha/\mu : x \to r\) such that \(\alpha = \alpha/\mu \circ \mu\).

- A cone \(\alpha : K \rightsquigarrow r : E^\ast \rightsquigarrow M\) is universal if and only if to every cone \(\alpha : K \rightsquigarrow a : E^\ast \rightsquigarrow M\) there is a unique \(A\)-arrow \(\mu/\alpha : r \to a\) such that \(\alpha = \mu \circ \mu/\alpha\).

Definition 7.1.3. A module \(M : X \to A\) is called

- inverse \(E\)-complete for a category \(E\), if every functor \(E \to A\) has a limit inverse along \(M\),

- direct \(E\)-complete for a category \(E\), if every functor \(E \to X\) has a limit direct along \(M\),

and called

- inverse complete if it is inverse \(E\)-complete for any small category \(E\).

- direct complete if it is direct \(E\)-complete for any small category \(E\).

Remark 7.1.4. By Corollary 6.3.11,

- the following conditions are equivalent:
7. Limits

1. $\mathcal{M}$ is inverse $E$-complete;
2. the module $(E^a, \mathcal{M})$ has a counit.

The following conditions are equivalent:

1. $\mathcal{M}$ is direct $E$-complete;
2. the module $(E^b, \mathcal{M})$ has a unit.

**Definition 7.1.5.** A cell

\[
\begin{array}{c}
X \xrightarrow{\mu} A \\
\Phi \downarrow \\
Y \xrightarrow{\nu} B
\end{array}
\]

is said to

- preserve (resp. reflect, create) inverse limits for functors $E \to A$ if the postcomposition cell $(E^a, \Phi)$ (see Definition 4.6.13) preserves (resp. reflects, creates) inverse universals,
- preserve (resp. reflect, create) direct limits for functors $E \to X$ if the postcomposition cell $(E^b, \Phi)$ (see Definition 4.6.13) preserves (resp. reflects, creates) direct universals,

and said to

- preserve (resp. reflect, create) inverse limits (resp. small inverse limits) if it does so for functors $E \to A$ for any category (resp. small category ) $E$.
- preserve (resp. reflect, create) direct limits (resp. small direct limits) if it does so for functors $E \to X$ for any category (resp. small category ) $E$.

**Remark 7.1.6.** Namely, $\Phi$ is said to

1. preserve
   - inverse limits for functors $E \to A$ if every universal cone $\mu : r \to K : E^a \to \mathcal{M}$ yields a universal cone $\mu \circ \Phi : r : P \to Q \circ K : E^a \to \mathcal{N}$.
   - direct limits for functors $E \to X$ if every universal cone $\mu : K \to r : E^b \to \mathcal{M}$ yields a universal cone $\mu \circ \Phi : K \circ P \to Q \circ r : E^b \to \mathcal{N}$.

2. reflect
   - inverse limits for functors $E \to A$ if a cone $\mu : r : K : E^a \to \mathcal{M}$ is universal whenever the cone $\mu \circ \Phi : r : P \to Q \circ K : E^a \to \mathcal{N}$ is universal.
   - direct limits for functors $E \to X$ if a cone $\mu : K \to r : E^b \to \mathcal{M}$ is universal whenever the cone $\mu \circ \Phi : K \circ P \to Q \circ r : E^b \to \mathcal{N}$ is universal.

3. create
   - inverse limits for functors $E \to A$ if for every functor $K : E \to A$ and for every universal cone $\nu : s \to Q \circ K : E^a \to \mathcal{N}$ there is exactly one cone $\mu : r \to K : E^a \to \mathcal{M}$ with $\mu \circ \Phi = \nu$, and if this $\mu$ is a universal cone.
   - direct limits for functors $E \to X$ if for every functor $K : E \to X$ and for every universal cone $\nu : K \circ P \to s : E^b \to \mathcal{N}$ there is exactly one cone $\mu : K \to r : E^b \to \mathcal{M}$ with $\mu \circ \Phi = \nu$, and if this $\mu$ is a universal cone.
Proposition 7.1.7. Consider a cell $\Phi$ as in Definition 7.1.5.

- If $\Phi$ is fully faithful and $P$ is iso, then $\Phi$ preserves, reflects, and creates inverse limits for functors $E \rightarrow A$.
- If $\Phi$ is fully faithful and $Q$ is iso, then $\Phi$ preserves, reflects, and creates direct limits for functors $E \rightarrow X$.

Proof. The assertion says that if $\Phi$ is fully faithful then the cell

$$
\begin{array}{c}
[E, X] \xrightarrow{(E^e, M)} A \\
\downarrow \downarrow \\
(E, P) \xrightarrow{(E^e, \Phi)} Q \\
\downarrow \downarrow \\
[E, Y] \xrightarrow{(E^e, \mathcal{N})} B
\end{array}
$$

preserves, reflects, and creates inverse universals. This is indeed the case by Proposition 6.2.16 on noting that the functorial operations $[E, -]$ and $(E^e, -)$ preserve isomorphisms. □

Proposition 7.1.8. Consider a cell $\Phi$ as in Definition 7.1.5.

- Suppose that $\Phi$ creates inverse limits for functors $E \rightarrow A$. In this situation, if $\mathcal{N}$ is inverse $E$-complete, so is $\mathcal{M}$.
- Suppose that $\Phi$ creates direct limits for functors $E \rightarrow X$. In this situation, if $\mathcal{N}$ is direct $E$-complete, so is $\mathcal{M}$.

Proof. Let $K : E \rightarrow A$ be a functor. Since $\mathcal{N}$ is $E$-complete, $Q \circ K$ has a limit inverse along $\mathcal{N}$, from which $\Phi$ creates a limit of $K$ inverse along $\mathcal{M}$. □

7.2. Universal wedges

Definition 7.2.1. Let $E$ and $D$ be categories and $\mathcal{M} : X \rightarrow A$ be a module.

- A cone

$$
\begin{array}{c}
* \\
\downarrow R \\
[E, X] \xrightarrow{(E, \mathcal{M})} [E, A]
\end{array}
$$

along the module $(E, \mathcal{M})$ (see Definition 4.3.6) is called pointwise universal if its transpose

$$
\begin{array}{c}
E \\
\downarrow R \\
X \xrightarrow{(D^e, \mathcal{M})} [D, A]
\end{array}
$$

(see Remark 4.7.10(3)) is a pointwise inverse universal cylinder; that is, if the component

$$
\begin{array}{c}
* \\
\downarrow e \circ R \\
[X, -] \xrightarrow{[\mu]^t} [D, A]
\end{array}
$$

(see Remark 4.7.10(3)) is an inverse universal cylinder; that is, if the component

$$
\begin{array}{c}
* \\
\downarrow e \circ R \\
[X, -] \xrightarrow{[\mu]^t} [D, A]
\end{array}
$$
7. Limits

of $\mu^+$ at each $e \in \parallel E \parallel$ is a universal cone. Given a functor $K : D \to [E, A]$, a pointwise universal cone $\mu : R \sim K : D^\downarrow \sim (E, M)$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a pointwise limit of $K$ inverse along $(E, M)$.

- A cone

\[
\begin{array}{c}
D \xrightarrow{\Delta_D} * \\
\downarrow K \quad \downarrow \mu \\
[E, X] \xrightarrow{(E, M)} [E, A]
\end{array}
\]

along the module $(E, M)$ (see Definition 4.3.6) is called pointwise universal if its transpose

\[
\begin{array}{c}
E \xrightarrow{\Delta_E} E \\
\downarrow K^\top \quad \downarrow \mu^\top \\
[D, X] \xrightarrow{(D, M)} A
\end{array}
\]

(see Remark 4.7.10(3)) is a pointwise direct universal cylinder; that is, if the component

\[
\begin{array}{c}
D \xrightarrow{\Delta_D} * \\
\downarrow e \cdot K^\top \quad \downarrow (\mu^\top)_e \\
X \xrightarrow{(X, M)} A
\end{array}
\]

of $\mu^+$ at each $e \in \parallel E \parallel$ is a universal cone. Given a functor $K : D \to [E, X]$, a pointwise universal cone $\mu : K \sim R : D^p \sim (E, M)$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a pointwise limit of $K$ direct along $(E, M)$.

Proposition 7.2.2.

- A cone $\mu : R \sim K : D^\downarrow \sim (E, M)$ is universal (resp. pointwise universal) if and only if the cylinder $\mu^+ : R \sim K^\top : E \sim (D^q, M)$ is inverse universal (resp. pointwise inverse universal).

- A cone $\mu : K \sim R : D^p \sim (E, M)$ is universal (resp. pointwise universal) if and only if the cylinder $\mu^+ : K^\top \sim R : E \sim (D^p, M)$ is direct universal (resp. pointwise direct universal).

Proof. By the isomorphism in Remark 4.7.10(3), $\mu$ is universal iff $\mu^+$ is inverse universal. By definition, $\mu$ is pointwise universal iff $\mu^+$ is pointwise inverse universal. \qed

Proposition 7.2.3.

- If a cone $\mu : R \sim K : D^\downarrow \sim (E, M)$ is pointwise universal then it is universal.

- If a cone $\mu : K \sim R : D^p \sim (E, M)$ is pointwise universal then it is universal.

Proof. The assertion is reduced to Proposition 6.4.5 by Proposition 7.2.2. \qed

Proposition 7.2.4. Let $E$ and $D$ be categories and let $M : X \to A$ be a module.

- If a functor $K : D \to [E, A]$ has a pointwise limit inverse along $(E, M)$, then every limit of $K$ inverse along $(E, M)$ is pointwise.
7. Limits

- If a functor $K : D \to [E, X]$ has a pointwise limit direct along $(E, M)$, then every limit of $K$ direct along $(E, M)$ is pointwise.

Proof. The assertion is reduced to Corollary 6.4.10 by Proposition 7.2.2.

Note. The following is an instance of Theorem 6.4.7 where $M$ is given by $(D^\circ, M)$ (resp. $(D^p, M)$).

**Theorem 7.2.5.** Let $E$ and $D$ be categories and let $M : X \to A$ be a module.

- Given a functor $K : E \to [D, A]$, suppose that, for each object $e \in \|E\|$, the functor $K : D \to A$ has a limit inverse along $M$ and a universal cone $\mu_e : r_e \Rightarrow K : D \to A$ is chosen. Then there is a unique functor $R : E \to X$ with $e \colon R = r_e$ such that the family of cones $\mu_e$, one for each $e \in \|E\|$, forms a cylinder $\mu : R \Rightarrow K : E \Rightarrow (D^\circ, M)$; moreover, $\mu$ is pointwise inverse universal.

- Given a functor $K : E \to [D, X]$, suppose that, for each object $e \in \|E\|$, the functor $e : K : D \to X$ has a limit direct along $M$ and a universal cone $\mu_e : e \colon K \Rightarrow r_e : D^\circ \Rightarrow M$ is chosen. Then there is a unique functor $R : E \to A$ with $e \colon R = r_e$ such that the family of cones $\mu_e$, one for each $e \in \|E\|$, forms a cylinder $\mu : R \Rightarrow K : E \Rightarrow (D^p, M)$; moreover, $\mu$ is pointwise direct universal.

Proof. See Note above.

**Theorem 7.2.6.** Let $E$ and $D$ be categories and let $M : X \to A$ be a module.

- The family of evaluations

$$
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, M)} & [E, A] \\
\xymatrix{[e, X] \ar[rr]_{(e, M)} & & [e, A] \\
X \ar[u]_{(e, M)} \ar[r]_{\delta e} & M \ar[r]_{\delta e} & A 
}
\end{array}
$$

(see Example 4.3.29), one for each $e \in \|E\|$, creates a pointwise limit for a functor $K : D \to [E, A]$ inverse along $(E, M)$ in the following sense: if, for each $e \in \|E\|$, $[e, A] \delta K : D \to A$ has a limit inverse along $M$ and a universal cone $\mu_e : r_e \Rightarrow [e, A] \delta K : D^\circ \Rightarrow M$ is chosen, then there is a unique cone $\mu : R \Rightarrow K : D^\circ \Rightarrow (E, M)$ such that $\delta \mu (e, M) = \mu_e$; moreover, $\mu$ is pointwise universal.

- The family of evaluations

$$
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, M)} & [E, A] \\
\xymatrix{[e, X] \ar[rr]_{(e, M)} & & [e, A] \\
X \ar[u]_{(e, M)} \ar[r]_{\delta e} & M \ar[r]_{\delta e} & A 
}
\end{array}
$$

(see Example 4.3.29), one for each $e \in \|E\|$, creates a pointwise limit for a functor $K : D \to [E, X]$ direct along $(E, M)$ in the following sense: if, for each $e \in \|E\|$, $K \delta [e, X] : D \to X$ has a limit direct along $M$ and a universal cone $\mu_e : K \delta [e, X] \Rightarrow r_e : D^p \Rightarrow M$ is chosen, then there is a unique cone $\mu : K \Rightarrow R : D^p \Rightarrow (E, M)$ such that $\delta \mu (e, M) = \mu_e$; moreover, $\mu$ is pointwise universal.

Proof. The assertion is reduced to Theorem 7.2.5 by transposition on noting Remark 4.7.10 (4).
Note. The axiom of choice is used in the proof of the following.

**Corollary 7.2.7.** Let $E$ and $D$ be categories and let $M : X \to A$ be a module. If $M$ is inverse (resp. direct) $D$-complete, so is the module $(E, M)$, and all limits are given pointwise. As a consequence, if $M$ is inverse (resp. direct) complete, so is the module $(E, M)$.

**Proof.** Let $K : D \to [E, A]$ be a functor. Since $M$ is inverse $D$-complete, for each $e \in [E]$, the functor $[e, A] : D \to A$ has a limit inverse along $M$. Hence, by Theorem 7.2.6, if we chose a universal cone for each $e$, a pointwise limit of $K$ is obtained inverse along $(E, M)$, and, by Proposition 7.2.4, all limits of $K$ are pointwise.

**Corollary 7.2.8.** Let $D$ be a category and $F : E' \to E$ be a functor. If a module $M : X \to A$ is inverse (resp. direct) $D$-complete, then the precomposition cell

$$
\begin{array}{c}
\left[D, (E, M)\right] \\
\left[D, (F, M)\right] \\
\left[D, (E', M)\right]
\end{array}
\xrightarrow{\left[D, (E, M)\right]} \left[D, (F, M)\right] \xrightarrow{\left[D, (E', M)\right]}
$$

(see Definition 4.3.26) preserves inverse (resp. direct) limits for functors $D \to [E, A]$ (resp. $D \to [E, X]$).

**Proof.** Since the isomorphism $(D, (E, M)) \sim (E, (D, M))$ in Remark 4.7.10(3) is natural in $E$, the diagram

$$
\begin{array}{ccc}
(D, (E, M)) & \sim & (E, (D, M)) \\
\downarrow & & \downarrow \\
(D, (F, M)) & \sim & (E, (D, M))
\end{array}
$$

commutes. By Corollary 7.2.7, every universal cone $D \sim (E, M)$ is pointwise universal. Hence, by the commutative diagram above, it suffices to show that the precomposition cell $(F, (D, M))$ preserves pointwise inverse universal cylinders $E \sim (D, M)$. But this is the case by Proposition 6.4.6.

**Definition 7.2.9.**

- A wedge

$$
\begin{array}{cccc}
E & \leftarrow & E \times D & \rightarrow \\
\mu & & & K \\
\downarrow & & \downarrow & \downarrow \\
X & \leftarrow & M & \rightarrow A
\end{array}
$$

is called

1. universal if it is an inverse universal $(E \times D, M)$-arrow (see Definition 4.7.7);  
2. pointwise universal if the slice

$$
\begin{array}{cccc}
* & \leftarrow & D & \rightarrow \\
\downarrow & & \downarrow & \downarrow \\
e \cdot R & \rightarrow & \left[\mu\right] & \rightarrow A
\end{array}
$$

(see Remark 4.7.10(1)) of $\mu$ at each $e \in [E]$ is a universal cone.
7. Limits

- A wedge

\[ \begin{array}{ccc}
E \times D & \xrightarrow{\mu} & E \\
K & \searrow & \downarrow R \\
X & \xrightarrow{\mu} & A
\end{array} \]

is called

1. universal if it is a direct universal \((E \times D^r, M)\)-arrow (see Definition 4.7.7);
2. pointwise universal if the slice

\[ \begin{array}{ccc}
D & \xrightarrow{\Delta_D} & * \\
e \cdot K & \searrow & \downarrow R \cdot e \\
X & \xrightarrow{\mu} & A
\end{array} \]

(see Remark 4.7.10(1)) of \(\mu\) at each \(e \in \|E\|\) is a universal cone.

Remark 7.2.10.

- A wedge \(\alpha : R \to K : E \times D^s \to M\) is universal if and only if to every wedge \(\alpha : S \to K : E \times D^s \to M\) there is a unique natural transformation \(\alpha / \mu : S \to R\) such that \(\alpha = \alpha / \mu \circ \mu\).

- A wedge \(\alpha : K \to R : E \times D^r \to M\) is universal if and only if to every wedge \(\alpha : K \to T : E \times D^r \to M\) there is a unique natural transformation \(\mu / \alpha : R \to T\) such that \(\alpha = \mu / \alpha \circ \alpha\).

Proposition 7.2.11.

- For a wedge

\[ \begin{array}{ccc}
E & \xleftarrow{E \times \Delta_D} & E \times D \\
R & \searrow & \downarrow K \\
X & \xleftarrow{\mu} & A
\end{array} \]

, the following conditions are equivalent:

1. \(\mu\) is universal (resp. pointwise universal);
2. the cylinder

\[ \begin{array}{ccc}
E & \xrightarrow{\mu^r} & E \\
R \cdot K & \searrow & \downarrow K \cdot e \\
X & \xrightarrow{\mu^r} & [D, A]
\end{array} \]

is inverse universal (resp. pointwise inverse universal);
3. the cone

\[ \begin{array}{ccc}
* & \xleftarrow{\Delta_D} & D \\
R & \searrow & \downarrow K \cdot e \\
[X, E] & \xleftarrow{\mu^r} & [E, A]
\end{array} \]

is universal (resp. pointwise universal).
7. Limits

For a wedge

\[
\begin{array}{c}
E \times D \\
\downarrow \mu \\
X \to A
\end{array}
\begin{array}{c}
E \to E \\
\downarrow \mu^r \\
[D, X] \to A
\end{array}
\begin{array}{c}
E \to E \\
\downarrow \mu^* \\
\{E, X\} \to \{E, A\}
\end{array}
\]

the following conditions are equivalent:

1. \(\mu\) is universal (resp. pointwise universal);
2. the cylinder

\[
\begin{array}{c}
E \to E \\
\downarrow \mu^r \\
[D, X] \to A
\end{array}
\begin{array}{c}
E \to E \\
\downarrow \mu^* \\
\{E, X\} \to \{E, A\}
\end{array}
\]

is direct universal (resp. pointwise direct universal);
3. the cone

\[
\begin{array}{c}
D \to \Delta D \\
\downarrow \mu^r \\
\{E, X\} \to \{E, A\}
\end{array}
\begin{array}{c}
D \to \Delta D \\
\downarrow \mu^* \\
\{E, X\} \to \{E, A\}
\end{array}
\]

is universal (resp. pointwise universal).

Proof. By the isomorphisms in Remark 4.7.10(2), \(\mu\) is universal iff \(\mu^r\) (resp. \(\mu^*\)) is universal. Since the slice of \(\mu\) at each \(e \in \|E\|\) is given by the component of \(\mu^r\) at \(e\), \(\mu\) is pointwise universal iff the cylinder \(\mu^r\) is pointwise inverse universal. By Definition 7.2.1, \(\mu^*\) is pointwise universal iff \([\mu^r]^\top\) is pointwise inverse universal. But \([\mu^r]^\top = \mu^r\) by Remark 4.7.10(3). \(\square\)

Proposition 7.2.12.

- If a wedge \(\mu : R \to K : E \times D^\delta \to M\) is pointwise universal then, it is universal.
- If a wedge \(\mu : K \to R : E \times D^p \to M\) is pointwise universal then, it is universal.

Proof. The assertion is reduced to Proposition 6.4.5 by Proposition 7.2.11. \(\square\)

Proposition 7.2.13.

- If a wedge \(\mu : R \to K : E \times D^\delta \to M\) is universal (resp. pointwise inverse universal), then a natural transformation \(\tau : S \to R\) is iso if and only if the wedge \(\tau \circ \mu : S \to K : E \times D^\delta \to M\) is universal (resp. pointwise inverse universal).
- If a wedge \(\mu : K \to R : E \times D^p \to M\) is universal (resp. pointwise direct universal), then a natural transformation \(\tau : R \to S\) is iso if and only if the wedge \(\mu \circ \tau : K \to S : E \times D^p \to M\) is universal (resp. pointwise direct universal).

Proof. The assertion is reduced to Theorem 6.4.9 by Proposition 7.2.11. \(\square\)
7. Limits

7.3. Limits in a category

Note. The following definition is a special case of Definition 7.1.1 where \( M \) is given by the hom of a category and coincides with the usual definition of a universal cone (a limit) in the literature.

**Definition 7.3.1.**

- A cone \( \mu : r \sim K : E^\circ \to C \) is called universal if it is an inverse universal \((E^\circ, C)\)-arrow (see Definition 4.8.3). Given a functor \( K : E \to C \), a universal cone \( \mu : r \sim K : E^\circ \to C \) or the pair \((r, \mu)\), or the object \( r \) itself, is called an inverse limit of \( K \) in \( C \).

- A cone \( \mu : K \leadsto r : E^\circ \to C \) is called universal if it is a direct universal \((E^\circ, C)\)-arrow (see Definition 4.8.3). Given a functor \( K : E \to C \), a universal cone \( \mu : K \leadsto r : E^\circ \to C \) or the pair \((r, \mu)\), or the object \( r \) itself, is called a direct limit of \( K \) in \( C \).

**Remark 7.3.2.**

- A cone \( \mu : r \sim K : E^\circ \to C \) is universal if and only if to every cone \( \alpha : s \sim K : E^\circ \to C \) there is a unique \( C \)-arrow \( \alpha/\mu : s \to r \) such that \( \alpha = \alpha/\mu \circ \mu \).

- A cone \( \mu : K \leadsto r : E^\circ \to C \) is universal if and only if to every cone \( \alpha : K \leadsto t : E^\circ \to C \) there is a unique \( C \)-arrow \( \mu/\alpha : r \to t \) such that \( \alpha = \mu \circ \mu/\alpha \).

**Definition 7.3.3.** A category \( C \) is called inverse (resp. direct) \( E \)-complete for a category \( E \) if every functor \( E \to C \) has an inverse (resp. direct) limit in \( C \), and called inverse (resp. direct) complete if it is inverse (resp. direct) \( E \)-complete for any small category \( E \).

**Remark 7.3.4.**

1. A category \( C \) is inverse (resp. direct) complete if and only if the hom \((C)\) is inverse (resp. direct) complete in the sense of Definition 7.1.3.

2. Remark 7.1.4 holds with \( M \) replaced with \( C \).

Note. The following definition is a special case of Definition 7.1.5 where \( \Phi \) is given by the hom of a functor and coincides with the usual notion of preservation (resp. reflection, creation) of limits defined in the literature.

**Definition 7.3.5.** A functor \( H : C \to B \) is said to

- preserve (resp. reflect, create) inverse limits for functors \( E \to C \) if the the postcomposition cell \((E^\circ, H)\) (see 4.8.5) preserves (resp. reflects, creates) inverse universals.

- preserve (resp. reflect, create) direct limits for functors \( E \to C \) if the the postcomposition cell \((E^\circ, H)\) (see 4.8.5) preserves (resp. reflects, creates) direct universals.

and said to

- preserve (resp. reflect, create) inverse limits (resp. small inverse limits) if it does so for functors \( E \to C \) for any category (resp. small category ) \( E \).

- preserve (resp. reflect, create) direct limits (resp. small direct limits) if it does so for functors \( E \to C \) for any category (resp. small category ) \( E \).
7. Limits

**Remark 7.3.6.** A functor $H$ preserves (resp. reflects, creates) limits if and only if the hom cell $(H)$ does the same in the sense of 7.1.5.

**Note.** The following is a special case of Proposition 7.1.8 where $\Phi$ is given by the hom of a functor.

**Proposition 7.3.7.** Let $H : C \to B$ be a functor.

- Suppose that $H$ creates inverse limits for functors $E \to C$. In this situation, if $B$ is inverse $E$-complete, so is $C$.

- Suppose that $H$ creates direct limits for functors $E \to C$. In this situation, if $B$ is direct $E$-complete, so is $C$.

**Proof.** See Note above.

**Theorem 7.3.8.** Let $M : X \to A$ be a module.

- Any corepresentation

\[
\begin{array}{c}
X & \xrightarrow{M} & A \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{\delta} & \Lambda \\
\downarrow & & \downarrow \\
X & \xrightarrow{\gamma} & X
\end{array}
\]

of $M$ preserves, reflects, and creates inverse limits for functors $E \to A$.

- Any representation

\[
\begin{array}{c}
A & \xrightarrow{M} & A \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{\delta} & \Lambda \\
\downarrow & & \downarrow \\
A & \xrightarrow{\gamma} & A
\end{array}
\]

of $M$, preserves, reflects, and creates direct limits for functors $E \to X$.

**Proof.** Immediate from Proposition 7.1.7.

**Corollary 7.3.9.** Let $E$ be a category and $M : X \to A$ be a module.

- Suppose that $M$ is corepresentable. In this situation, if $X$ is inverse $E$-complete, so is $M$.

- Suppose that $M$ is representable. In this situation, if $A$ is direct $E$-complete, so is $M$.

**Proof.** Since a corepresentation of $M$ creates inverse limits for functors $E \to A$ by Theorem 7.3.8, the assertion follows from Proposition 7.1.8.

**Corollary 7.3.10.** Let $M : X \to A$ be a module.

- Suppose that $M$ is corepresentable. In this situation, if $X$ is inverse complete, so is $M$.

- Suppose that $M$ is representable. In this situation, if $A$ is direct complete, so is $M$.

**Proof.** Immediate from Corollary 7.3.9.

**Note.** The following definition is a special case of Definition 7.2.1 where $M$ is given by the hom of a category.
7. Limits

**Definition 7.3.11.** Let $E$, $D$, and $C$ be categories.

- A cone

\[
\begin{array}{ccc}
\ast & \xleftarrow{\Delta_D} & D \\
R & \mu & K \\
\downarrow & \downarrow & \downarrow \\
[E, C] & \xrightarrow{(E, C)} & [E, C]
\end{array}
\]

in the functor category $[E, C]$ is called pointwise universal if its transpose

\[
\begin{array}{ccc}
E & \xleftarrow{E} & E \\
R & \mu^\top & K^\top \\
\downarrow & \downarrow & \downarrow \\
C & \xrightarrow{(D^\top, C)} & [D, C]
\end{array}
\]

(see Remark 4.8.15(3)) is a pointwise inverse universal cylinder; that is, if the component of $\mu^\top$ at each $e \in \|E\|$ is a universal cone in $C$. Given a functor $K : D \to [E, C]$, a pointwise universal cone $\mu : R \to K : D^\top \to [E, C]$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a pointwise inverse limit of $K$ in $[E, C]$.

- A cone

\[
\begin{array}{ccc}
D & \xrightarrow{\Delta_D} & \ast \\
K & \mu & R \\
\downarrow & \downarrow & \downarrow \\
[E, C] & \xrightarrow{(E, C)} & [E, C]
\end{array}
\]

in the functor category $[E, C]$ is called pointwise universal if its transpose

\[
\begin{array}{ccc}
E & \xleftarrow{E} & E \\
K^\top & \mu^\top & R \\
\downarrow & \downarrow & \downarrow \\
[D, C] & \xrightarrow{(D^\top, C)} & [D, C]
\end{array}
\]

(see Remark 4.8.15(3)) is a pointwise direct universal cylinder; that is, if the component of $\mu^\top$ at each $e \in \|E\|$ is a universal cone in $C$. Given a functor $K : D \to [E, C]$, a pointwise universal cone $\mu : K \sim R : D^\top \to [E, C]$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a pointwise direct limit of $K$ in $[E, C]$.
7. Limits

Remark 7.3.12. Proposition 7.2.2, Proposition 7.2.3, and Proposition 7.2.4 hold also for cones in a functor category.

Note. The following is a special case of Theorem 7.2.6 where $\mathcal{M}$ is given by by the hom of a category.

Theorem 7.3.13. Let $\textbf{E}$, $\textbf{D}$, and $\textbf{C}$ be categories.

- The family of evaluations $[e, \textbf{C}] : [\textbf{E}, \textbf{C}] \to \textbf{C}$ (see Preliminaries(13)), one for each $e \in [\textbf{E}]$, creates a pointwise inverse limit (see Definition 7.3.11) of a functor $K : \textbf{D} \to [\textbf{E}, \textbf{C}]$ in the following sense: if, for each $e \in [\textbf{E}]$, $[e, \textbf{C}] \circ K : \textbf{D} \to \textbf{C}$ has an inverse limit in $\textbf{C}$ and a universal cone $\mu_e : r_e \circ [e, \textbf{C}] \circ K : \textbf{D}^\bullet \to \textbf{C}$ is chosen, then there is a unique cone $\mu : R \to K : \textbf{D}^\bullet \to [\textbf{E}, \textbf{C}]$ such that $\mu \circ [e, \textbf{C}] = \mu_e$; moreover, $\mu$ is pointwise universal.

- The family of evaluations $[e, \textbf{C}] : [\textbf{E}, \textbf{C}] \to \textbf{C}$ (see Preliminaries(13)), one for each $e \in [\textbf{E}]$, creates a pointwise direct limit (see Definition 7.3.11) of a functor $K : \textbf{D} \to [\textbf{E}, \textbf{C}]$ in the following sense: if, for each $e \in [\textbf{E}]$, $K \circ [e, \textbf{C}] : \textbf{D} \to \textbf{C}$ has a direct limit in $\textbf{C}$ and a universal cone $\mu_e : K \circ [e, \textbf{C}] \circ r_e : \textbf{D}^\circ \to \textbf{C}$ is chosen, then there is a unique cone $\mu : R \to K : \textbf{D}^\circ \to [\textbf{E}, \textbf{C}]$ such that $\mu \circ [e, \textbf{C}] = \mu_e$; moreover, $\mu$ is pointwise universal.

Proof. See Note above.

Note. The following is a special case of Corollary 7.2.7 where $\mathcal{M}$ is given by by the hom of a category.

Corollary 7.3.14. Let $\textbf{E}$, $\textbf{D}$, and $\textbf{C}$ be categories. If $\textbf{C}$ is inverse (resp. direct) $\textbf{D}$-complete, so is the functor category $[\textbf{E}, \textbf{C}]$, and all limits are given pointwise. As a consequence, if a category $\textbf{C}$ is inverse (resp. direct) complete, so is the functor category $[\textbf{E}, \textbf{C}]$.

Proof. See Note above.

Example 7.3.15. Since $\textbf{Set}$ is inverse and direct complete, so is the module category $[X : A]$ for any pair of categories $X$ and $A$.

Note. The following is a special case of Corollary 7.2.8 where $\mathcal{M}$ is given by by the hom of a category.

Corollary 7.3.16. Let $\textbf{D}$ be a category and $F : \textbf{E}' \to \textbf{E}$ be a functor. If a category $\textbf{C}$ is inverse (resp. direct) $\textbf{D}$-complete, then the precomposition functor $[F, \textbf{C}] : [\textbf{E}, \textbf{C}] \to [\textbf{E}', \textbf{C}]$ preserves inverse (resp. direct) limits for functors $\textbf{D} \to [\textbf{E}, \textbf{C}]$.

Proof. See Note above.

Example 7.3.17. Since $\textbf{Set}$ is complete, for any pair of functors $P : X \to Y$ and $Q : A \to B$, the precomposition functor $[P, Q] : [Y : B] \to [X : A]$ preserves small limits.

7.4. Limits of modules

Theorem 7.4.1. Let $\textbf{E}$ be a small category.

- The universal cone $E^\bullet_M$ of a left module $\mathcal{M} : * \to \textbf{E}$ defined in Definition 4.9.5 is indeed universal.
7. Limits

- The universal cone $E^\alpha_M$ of a right module $M : E \to \ast$ defined in Definition 4.9.5 is indeed universal.

Proof. Given a set $S$ and a cone $\alpha : S \to M$, the unique function $\alpha' : S \to \prod_{E^\ast} M$ making the triangle

$$
\begin{array}{c}
S \\
\alpha' \\
\prod_{E^\ast} M \\
\alpha \\
E^\alpha_M \\
M
\end{array}
$$

commute is defined by

$$(s \cdot \alpha')_e = s \cdot (\alpha) e$$

for $s \in S$ and $e \in \|E\|$.

Remark 7.4.2. The inverse completeness of Set follows from Theorem 7.4.1.

Corollary 7.4.3. Let $E$ be a small category.

- The right cylinder $E^a$ in Proposition 4.9.7 is a counit of the module $\langle : E^\ast \rangle$.
- The right cylinder $E^b$ in Proposition 4.9.7 is a counit of the module $\langle E^\ast : \rangle$.

Proof. Immediate from Theorem 7.4.1.

Corollary 7.4.4. Let $E$ be a small category.

- The right cylinder $[X : E]^a$ defined in Definition 4.9.8 is a counit of the module $\langle X : E^\ast \rangle$; in fact, the component of $[X : E]^a$ at each module $M : X \to E$ is a pointwise universal wedge.
- The right cylinder $[E : A]^b$ defined in Definition 4.9.8 is a counit of the module $\langle E^\ast : A \rangle$; in fact, the component of $[E : A]^b$ at each module $M : E \to A$ is a pointwise universal wedge.

Proof. The slice of the wedge $[X : E]^a_M : X \to E^\ast$ at each $x \in \|X\|$ is the universal cone of the left module $x(M) : \ast \to E$.

Corollary 7.4.5. Let $E$ be a small category and $M : X \to A$ be a module.

- The cylinder $[M \cdot E]^a$ defined in Definition 4.9.11 is pointwise inverse universal; in fact, the component of $[M \cdot E]^a$ at each functor $K : E \to A$ is a pointwise universal wedge.
- The cylinder $[E \cdot M]^b$ defined in Definition 4.9.11 is pointwise direct universal; in fact, the component of $[E \cdot M]^b$ at each functor $K : E \to X$ is a pointwise universal wedge.

Proof. The slice of the wedge $[M \cdot E]^a_M : X \to E^\ast$ at each $x \in \|X\|$ is the universal cone of the left module $x(M) K : \ast \to E$ (see Remark 4.9.12).

Theorem 7.4.6. Let $E$ be a category and $M : X \to A$ be a module.
7. Limits

- The right general Yoneda morphism for \((E^a, M)\)

\[
\begin{array}{c}
X \ar[r]^{(E^a, M)} & [E, A] \\
\ar[d]^{(X|M)\triangleright E^a} & \ar[d]^{M\triangleright E} \\
[X:] & [X:E]
\end{array}
\]

(see Definition 5.4.1) preserves and reflects universals in the following sense: a cone \(\mu : r \rightharpoonup K : E^a \rightharpoonup M\) is universal if and only if the wedge \((X|M{\mu}) : (X) \rightharpoonup (M) K : X \rightharpoonup E^*\) is pointwise universal; that is, if and only if the cone \(x(X|M{\mu}) : x(X) \rightharpoonup x(M) K : * \rightharpoonup E^*\) is universal for each \(x \in |X|\).

- The left general Yoneda morphism for \((E^p, M)\)

\[
\begin{array}{c}
[E, X] \ar[r]^{(E^p, M)} & A \\
\ar[d]^{E\triangleleft X} & \ar[d]^{\triangleleft A} \\
[E : A] & [E : A]^-
\end{array}
\]

(see Definition 5.4.1) preserves and reflects universals in the following sense: a cone \(\mu : K \rightharpoonup r : E^p \rightharpoonup M\) is universal if and only if the wedge \(\mu(M|A) : r(A) \rightharpoonup K(M) : E^* \rightharpoonup A\) is pointwise universal; that is, if and only if the cone \(\mu(M|A) a : r(A) a \rightharpoonup K(M) a : E^* \rightharpoonup *\) is universal for each \(a \in |A|\).

**Proof.** First choose a universe so that \(M\) become small and \(M\) become locally small. Let \(\mu : r \rightharpoonup K\) be an \((E^a, M)-}\)-arrow. By Theorem 5.47, \((X|M)\mu\) is given by the composition of \(\langle X |(E^a, M)\rangle\mu\) and \([M\triangleright E]\rangle\). By Corollary 7.4.5, \([M\triangleright E]\rangle\) is a pointwise universal wedge. Hence, by Proposition 7.2.13, \((X|M)\mu\) is pointwise universal iff \(\langle X |(E^a, M)\rangle\mu\) is iso, i.e. iff \(\mu\) is inverse universal.

**Corollary 7.4.7.** Let \(C\) and \(E\) be a categories.

- The right general Yoneda morphism for \((E^a, C)\)

\[
\begin{array}{c}
C \ar[r]^{(E^a, C)} & [E, C] \\
\ar[d]^{C\triangleright E^a} & \ar[d]^{C\triangleright E} \\
[C:] & [C : E]
\end{array}
\]

(see Definition 5.4.4) preserves and reflects universals in the following sense: a cone \(\mu : r \rightharpoonup K : E^a \rightharpoonup C\) is universal if and only if the wedge \((C)\mu : (C) r \rightharpoonup (C) K : C \rightharpoonup E^*\) is pointwise universal; that is, if and only if the cone \(c(C)\mu : c(C) r \rightharpoonup c(C) K : * \rightharpoonup E^*\) is universal for each \(c \in |C|\).

- The left general Yoneda morphism for \((E^p, C)\)

\[
\begin{array}{c}
[E, C] \ar[r]^{(E^p, C)} & C \\
\ar[d]^{E\triangleleft C} & \ar[d]^{\triangleleft C} \\
[E : C] & [E : C]^-
\end{array}
\]

\[
\begin{array}{c}
\parallel.alt1
\end{array}
\]

223
7. Limits

(see Definition 5.4.4) preserves and reflects universals in the following sense: a cone \( \mu : K \to r : E^* \to C \) is universal if and only if the wedge \( \mu(C) : r(C) \to K(C) : E^* \to C \) is pointwise universal; that is, if and only if the cone \( \mu(C) c : r(C) c \to K(C) c : E^* \to * \) is universal for each \( c \in \|C\| \).

Proof. The is a special case of Theorem 7.4.6 where \( M \) is given by the hom of \( C \). □

Corollary 7.4.8.

- A representable left module \( M : * \to A \) preserves inverse limits; that is, if a cone \( \mu : r \to K : E^a \to A \) is universal, so is the cone \( \langle M \rangle \mu : (M) r \to (M) K : * \to E^* \).

- A representable right module \( M : X \to * \) preserves direct limits; that is, if a cone \( \mu : K \to r : E^b \to X \) is universal, so is the cone \( \mu(M) : r(M) \to K(M) : E^* \to * \).

Proof. Since \( M \) is representable, \( M \cong a(A) \) for some \( a \in \|A\| \). But, by Corollary 7.4.7, if a cone \( \mu : r \to K : E^a \to A \) is universal, so is the cone \( a(A) \mu : a(A) r \to a(A) K : * \to E^* \). □

Corollary 7.4.9. Given a module \( M \), the right (resp. left) Yoneda morphism for \( M \) preserves inverse (resp. direct) limits.

Proof. Consider the right Yoneda morphism \( \langle X, M \rangle \to \) for a module \( M : X \to A \). Given a category \( E \), we need to show that the postcomposition cell \( \langle E^a, \langle X, M \rangle \to \rangle \) preserves inverse universals. But, by Proposition 5.4.6, this is the case if and only if the right general Yoneda morphism \( \langle X, M \rangle \to E^a \) does the same. The assertion thus follows from Theorem 7.4.6. □

Corollary 7.4.10.

- A functor \( H : C \to B \) preserves (resp. reflects) inverse limits for functors \( E \to C \) if and only if the left module \( b(B) H : * \to C \) does so for each \( b \in \|B\| \).

- A functor \( H : C \to B \) preserves (resp. reflects) direct limits for functors \( E \to C \) if and only if the right module \( H(b) b : C \to * \) does so for each \( b \in \|B\| \).

Proof. By Corollary 7.4.7, for any cone \( \mu : r \to K : E^a \to C \), \( H \circ \mu \) is universal iff \( b(B)[H \circ \mu] = (b(B) H) \mu \) is universal for each \( b \in \|B\| \). □

Note. In Corollary 7.4.11 and Corollary 7.4.12, a module is freely identified with its collage (see Remark 3.1.16(1)).

Corollary 7.4.11. Let \( M : X \to A \) be a module.

- The inclusion \( M_A : A \to \|M\| \) preserves inverse limits for functors \( E \to A \) if and only if the left module \( x(M) : \ast \to A \) does so for every \( x \in \|X\| \).

- The inclusion \( M_X : X \to \|M\| \) preserves direct limits for functors \( E \to X \) if and only if the right module \( a(M) : X \to * \) does so for every \( a \in \|A\| \).

Proof. By Corollary 7.4.10, the inclusion \( M_A : A \to \|M\| \) preserves inverse limits for functors \( E \to A \) iff the following conditions hold:

1. for each \( x \in \|X\| \), the left module \( x(\|M\|) A = x(M) : \ast \to A \) preserves inverse limits for functors \( E \to A \);
2. for each \( a \in \|A\| \), the left module \( a([M])A = a(A) : * \to A \) preserves inverse limits for functors \( E \to A \).

Since the second condition always holds by Corollary 7.4.8, the assertion results. \( \square \)

**Corollary 7.4.12.**

- *Given a left module \( M : * \to A \), the inclusion \( M : A \to [M] \) preserves inverse limits for functors \( E \to A \) if and only if \( M \) does so.*

- *Given a right module \( M : X \to * \), the inclusion \( M : X \to [M] \) preserves direct limits for functors \( E \to X \) if and only if \( M \) does so.*

*Proof. *This is a special case of Corollary 7.4.11 where \( X \) is the terminal category. \( \square \)
8. Adjunctions

8.1. Adjunctions for categories

Definition 8.1.1. Given a pair of functors $G : A \to X$ and $F : X \to A$,

- an adjunction $\Upsilon : G \leadsto F : X \to A$, written graphically as

  \[
  \begin{array}{c}
  X \\
  \downarrow \Upsilon \\
  F \\
  \end{array}
  \begin{array}{c}
  \leftarrow \\
  \Upsilon \\
  \leftarrow \ \ \\
  A \\
  \end{array}
  \]

  is defined by a module isomorphism

  \[
  \Upsilon : (X) \cong F(A) : X \to A
  \]

  If $\Upsilon : G \leadsto F : X \to A$ is an adjunction, then the pair $(G, \Upsilon)$, or the functor $G$ itself, is called a right adjoint of $F$.

- an adjunction $\Upsilon : F \leadsto G : A \to X$, written graphically as

  \[
  \begin{array}{c}
  A \\
  \downarrow \Upsilon \\
  F \\
  \end{array}
  \begin{array}{c}
  \leftarrow \\
  \Upsilon \\
  \leftarrow \ \ \\
  X \\
  \end{array}
  \]

  is defined by a module isomorphism

  \[
  \Upsilon : F(A) \cong (X) G : X \to A
  \]

  If $\Upsilon : F \leadsto G : A \to X$ is an adjunction, then the pair $(F, \Upsilon)$, or the functor $F$ itself, is called a left adjoint of $G$.

Remark 8.1.2.

1. If a pair $(G, \Upsilon)$ is a right adjoint of $F$, then the pair $(G, \Upsilon^{-1})$ is a corepresentation of the representable module $F(A)$. Conversely if a pair $(G, \Upsilon)$ is a corepresentation of $F(A)$, then the pair $(G, \Upsilon^{-1})$ is a right adjoint of $F$.

2. If a pair $(F, \Upsilon)$ is a left adjoint of $G$, then the pair $(F, \Upsilon^{-1})$ is a representation of the corepresentable module $(X) G$. Conversely if a pair $(F, \Upsilon)$ is a representation of $(X) G$, then the pair $(F, \Upsilon^{-1})$ is a left adjoint of $G$.

2. The two forms of an adjunctions, $X \to A$ and $A \to X$, are referred to as the right and left forms. If $\Upsilon$ is the left form of an adjunction, then the inverse $\Upsilon^{-1}$ gives the right form of the adjunction. We mainly deal with the right form for each adjunction.

Note. Theorem 5.5.1 justifies the following definition.
8. Adjunctions

Definition 8.1.3.

- If \( \Upsilon : G \rightarrow F : X \rightarrow A \) is an adjunction, then
  - the counit of \( \Upsilon \) is the natural transformation \( \epsilon : G \circ F \rightarrow 1_A \) such that \( X \epsilon = \Upsilon \);
  - the unit of \( \Upsilon \) is the natural transformation \( \eta : 1_X \rightarrow G \circ F \) such that \( \eta A = \Upsilon^{-1} \).

- If \( \Upsilon : F \rightarrow G : A \rightarrow X \) is an adjunction, then
  - the unit of \( \Upsilon \) is the natural transformation \( \eta : 1_X \rightarrow G \circ F \) such that \( \eta A = \Upsilon \);
  - the counit of \( \Upsilon \) is the natural transformation \( \epsilon : G \circ F \rightarrow 1_A \) such that \( X \epsilon = \Upsilon^{-1} \).

Remark 8.1.4.

1. Theorem 5.5.1 shows how the counit and unit of an adjunction are obtained and how an adjunction is recovered from its counit and unit.

2. An adjunction \( \Upsilon : G \rightarrow F : X \rightarrow A \) is also written as \((\eta, \epsilon) : G \rightarrow F : X \rightarrow A\), or graphically as

\[
\begin{array}{ccc}
X & \xrightarrow{(\eta, \epsilon)} & A \\
\downarrow F & & \\
\end{array}
\]

, showing its unit and counit.

- An adjunction \( \Upsilon : F \rightarrow G : A \rightarrow X \) is also written as \((\epsilon, \eta) : F \rightarrow G : A \rightarrow X\), or graphically as

\[
\begin{array}{ccc}
A & \xleftarrow{(\epsilon, \eta)} & X \\
\downarrow F & & \\
\end{array}
\]

, showing its counit and unit.

3. The counit \( \epsilon \) of an adjunction \( \Upsilon : G \rightarrow F : X \rightarrow A \) satisfies the equivalent conditions in Proposition 8.1.5. Conversely, if a natural transformation \( \epsilon : G \circ F \rightarrow 1_A \) satisfies those equivalent conditions, then the pair \((G, X \epsilon)\) forms a right adjoint of \( F \).

- The unit \( \eta \) of an adjunction \( \Upsilon : G \rightarrow F : X \rightarrow A \) satisfies the equivalent conditions in Proposition 8.1.5. Conversely, if a natural transformation \( \eta : 1_X \rightarrow G \circ F \) satisfies those equivalent conditions, then the pair \((F, \eta A)\) forms a left adjoint of \( G \).

Proposition 8.1.5. Consider a pair of functors as in

\[
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
\downarrow F & & \\
\end{array}
\]

- For a natural transformation \( \epsilon : G \circ F \rightarrow 1_A \), the following conditions are equivalent:
  1. \( \epsilon \) forms a pointwise right Kan lift of the identity \( 1_A \) along \( F \);
  2. each component \( \epsilon_a : a \circ G : F \rightarrow a \) is universal from \( F \) to \( a \);
8. Adjunctions

3. \( \epsilon \) regarded as a two-sided cylinder

\[
\begin{array}{c}
A \\
\text{G} \\
\text{F(A)}
\end{array}
\quad
\begin{array}{c}
\epsilon \\
1
\end{array}
\quad
\begin{array}{c}
X \\
\text{F(A)}
\end{array}
\]

forms a pointwise lift of the identity \( 1_A \) inverse along the representable module \( F(A) \).

4. \( \epsilon \) regarded as a right cylinder

\[
\begin{array}{c}
X \\
\text{F(A)}
\end{array}
\quad
\begin{array}{c}
\epsilon \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
A \\
\rightarrow
\end{array}
\]

forms a counit of the representable module \( F(A) \);

5. the module morphism \( X\epsilon : (X)G \rightarrow F(A) : X \rightarrow A \) is iso;

\[ \text{For a natural transformation } \eta : 1_X \rightarrow G \circ F, \text{ the following conditions are equivalent:} \]
1. \( \eta \) forms a pointwise left Kan lift of the identity \( 1_X \) along \( G \);
2. each component \( \eta_x : x \rightarrow G \cdot F \cdot x \) is universal from \( x \) to \( G \);
3. \( \eta \) regarded as a two-sided cylinder

\[
\begin{array}{c}
X \\
\text{1} \\
\text{F}
\end{array}
\quad
\begin{array}{c}
\eta \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
A \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
(X)G \\
\rightarrow
\end{array}
\]

forms a pointwise lift of the identity \( 1_X \) direct along the corepresentable module \( (X)G \);

4. \( \eta \) regarded as a left cylinder

\[
\begin{array}{c}
X \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
\eta \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
(X)G \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
\text{F} \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
A \\
\rightarrow
\end{array}
\]

forms a unit of the corepresentable module \( (X)G \);

5. the module morphism \( \eta|A : F(A) \rightarrow (X)G : X \rightarrow A \) is iso.

Proof. The conditions (1)-(3) are equivalent by definition. The equivalence of (3) and (4) is stated in Remark 6.4.2(2). The conditions (4) and (5) are equivalent by definition.

Note. The following is a special case of Theorem 6.3.10 where \( \mathcal{M} \) is given by a representable (resp. corepresentable) module.

Theorem 8.1.6.

- Let \( F : X \rightarrow A \) be a functor. Suppose that every object \( a \in \|A\| \) has a universal inverse along \( F(A) \) and an \( A \)-arrow \( \epsilon_a : r_a : F \rightarrow a \) universal from \( F \) to \( a \) is chosen. Then there is a unique functor \( G : A \rightarrow X \) with \( a \cdot G = r_a \) such that the family of \( A \)-arrows \( \epsilon_a \), one for each \( a \in \|A\| \), forms a natural transformation \( \epsilon : G \circ F \rightarrow 1_A \); moreover, \( G \) is a right adjoint of \( F \) with \( \epsilon \) the counit of the adjunction.
8. Adjunctions

Let \( G : A \to X \) be a functor. Suppose that every object \( x \in \|X\| \) has a universal direct along \((X)G\) and an \( X\)-arrow \( \eta_x : x \to G \cdot r_x \) universal from \( x \) to \( G \) is chosen. Then there is a unique functor \( F : X \to A \) with \( x \cdot F = r_x \) such that the family of \( X\)-arrows \( \eta_x \), one for each \( x \in \|X\| \), forms a natural transformation \( \eta : 1_X \to G \circ F \); moreover, \( F \) is a left adjoint of \( G \) with \( \eta \) the unit of the adjunction.

Proof. See Note above.

Note. The axiom of choice is used in the proof of the following.

Corollary 8.1.7.

The following conditions are equivalent for a functor \( F : X \to A \):

1. \( F \) has a right adjoint;
2. for each object \( a \in \|A\| \) there is an object \( r_a \in \|X\| \) and an \( A\)-arrow \( \epsilon_a : r_a \cdot F \to a \) universal from \( F \) to \( a \).

The following conditions are equivalent for a functor \( G : A \to X \):

1. \( G \) has a left adjoint;
2. for each object \( x \in \|X\| \) there is an object \( r_x \in \|A\| \) and an \( X\)-arrow \( \eta_x : x \to G \cdot r_x \) universal from \( x \) to \( G \).

Proof.

\((1) \Rightarrow (2)\) If \( F \) has a right adjoint \((G, \Upsilon)\), then the counit of the adjunction \( \Upsilon \) satisfies the condition (2) (see Remark 8.1.4(3)).

\((2) \Rightarrow (1)\) A family of universal \( A\)-arrows \( \epsilon_a : r_a \cdot F \to a \), one chosen for each \( a \in \|A\| \), yields a right adjoint of \( F \) by Theorem 8.1.6.

Theorem 8.1.8. Given functors

\[
\begin{array}{ccc}
X & \xleftarrow{G} & A \\
\downarrow{F} & & \downarrow{1} \\
A & \xrightarrow{F} & X
\end{array}
\]

, a pair of natural transformations \( \eta : 1_X \to G \circ F \) and \( \epsilon : G \circ F \to 1_A \) form an adjunction \((\eta, \epsilon) : G \dashv F : X \to A\) if and only if the pasting compositions

\[
\begin{array}{c}
X \\
\downarrow{G} \\
A
\end{array}
\begin{array}{c}
\downarrow{F} \\
\uparrow{1}
\end{array}
\begin{array}{c}
X \\
\downarrow{\epsilon} \\
A
\end{array}
\quad
\begin{array}{c}
A \\
\downarrow{1} \\
\downarrow{F}
\end{array}
\begin{array}{c}
\downarrow{\eta} \\
\uparrow{G}
\end{array}
\begin{array}{c}
A \\
\downarrow{1} \\
X
\end{array}
\]

yield the identity natural transformations \( F \to F \) and \( G \to G \); that is, if and only if the triangles

\[
\begin{array}{c}
F \\
\downarrow{\eta \circ F} \\
F \circ G \circ F
\end{array}
\begin{array}{c}
F \circ G \circ F \\
\downarrow{F \circ \epsilon} \\
1
\end{array}
\begin{array}{c}
G \\
\downarrow{\epsilon \circ G} \\
G \circ F \circ G
\end{array}
\begin{array}{c}
G \circ F \circ G \\
\downarrow{1} \\
G
\end{array}
\]

commute.
8. Adjunctions

Proof. By Corollary 5.5.10, the triangles

\[
\begin{array}{ccc}
F(A) & \xrightarrow{[\eta \circ F \circ \varepsilon]} & F(A) \\
\downarrow \eta & & \downarrow \eta \\
(X) G & \xrightarrow{X \cdot \varepsilon} & (X) G
\end{array}
\quad \begin{array}{ccc}
(X) G & \xrightarrow{[(X \circ \eta) \circ [\varepsilon \circ G]]} & (X) G \\
\downarrow X \cdot \varepsilon & & \downarrow X \cdot \varepsilon \\
F(A) & \xrightarrow{\eta \circ A} & F(A)
\end{array}
\]

commute. Since the general Yoneda functor is fully faithful, \([[(\eta \circ F) \circ [\varepsilon \circ G]]) \) (resp. \([[(X \circ \eta) \circ [\varepsilon \circ G]]) \) is the identity if and only if \([\eta \circ F \circ [\varepsilon \circ G]]) \) (resp. \([\eta \circ \varepsilon \circ G]]) \) hold if and only if \(\eta \circ A \) and \(X \cdot \varepsilon \) are the inverse of each other.

\[\begin{array}{c}
\eta \circ A \\
\downarrow \\
X \cdot \varepsilon \\
\eta \circ A
\end{array}
\quad \begin{array}{c}
\eta \circ A \\
\downarrow \\
X \cdot \varepsilon \\
\eta \circ A
\end{array}
\]

\[\begin{array}{ccc}
\eta \circ A & \xrightarrow{\varepsilon \circ G} & (X) G \\
\uparrow & & \uparrow \\
\eta \circ A & \xrightarrow{\varepsilon \circ G} & (X) G
\end{array}
\quad \begin{array}{ccc}
\eta \circ A & \xrightarrow{\varepsilon \circ G} & (X) G \\
\uparrow & & \uparrow \\
\eta \circ A & \xrightarrow{\varepsilon \circ G} & (X) G
\end{array}
\]

Theorem 8.1.9. Given an adjunction \((\eta, \varepsilon) : G \leadsto F : X \rightarrow A\),

- the following conditions are equivalent:
  1. the functor \(G : A \rightarrow X\) is fully faithful;
  2. the counit \(\varepsilon : G \circ F \rightarrow 1_A\) is a natural isomorphism.

- the following conditions are equivalent:
  1. the functor \(F : X \rightarrow A\) is fully faithful;
  2. the unit \(\eta : 1_X \rightarrow F \circ G\) is a natural isomorphism.

Proof. Since \(G\) and \(\varepsilon\) form a counit of \(F(A)\) (see Remark 8.1.4(3)), by Theorem 6.3.12, \(G\) is fully faithful iff for each \(a \in \|A\|\) the \(F(A)\)-arrow \(\epsilon_a : a \cdot G \rightarrow a\) is not only inverse universal but direct universal. By Corollary 6.2.10, the \(F(A)\)-arrow \(\epsilon_a : a \cdot G \rightarrow a\) is direct universal iff the \(A\)-arrow \(\epsilon_a : a \cdot G : F \rightarrow a\) is invertible.

Theorem 8.1.10. Suppose that a module \(\mathcal{M} : X \rightarrow A\) has a counit and a unit as depicted in

\[
\begin{array}{c}
X \xrightarrow{\rho} \mathcal{M} \xrightarrow{\lambda} A \\
\downarrow 1 \\
F(A)
\end{array}
\quad \begin{array}{c}
X \xrightarrow{\lambda} A \\
\downarrow 1 \\
\mathcal{M}
\end{array}
\]

Then there is an adjunction \(\Upsilon : G \leadsto F : X \rightarrow A\) with the module isomorphism \(\Upsilon : X(A) \rightarrow F(A) : X \rightarrow A\) defined by the composition

\[
X(A) \xrightarrow{X(\rho)} \mathcal{M} \xrightarrow{(\lambda(A)^{-1})} F(A)
\]

and the counit \(\varepsilon : G \circ F \rightarrow 1_A\) and the unit \(\eta : 1_X \rightarrow F \circ G\) of the adjunction are given by the compositions

\[
\begin{array}{cccc}
A & \xrightarrow{\rho} & 1 & \xrightarrow{1} \\
\downarrow & & \downarrow & \downarrow \\
F(A) & \xrightarrow{\lambda(A)^{-1}} & \mathcal{M} & \xrightarrow{\rho} \\
\downarrow & & \downarrow & \downarrow \\
A & \xrightarrow{\lambda} & A & \xrightarrow{\lambda} A
\end{array}
\quad \begin{array}{cccc}
1 & \xrightarrow{1} & \xrightarrow{1} & \xrightarrow{1} \\
\downarrow & & \downarrow & \downarrow \\
\mathcal{X} & \xrightarrow{(\lambda(\rho)^{-1})^{-1}} & G & \xrightarrow{G} \\
\downarrow & & \downarrow & \downarrow \\
\mathcal{X} & \xrightarrow{\lambda} & A & \xrightarrow{\lambda} A
\end{array}
\]

That is, the component of \(\varepsilon\) at \(a \in \|A\|\) is given by the adjunct of \(\lambda(A)^{-1}\) along \(\lambda(A)\) and the component of \(\eta\) at \(x \in \|X\|\) is given by the adjunct of \(\lambda_x\) along \(\lambda(\rho)\) as indicated in

\[
\begin{array}{c}
a : G \xrightarrow{\lambda(a : G)} F : G \\
\downarrow \rho_a \\
a
\end{array} \quad \begin{array}{c}
x : F \xrightarrow{\lambda(\rho(x : F))} F : X \\
\downarrow \eta_x \\
x
\end{array}
\quad \begin{array}{c}
x : F \xrightarrow{\lambda(\rho(x : F))} F : X \\
\downarrow \eta_x \\
x
\end{array}
\]

230
8. Adjunctions

Proof. The first assertion is obvious since $X \uparrow \rho$ and $\lambda \uparrow A$ are isomorphisms by the definition of units. The counit and unit are given by

$$
\epsilon = [\mathcal{T}] = [(X \uparrow \rho) \circ (\lambda \uparrow A)^{-1}] = [X \uparrow \rho] \circ (\lambda \uparrow A)^{-1} = \rho \circ (\lambda \uparrow A)^{-1}, \quad \eta = [\mathcal{T}^{-1}] = [(\lambda \uparrow A) \circ (X \uparrow \rho)^{-1}] = [\lambda \uparrow A] \circ (X \uparrow \rho)^{-1} \quad \ast \tag{\ast \text{ i by Corollary 5.3.19)}.
$$

8.2. Transformations of adjoints

**Definition 8.2.1.** Given a pair of categories $X$ and $A$,

- the module

$$
\langle X \uparrow A \rangle : \{A, X\} \rightarrow [X, A]^-
$$

is defined by the composition

$$
\xymatrix{[A, X] & [X : A] \ar[l]^{X \times A} & [X : A] \ar[r]^{- (X : A)} & [X : A] \ar[r]^{X \times A} & [X, A]^-
}
$$

where $X \times A$ is the right general Yoneda functor for the functor category $[A, X]$ and $X \times A$ is the left general Yoneda functor for the functor category $[X, A]$.

- the module

$$
\langle A \uparrow X \rangle : \{X, A\}^- \rightarrow [A, X]
$$

is defined by the composition

$$
}
$$

where $X \times A$ is the left general Yoneda functor for the functor category $[X, A]$ and $X \times A$ is the right general Yoneda functor for the functor category $[A, X]$.

**Remark 8.2.2.**

1. For each pair of functors

$$
X \xrightarrow{G} A \xleftarrow{F}
$$

- the set $(G) \langle X \uparrow A \rangle (F)$ consists of all module morphisms $(X) G \rightarrow F \langle A \rangle : X \rightarrow A$; that is,

$$
(G) \langle X \uparrow A \rangle (F) = (G \langle X \rangle, (X) G) (F \langle A \rangle)
$$

- the set $(F) \langle A \uparrow X \rangle (G)$ consists of all module morphisms $F \langle A \rangle \rightarrow (X) G : X \rightarrow A$; that is,

$$
(F) \langle A \uparrow X \rangle (G) = (F \langle A \rangle) (X) (G)
$$
8. Adjunctions

2. The bijections in Theorem 5.5.1 are now written as

\[(G \circ F)(A, A)(1_A) \cong (G)(X, A)(F) \quad (1_X)(X, X)(G \circ F) \cong (G)(A, X)(F)\]

3. For any categories \(X\) and \(A\),

\[\langle A, X \rangle \cong \langle A^\sim, X^\sim \rangle\]

**Proposition 8.2.3.** Let \(X\) and \(A\) be categories.

- The right form \(\Upsilon : G \Rightarrow F : X \rightarrow A\) of an adjunction is a two-way universal \((X, A)\)-arrow.
- The left form \(\Upsilon : F \Rightarrow G : A \rightarrow X\) of an adjunction is a two-way universal \((A, X)\)-arrow.

**Proof.** Since the Yoneda functors are fully faithful and \(\Upsilon\) is invertible in \([X : A]\), the assertion follows from Theorem 6.2.13.\(\Box\)

**Definition 8.2.4.**

- Given two adjunctions \(\Upsilon : G \Rightarrow F : X \rightarrow A\) and \(\Upsilon' : G' \Rightarrow F' : X \rightarrow A\), a pair of natural transformations \(\tau : G \rightarrow G' : A \rightarrow X\) and \(\sigma : F' \rightarrow F : X \rightarrow A\) are called conjugate if the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{(X)} & F(A) \\
\downarrow_{\tau} & & \downarrow_{\sigma(A)} \\
G' & \xrightarrow{(X)} & F'(A)
\end{array}
\]

commutes; that is, if the quadrangle

\[
\begin{array}{ccc}
G & \xrightarrow{\Upsilon} & F \\
\downarrow_{\tau} & & \downarrow_{\sigma} \\
G' & \xrightarrow{\Upsilon'} & F'
\end{array}
\]

is a two-way conjugation along the module \((X, A)\) in the sense of Definition 6.2.18.

- Given two adjunctions \(\Upsilon : F \Rightarrow G : A \rightarrow X\) and \(\Upsilon' : F' \Rightarrow G' : A \rightarrow X\), a pair of natural transformations \(\tau : G \rightarrow G' : A \rightarrow X\) and \(\sigma : F' \rightarrow F : X \rightarrow A\) are called conjugate if the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{(X)} & G \\
\downarrow_{\sigma} & & \downarrow_{\tau} \\
F' & \xrightarrow{(X)} & G'
\end{array}
\]

commutes; that is, if the quadrangle

\[
\begin{array}{ccc}
F & \xrightarrow{\Upsilon} & G \\
\downarrow_{\sigma} & & \downarrow_{\tau} \\
F' & \xrightarrow{\Upsilon'} & G'
\end{array}
\]

is a two-way conjugation along the module \((A, X)\) in the sense of Definition 6.2.18.

232
8. Adjunctions

**Proposition 8.2.5.** Consider adjunctions and natural transformations as in Definition 8.2.4 and let $\epsilon$ and $\epsilon'$ (resp. $\eta$ and $\eta'$) be the counits (resp. units) of $\Upsilon$ and $\Upsilon'$.

1. $\triangleright$ $\tau$ and $\sigma$ are conjugate if and only if the quadrangle

\[
\begin{array}{c}
G \circ F' \xrightarrow{G \circ \sigma} G \circ F \\
\downarrow \tau \circ F' & \downarrow \epsilon' \\
G' \circ F' & \downarrow 1_A
\end{array}
\]

commutes.

$\triangleright$ $\tau$ and $\sigma$ are conjugate if and only if the quadrangle

\[
\begin{array}{c}
1_X \xrightarrow{\eta} G \circ F \\
\downarrow \eta' & \downarrow \tau \circ F' \\
G' \circ F' & \downarrow G' \circ F
\end{array}
\]

commutes.

2. $\triangleright$ Given $\tau$, its conjugate $\sigma$ is obtained by the composition

\[
\begin{array}{c}
F' \xrightarrow{\sigma} F \\
\downarrow \eta \circ F' & \downarrow F \circ \epsilon' \\
F \circ G \circ F' & \downarrow F \circ G' \circ F'
\end{array}
\]

; that is, by the pasting composition

\[
\begin{array}{c}
X \xrightarrow{1} \xrightarrow{\eta} F \\
\downarrow G & \downarrow \tau \\
G' & \xrightarrow{\epsilon'} 1_A
\end{array}
\]

$\triangleright$ Given $\sigma$, its conjugate $\tau$ is obtained by the composition

\[
\begin{array}{c}
G \xrightarrow{\tau} G' \\
\downarrow G \circ \eta' & \downarrow \epsilon \circ G' \\
G \circ F' \circ G' & \downarrow G \circ F \circ G'
\end{array}
\]

; that is, by the pasting composition

\[
\begin{array}{c}
A \xrightarrow{1} \xrightarrow{\epsilon} F \\
\downarrow \sigma & \downarrow \tau \\
\xrightarrow{\sigma} \xrightarrow{\eta'} A
\end{array}
\]
8. Adjunctions

Proof.

1. Since the bijection \((G \circ F)(A, A)(1_A) \cong (G)(X \downarrow A)(F)\) (see Remark 8.2.2(2)) is natural in \(G\) and \(F\), the quadrangle

\[
\begin{array}{ccc}
G & \sim & F \\
\updownarrow_{\tau} & & \updownarrow_{\sigma} \\
G' & \sim & F'
\end{array}
\]

commutes iff the quadrangle

\[
\begin{array}{ccc}
G \circ F' & \overset{G \circ \sigma}{\longrightarrow} & G \circ F \\
\downarrow_{\tau \circ F'} & & \downarrow_{\epsilon} \\
G' \circ F' & \overset{\epsilon'}{\longrightarrow} & 1_A
\end{array}
\]

commutes.

2. The commutativity of the quadrangle above, i.e.

\[\tau \circ F' \circ \epsilon' = G \circ \sigma \circ \epsilon\]

, says that the two pasting compositions

\[
\begin{array}{ccc}
X & \overset{G}{\longrightarrow} & A \\
\downarrow_{F'} & & \downarrow_{\epsilon'} \\
A & \overset{1}{\longrightarrow} & 1
\end{array}
\quad
\begin{array}{ccc}
X & \overset{G}{\longrightarrow} & A \\
\downarrow_{F'} & & \downarrow_{\epsilon'} \\
A & \overset{1}{\longrightarrow} & 1
\end{array}
\]

yield the same natural transformation \(G \circ F' \rightarrow 1_A\). Hence the two pasting compositions

\[
\begin{array}{ccc}
X & \overset{1}{\longrightarrow} & A \\
\uparrow_{\eta} & & \downarrow_{F} \\
X & \overset{G}{\longrightarrow} & A \\
\downarrow_{F'} & & \downarrow_{\epsilon'} \\
A & \overset{1}{\longrightarrow} & 1
\end{array}
\quad
\begin{array}{ccc}
X & \overset{1}{\longrightarrow} & A \\
\uparrow_{\eta} & & \downarrow_{F} \\
X & \overset{G}{\longrightarrow} & A \\
\downarrow_{F'} & & \downarrow_{\epsilon'} \\
A & \overset{1}{\longrightarrow} & 1
\end{array}
\]

yields the same natural transformation, which is \(\sigma\) by Theorem 8.1.8.

\[\square\]

Definition 8.2.6. Given a pair of bifunctors \(F : X \times E^\rightarrow \rightarrow A\) and \(G : A \times E \rightarrow X\), an \(E\)-parameterized adjunction \(\Upsilon : G \Rightarrow F : X \rightarrow A\) is defined by a cylinder

\[
\begin{array}{ccc}
E & \overset{G^\rightarrow}{\longrightarrow} & [A, X] \\
\uparrow_{\Upsilon} & & \downarrow_{(X \downarrow A)} \\
[X, A] & \overset{F^\rightarrow}{\longrightarrow} & [X, A]
\end{array}
\]

such that each component forms an adjunction \(\Upsilon_e : G^\rightarrow e \Rightarrow F^\rightarrow e : X \rightarrow A\).

Remark 8.2.7.
8. Adjunctions

1. If \( \Upsilon : G \dashv F : X \to A \) is an \( E \)-parameterized adjunction, then, by Proposition 8.2.3, the cylinder \( \Upsilon : G^\dagger \dashv F^\dagger : E \to (X \uparrow A) \) is two-way pointwise universal.

2. \( \Upsilon \) forms an \( E \)-parameterized adjunction if and only if the quadrangle

\[
\begin{array}{ccc}
\quad & \quad & \quad \\
\e \cdot G^\dagger & \xrightarrow{\Upsilon_e} & e \cdot F^\dagger (A) & \xrightarrow{F^\dagger \cdot e} \\
\downarrow h \cdot G^\dagger & \quad & \downarrow [h \cdot F^\dagger](A) & \quad & \downarrow F^\dagger \cdot h \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\end{array}
\]

commutes (i.e. \( G^\dagger \cdot h \) and \( F^\dagger \cdot h \) are conjugate) for every \( E \)-arrow \( h : e \to e' \).

Proposition 8.2.8.

- Let \( F : X \times E^\dagger \to A \) be a bifunctor and assume that, for each object \( e \in \mathcal{E} \), \( F^\dagger \cdot e : X \to A \) has a right adjoint \( G_e : A \to X \) via an adjunction \( \Upsilon_e : G_e \dashv F^\dagger \cdot e : X \to A \). Then there is a unique bifunctor \( G : A \times E \to X \) with \( G^\dagger \cdot e = G_e \) such that the family of adjunctions \( \Upsilon_e \), one for each \( e \in \mathcal{E} \), forms a cylinder \( \Upsilon : G^\dagger \dashv F^\dagger : E \to (X \uparrow A) \); moreover, \( \Upsilon \) is an \( E \)-parameterized adjunction.

- Let \( G : A \times E \to X \) be a bifunctor and assume that, for each object \( e \in \mathcal{E} \), \( G^\dagger \cdot e : A \to X \) has a left adjoint \( F_e : X \to A \) via an adjunction \( \Upsilon_e : G^\dagger \cdot e \sim F_e : X \to A \). Then there is a unique bifunctor \( F : X \times E^\dagger \to A \) with \( F^\dagger \cdot e = F_e \) such that the family of adjunctions \( \Upsilon_e \), one for each \( e \in \mathcal{E} \), forms a cylinder \( \Upsilon : G^\dagger \dashv F^\dagger : E \to (X \uparrow A) \); moreover, \( \Upsilon \) is an \( E \)-parameterized adjunction.

Proof. The assertion follows from Theorem 6.4.7 on noting Proposition 8.2.3.

\[ \square \]

8.3. Adjunctions for modules

Definition 8.3.1. Let \( \mathcal{M} : X \to A \) and \( \mathcal{N} : Y \to B \) be modules. Given a pair of functors \( P : X \to Y \) and \( Q : B \to A \),

- a right slant cell \( \Phi : Q \dashv P : M \to N \), depicted as

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow P & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \Quad
8. Adjunctions

**Remark 8.3.2.**

1. Given a pair of objects \( x \in \|X\| \) and \( b \in \|B\| \),

   - a right slant cell \( \Phi : Q \twoheadrightarrow P : M \rightarrow N \) sends each \( M \)-arrow \( m : x \twoheadrightarrow Q \cdot b \) to the \( N \)-arrow \( m : x \twoheadrightarrow P \twoheadrightarrow b \), the image of \( m \) under the function

     \[
     x \langle M \rangle \langle Q \cdot b \rangle = x \langle (M) Q \rangle b \xrightarrow{x(\Phi)b} x \langle (P \langle N \rangle) b = (x : P) \langle N \rangle b
     \]

   - a left slant cell \( \Phi : P \twoheadrightarrow Q : N \rightarrow M \) sends each \( N \)-arrow \( n : x \twoheadrightarrow Q \cdot b \) to the \( M \)-arrow \( n : x \twoheadrightarrow P \twoheadrightarrow b \), the image of \( n \) under the function

     \[
     (x : P) \langle N \rangle b = x \langle P \langle N \rangle \rangle b \xrightarrow{x(\Phi)b} x \langle (M) Q \rangle b = x \langle (M) \langle Q \cdot b \rangle
     \]

2. The identity module morphism \( M \rightarrow M \) yields the identity right and left slant cells

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow 1 & & \downarrow 1 \\
X & \xrightarrow{M} & A
\end{array}
\]

\[
\begin{array}{ccc}
X & \xleftarrow{M} & A \\
\downarrow 1 & & \downarrow 1 \\
X & \xleftarrow{M} & A
\end{array}
\]

**Definition 8.3.3.**

- Given a pair of right slant cells as in

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow P & & \downarrow Q \\
X' & \xrightarrow{M'} & A'
\end{array}
\]

\[
\begin{array}{ccc}
X' & \xleftarrow{M'} & A' \\
\downarrow P' & & \downarrow Q' \\
X'' & \xleftarrow{M''} & A''
\end{array}
\]

their composite \( \Phi \circ \Phi' = \Phi' \circ \Phi \) is the right slant cell

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow P \circ P' & & \downarrow Q \circ Q' \\
X'' & \xleftarrow{M''} & A''
\end{array}
\]

defined by the module morphism \( \Phi \circ \Phi' : \langle M \rangle [Q \circ Q'] \rightarrow [P \circ P'] \langle M'' \rangle : X \rightarrow A'' \) given by the composition

\[
\langle M \rangle [Q \circ Q'] = \langle (M) Q \rangle [Q'] \xrightarrow{(P \circ P')(\Phi')} Q' = P \langle (M') Q' \rangle \xrightarrow{P(\Phi')} P' \langle (M'') \rangle = [P \circ P'] \langle M'' \rangle
\]
Given a pair of left slant cells as in
\[
\begin{array}{ccc}
X'' \xrightarrow{\varepsilon} & M'' \xrightarrow{\mu''} & A'' \\
\downarrow{P} & \xrightarrow{\Phi'} & \downarrow{Q'} \\
X' \xrightarrow{\varepsilon} & M' \xrightarrow{\mu'} & A' \\
\downarrow{P} & \xrightarrow{\Phi} & \downarrow{Q} \\
X \xrightarrow{\varepsilon} & M \xrightarrow{\mu} & A
\end{array}
\]
their composite \(\Phi' \circ \Phi = \Phi \circ \Phi'\) is the left slant cell
\[
\begin{array}{ccc}
X'' \xrightarrow{\varepsilon} & M'' \xrightarrow{\mu''} & A'' \\
\downarrow{P \circ P'} & \xrightarrow{\Phi \circ \Phi'} & \downarrow{Q \circ Q'} \\
X \xrightarrow{\varepsilon} & M \xrightarrow{\mu} & A
\end{array}
\]
defined by the module morphism \(\Phi \circ \Phi' : [P \circ P'] \langle M'' \rangle \to \langle M \rangle [Q \circ Q'] : X \to A''\) given by the composition
\[
[P \circ P'] \langle M'' \rangle = P \langle P' \langle M'' \rangle \rangle \xrightarrow{P(\Phi')} P(\langle M' \rangle Q') = P(\langle M' \rangle) Q' \xrightarrow{(\Phi)Q'} (\langle M \rangle) Q' = \langle M \rangle [Q \circ Q']
\].

**Proposition 8.3.4.** Modules and right (resp. left) slant cells among them form a category with the composition given in Definition 8.3.3 and the identities given in Remark 8.3.2(2).

**Proof.** The only non-trivial part is the verification of the associativity of the composition. Consider cells as in
\[
\begin{array}{ccc}
X \xrightarrow{\varepsilon} & M \xrightarrow{\mu} & A \\
\downarrow{P} & \xrightarrow{\Phi} & \downarrow{Q} \\
X' \xrightarrow{\varepsilon} & M' \xrightarrow{\mu'} & A' \\
\downarrow{P'} & \xrightarrow{\Phi'} & \downarrow{Q'} \\
X'' \xrightarrow{\varepsilon} & M'' \xrightarrow{\mu''} & A'' \\
\downarrow{P''} & \xrightarrow{\Phi''} & \downarrow{Q''} \\
X''' \xrightarrow{\varepsilon} & M''' \xrightarrow{\mu'''} & A'''
\end{array}
\]
The composites \(\langle \Phi \circ \Phi' \rangle \circ \Phi''\) and \(\Phi \circ \langle \Phi' \circ \Phi'' \rangle\) are defined by the module morphisms \(\langle \langle \Phi \rangle \rangle Q' \circ P(\langle \Phi' \rangle) Q'' \circ [P \circ P'] \langle \Phi'' \rangle\) and \(\langle \Phi \rangle [Q' \circ Q'' \circ P \circ P' \langle \Phi'' \rangle]\) respectively. But, by the functoriality and associativity of the composition, we have
\[
\langle \langle \Phi \rangle \rangle Q' \circ P(\langle \Phi' \rangle) Q'' \circ [P \circ P'] \langle \Phi'' \rangle = \langle \langle \Phi \rangle \rangle Q' \circ P(\langle \Phi' \rangle) Q'' \circ P(\langle P' \langle \Phi'' \rangle \rangle) = \langle \Phi \rangle [Q' \circ Q'' \circ P \circ P' \langle \Phi'' \rangle]
\].

**Definition 8.3.5.** Given a pair of modules \(M : X \to A\) and \(N : Y \to B\),
8. Adjunctions

- The module of right slant cells \( M \to N \),

\[
\langle M \downarrow N \rangle : [B, A] \to [X, Y]^
\]

, is defined by the composition

\[
[B, A] \xrightarrow{M \cdot B} [X : B] \xrightarrow{(X : B)} [X : B] \xrightarrow{X \cdot N} [X, Y]^
\]

, where \( M \cdot B \) is the right action of \( M \) on the functor category \([B, A]\) and \( X \cdot N \) is the left action of \( N \) on the functor category \([X, Y]\).

- The module of left slant cells \( N \to M \),

\[
\langle N \uparrow M \rangle : [X, Y]^
\to [B, A]
\]

, is defined by the composition

\[
[X, Y]^
\xrightarrow{X \cdot N} [X : B] \xrightarrow{(X : B)} [X : B] \xrightarrow{M \cdot B} [B, A]
\]

, where \( X \cdot N \) is the left action of \( N \) on the functor category \([X, Y]\) and \( M \cdot B \) is the right action of \( M \) on the functor category \([B, A]\).

**Remark 8.3.6.**

1. For each pair of functors \( Q : B \to A \) and \( P : X \to Y \),

   - the set \( (Q) \langle M \downarrow N \rangle (P) \) consists of all module morphisms \( \langle M \rangle Q \to P \langle N \rangle : X \to B \), i.e. all right slant cells \( Q \to P : M \to N \).
   - the set \( (P) \langle N \uparrow M \rangle (Q) \) consists of all module morphisms \( P \langle N \rangle \to \langle M \rangle Q : X \to B \), i.e. all left slant cells \( P \to Q : N \to M \).

2. The module \( \langle X \downarrow A \rangle : [A, X] \to [X, A]^
\) defined in Definition 8.2.1 is nothing but the module of right slant cells \( \langle X \rangle \to \langle A \rangle \) from the hom of \( X \) to the hom of \( A \); that is,

\[
\langle X \downarrow A \rangle = \langle \langle X \rangle \downarrow \langle A \rangle \rangle
\]

. The module \( \langle A \uparrow X \rangle : [X, A]^
\to [A, X] \) defined in Definition 8.2.1 is nothing but the module of left slant cells \( \langle A \rangle \to \langle X \rangle \) from the hom of \( A \) to the hom of \( X \); that is,

\[
\langle A \uparrow X \rangle = \langle \langle A \rangle \uparrow \langle X \rangle \rangle
\]

. The module \( \langle N \downarrow M \rangle \) is defined in Definition 8.2.1 is nothing but the module of right slant cells \( \langle N \rangle \to \langle M \rangle \) from the hom of \( A \) to the hom of \( X \); that is,

\[
\langle N \downarrow M \rangle = \langle \langle N \rangle \downarrow \langle M \rangle \rangle
\]

3. For any modules \( M \) and \( N \),

\[
\langle N \downarrow M \rangle \cong \langle N \downarrow M \rangle^
\]

.

**Definition 8.3.7.**
8. Adjunctions

- A right slant cell

\[
\begin{array}{c}
\text{X} \xrightarrow{\mathcal{M}} \text{A} \\
\text{P} \xrightarrow{\Phi} \text{Q} \\
\text{Y} \xrightarrow{\mathcal{N}} \text{B}
\end{array}
\]

is called an adjunction if the module morphism \(\Phi : \langle \mathcal{M} \rangle \text{Q} \rightarrow \mathcal{P} \langle \mathcal{N} \rangle : \text{X} \rightarrow \text{B} \) is iso. If \(\Phi : \mathcal{Q} \sim \mathcal{P} : \mathcal{M} \rightarrow \mathcal{N} \) is an adjunction, then the pair \((\mathcal{Q}, \Phi)\), or the functor \(\mathcal{Q}\) itself, is called a right adjoint of \(\mathcal{P}\) along \(\mathcal{M}\) and \(\mathcal{N}\).

- A left slant cell

\[
\begin{array}{c}
\text{Y} \xrightarrow{\mathcal{N}} \text{B} \\
\text{P} \xrightarrow{\Phi} \text{Q} \\
\text{X} \xrightarrow{\mathcal{M}} \text{A}
\end{array}
\]

is called an adjunction if the module morphism \(\Phi : \mathcal{P} \langle \mathcal{N} \rangle \rightarrow \langle \mathcal{M} \rangle \text{Q} : \text{X} \rightarrow \text{B} \) is iso. If \(\Phi : \mathcal{P} \sim \mathcal{Q} : \mathcal{N} \rightarrow \mathcal{M} \) is an adjunction, then the pair \((\mathcal{P}, \Phi)\), or the functor \(\mathcal{P}\) itself, is called a left adjoint of \(\mathcal{Q}\) along \(\mathcal{N}\) and \(\mathcal{M}\).

Remark 8.3.8.

1. The two forms of an adjunctions, \(\mathcal{M} \rightarrow \mathcal{N}\) and \(\mathcal{N} \rightarrow \mathcal{M}\), are referred to as the right and left forms. If \(\Phi\) is the left form of an adjunction, then the inverse \(\Phi^{-1}\) gives the right form of the adjunction.

2. The right and left forms of an adjunction defined in Definition 8.1.1 are the same things as these adjunctions

\[
\begin{array}{c}
\text{X} \xrightarrow{(X)} \text{X} \\
\text{A} \xrightarrow{(A)} \text{A}
\end{array}
\]

along the hom of \(\text{X}\) and the hom of \(\text{A}\).

Proposition 8.3.9.

1. The identity right (resp. left) slant cell is an adjunction.

2.

- If two right slant cells \(\Phi : \mathcal{M} \rightarrow \mathcal{M}'\) and \(\Phi' : \mathcal{M}' \rightarrow \mathcal{M}''\) are adjunctions, so is the composite \(\Phi \circ \Phi' : \mathcal{M} \rightarrow \mathcal{M}''\).

- If two left slant cells \(\Phi' : \mathcal{M}'' \rightarrow \mathcal{M}'\) and \(\Phi : \mathcal{M}' \rightarrow \mathcal{M}\) are adjunctions, so is the composite \(\Phi \circ \Phi' : \mathcal{M}'' \rightarrow \mathcal{M}\).

Proof.

1. Evident.

2. Consider a pair of right slant cells as in Definition 8.3.3. Since the cell \(\Phi \circ \Phi' : \mathcal{M} \rightarrow \mathcal{M}''\) is defined by the module morphism \(\langle \Phi \rangle \mathcal{Q}' \circ \mathcal{P} \langle \Phi' \rangle\), it suffices to show that \(\langle \Phi \rangle \mathcal{Q}'\) and \(\mathcal{P} \langle \Phi' \rangle\) are isomorphisms. But this is the case by Proposition 1.1.19.
Proposition 8.3.10. Modules and the right (resp. left) forms of adjunctions among them constitute a subcategory of the category of modules and right (resp. left) slant cells.

Proof. Immediate from Proposition 8.3.9.

Note. Proposition 1.2.9 allows the following definition (cf. Definition 1.2.21).

Definition 8.3.11. Let \( J : E \rightarrow D \) be a module.

- Given a right slant cell

\[
\begin{array}{c}
X \xrightarrow{M} A \\
P \downarrow \quad \phi \\
Y \xleftarrow{N} B
\end{array}
\]

the right slant cell

\[
\begin{array}{c}
[E, X] \xrightarrow{(J, M)} [D, A] \\
[E,P] \quad (J, \phi) \\
[E,Y] \xleftarrow{(J, N)} [D,B]
\end{array}
\]

is defined by the postcomposition module morphism

\[
(J, M)[D, Q] = (J, (M) Q) \xrightarrow{(J, \phi)} (J, P (N)) = [E, P] (J, N)
\]

with \( \Phi : (M) Q \rightarrow P (N) \).

- Given a left slant cell

\[
\begin{array}{c}
Y \xleftarrow{N} B \\
P \downarrow \quad \phi \\
X \xrightarrow{M} A
\end{array}
\]

the left slant cell

\[
\begin{array}{c}
[E, Y] \xleftarrow{(J, N)} [D, B] \\
[E,P] \quad (J, \phi) \\
[E,X] \xrightarrow{(J, M)} [D,A]
\end{array}
\]

is defined by the postcomposition module morphism

\[
[E, P] (J, N) = (J, P (N)) \xrightarrow{(J, \phi)} (J, (M) Q) = (J, M)[D, Q]
\]

with \( \Phi : P (N) \rightarrow (M) Q \).

Remark 8.3.12. Given a pair of functors \( S : E \rightarrow X \) and \( T : D \rightarrow B \),
the right slant cell \( \langle J, \Phi \rangle \) sends each cell \( \Theta : S \sim Q \to T : J \to M \) to the cell \( \Theta \circ \Phi : S \circ P \sim T : J \to N \) defined by the module morphism \( \Theta \circ \Phi : J \to [S \circ P] (N) T \) given by the composition

\[
J \xrightarrow{\Theta} S (M) T = S (P (N)) T = [S \circ P] (N) T
\]

the left slant cell \( \langle J, \Phi \rangle \) sends each cell \( \Theta : S \circ P \sim T : J \to N \) to the cell \( \Theta \circ \Phi : S \sim Q \circ T : J \to M \) defined by the module morphism \( \Theta \circ \Phi : J \to S (M) [Q \circ T] \) given by the composition

\[
J \xrightarrow{\Theta} [S \circ P] (N) T = S (P (N)) T = S (M) [Q \circ T]
\]

Proposition 8.3.13. If a cell \( \Phi \) is an adjunction, so is the cell \( \langle J, \Phi \rangle \).

Proof. Since the operation \( \langle J, - \rangle \) is functorial, it preserves isomorphisms.

\[ \square \]

Note. Proposition 4.3.8 allows the following definition (cf. Definition 4.3.15).

Definition 8.3.14. Let \( E \) be a category.

- Given a right slant cell

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow{P} & & \downarrow{Q} \\
Y & \xrightarrow{N} & B
\end{array}
\]

the right slant cell

\[
\begin{array}{ccc}
[E, X] & \xrightarrow{(E, M)} & [E, A] \\
\downarrow{(E, P)} & & \downarrow{(E, Q)} \\
[E, Y] & \xrightarrow{(E, N)} & [E, B]
\end{array}
\]

is defined by the postcomposition module morphism

\[
\langle E, M \rangle [E, Q] = \langle E, (M) Q \rangle \xrightarrow{(E, \Phi)} \langle E, P (N) \rangle = \langle E, P \rangle \langle E, N \rangle
\]

with \( \Phi : (M) Q \to P (N) \).

- Given a left slant cell

\[
\begin{array}{ccc}
Y & \xrightarrow{N} & B \\
\downarrow{P} & & \downarrow{Q} \\
X & \xrightarrow{M} & A
\end{array}
\]

the left slant cell

\[
\begin{array}{ccc}
[E, Y] & \xrightarrow{(E, N)} & [E, B] \\
\downarrow{(E, P)} & & \downarrow{(E, Q)} \\
[E, X] & \xrightarrow{(E, M)} & [E, A]
\end{array}
\]
is defined by the postcomposition module morphism

\[ [E, P] (E, N) = (E, P(N)) \xrightarrow{(E, \Phi)} (E, (M) Q) = (E, M) [E, Q] \]

with \( \Phi : P(N) \to (M) Q \).

**Remark 8.3.15.** Given a pair of functors \( S : E \to X \) and \( T : E \to B \),

- the right slant cell \( (E, \Phi) \) sends each cylinder \( \alpha : S \xrightarrow{\gamma} Q \xrightarrow{\eta} T : E \xrightarrow{\delta} M \) to the cylinder \( \alpha \circ \Phi : S \xrightarrow{\gamma} P \xrightarrow{\eta} T : E \xrightarrow{\delta} N \) defined by

  \[ \alpha \circ \Phi = \alpha \circ S(\Phi) T = \alpha \cdot \prod_E S(\Phi) T \]

  , the image of the frame \( \alpha \in \prod_E S(M)[Q \circ T] \) under the function

  \[ \Pi_E S(M)[Q \circ T] = \Pi_E S((M) Q) T \xrightarrow{\Pi_E S(\Phi) T} \Pi_E S(P(N)) T = \Pi_E [S \circ P](N) T \]

- the left slant cell \( (E, \Phi) \) sends each cylinder \( \alpha : S \xrightarrow{\eta} Q \xrightarrow{\gamma} T : E \xrightarrow{\delta} N \) to the cylinder \( \alpha \circ \Phi : S \xrightarrow{\eta} P \xrightarrow{\gamma} T : E \xrightarrow{\delta} M \) defined by

  \[ \alpha \circ \Phi = \alpha \circ S(\Phi) T = \alpha \cdot \prod_E S(\Phi) T \]

  , the image of the frame \( \alpha \in \prod_E S(M)[Q \circ T] \) under the function

  \[ \Pi_E [S \circ P](N) T = \Pi_E S(P(N)) T \xrightarrow{\Pi_E S(\Phi) T} \Pi_E S((M) Q) T = \Pi_E S(M)[Q \circ T] \]

**Proposition 8.3.16.** If a cell \( \Phi \) is an adjunction, so is the cell \( (E, \Phi) \).

**Proof.** Since the operation \( (E, -) \) is functorial, it preserves isomorphisms. \( \square \)

**Definition 8.3.17.** Given a cell

\[
\begin{array}{c}
X \\ \xrightarrow{\mathcal{M}} \\ P \\
\xrightarrow{\Phi} \\
Y \xrightarrow{\mathcal{N}} \xrightarrow{\mathcal{A}} \\
\xrightarrow{\mathcal{B}} \xrightarrow{Q} \\
\end{array}
\]

- a right adjoint \( (R, \Upsilon) \) of the collage functor \( [\Phi] \) (see Definition 3.1.10) along the left hom (see Definition 5.2.1) of \( \mathcal{M} \) and the left hom of \( \mathcal{N} \) as depicted in

\[
\begin{array}{c}
[M] \\ \xrightarrow{[\mathcal{M}(\mathcal{A})]} \\
\xrightarrow{[\Phi]} \\
[N] \xrightarrow{[\mathcal{N}(\mathcal{B})]} \xrightarrow{\mathcal{A}} \\
\xrightarrow{\mathcal{B}} \xrightarrow{\mathcal{R}} \\
\end{array}
\]

is called a right adjoint of \( \Phi \).
8. Adjunctions

- a left adjoint \((R, \Upsilon)\) of the collage functor \([\Phi]\) (see Definition 3.1.10) along the right hom (see Definition 5.2.1) of \(\mathcal{M}\) and the right hom of \(\mathcal{N}\) as depicted in

\[
\begin{array}{c}
X \xrightarrow{(X|M)} [\mathcal{M}] \\
\Upsilon \downarrow \\
Y \xrightarrow{(Y|N)} [\mathcal{N}]
\end{array}
\]

is called a left adjoint of \(\Phi\).

Remark 8.3.18.

1. If \((R, \Upsilon)\) is a right adjoint of \(\Phi\), then the restriction of \(\Upsilon\) to \(X\) and \(A\) yields right adjoints of \(P\) and \(Q\) as depicted in

\[
\begin{array}{c}
X \xrightarrow{M} A \\
P \downarrow \Upsilon \\
Y \xrightarrow{N} B
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{(A)} A \\
Q \downarrow \Upsilon \\
B \xrightarrow{(B)} B
\end{array}
\]

Conversely, a pair of right adjoints as above forms a right adjoint of \(\Phi\) if it satisfies the following naturality condition for each object \(b \in [B]\) and each \(\mathcal{M}\)-arrow \(m : x \leadsto a\): for any \(A\)-arrow \(f : a \to R \cdot b\) as in

\[
\begin{array}{c}
x \xrightarrow{m} \downarrow \Upsilon \\
R \cdot b
\end{array}
\]

the triangle

\[
\begin{array}{c}
x : P \xrightarrow{m \cdot \Upsilon} Q \cdot a \\
(f \cdot \Upsilon) \downarrow \Upsilon \\
b
\end{array}
\]

commutes.

- If \((R, \Upsilon)\) is a left adjoint of \(\Phi\), then the restriction of \(\Upsilon\) to \(A\) and \(X\) yields left adjoints of \(Q\) and \(P\) as depicted in

\[
\begin{array}{c}
X \xrightarrow{M} A \\
R \downarrow \Upsilon \\
Y \xrightarrow{N} B
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{(X)} X \\
R \downarrow \Upsilon \\
Y \xrightarrow{(Y)} Y
\end{array}
\]

Conversely, a pair of left adjoints as above forms a left adjoint of \(\Phi\) if it satisfies the following naturality condition for each object \(y \in [Y]\) and each \(\mathcal{M}\)-arrow \(m : x \leadsto a\): for any \(X\)-arrow \(f : y \cdot R \to x\) as in

\[
\begin{array}{c}
y \cdot R \\
\downarrow f \cdot m \\
x \xrightarrow{m} \downarrow \Upsilon \\
a
\end{array}
\]
8. Adjunctions

The triangle

\[
\begin{array}{c}
X \xrightarrow{f \circ m \cdot \Phi} Y \\
\downarrow f \cdot \Upsilon \\
Q \cdot a \xrightarrow{\alpha} P
\end{array}
\]

commutes.

2. A right (resp. left) adjoint of a functor \( H \) is the same thing as a right (resp. left) adjoint of the hom cell \( (H) \).

**Theorem 8.3.19.** If a cell has a left (resp. right) adjoint, then it preserves inverse (resp. direct) universals.

**Proof.** A left adjoint \((R, \Upsilon)\) of \( \Phi \) in Definition 8.3.17 is depicted more elaborately as

\[
\begin{array}{c}
X \\
\downarrow (X|M) \\
Y \\
\downarrow (Y|N) \\
\Phi \\
\end{array}
\]

\[
\begin{array}{c}
(R(X|M)) \\
\Upsilon \\
(\gamma|N) \\
\Phi
\end{array}
\]

, and the right exponential transposition yields the natural isomorphism \( \Upsilon \cdot \) as in

\[
\begin{array}{c}
[X:] \\
\downarrow \lambda (X|M) \cdot \\
[Y:]
\end{array}
\]

(see Proposition 2.1.5 and Proposition 2.1.6). Let \( u : r \to a \) be an inverse universal \( M \)-arrow. Then, by the very definition of inverse universality, \( (X|M) \cdot u \) is an isomorphism, and preserved by \( [R:] \). Since \( \Upsilon \cdot \) is a natural isomorphism and \( [R:] \cdot (X|M) \cdot u \) is an isomorphism, \( [(Y|N) \cdot \Phi] \cdot u \) is also an isomorphism; that is, \( [\Phi] \cdot u = \Phi \cdot u \) is inverse universal.

**Theorem 8.3.20.** If a cell

\[
\begin{array}{c}
X \xrightarrow{M} A \\
\downarrow P \\
Y \xrightarrow{N} B
\end{array}
\]

has a right (resp. left) adjoint, so does the postcomposition cell

\[
\begin{array}{c}
[E, X] \xrightarrow{(J,M)} [D, A] \\
\downarrow [E,P] \cdot (J,\Phi) \\
[E,Y] \xrightarrow{(J,N)} [D, B]
\end{array}
\]

(see Definition 1.2.21) for any module \( J \).
8. Adjunctions

Proof. Suppose that $\Phi$ has a right adjoint as in Remark 8.3.18(1). Given a module $J : E \to D$, the postcomposition with $\Upsilon$ yields the adjunctions

$$\begin{align*}
\text{[E, X]} & \cong\quad \text{[D, A]} \quad & \text{[D, A]} & \cong\quad \text{[D, A]} \\
\text{[E, Y]} & \cong\quad \text{[D, B]} \quad & \text{[D, B]} & \cong\quad \text{[D, B]}
\end{align*}$$

by Proposition 8.3.13 and Proposition 8.3.16. The proof will be complete if we show that this pair or right adjoints satisfies the naturality condition in Remark 8.3.18(1) for each functor $K : D \to B$ and each cell $\Theta : S \to T : J \to M$, i.e. that, for any natural transformation $\tau : T \to R \circ K : D \to A$ as in

$$S \xrightarrow{\Theta} T \xrightarrow{\tau} R \circ K$$

the triangle

$$S \circ P \xrightarrow{(\Theta \circ \eta) \circ \tau} Q \circ T \xrightarrow{\tau \circ \eta} K$$

commutes. But this is reduced to the commutativity of the triangle

$$\begin{align*}
e: S & \xrightarrow{i' \circ \Phi} Q \circ T \xrightarrow{\tau \circ \eta} K \\
& \xrightarrow{\tau \circ \eta}
\end{align*}$$

for each $J$-arrow $j : e \sim d$, i.e. to the naturality of $\Upsilon$. □

Corollary 8.3.21. If a cell $\Phi$ has a right (resp. left) adjoint, so does the cell $\langle E, \Phi \rangle$ (see Definition 4.3.15) for any category $E$.

Proof. By Corollary 5.5.4, the cell $\langle E, \Phi \rangle$ is identified with the cell $\langle \langle E \rangle, \Phi \rangle$. Hence the assertion follows from Theorem 8.3.20. □

Corollary 8.3.22. If a cell $\Phi$ has a right (resp. left) adjoint, then so do the cells $\langle E^e, \Phi \rangle$ and $\langle E^p, \Phi \rangle$ (see Definition 4.6.13) for any category $E$.

Proof. By Corollary 5.5.5, the cell $\langle E^e, \Phi \rangle$ (resp. $\langle E^p, \Phi \rangle$) is identified with the cell $\langle \star E, \Phi \rangle$ (resp. $\langle \star E, \Phi \rangle$). Hence the assertion follows from Theorem 8.3.20. □

Corollary 8.3.23. If a cell has a left (resp. right) adjoint, then it preserves inverse (resp. direct) limits.

Proof. Immediate from Corollary 8.3.22 and Theorem 8.3.19. □

Corollary 8.3.24. If a functor has a left (resp. right) adjoint, then it preserves inverse (resp. direct) limits.

Proof. This is a special case of Corollary 8.3.23 on noting Remark 8.3.18(2) and Remark 7.3.6. □
8.4. Equivalence of categories

Definition 8.4.1. A functor $F : E \to D$ is called an equivalence provided that it is fully faithful and essentially surjective on objects in the sense that for each object $d \in \|D\|$ there exists an object $e \in \|E\|$ such that $e \cdot F$ is isomorphic to $d$.

Proposition 8.4.2. If a functor $F : E \to D$ is an equivalence, so is any functor $E \to D$ isomorphic to $F$.

Proof. Evident. □

Definition 8.4.3. A module $M : X \to A$ is called an equivalence provided that

- for each object $a \in \|A\|$ there exist an object $x \in \|X\|$ and a two-way universal $M$-arrow $u : x \to a$;
- for each object $x \in \|X\|$ there exist an object $a \in \|A\|$ and a two-way universal $M$-arrow $u : x \to a$.

Proposition 8.4.4. If a module $M : X \to A$ is an equivalence, so is any module $X \to A$ isomorphic to $M$.

Proof. Evident since any module isomorphism preserves universals. □

Theorem 8.4.5.

If a functor $G : A \to X$ is an equivalence, then its corepresentable module $(X) G : X \to A$ is an equivalence.

If a functor $F : X \to A$ is an equivalence, then its representable module $F(A) : X \to A$ is an equivalence.

Proof. By Theorem 6.2.13, for each $a \in \|A\|$, the identity $X$-arrow $a : G \to G \cdot a$ gives a two-way universal $(X) G$-arrow $a : G \cdot a$, and, for each $x \in \|X\|$, an invertible $X$-arrow $i : x \to G \cdot a$ gives a two-way universal $(X) G$-arrow $i : x \to a$. □

Note. The axiom of choice is used in the proof of the following.

Theorem 8.4.6. Any equivalence module $M : X \to A$ has

- a counit $\rho : G \sim M$ with $G : A \to X$ an equivalence functor.
- a unit $\lambda : M \sim F$ with $F : X \to A$ an equivalence functor.

Proof. Choose a two-way universal $M$-arrow $\rho_a : r_a \to a$ for each $a \in \|A\|$. By Theorem 6.3.10, there is a functor $G : A \to X$ such that $\rho := (\rho_a)_{a \in \|A\|}$ forms a counit $\rho : G \sim M$. We claim that $G$ is an equivalence functor. By Theorem 6.3.12, $G$ is fully faithful. It remains to show that $G$ is essentially surjective on objects. Let $x \in \|X\|$. Since $M$ is an equivalence, there exist $a \in \|A\|$ and a two-way universal $M$-arrow $u : x \to a$. Since $x$ and $G \cdot a$ are universals of $a$ inverse along $M$, $x \cong G \cdot a$ by Theorem 6.2.6. □

Corollary 8.4.7.
8. Adjunctions

- Suppose that a module $\mathcal{M} : X \rightarrow A$ has a counit

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & A \\
\downarrow{\rho} & & \downarrow{\ } \\
\end{array}
\]

Then $\mathcal{M}$ is an equivalence if and only if $G$ is an equivalence. Moreover, if these equivalent conditions hold, then each component of $\rho$ is two-way universal.

- Suppose that a module $\mathcal{M} : X \rightarrow A$ has a unit

\[
\begin{array}{ccc}
X & \xleftarrow{\lambda} & A \\
\downarrow{\ } & & \downarrow{\mathcal{F}} \\
\end{array}
\]

Then $\mathcal{M}$ is an equivalence if and only if $F$ is an equivalence. Moreover, if these equivalent conditions hold, then each component of $\lambda$ is two-way universal.

**Proof.** Suppose that $\mathcal{M}$ is an equivalence. Then, by Theorem 8.4.6, there is a counit $\rho' : G' \rightarrow \mathcal{M}$ with $G'$ an equivalence. Since $G \cong G'$ by Theorem 6.3.8, $G$ is an equivalence by Proposition 8.4.2. Conversely, suppose that $G$ is an equivalence. Then $(X) G$ is an equivalence by Theorem 8.4.5. Since $\rho$ yields $\mathcal{M} \cong (X) G$, $\mathcal{M}$ is an equivalence by Proposition 8.4.4. The second assertion follows from Theorem 6.3.12.

**Corollary 8.4.8.**

- Suppose that a module $\mathcal{M} : X \rightarrow A$ is corepresented by a functor $G : A \rightarrow X$. Then $\mathcal{M}$ is an equivalence if and only if $G$ is an equivalence.

- Suppose that a module $\mathcal{M} : X \rightarrow A$ is represented by a functor $F : X \rightarrow A$. Then $\mathcal{M}$ is an equivalence if and only if $F$ is an equivalence.

**Proof.** The assertion follows from Corollary 8.4.7 on noting Remark 6.3.2.

**Definition 8.4.9.** An equivalence of categories $X$ and $A$, written $(F, G, \eta, \epsilon) : X \simeq A$, consists of a pair of functors

\[
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
\xleftarrow{F} & & \\
\end{array}
\]

and a pair of natural isomorphisms

\[\eta : 1_X \rightarrow G \circ F : X \rightarrow X\]
\[\epsilon : G \circ F \rightarrow 1_A : A \rightarrow A\]

In this situation, the functors $F$ and $G$ are said to be quasi-inverse to each other. Two categories $X$ and $A$ are called equivalent, written $X \simeq A$, when there is an equivalence of categories between them.

**Proposition 8.4.10.** If $(F, G, \eta, \epsilon) : X \simeq A$ is an equivalence of categories, then $F$ and $G$ are equivalences.

**Proof.** Since $\epsilon_a : a : G : F \rightarrow a$ (resp. $\eta_x : x \rightarrow G : F : x$) is invertible for each $a \in |A|$ (resp. $x \in |X|$), $F$ (resp. $G$) is essentially surjective on objects. Hence the proof is complete if we
The following conditions are equivalent:

1. \( \eta \) and \( \epsilon \) are natural isomorphisms, i.e. \( (F, G, \eta, \epsilon) \) forms an equivalence of categories \( X \cong A \).
2. \( G \) and \( F \) are fully faithful;
3. \( G \) and \( F \) are equivalences;
4. \( G \) (resp. \( F \)) is an equivalence.

Proof.

(1) \( \Leftrightarrow \) (2): Immediate from Theorem 8.1.9.

(1) \( \Rightarrow \) (3): This is Proposition 8.4.10.

(3) \( \Rightarrow \) (2): Immediate by definition.

(3) \( \Rightarrow \) (4): A well-known tautology.

(4) \( \Rightarrow \) (3): By Corollary 8.4.8, \( G \) (resp. \( F \)) is an equivalence iff \( \langle X \rangle G \) (resp. \( F \langle A \rangle \)) is an equivalence. Since \( \langle X \rangle G \cong F \langle A \rangle \), by Proposition 8.4.4, \( \langle X \rangle G \) is an equivalence iff \( F \langle A \rangle \) is an equivalence. Hence if \( G \) (resp. \( F \)) is an equivalence, so is \( F \) (resp. \( G \)).
8. Adjunctions

**Definition 8.4.12.** An adjoint equivalence of categories $X$ and $A$ is an adjunction $(\eta, \epsilon) : G \sim F : X \to A$ satisfying the equivalent conditions in Theorem 8.4.11.

**Theorem 8.4.13.** Given a functor $G : A \to X$ (resp. $F : X \to A$), the following conditions are equivalent:

1. $G$ (resp. $F$) is an equivalence;
2. $G$ (resp. $F$) is a part of an adjoint equivalence $(\eta, \epsilon) : G \sim F : X \to A$;
3. $G$ (resp. $F$) is a part of an equivalence of categories $(F, G, \eta, \epsilon) : X \simeq A$.

**Proof.**

$(1) \Rightarrow (2)$ By Theorem 8.4.5, $(X)G : X \to A$ is an equivalence module. Hence, by Theorem 8.4.6, it has a unit $\eta : (X)G \sim F$, yielding an adjoint equivalence $(\eta, \epsilon) : G \sim F : X \to A$.

$(2) \Rightarrow (3)$ Immediate by definition.

$(3) \Rightarrow (1)$ This is Proposition 8.4.10.

\[\square\]

**8.5. Equivalence of modules**

**Definition 8.5.1.** A cell

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow^{P} & \Phi & \downarrow^{Q} \\
Y & \xrightarrow{N} & B
\end{array}
\]

is called an equivalence if $P$ and $Q$ are equivalence functors and $\Phi$ is fully faithful.

**Remark 8.5.2.** A functor $F : X \to A$ is an equivalence if and only if the hom cell $(F)$ is an equivalence.

**Proposition 8.5.3.** A cell $\Phi : M \to N$ is an equivalence if and only if the collage functor $[\Phi] : [M] \to [N]$ (see Definition 3.1.10) is an equivalence.

**Proof.** By the construction of $[\Phi]$ it is easily seen that:

1. $[\Phi]$ is fully faithful iff so are $P$, $Q$, and $\Phi$;
2. $[\Phi]$ is essentially surjective on objects iff so are $P$ and $Q$.

\[\square\]

**Definition 8.5.4.** An equivalence of modules $M$ and $N$, written $(\Phi, \Psi, \eta, \epsilon) : M \simeq N$, consists of a pair of cells

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow^{P} & \Phi & \downarrow^{Q} \\
Y & \xrightarrow{N} & B
\end{array}
\]

\[
\begin{array}{ccc}
Y & \xrightarrow{N} & B \\
\downarrow^{S} & \Psi & \downarrow^{T} \\
X & \xrightarrow{M} & A
\end{array}
\]
and a pair of cell isomorphisms

\[ \eta : 1_M \Rightarrow \Psi \circ \Phi : M \Rightarrow M \quad \epsilon : \Psi \circ \Phi \Rightarrow 1_N : N \Rightarrow N \]

In this situation, the cells \( \Phi \) and \( \Psi \) are said to be quasi-inverse to each other. Two modules \( M \) and \( N \) are called equivalent, written \( M \approx N \), if there is an equivalence of modules between them.

**Proposition 8.5.5.** Consider cells and cell morphisms as in Definition 8.5.4. The following conditions are equivalent:

1. \( (\Phi, \Psi, \eta, \epsilon) \) forms an equivalence of modules \( M \approx N \);
2. \( ([\Phi], [\Psi], [\eta], [\epsilon]) \) forms an equivalence of categories \( [M] \approx [N] \);
3. \( (P, S, \eta^1, \epsilon^1) \) forms an equivalence of categories \( X \approx Y \) and \( (Q, T, \eta^2, \epsilon^2) \) forms an equivalence of categories \( A \approx B \)

**Proof.** The equivalence of the conditions (1) and (3) is evident by recalling Remark 1.3.6. The equivalence of the conditions (1) and (2) is also evident by the construction of collages in Definition 3.1.10.

**Note.** The axiom of choice is used in the proof of the following.

**Theorem 8.5.6.** Given a cell as in Definition 8.5.1, the following conditions are equivalent:

1. \( \Phi \) is an equivalence;
2. \( \Phi \) is a part of an equivalence of modules \( (\Phi, \Psi, \eta, \epsilon) : M \approx N \).

**Proof.**

1 \( \Rightarrow \) 2 First note that if \( \Phi \) is an equivalence, then \( P, Q, \) and \( [\Phi] \) are equivalences by definition and Proposition 8.5.3. Since \( P \) and \( Q \) are essentially surjective on objects, we may choose, for each object \( y \in [Y] \), an object \( r_y \in [X] \) and an invertible \( Y \)-arrow \( \epsilon_y : r_y \cdot P \to y \), and, for each object \( b \in [B] \), an object \( r_b \in [A] \) and an invertible \( B \)-arrow \( \epsilon_b : r_b \cdot Q \to b \). Since \( [\Phi] \) is fully faithful, by Corollary 6.2.12, each \( \epsilon_y \) is universal from \( [\Phi] \) to \( y \) and each \( \epsilon_b \) is universal from \( [\Phi] \) to \( b \). Hence, by Theorem 8.1.6, \( [\Phi] \) has a right adjoint \( G : [N] \to [M] \) with \( y \cdot G = r_y \) and \( b : b \cdot G = r_b \); clearly, \( G \) defines a cell \( \Psi : N \Rightarrow M \) such that \( G = [\Psi] \). Let \( [\eta] : 1_{[M]} \Rightarrow [\Psi] \circ [\Phi] \) and \( [\epsilon] : [\Psi] \circ [\Phi] \Rightarrow 1_{[N]} \) be the collage natural transformations forming the unit and counit of the adjunction. Since \( [\Phi] \) is an equivalence, \( ([\Phi], [\Psi], [\eta], [\epsilon]) \) forms an equivalence of categories \( [M] \approx [N] \) (see Theorem 8.4.11). Hence, by Proposition 8.5.5, \( (\Phi, \Psi, \eta, \epsilon) \) forms an equivalence of modules \( M \approx N \).

2 \( \Rightarrow \) 1 By Proposition 8.5.5, \( ([\Phi], [\Psi], [\eta], [\epsilon]) \) is an equivalence of categories \( [M] \approx [N] \). Hence, by Theorem 8.4.13, \( [\Phi] \) is an equivalence. \( \Phi \) is thus an equivalence by Proposition 8.5.3.

**Theorem 8.5.7.** In the situation of Theorem 8.1.10, suppose that the following equivalent conditions (see Corollary 8.4.7) hold:

1. \( M \) is an equivalence;
2. G and F are equivalences.

Then

\[ (\eta, 1_A): 1_M \Rightarrow (\rho \circ \lambda) \circ (\lambda \circ A)^{-1}, \quad (\epsilon, 1_A): (\rho \circ \lambda) \circ (\lambda \circ A)^{-1} \Rightarrow 1_M \]

given by the unit and counit of the adjunction in Theorem 8.1.10.

Claim.

1. The pair \((1_X, \eta)\) forms a cell morphism \(1_X \Rightarrow (X \circ \rho)^{-1} \circ (X \circ \lambda)\); that is, the diagram

\[
\begin{array}{ccc}
\text{y} & \xrightarrow{f} & \text{x} \\
\downarrow{1_y} & & \downarrow{\eta_x} \\
\text{y} & \xrightarrow{f \circ (X \circ \lambda)^{-1}} & G \circ F \circ \text{x}
\end{array}
\]

commutes for every \(X\)-arrow \(f: y \Rightarrow x\).

2. The pair \((1_X, \epsilon)\) forms a cell morphism \((X \circ \rho)^{-1} \circ (X \circ \lambda) \Rightarrow 1_M\); that is, the diagram

\[
\begin{array}{ccc}
\text{x} & \xrightarrow{m \circ (X \circ \rho)^{-1} \circ (X \circ \lambda)} & F \circ G \circ \text{a} \\
\downarrow{1_x} & & \downarrow{\epsilon_a} \\
\text{x} & \xrightarrow{m} & \text{a}
\end{array}
\]

commutes for every \(M\)-arrow \(m: x \Rightarrow a\).
8. Adjunctions

Proof.

1. By Theorem 8.1.10, the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\lambda} & F \cdot X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\eta} & F \cdot G \\
\end{array}
\]

commutes; that is,

\[f \circ \lambda_x = f \circ \eta_x \circ \rho_{(F \cdot X)}\]

But

\[f \circ \lambda_x = f : (X \uparrow \lambda)\]

and

\[f \circ \eta_x \circ \rho_{(F \cdot X)} = (f \circ \eta_x) : (X \uparrow \rho)\]

Hence,

\[f : (X \uparrow \lambda) = (f \circ \eta_x) : (X \uparrow \rho)\]

i.e.

\[f : (X \uparrow \lambda) : (X \uparrow \rho)^{-1} = f \circ \eta_x\]

as required.

2. By Theorem 8.1.10, the diagram

\[
\begin{array}{ccc}
a : G & \xrightarrow{\lambda_{(a : G)}} & F \cdot a \\
\downarrow & \downarrow & \downarrow \\
x & \xleftarrow{m/\rho_a} & a \\
\end{array}
\]

commutes, where \(m/\rho_a\) is the adjunct of \(m\) along \(\rho_a\). Since

\[m/\rho_a \circ \lambda_{(a : G)} = (m : (X \uparrow \rho)^{-1} \circ \lambda_{(a : G)}) = m : (X \uparrow \rho)^{-1} : (X \uparrow \lambda)\]

we have

\[m = m/\rho_a \circ \lambda_{(a : G)} \circ \epsilon_a = (m : (X \uparrow \rho)^{-1} : (X \uparrow \lambda)) \circ \epsilon_a\]

as required.

\[\square\]

8.6. Adjoint functor theorem

Theorem 8.6.1. Consider the comma and collage of a module \(M : X \to A\) and a functor \(K\) as in

\[
\begin{array}{ccc}
D & \xrightarrow{K} & A \\
\downarrow & \downarrow & \downarrow \\
[M] & \xleftarrow{M_a} & A \\
\end{array}
\]
8. Adjunctions

- If the inclusion $\mathcal{M}_A$ preserves inverse limit for functors $D \to A$, then the pair of fibrations $\mathcal{M}_X$ and $\mathcal{M}_A$ creates inverse limits for $K$ in the following sense: if $\mathcal{M}_X \circ K$ and $\mathcal{M}_A \circ K$ have inverse limits $\rho: r \to \mathcal{M}_X \circ K$ and $\sigma: s \to \mathcal{M}_A \circ K$, then there is a unique cone $\mu: (m: r \to s) \to K$ in $[\mathcal{M}]$ such that $\mu \circ \mathcal{M}_X = \rho$ and $\mu \circ \mathcal{M}_A = \sigma$, and $\mu$ is an inverse limit of $K$.

- If the inclusion $\mathcal{M}_X$ preserves direct limit for functors $D \to X$, then the pair of fibrations $\mathcal{M}_X$ and $\mathcal{M}_A$ creates direct limits for $K$ in the following sense: if $K \circ \mathcal{M}_X$ and $K \circ \mathcal{M}_A$ have direct limits $\rho: K \circ \mathcal{M}_X \to r$ and $\sigma: K \circ \mathcal{M}_A \to s$, then there is a unique cone $\mu: K \to (m: r \to s)$ in $[\mathcal{M}]$ such that $\mu \circ \mathcal{M}_X = \rho$ and $\mu \circ \mathcal{M}_A = \sigma$, and $\mu$ is a direct limit of $K$.

Proof. First note that the inclusion $\mathcal{M}_X$ preserves inverse limits unconditionally. Now, by recalling Definition 3.2.21(1), the assertion follows from Theorem 7.3.13 by replacing $[E, C]$ with $[2, [\mathcal{M}]]$. □

Corollary 8.6.2.

- If a left module $\mathcal{M}: * \to A$ preserves inverse limits for functors $D \to A$, then the comma fibration $\mathcal{M}^i: [\mathcal{M}] \to A$ creates inverse limits for a functor $K: D \to [\mathcal{M}]$.

- If a right module $\mathcal{M}: X \to *$ preserves direct limits for functors $D \to X$, then the comma fibration $\mathcal{M}^i: [\mathcal{M}] \to X$ creates direct limits for a functor $K: D \to [\mathcal{M}]$.

Proof. By noting Corollary 7.4.12, the assertion is a special case of Theorem 8.6.1 where $X$ is the terminal category. □

Corollary 8.6.3.

- If a left module $\mathcal{M}: * \to A$ preserves inverse limits and $A$ is inverse complete, then the comma category $[\mathcal{M}]$ is inverse complete.

- If a right module $\mathcal{M}: X \to *$ preserves direct limits and $X$ is direct complete, then the comma category $[\mathcal{M}]$ is direct complete.

Proof. The assertion follows from Corollary 8.6.2 and Proposition 7.3.7. □

Definition 8.6.4. Given a category $E$, a subset $I \subseteq [E]$ of the objects of $E$ is called

- weakly initial in $E$ if to every object $e \in [E]$ there is an $E$-arrow $i \to e$ with $i \in I$.

- weakly terminal in $E$ if to every object $e \in [E]$ there is an $E$-arrow $e \to i$ with $i \in I$.

, and called

- initial in $E$ if it is weakly initial and satisfies the following condition: for every object $e \in [E]$ and for every pair of $E$-arrows $f: j \to e$ and $g: k \to e$ with $j, k \in I$, there is an object $i \in I$ and a pair of $E$-arrows $p: i \to j$ and $q: i \to k$ making the quadrangle

\[
\begin{array}{ccc}
  p & j & f \\
  q & k & g \\
  i & e & \\
\end{array}
\]

commute.
8. Adjunctions

- terminal in \( E \) if it is weakly terminal and satisfies the following condition: for every object \( e \in \| E \| \) and for every pair of \( E \)-arrows \( f : e \to j \) and \( g : e \to k \) with \( j, k \in \mathcal{I} \), there is an object \( i \in \mathcal{I} \) and a pair of \( E \)-arrows \( p : j \to i \) and \( q : k \to i \) making the quadrangle

\[
\begin{array}{ccc}
& j & \\
i & \searrow & \\
p & & f \\
& k & \swarrow & \\
q & & g \\
& e & \\
\end{array}
\]

commute.

Remark 8.6.5.

1. A set of objects of a category \( E \) is terminal (resp. weakly terminal) in \( E \) if and only if it is initial (resp. weakly initial) in the opposite category \( E^{\text{op}} \).

2. A subcategory \( I \) of \( E \) is called
   - initial (resp. weakly initial) in \( E \) if the set of objects \( \| I \| \) is initial (resp. weakly initial) in \( E \).
   - terminal (resp. weakly terminal) in \( E \) if the set of objects \( \| I \| \) is terminal (resp. weakly terminal) in \( E \).

Proposition 8.6.6.

- If \( i \) is an initial object of a category \( E \), then the set \( \{ i \} \) is initial in \( E \).

- If \( i \) is a terminal object of a category \( E \), then the set \( \{ i \} \) is terminal in \( E \).

Proof. Evident. \( \square \)

Proposition 8.6.7.

- If a category \( E \) is inverse complete, then the two notions “weakly initial” and “initial” in \( E \) coincide.

- If a category \( E \) is direct complete, then the two notions “weakly terminal” and “terminal” in \( E \) coincide.

Proof. Suppose that \( \mathcal{I} \subseteq \| E \| \) is weakly initial. Since \( E \) is complete, given \( e \in \| E \| \) and a pair of \( E \)-arrows \( f : j \to e \) and \( g : k \to e \), there is a pullback

\[
\begin{array}{ccc}
& j & \\
r & \searrow & \\
p & & f \\
& k & \swarrow & \\
q & & g \\
& e & \\
\end{array}
\]

Then, since \( \mathcal{I} \) is weakly initial, there is an arrow \( h : i \to r \) with \( i \in \mathcal{I} \), yielding the commutative quadrangle

\[
\begin{array}{ccc}
& j & \\
i & \searrow & \\
h \circ p & & f \\
& k & \swarrow & \\
h \circ q & & g \\
& e & \\
\end{array}
\]

\( \square \)
Theorem 8.6.8.

- Let $\mathcal{M} : * \to \mathbf{E}$ be a left module and $\mathbf{I}$ be a full subcategory of $\mathbf{E}$ and let $(\mathcal{M})\mathbf{I} : * \to \mathbf{I}$ denote the restriction of $\mathcal{M}$ to $\mathbf{I}$. If $\mathbf{I}$ is initial in $\mathbf{E}$, then a frame of $(\mathcal{M})\mathbf{I}$ uniquely extends to a frame of $\mathcal{M}$.

- Let $\mathcal{M} : \mathbf{E} \to *$ be a right module and $\mathbf{I}$ be a full subcategory of $\mathbf{E}$ and let $\mathbf{I}(\mathcal{M}) : \mathbf{I} \to *$ denote the restriction of $\mathcal{M}$ to $\mathbf{I}$. If $\mathbf{I}$ is terminal in $\mathbf{E}$, then a frame of $\mathbf{I}(\mathcal{M})$ uniquely extends to a frame of $\mathcal{M}$.

Proof.

1. Let $\alpha$ be a frame of $(\mathcal{M})\mathbf{I}$. Given $e \in |\mathbf{E}|$, choose $i \in |\mathbf{I}|$ and an $\mathbf{E}$-arrow $f : i \to e$ and define the $\mathcal{M}$-arrow $\beta_e : * \to e$ by $\beta_e = \alpha_i \circ f$. We claim:

   a) the definition of $\beta_e$ is independent of the choices of $i$ and $f$;
   b) $\beta := (\beta_e)_{e \in |\mathbf{E}|}$ forms a frame of $\mathcal{M}$;
   c) $\beta$ is the only frame of $\mathcal{M}$ that extends $\alpha$.

Let $\beta_e = \alpha_i \circ f$ and $\beta_e' = \alpha_{i'} \circ f'$ be two $\mathcal{M}$-arrows defined as above and let $h : e \to e'$ be an $\mathbf{E}$-arrow. By the initiality of $\mathbf{I}$, there is an object $r \in |\mathbf{I}|$ and a pair of $\mathcal{M}$-arrows $p : r \to i$ and $p' : r \to i'$ making the diagram

\[
\begin{array}{ccc}
  i & \xrightarrow{f} & e \\
  p & \downarrow & \downarrow h \\
  r & \xrightarrow{p} & e'
\end{array}
\]

commute. Since $\mathbf{I}$ is a full subcategory, $p$ and $p'$ belong to $\mathbf{I}$. Hence, by the naturality of $\alpha$, the diagram

\[
\begin{array}{ccc}
  * & \xrightarrow{\alpha_i} & i \\
  & \downarrow & \downarrow p \\
  r & \xrightarrow{\alpha_{i'}} & i'
\end{array}
\]

commutes. Combining the two commutative diagrams above, we have the commutative diagram

\[
\begin{array}{ccc}
  * & \xrightarrow{\alpha_i} & i \\
  & \downarrow & \downarrow p \\
  i' & \xrightarrow{\alpha_{i'}} & i'
\end{array}
\]

\[
\begin{array}{ccc}
  i & \xrightarrow{f} & e \\
  & \downarrow & \downarrow h \\
  i' & \xrightarrow{p'} & e'
\end{array}
\]

. The second claim is read off from this commutative diagram. By setting $e' = e$ and $h = 1_e$, the first claim follows. The third claim is now obvious.

\[\square\]

Remark 8.6.9. Let $\mathbf{E}$ and $\mathbf{C}$ be categories.
If \( I \subseteq E \) is an initial full subcategory, then the cell

\[
\begin{array}{ccc}
C & \overset{(E^\circ, C)}{\longrightarrow} & [E, C] \\
\downarrow & & \downarrow \\
[I^*, C] & \overset{(I^*_r, C)}{\longrightarrow} & [I, C]
\end{array}
\]

is a terminal full subcategory, then the cell

\[
\begin{array}{ccc}
[E, C] & \overset{(E^\circ, C)}{\longrightarrow} & C \\
\downarrow & \downarrow & \downarrow \\
[I, C] & \overset{(I^*_r, C)}{\longrightarrow} & C
\end{array}
\]

Remark 8.6.11.

Let \( M : * \rightarrow E \) be a left module. If \( i \) is an initial object of \( E \), then for any \( M \)-arrow \( m : * \twoheadrightarrow i \) there is a unique frame \( \alpha \) of \( M \) with \( \alpha_i = m \).

Let \( M : E \rightarrow * \) be a right module. If \( i \) is a terminal object of \( E \), then for any \( M \)-arrow \( m : i \twoheadrightarrow * \) there is a unique frame \( \alpha \) of \( M \) with \( \alpha_i = m \).

Proof. This is a special case of Theorem 8.6.8 on noting Proposition 8.6.6; the unique frame \( \alpha \) is defined by

\[ \alpha_e = m \circ i_e \]

for \( e \in \|E\| \), where \( i_e \) is the unique \( E \)-arrow \( i \rightarrow e \).

\[ \square \]

Remark 8.6.11.
If \(i\) is an initial object of \(E\), then the cell
\[
\begin{array}{c}
\frac{\mathcal{C}}{\rightarrow} \\
\downarrow \\
\frac{\mathcal{C}}{\rightarrow}
\end{array}
\xrightarrow{\{i^*, C\}} \\
\frac{\mathcal{E}, C}{\rightarrow}
\xrightarrow{\{i, C\}} \\
\frac{\mathcal{C}}{\rightarrow}
\]

, "evaluation at \(i\)" (see Example 4.8.9(2)), is fully faithful. Indeed, by Corollary 8.6.10, for every \(C\)-arrow \(f : c \to K \cdot i\) with \(K\) a functor \(E \to C\), there is a unique cone \(\alpha : c \to K : E^c \to C\) with \(\alpha_1 = f\). By Proposition 6.2.16, the cell preserves, reflects, and creates inverse universals; that is:

1. a cone \(\mu : R \to K : E^c \to C\) is universal if and only if the \(C\)-arrow \(\mu_1 : R \to K \cdot i\) is invertible (see Proposition 6.2.5);
2. given a functor \(K : E \to C\) and an invertible \(C\)-arrow \(u : R \to K \cdot i\), the unique cone \(\mu : R \to K : E^c \to C\) with \(\mu_1 = u\) is an inverse limit of \(K\).

If \(i\) is a terminal object of \(E\), then the cell
\[
\begin{array}{c}
\frac{\mathcal{C}}{\rightarrow} \\
\downarrow \\
\frac{\mathcal{C}}{\rightarrow}
\end{array}
\xrightarrow{\{i^*, C\}} \\
\frac{\mathcal{E}, C}{\rightarrow}
\xrightarrow{\{i, C\}} \\
\frac{\mathcal{C}}{\rightarrow}
\]

, "evaluation at \(i\)" (see Example 4.8.9(2)), is fully faithful. Indeed, by Corollary 8.6.10, for every \(C\)-arrow \(f : i \cdot K \to c\) with \(K\) a functor \(E \to C\), there is a unique cone \(\alpha : K \cdot c \to E^c \to C\) with \(\alpha_1 = f\). By Proposition 6.2.16, the cell preserves, reflects, and creates direct universals; that is:

1. a cone \(\mu : K \to R : E^c \to C\) is universal if and only if the \(C\)-arrow \(\mu_1 : i \cdot K \to R\) is invertible (see Proposition 6.2.5);
2. given a functor \(K : E \to C\) and an invertible \(C\)-arrow \(u : i \cdot K \to R\), the unique cone \(\mu : K \to R : E^c \to C\) with \(\mu_1 = u\) is a direct limit of \(K\).

Corollary 8.6.12.

If a category \(E\) has an initial object \(i\), then any functor \(K : E \to C\) has an inverse limit \(\mu : i \cdot K \to K : E^c \to C\).

Conversely, if the identity functor \(E \to E\) has an inverse limit \(\iota : i \to 1_E : E^c \to E\), then \(i\) is an initial object.

Proof. The cell \(\{i^*, C\}\) in Remark 8.6.11 creates an inverse limit of \(K\) from the identity \(i \cdot K \to K \cdot i\).

Theorem 8.6.13.

If \(E\) has an initial object \(i\), then the identity functor \(E \to E\) has an inverse limit \(\iota : i \to 1_E : E^c \to E\). Conversely, if the identity functor \(E \to E\) has an inverse limit \(\iota : i \to 1_E : E^c \to E\), then \(i\) is an initial object.
8. Adjunctions

* If $E$ has a terminal object $i$, then the identity functor $E \to E$ has a direct limit $i : 1_E \to i : E \to E$. Conversely, if the identity functor $E \to E$ has a direct limit $i : 1_E \to i$, then $i$ is a terminal object.

**Proof.** The first assertion follows from Corollary 8.6.12. For the second assertion, see [Ma] p234 Lemma 1.

**Theorem 8.6.14.**

- An inverse complete category $E$ has an initial object if and only if it has a small weakly initial set.

- A direct complete category $E$ has a terminal object if and only if it has a small weakly terminal set.

**Proof.** If $E$ has an initial object $i$, then it has a small initial set $\{i\}$. Conversely, suppose that $E$ has a small weakly initial set $I$, which is in fact initial by Proposition 8.6.7. Let $I$ be the full subcategory of $E$ generated by $I$. Since $E$ is complete and $I$ is small, the inclusion $I \to E$ has an inverse limit, and this limit extends to that of the identity $E \to E$ (see Remark 8.6.9). Hence $E$ has an initial object by Theorem 8.6.13.

**Theorem 8.6.15.**

- A left module $M : * \to A$ is representable if and only if the following conditions hold:
  1. $M$ preserves inverse limits;
  2. the comma category $M^* : [M^1] \to A$ has an inverse limit.

- A right module $M : X \to *$ is representable if and only if the following conditions hold:
  1. $M$ preserves direct limits;
  2. the comma category $M^* : [M^1] \to X$ has a direct limit.

**Proof.** Suppose that $M$ is representable. By Corollary 7.4.8, $M$ preserves inverse limits. By Proposition 6.1.4, $[M^1]$ has an initial object; hence, by Corollary 8.6.12, $M^* : [M^1] \to A$ has an inverse limit. Conversely, suppose that the conditions (1) and (2) hold. Then the identity $[M^1] \to [M^1]$ has a limit by Corollary 8.6.2. Hence $[M^1]$ has an initial object by Theorem 8.6.13, and $M$ is representable by Proposition 6.1.4.

**Theorem 8.6.16.**

- A left module $M : * \to A$ over an inverse complete category $A$ is representable if and only if the following conditions hold:
  1. $M$ preserves inverse limits;
  2. the comma category $[M^1]$ has a small weakly initial set, i.e. a small set $\{m_i : * \to a_i \mid i \in I\}$ of $M$-arrows such that every $M$-arrow $m : * \to a$ has a factorization $m = m_i \circ h$ for some $i \in I$ and an $A$-arrow $h : a_i \to a$ as indicated in

\[
\begin{array}{c}
\ast \\
\downarrow m \\
\downarrow h \\
\ast \\
\end{array}
\]

\[a_i \\
\downarrow a \\
\]
A right module $M : X \to \ast$ over a direct complete category $X$ is representable if and only if the following conditions hold:

1. $M$ preserves direct limits;
2. the comma category $[M^{\downarrow}]$ has a small weakly terminal set, i.e., a small set $\{m_i : x_i \to \ast \mid i \in I\}$ of $M$-arrows such that every $M$-arrow $m : x \to \ast$ has a factorization $m = h \circ m_i$ for some $i \in I$ and an $X$-arrow $h : x \to x_i$ as indicated in

\[
\begin{array}{c}
x \\
\downarrow^h \\
x_i \\
\downarrow^m \\
\ast
\end{array}
\]

Proof. Suppose that $M$ is representable. Then $M$ preserves inverse limits by Corollary 7.4.8, and $[M^{\downarrow}]$ has an initial object $i$ by Proposition 6.1.4, hence a small initial set $\{i\}$. Conversely, suppose that the conditions (1) and (2) hold. Then $[M^{\downarrow}]$ is inverse complete by Corollary 8.6.3. Hence $[M^{\downarrow}]$ has an initial object by Theorem 8.6.14, and thus be representable by Proposition 6.1.4.

Corollary 8.6.17.

A module $M : X \to A$ with $A$ inverse complete is representable if and only if for each object $x \in \|X\|$ the following conditions hold:

1. the left module $x(M) : \ast \to A$ preserves inverse limits;
2. the comma category $[x^{\downarrow}M]$ (see Remark 3.2.32(1)) has a small weakly initial set, i.e., a small set $\{m_i : x \to a_i \mid i \in I\}$ of $M$-arrows such that every $M$-arrow $m : x \to a$ has a factorization $m = m_i \circ h$ for some $i \in I$ and an $A$-arrow $h : a_i \to a$ as indicated in

\[
\begin{array}{c}
x \\
\downarrow^m \\
\ast \\
\downarrow^a \\
\downarrow^h
\end{array}
\]

Proof. Since a module $M : X \to A$ is representable if the left module $x(M) : \ast \to A$ is representable for each $x \in \|X\|$ (see Corollary 6.3.11), the assertion follows from Theorem 8.6.16.
Corollary 8.6.18. (Adjoint Functor Theorem).

- A functor $G : A \to X$ with $A$ inverse complete has a left adjoint if and only if the following conditions hold:
  1. $G$ preserves inverse limits;
  2. for each object $x \in |X|$, the comma category $[x \downarrow G]$ (see Remark 3.2.32(2)) has a small weakly initial set, i.e. a small set $\{f_i : x \to G \cdot a_i \mid i \in I\}$ of $X$-arrows such that every $X$-arrow $f : x \to G \cdot a$ has a factorization $f = f_i \circ (G \cdot h)$ for some $i \in I$ and an $A$-arrow $h : a_i \to a$ as indicated in

\[
\begin{array}{c}
  x \xrightarrow{f_i} G \cdot a_i \\
  \downarrow f \\
  G \cdot a \\
\end{array}
\]

- A functor $F : X \to A$ with $X$ direct complete has a right adjoint if and only if the following conditions hold:
  1. $F$ preserves direct limits;
  2. for each object $a \in |A|$, the comma category $[F \downarrow a]$ (see Remark 3.2.32(2)) has a small weakly terminal set, i.e. a small set $\{f_i : x_i \cdot F \to a \mid i \in I\}$ of $A$-arrows such that every $A$-arrow $f : x \cdot F \to a$ has a factorization $f = (h : F) \circ f_i$ for some $i \in I$ and an $X$-arrow $h : x_i \to x_i$ as indicated in

\[
\begin{array}{c}
  x \xrightarrow{h} x_i \\
  x_i \xrightarrow{f_i} a \\
\end{array}
\]

Proof. Since a functor $G : A \to X$ has a left adjoint iff the module $(X) G : X \to A$ is representable, the assertion follows from Corollary 8.6.17 on noting Corollary 7.4.10. \qed
9. Collages and Commas (continued)

9.1. Cylinder modules

Note. The following modules are defined in Definition 9.1.1:

- $\downarrow\text{CYL} : \text{COM} \to \text{CAT}$
- $\uparrow\text{CYL} : \text{CAT} \to \text{CLG}$
- $\uparrow\text{CYL} : \text{CAT} \to \text{MOD}$

Although $\uparrow\text{CYL} : \text{CAT} \to \text{MOD}$ and $\uparrow\text{CYL} : \text{CAT} \to \text{CLG}$ are regarded as the same thing under the identification $\text{CLG} \cong \text{MOD}$, paralleling definitions are presented for the sake of reference.

Definition 9.1.1.

1. The module $\downarrow\text{CYL} : \text{COM} \to \text{CAT}$ is defined in the following way:

   a) a $\downarrow\text{CYL}$-arrow from a comma $\mathcal{K} : X \to A$ to a category $E$, written $\alpha : S \Rightarrow T : \mathcal{K} \Rightarrow E$, is a triple $(S, \alpha, T)$ consisting of a functor $S : X \to E$, a second functor $T : A \to E$, and a natural transformation

   \[
   \begin{array}{ccc}
   X & \rightarrow & [\mathcal{K}] \rightarrow [\mathcal{K}_A] \rightarrow A \\
   S & \alpha & T \\
   \downarrow & & \downarrow \\
   E & & E
   \end{array}
   \]

   from $\mathcal{K}_X \circ S$ to $T \circ \mathcal{K}_A$.

   b) for a comma cell $\Phi : P \Rightarrow Q : J \Rightarrow \mathcal{K}$ and a $\downarrow\text{CYL}$-arrow $\alpha : S \Rightarrow T : \mathcal{K} \Rightarrow E$ as in

   \[
   \begin{array}{ccc}
   Y & \rightarrow & [J] \rightarrow [J_B] \rightarrow B \\
   J & \downarrow \Phi & \downarrow \\
   P & & Q \\
   \downarrow & & \downarrow \\
   X & \rightarrow & [\mathcal{K}] \rightarrow [\mathcal{K}_A] \rightarrow A \\
   S & \alpha & T \\
   \downarrow & & \downarrow \\
   E & & E
   \end{array}
   \]

   , their composite is the $\downarrow\text{CYL}$-arrow $\Phi \circ \alpha : P \circ S \Rightarrow T \circ Q : J \Rightarrow E$ with the natural transformation $\Phi \circ \alpha : [J_Y] \circ P \circ S \Rightarrow T \circ Q \circ [J_B] : [J] \Rightarrow E$ defined by

   \[
   \Phi \circ \alpha = [\Phi] \circ \alpha
   \]

   , the usual composite of a functor and a natural transformation.
9. Collages and Commas (continued)

c) for a ↓CYL-arrow \( \alpha : S \to T : [K] \to E \) and a functor \( H : E \to D \) as in

\[
\begin{array}{c}
X & \xrightarrow{K_X} & [K] & \xrightarrow{K_A} & A \\
\downarrow{S} & & \downarrow{E} & & \downarrow{H} \\
E & & \downarrow{H} & & D
\end{array}
\]

, their composite is the ↓CYL-arrow \( \alpha \circ H : S \circ H \to H \circ T : [K] \to D \) with the natural transformation \( \alpha \circ H : K_X \circ S \circ H \to H \circ T \circ K_A : [K] \to E \) given by the usual composition of a natural transformation and a functor.

2. The module ↑CYL : CAT → CLG is defined in the following way:

a) a ↑CYL-arrow from a category \( E \) to a collage \( M : X \to A \), written \( \alpha : S \sim T : E \sim M \), is a triple \((S, \alpha, T)\) consisting of a functor \( S : E \to X \), a second functor \( T : E \to A \), and a natural transformation

\[
\begin{array}{c}
E & \xleftarrow{S} & X & \xrightarrow{\alpha} & T & \xrightarrow{\alpha} & A \\
\downarrow{\alpha} & & \downarrow{M_X} & & \downarrow{M_A} & & \downarrow{M_A}
\end{array}
\]

from \( S \circ M_X \) to \( M_A \circ T \).

b) for a functor \( H : D \to E \) and a ↑CYL-arrow \( \alpha : S \sim T : E \sim M \) as in

\[
\begin{array}{c}
D & \xrightarrow{H} & E & \xrightarrow{T} & A \\
\downarrow{S} & & \downarrow{\alpha} & & \downarrow{\alpha} \\
X & \xleftarrow{M_X} & [M] & \xrightarrow{T} & [M] & \xrightarrow{M_A} & A
\end{array}
\]

, their composite is the ↑CYL-arrow \( H \circ \alpha : H \circ S \sim T \circ H : D \sim M \) with the natural transformation \( H \circ \alpha : H \circ S \circ M_X \to M_A \circ T \circ H : D \to [M] \) given by the usual composition of a functor and a natural transformation.

c) for a ↑CYL-arrow \( \alpha : S \sim T : E \sim M \) and a collage cell \( \Phi : P \sim Q : M \to N \) as in

\[
\begin{array}{c}
E & \xleftarrow{S} & X & \xrightleftharpoons{\alpha} & T & \xrightarrow{\alpha} & A \\
\downarrow{\alpha} & & \downarrow{M_X} & & \downarrow{M_A}
\end{array}
\]

\[
\begin{array}{c}
P & \xrightarrow{\Phi} & Q \\
\downarrow{\Phi} & & \downarrow{\Phi}
\end{array}
\]

\[
\begin{array}{c}
Y & \xrightarrow{N_Y} & N & \xrightarrow{N_B} & B
\end{array}
\]

, their composite is the ↑CYL-arrow \( \alpha \circ \Phi : S \circ P \sim Q \circ T : E \sim N \) with the natural transformation \( \alpha \circ \Phi : S \circ P \circ N_Y \sim N_B \circ Q \circ T : E \sim [N] \) defined by \( \alpha \circ \Phi = \alpha \circ [\Phi] \)

, the usual composite of a natural transformation and a functor.
3. The module $\uparrow\text{CYL} : \text{CAT} \to \text{MOD}$ is defined in the following way:

a) a $\uparrow\text{CYL}$-arrow from a category $E$ to a module $M : X \to A$ is a cylinder $\alpha : S \Rightarrow T : E \Rightarrow M$.

b) for a functor $H : D \to E$ and a $\uparrow\text{CYL}$-arrow $\alpha : S \Rightarrow T : E \Rightarrow M$ as in

```
\begin{tikzcd}
D & E \\
\downarrow{H} & \downarrow{\alpha} \\
X & A
\end{tikzcd}
```

, their composite is the $\uparrow\text{CYL}$-arrow $H \circ \alpha : H \circ S \Rightarrow T \circ H : D \Rightarrow M$, the usual composite of a functor and a cylinder (see Definition 4.3.22).

c) for a $\uparrow\text{CYL}$-arrow $\alpha : S \Rightarrow T : E \Rightarrow M$ and a module cell $\Phi : P \Rightarrow Q : M \Rightarrow N$ as in

```
\begin{tikzcd}
E & X \\
\downarrow{\alpha} & \downarrow{\Phi} \\
A & B
\end{tikzcd}
```

, their composite is the $\uparrow\text{CYL}$-arrow $\alpha \circ \Phi : S \circ P \Rightarrow T \circ Q : E \Rightarrow N$, the usual composite of a cylinder and a cell (see Definition 4.3.13).

**Remark 9.1.2.** By Remark 4.3.4(3), the identity

```
\begin{tikzcd}
\text{CAT} \ar[rr, \uparrow\text{CYL}] \ar[d, 1] & & \text{MOD} \ar[d, 1] \\
\text{CAT} \ar[u, \uparrow\text{CYL}] & & \text{CLG}
\end{tikzcd}
```

holds.

**Definition 9.1.3.**

1. The unit cylinder of a module $M : X \to A$ is the cylinder

```
\begin{tikzcd}
X & A \\
\downarrow{1_M} & \downarrow{1_M^A} \\
\downarrow{[M^1]} & \downarrow{1_M}
\end{tikzcd}
```

defined by

```
[1_M]^1_m = m
```

for $m$ an arrow of $M$. 
2. The unit cylinder of a comma \( \mathbb{K} : X \to A \) is the cylinder

\[
\begin{array}{c}
\mathbb{K} \\
\downarrow \mathbb{K}_X \\
X \\
\xleftarrow{1^\downarrow_X} \\
A
\end{array}
\]

defined by

\[
[1^\downarrow_X]_k = k
\]

for \( k \) an object of \([\mathbb{K}]\).

**Remark 9.1.4.**

1. By Remark 4.3.4(3), the unit cylinders \( 1^\uparrow_M \) and \( 1^\uparrow_K \) are also written as

\[
\begin{array}{c}
\mathbb{M} \\
\downarrow \mathbb{M}_X \\
X \\
\xleftarrow{1^\downarrow_X} \\
\mathbb{A}
\end{array}
\quad \begin{array}{c}
\mathbb{K} \\
\downarrow \mathbb{K}_X \\
X \\
\xleftarrow{1^\downarrow_X} \\
\mathbb{A}
\end{array}
\]

using the collages of \( \mathbb{M} \) and \( \mathbb{K} \).

2. The unit cylinder \( 1^\uparrow_M \) forms a \( \uparrow \text{CYL-arrow} \) \( 1^\uparrow_M : \mathbb{M}^\uparrow_X \sim \mathbb{M}^\uparrow_A : [\mathbb{M}] \sim \mathbb{M} \).

3. The unit cylinder \( 1^\uparrow_K \) forms a \( \downarrow \text{CYL-arrow} \) \( 1^\downarrow_K : \mathbb{K}^\downarrow_X \sim \mathbb{K}^\downarrow_A : [\mathbb{K}] \sim [\mathbb{K}^\uparrow] \).

**Proposition 9.1.5.**

1. For any module cell \( \Phi : \mathbb{M} \to \mathbb{N} \), the quadrangle

\[
\begin{array}{c}
[\mathbb{M}] \\
\downarrow \Phi \\
[\mathbb{N}]
\end{array}
\]

commutes.

2. For any comma cell \( \Phi : \mathbb{J} \to \mathbb{K} \), the quadrangle

\[
\begin{array}{c}
[\mathbb{J}] \\
\downarrow \Phi \\
[\mathbb{K}]
\end{array}
\]

commutes.

**Proof.** Immediate from the definitions of unit cylinders and the constructions of \( \Phi^\uparrow \) and \( \Phi^\downarrow \) (see Definition 3.2.21, and Definition 3.2.25). \( \square \)

**Proposition 9.1.6.**
9. Collages and Commas (continued)

1. Given a module \( \mathcal{M} : X \rightarrow A \), the triangle

\[
\begin{array}{c}
\mathcal{M}^\dagger \\
\downarrow \phi_M \\
\mathcal{M}
\end{array}
\]

commutes, where \( \phi_M \) is the isomorphism in Theorem 3.2.29.

2. Given a comma \( \mathbb{K} : X \rightarrow A \), the triangle

\[
\begin{array}{c}
\mathbb{K}^\dagger \\
\downarrow \eta_{\mathbb{K}} \\
\mathbb{K}
\end{array}
\]

commutes, where \( \eta_{\mathbb{K}} \) is the isomorphism in Theorem 3.2.29.

**Proof.** Immediate from the definitions. \( \square \)

**Definition 9.1.7.**

1. The comma adjunct of a \( \uparrow \)CYL-arrow \( \alpha : S \sim T : E \sim \mathcal{M} \), i.e. a cylinder

\[
\begin{array}{c}
E \\
\downarrow \alpha \\
X \rightarrow A
\end{array}
\]

is the functor \( [\alpha^\dagger] : E \rightarrow [\mathcal{M}^\dagger] \) defined by

\[
[\alpha^\dagger] : e = \alpha_e \quad [\alpha^\dagger] : h = (S : h, T : h)
\]

for \( e \) an object and \( h \) an arrow of \( E \).

2. The collage adjunct of a \( \downarrow \)CYL-arrow \( \alpha : S \sim T : \mathbb{K} \sim E \), i.e. a natural transformation

\[
\begin{array}{c}
X \leftarrow \mathbb{K}_X \\
\downarrow \alpha \\
E \rightarrow A
\end{array}
\]

is the functor \( [\alpha^\dagger] : [\mathbb{K}^\dagger] \rightarrow E \) adjunct (see Theorem 3.1.17) to the cell

\[
\begin{array}{c}
\mathbb{K}_X \\
\mathbb{K}_A \\
\mathbb{E} \rightarrow E
\end{array}
\]

defined by

\[
[\alpha^\dagger] : k = \alpha_k
\]

for \( k \) an object of \([\mathbb{K}]\).
Proposition 9.1.8.

1. The assignment \( \alpha \mapsto [\alpha^t] \) yields a bijection from the set of \( \uparrow \)CYL-arrows \( E \rightsquigarrow \mathcal{M} \) to the set of functors \( E \to [\mathcal{M}^t] \), and the triangle

\[
\begin{array}{c}
\mathcal{E} \\
[\alpha^t] \\
\downarrow^\alpha \\
[\mathcal{M}^t] \\
\downarrow_{1_\mathcal{M}} \\
\mathcal{M}
\end{array}
\]

commutes; the unit cylinder \( 1^t_{\mathcal{M}} \) thus forms an inverse universal \( \uparrow \)CYL-arrow.

2. The assignment \( \alpha \mapsto [\alpha^t] \) yields a bijection from the set of \( \downarrow \)CYL-arrows \( \mathbb{K} \rightsquigarrow E \) to the set of functors \( \mathbb{K}^t \to E \), and the triangle

\[
\begin{array}{c}
\mathbb{K} \\
1^t_{\mathbb{K}} \\
\downarrow^\alpha \\
[\mathbb{K}^t] \\
\downarrow_{[\alpha^t]} \\
E
\end{array}
\]

commutes; the unit cylinder \( 1^t_{\mathbb{K}} \) thus forms a direct universal \( \downarrow \)CYL-arrow.

Proof. The commutativity of each triangle is easily verified. It is also easy to verify that

\[
\left[ \left[ H \circ 1^t_{\mathcal{M}} \right]^t \right] = H
\]

for any functor \( H : E \to [\mathcal{M}^t] \), and verify that

\[
\left[ \left[ 1^t_{\mathbb{K}} \circ H \right]^t \right] = H
\]

for any functor \( H : [\mathbb{K}^t] \to E \).

\[\square\]

Theorem 9.1.9.

1. The functor \( M \mapsto [M^t] : \text{MOD} \to \text{CAT} \) and the family of unit cylinders \( 1^t_M : [M^t] \rightsquigarrow \mathcal{M} \), one for each locally small module \( M \), form a counit of the module \( \uparrow \)CYL : \( \text{CAT} \to \text{MOD} \).

2. The functor \( K \mapsto [K^t] : \text{COM} \to \text{CAT} \) and the family of unit cylinders \( 1^t_K : \mathbb{K} \rightsquigarrow [\mathbb{K}^t] \), one for each locally small comma \( \mathbb{K} \), form a unit of the module \( \downarrow \)CYL : \( \text{COM} \to \text{CAT} \).

Proof. Immediate from Proposition 9.1.5 and Proposition 9.1.8. \[\square\]

Definition 9.1.10. The unit cylinder of a category \( E \) is the cylinder

\[
\begin{array}{c}
\mathcal{E} \\
1 \\
\downarrow^{[1_{(E)}]} \\
\mathcal{E} \\
\downarrow_{(E)} \\
\mathcal{E}
\end{array}
\]

defined by

\[
[1_{(E)}]_e = 1_e
\]

for \( e \in \|E\| \), i.e. by the identity natural transformation \( 1_E \to 1_E \).
9. Collages and Commas (continued)

Proposition 9.1.11. For any functor $F : E \to D$, the quadrangle

\[
\begin{array}{ccc}
E & \xrightarrow{[1_E]} & (E) \\
\downarrow F & & \downarrow (F) \\
D & \xrightarrow{[1_D]} & (D)
\end{array}
\]

commutes.

Proof. Easily verified. \qed

Proposition 9.1.12. For any locally small category $E$, the unit cylinder $[1_E]$ forms a direct universal $\uparrow$CYL-arrow.

Proof. The family of module isomorphisms $\Psi^E : (E, M) \to (E, M)$ in Corollary 5.5.4, one for each locally small module $M$, yields a representation of the left slice of $\uparrow$CYL at $E$. The unit corresponding to this representation is given by the inverse image of identity cell $(E) \to (E)$ under $\Psi^E_{(E)}$, but it is immediately seen that this is nothing but the unit cylinder $[1_E]$ of $E$. \qed

Theorem 9.1.13. The hom operation $E \mapsto \langle E \rangle : CAT \to MOD$ and the family of unit cylinders $[1_E] : E \to \langle E \rangle$, one for each locally small category $E$, form a unit of the module $\uparrow$CYL : CAT $\to$ MOD.

Proof. Immediate from Proposition 9.1.11 and Proposition 9.1.12. \qed

Theorem 9.1.14. There is a canonical adjunction between the comma operation $M \mapsto [M^i] : MOD \to CAT$ and the hom operation $E \mapsto \langle E \rangle : CAT \to MOD$.

Proof. The assertion results by applying Theorem 8.1.10 to the counit and unit of the module $\uparrow$CYL given in Theorem 9.1.9(1) and Theorem 9.1.13. \qed

Remark 9.1.15. The comma operation $M \mapsto [M^i] : MOD \to CAT$ is thus a right adjoint of the hom operation $E \mapsto \langle E \rangle : CAT \to MOD$ (cf. Remark 3.1.18).

9.2. Equivalence COM $\simeq$ CLG

Note. The following modules are defined in Definition 9.2.1:

- $\uparrow$CYL : COM $\to$ CLG
- $\uparrow$CYL : COM $\to$ MOD

Although they are regarded as the same thing under the identification CLG $\simeq$ MOD, parallel definitions are presented for the sake of reference. After Remark 9.2.2(3) we will deal with only the module $\uparrow$CYL : COM $\to$ CLG; however, any result in this section stated for $\uparrow$CYL : COM $\to$ CLG also holds with CLG changed to MOD.

Definition 9.2.1.

1. The module $\uparrow$CYL : COM $\to$ CLG is defined in the following way:
9. Collages and Commas (continued)

a) a \( \downarrow \text{CYL} \)-arrow from a comma \( \mathbb{K} : Y \to B \) to a collage \( M : X \to A \), written \( \alpha : S \to T : \mathbb{K} \to M \), is a triple \( (S, \alpha, T) \) consisting of a functor \( S : Y \to X \), a second functor \( T : B \to A \), and a natural transformation

\[
\begin{array}{c}
Y \xleftarrow{K_Y} \mathbb{K} \xrightarrow{K_B} B \\
\downarrow \alpha \downarrow \downarrow T \\
X \xleftarrow{M_X} M \xrightarrow{M_A} A \\
\end{array}
\]

from \( K_Y \circ S \circ M_X \) to \( M_A \circ T \circ K_B \).

b) for a comma cell \( \Phi : P \to Q : J \to \mathbb{K} \) and a \( \downarrow \text{CYL} \)-arrow \( \alpha : S \to T : \mathbb{K} \to M \) as in

\[
\begin{array}{c}
Z \xleftarrow{J_Z} \mathbb{J} \xrightarrow{J_C} C \\
P \xleftarrow{K_Y} \mathbb{K} \xrightarrow{K_B} B \\
\downarrow \Phi \downarrow \downarrow T \\
Y \xleftarrow{M_X} M \xrightarrow{M_A} A \\
\end{array}
\]

their composite is the \( \downarrow \text{CYL} \)-arrow \( \Phi \circ \alpha : P \circ S \to T \circ Q : J \to \mathbb{M} \) with the natural transformation \( \Phi \circ \alpha : J_Z \circ P \circ S \circ M_X \to M_A \circ T \circ Q \circ J_C : \mathbb{J} \to \mathbb{M} \) defined by

\[
\Phi \circ \alpha = [\Phi] \circ \alpha
\]

, the usual composite of a functor and a natural transformation.

c) for a \( \downarrow \text{CYL} \)-arrow \( \alpha : S \to T : \mathbb{K} \to M \) and a collage cell \( \Phi : P \to Q : M \to N \) as in

\[
\begin{array}{c}
Y \xleftarrow{K_Y} \mathbb{K} \xrightarrow{K_B} B \\
\downarrow \alpha \downarrow \downarrow T \\
X \xleftarrow{M_X} M \xrightarrow{M_A} A \\
P \xleftarrow{K_Z} \mathbb{N} \xrightarrow{K_C} C \\
\downarrow \Phi \downarrow \downarrow Q \\
Z \xleftarrow{N_Z} N \xrightarrow{N_C} C \\
\end{array}
\]

their composite is the \( \downarrow \text{CYL} \)-arrow \( \alpha \circ \Phi : S \circ P \to Q \circ T : \mathbb{K} \to \mathbb{N} \) with the natural transformation \( \alpha \circ \Phi : K_Y \circ S \circ P \circ N_Z \to N_C \circ Q \circ T \circ K_B : \mathbb{K} \to \mathbb{N} \) defined by

\[
\alpha \circ \Phi = \alpha \circ [\Phi]
\]

, the usual composite of a natural transformation and a functor.

2. The module \( \downarrow \text{CYL} : \text{COM} \to \text{MOD} \) is defined in the following way:

a) a \( \downarrow \text{CYL} \)-arrow from a comma \( \mathbb{K} : Y \to B \) to a module \( M : X \to A \), written \( \alpha : S \to T : \mathbb{K} \to M \), is a triple \( (S, \alpha, T) \) consisting of a functor \( S : Y \to X \), a second functor \( T : B \to A \), and a cylinder

\[
\begin{array}{c}
Y \xleftarrow{K_Y} \mathbb{K} \xrightarrow{K_B} B \\
\downarrow \alpha \downarrow \downarrow T \\
X \xleftarrow{M_X} M \xrightarrow{M_A} A \\
\end{array}
\]
from $K_Y \circ S$ to $T \circ K_B$ along $M$.

b) for a comma cell $\Phi : P \rightarrow Q : J \rightarrow K$ and a $\uparrow \text{CYL}$-arrow $\alpha : S \rightarrow T : K \rightarrow M$ as in

\[
\begin{array}{c}
Z \xleftarrow{J_Z} \overset{J}{\rightarrow} \overset{J_C}{\rightarrow} C \\
\downarrow \Phi \\
Y \xleftarrow{KY} \overset{K}{\rightarrow} \overset{KB}{\rightarrow} B \\
\downarrow \alpha \\
X \xleftarrow{M} \rightarrow A \\
\end{array}
\]

, their composite is the $\uparrow \text{CYL}$-arrow $\Phi \circ \alpha : P \circ S \rightarrow T \circ Q : J \rightarrow M$ with the cylinder $\Phi \circ \alpha : J_Z \circ P \circ S \rightarrow T \circ Q \circ J_C : [J] \rightarrow M$ defined by

$\Phi \circ \alpha = [\Phi] \circ \alpha$

, the usual composite of a functor and a cylinder (see Definition 4.3.22).

c) for a $\uparrow \text{CYL}$-arrow $\alpha : S \rightarrow T : K \rightarrow M$ and a module cell $\Phi : P \rightarrow Q : M \rightarrow N$ as in

\[
\begin{array}{c}
Y \xleftarrow{KY} \overset{K}{\rightarrow} \overset{KB}{\rightarrow} B \\
\downarrow \alpha \\
X \xleftarrow{M} \rightarrow A \\
\downarrow \Phi \\
Z \xleftarrow{N} \rightarrow C \\
\end{array}
\]

, their composite is the $\uparrow \text{CYL}$-arrow $\alpha \circ \Phi : S \circ P \rightarrow Q \circ T : K \rightarrow N$ with the cylinder $\alpha \circ \Phi : K_Y \circ S \circ P \rightarrow Q \circ T \circ K_B : [K] \rightarrow N$ given by the usual composition of a cylinder and a cell (see Definition 4.3.13).

**Remark 9.2.2.**

1. The unit cylinder (see Definition 9.1.3(1)) of a module (resp. collage) $M : X \rightarrow A$ forms a $\uparrow \text{CYL}$-arrow $1_M : 1_X \rightarrow 1_A : M \rightarrow M$.

2. The unit cylinder (see Definition 9.1.3(2)) of a comma $K : X \rightarrow A$ forms a $\uparrow \text{CYL}$-arrow $1_K : 1_X \rightarrow 1_A : K \rightarrow K$.

3. Since a cylinder

\[
\begin{array}{c}
Y \xleftarrow{KY} \overset{K}{\rightarrow} \overset{KB}{\rightarrow} B \\
\downarrow \alpha \\
X \xleftarrow{M} \rightarrow A \\
\end{array}
\]

is the same thing as a natural transformation

\[
\begin{array}{c}
Y \xleftarrow{KY} \overset{K}{\rightarrow} \overset{KB}{\rightarrow} B \\
\downarrow \alpha \\
X \xleftarrow{M_X} \overset{M}{\rightarrow} \overset{M_A}{\rightarrow} A \\
\end{array}
\]

269
(see Remark 4.3.4(3)), the identity

\[
\begin{array}{c}
\text{COM} \rightarrow \text{COM} \\
\downarrow \quad \downarrow \\
\text{COM} \rightarrow \text{COM}
\end{array}
\]

holds.

4. Each \(\downarrow\text{Cyl}\)-arrow \(\alpha : S \rightarrow T : \mathbb{K} \rightarrow \mathbb{M}\) yields the following arrows:

- \(\uparrow\text{Cyl}\)-arrow \(\alpha : \mathbb{K}_Y \odot S \rightarrow T \odot \mathbb{K}_B : [\mathbb{K}] \rightarrow \mathbb{M}\)
- \(\downarrow\text{Cyl}\)-arrow \(\alpha : S \odot \mathbb{M}_X \rightarrow \mathbb{M}_A \odot T : \mathbb{K} \rightarrow [\mathbb{M}]\)

, defining the following cells:

\[
\begin{array}{c}
\text{COM} \rightarrow \text{COM} \\
\downarrow \quad \downarrow \\
\text{COM} \rightarrow \text{COM}
\end{array}
\]

\[
\begin{array}{c}
\text{CAT} \rightarrow \text{CAT} \\
\downarrow \quad \downarrow \\
\text{CAT} \rightarrow \text{CAT}
\end{array}
\]

**Definition 9.2.3.** Given a \(\downarrow\text{Cyl}\)-arrow \(\alpha : S \rightarrow T : \mathbb{K} \rightarrow \mathbb{M}\), i.e. a natural transformation

\[
\begin{array}{c}
Y \leftarrow \mathbb{K}_Y \\
\downarrow \quad \downarrow \\
X \rightarrow \mathbb{K}_X
\end{array}
\begin{array}{c}
\mathbb{K} \\
\downarrow \quad \downarrow \\
\mathbb{M}
\end{array}
\begin{array}{c}
\rightarrow B \\
\downarrow \quad \downarrow \\
\rightarrow A
\end{array}
\]

1. the comma cell \(\alpha^! : S \rightarrow T : \mathbb{K} \rightarrow \mathbb{M}^!\), depicted in

\[
\begin{array}{c}
Y \leftarrow \mathbb{K}_Y \\
\downarrow \quad \downarrow \\
X \rightarrow \mathbb{K}_X
\end{array}
\begin{array}{c}
\mathbb{K} \\
\downarrow \quad \downarrow \\
\mathbb{M}^!
\end{array}
\begin{array}{c}
\rightarrow B \\
\downarrow \quad \downarrow \\
\rightarrow A
\end{array}
\]

, is defined by the comma adjunct \([\alpha^!] : [\mathbb{K}] \rightarrow [\mathbb{M}^!]\) (see Definition 9.1.7(1)) of the \(\uparrow\text{Cyl}\)-arrow \(\alpha : \mathbb{K}_Y \odot S \rightarrow T \odot \mathbb{K}_B : [\mathbb{K}] \rightarrow \mathbb{M}\).

2. the collage cell \(\alpha^\uparrow : S \rightarrow T : \mathbb{K}^! \rightarrow \mathbb{M}\), depicted in

\[
\begin{array}{c}
Y \leftarrow \mathbb{K}_Y \\
\downarrow \quad \downarrow \\
X \rightarrow \mathbb{K}_X
\end{array}
\begin{array}{c}
\mathbb{K} \\
\downarrow \quad \downarrow \\
\mathbb{M}
\end{array}
\begin{array}{c}
\rightarrow B \\
\downarrow \quad \downarrow \\
\rightarrow A
\end{array}
\]

, is defined by the collage adjunct \([\alpha^\uparrow] : [\mathbb{K}^!] \rightarrow [\mathbb{M}]\) (see Definition 9.1.7(2)) of the \(\downarrow\text{Cyl}\)-arrow \(\alpha : S \odot \mathbb{M}_X \rightarrow \mathbb{M}_A \odot T : \mathbb{K} \rightarrow [\mathbb{M}]\).

**Proposition 9.2.4.**
1. The assignment \( \alpha \mapsto \alpha^↓ \) yields a bijection from the set of \( \downarrow \text{CYL} \)-arrows \( \mathbb{K} \rightharpoonup \mathcal{M} \) to the set of comma cells \( \mathbb{K} \rightharpoonup \mathcal{M}^↓ \), and the triangle

\[
\begin{array}{c}
\mathbb{K} \\
\alpha^↓ \\
\downarrow \alpha \\
\mathcal{M} \\
\longrightarrow \\
\mathcal{M}^↓ \\
\end{array}
\]

commutes; the unit cylinder \( 1^↓_\mathcal{M} \) thus forms an inverse universal \( \downarrow \text{CYL} \)-arrow.

2. The assignment \( \alpha \mapsto \alpha^↑ \) yields a bijection from the set of \( \downarrow \text{CYL} \)-arrows \( \mathbb{K} \rightharpoonup \mathcal{M} \) to the set of collage cells \( \mathbb{K}^↑ \rightharpoonup \mathcal{M} \), and the triangle

\[
\begin{array}{c}
\mathbb{K} \\
1^↑_\mathbb{K} \\
\Downarrow \alpha \\
\mathbb{K}^↑ \\
\longrightarrow \\
\mathcal{M} \\
\end{array}
\]

commutes; the unit cylinder \( 1^↑_\mathbb{K} \) thus forms a direct universal \( \downarrow \text{CYL} \)-arrow.

**Proof.** By the definitions of \( \alpha^↓ \) and \( \alpha^↑ \), the assertion is reduced to Proposition 9.1.8. \( \square \)

**Proposition 9.2.5.**

1. The functor \( \text{COM}^↓ \rightharpoonup \text{CLG} \) (see Remark 3.2.30(2)) and the family of unit cylinders \( 1^↓_\mathcal{M} : \mathcal{M}^↓ \rightharpoonup \mathcal{M} \), one for each collage \( \mathcal{M} \), form a counit of the module \( \uparrow \text{CYL} \);

2. The functor \( \text{COM}^↑ \rightharpoonup \text{CLG} \) (see Remark 3.2.30(2)) and the family of unit cylinders \( 1^↑_\mathbb{K} : \mathbb{K} \rightharpoonup \mathbb{K}^↑ \), one for each comma \( \mathbb{K} \), form a unit of the module \( \downarrow \text{CYL} \).

**Proof.** We have seen in Proposition 9.2.4 the universality of each unit cylinder. It remains to show that the family of unit cylinders \( 1^↓_\mathcal{M} \) (resp. \( 1^↑_\mathbb{K} \)) satisfies the naturality condition. But this follows immediately from Proposition 9.1.5. \( \square \)

**Remark 9.2.6.** The counit and unit in Proposition 9.2.5 are depicted as

\[
\begin{array}{c}
\text{COM}^↓ \rightharpoonup \text{CLG} \\
\downarrow 1^↓ \text{CYL} \\
\text{COM} \xrightarrow{\eta} \text{CLG} \\
\uparrow \text{CLG} \\
\end{array}
\]

with \( 1^↓ \) and \( 1^↑ \) defined by

\[
1^↓ = (1^↓_\mathcal{M})_{\mathcal{M} \in \text{CLG}}, \quad 1^↑ = (1^↑_\mathbb{K})_{\mathbb{K} \in \text{COM}}.
\]

**Theorem 9.2.7.** There exists an adjoint equivalence

\[
\begin{array}{c}
\text{COM} \xrightarrow{(\eta, \epsilon)} \text{CLG} \\
\end{array}
\]

with the unit \( \eta \) and the counit \( \epsilon \) given by the isomorphisms in Theorem 3.2.29.
Proof. The assertion results by applying Theorem 8.1.10 to the counit and unit of $\uparrow\text{CYL}$ in Remark 9.2.6 and noting Proposition 9.1.6.

Corollary 9.2.8. The module $\uparrow\text{CYL}$ is an equivalence and each unit cylinder is a two-way universal $\uparrow\text{CYL}$-arrow.

Proof. We have seen in Theorem 9.2.7 that the functors $\text{COM} \uparrow \text{CLG}$ and $\text{COM} \downarrow \text{CLG}$ are equivalences. Hence the assertion results by applying Corollary 8.4.7 to the counit and unit of $\uparrow\text{CYL}$ in Remark 9.2.6.

9.3. Equivalence $[X \downarrow A] \simeq [X \uparrow A]$

Note. Given a pair of categories $X$ and $A$, the following modules are defined in Definition 9.3.1:

- $(X \uparrow A) : [X \downarrow A] \to [X \uparrow A]$
- $(X \downarrow A) : [X \downarrow A] \to [X \uparrow A]$

Although they are regarded as the same thing under the identification $[X \uparrow A] \simeq [X : A]$, parallel definitions are presented for the sake of reference. After Remark 9.3.2 (3) we will deal with only the module $(X \uparrow A) : [X \downarrow A] \to [X \uparrow A]$; however, any result in this section stated for $(X \downarrow A) : [X \downarrow A] \to [X \uparrow A]$ also holds with $[X \uparrow A]$ changed to $[X : A]$. This section is analogous to the previous section; all definitions and results given for $[X \downarrow A]$ and $[X \uparrow A]$ in this section are the reflections of those given for $\text{COM}$ and $\text{CLG}$ in the previous section along the embedding $[X \downarrow A] \hookrightarrow \text{COM}$ and $[X \uparrow A] \hookrightarrow \text{CLG}$ (see Remark 3.2.18(2) and Remark 3.1.7(2)).

Definition 9.3.1. Let $X$ and $A$ be categories.

1. The module $(X \downarrow A) : [X \downarrow A] \to [X \uparrow A]$ is defined in the following way:

   a) an $(X \downarrow A)$-arrow $\alpha : K \rightsquigarrow M$ from a comma $K : X \to A$ to a collage $M : X \to A$ is given by a natural transformation

   \[
   \begin{array}{ccc}
   X & \to & K \mathllap{\xrightarrow{\alpha}} \mathllap{\to} \ A \\
   M_X & \searrow & M_A \\
   \downarrow \mathllap{\downarrow} & & \downarrow \mathllap{\downarrow} \\
   [M] & \to & [M] \\
   \end{array}
   \]

   from $K_X \circ M_X$ to $M_A \circ K_A$.

   b) for a comma morphism $\Phi : J \to K : X \to A$ and an $(X \uparrow A)$-arrow $\alpha : K \rightsquigarrow M$ as in

   \[
   \begin{array}{ccc}
   X & \to & K \mathllap{\xrightarrow{\alpha}} \mathllap{\to} \ A \\
   M_X & \searrow & M_A \\
   \downarrow \mathllap{\downarrow} & & \downarrow \mathllap{\downarrow} \\
   [M] & \to & [M] \\
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   J & \searrow & J_A \\
   \downarrow \mathllap{\downarrow} & & \downarrow \mathllap{\downarrow} \\
   K & \to & K_A \\
   \end{array}
   \]
9. Collages and Commas (continued)

, their composite is the \( \langle X \downarrow A \rangle \)-arrow \( \Phi \circ \alpha : J \to M \) with the natural transformation
\( \Phi \circ \alpha : J_X \circ M_X \to M_A \circ J_A : [J] \to [M] \) defined by
\[
\Phi \circ \alpha = [\Phi] \circ \alpha
\]
, the usual composite of a functor and a natural transformation.

c) for an \( \langle X \downarrow A \rangle \)-arrow \( \alpha : K \to M \) and a collage morphism \( \Phi : M \to N : X \to A \) as in

\[
\begin{array}{c}
\xymatrix{
K 
\ar[r]^α & A \\
X 
\ar[u]_{M_X} 
\ar[r]_{M_A} & [M] 
\ar[u]_{\Phi} \\
N_X 
\ar[r]_{N_A} & [N] 
\ar[u]_{\Phi}\circ\alpha}
\end{array}
\]

, their composite is the \( \langle X \downarrow A \rangle \)-arrow \( \alpha \circ \Phi : K \to N \) with the natural transformation
\( \alpha \circ \Phi : K_X \circ N_X \to N_A \circ K_A : [K] \to [N] \) defined by
\[
\alpha \circ \Phi = \alpha \circ [\Phi]
\]
, the usual composite of a natural transformation and a functor.

2. The module \( \langle X \downarrow A \rangle : [X \downarrow A] \to [X : A] \) is defined in the following way:

a) an \( \langle X \downarrow A \rangle \)-arrow \( \alpha : K \to M \) from a comma \( K : X \to A \) to a module \( M : X \to A \) is
given by a cylinder

\[
\begin{array}{c}
\xymatrix{
K 
\ar[r]^α & A \\
X 
\ar[u]_{K_X} 
\ar[r]_{K_A} & [K] 
\ar[u]_{\Phi}
\end{array}
\]

b) for a comma morphism \( \Phi : J \to K : X \to A \) and an \( \langle X \downarrow A \rangle \)-arrow \( \alpha : K \to M \) as in

\[
\begin{array}{c}
\xymatrix{
J 
\ar[r]^α & A \\
X 
\ar[u]_{J_X} 
\ar[r]_{J_A} & [J] 
\ar[u]_{\Phi}
\end{array}
\]

, their composite is the \( \langle X \downarrow A \rangle \)-arrow \( \Phi \circ \alpha : J \to M \) with the cylinder \( \Phi \circ \alpha : J_X \to J_A : [J] \to [M] \) defined by
\[
\Phi \circ \alpha = [\Phi] \circ \alpha
\]
, the usual composite of a functor and a cylinder (see Definition 4.3.22).

c) for an \( \langle X \downarrow A \rangle \)-arrow \( \alpha : K \to M \) and a module morphism \( \Phi : M \to N : X \to A \) as in

\[
\begin{array}{c}
\xymatrix{
K 
\ar[r]^α & A \\
X 
\ar[u]_{K_X} 
\ar[r]_{K_A} & [K] 
\ar[u]_{\Phi}
\end{array}
\]

273
9. Collages and Commas (continued)

, their composite is the \( (X \downarrow A) \)-arrow \( \alpha \circ \Phi : K \Rightarrow N \) with the cylinder \( \alpha \circ \Phi : K_X \Rightarrow K_A : [K] \Rightarrow N \) given by the usual composition of a cylinder and a module morphism (see Definition 4.3.9).

Remark 9.3.2.

1. The unit cylinder (see Definition 9.1.3(1)) of a module (resp. collage) \( \mathcal{M} : X \rightarrow A \) forms an \( (X \downarrow A) \)-arrow \( 1^M : \mathcal{M} \Rightarrow \mathcal{M} \).

2. The unit cylinder (see Definition 9.1.3(2)) of a comma \( \mathcal{K} : X \rightarrow A \) forms an \( (X \downarrow A) \)-arrow \( 1^\mathcal{K} : \mathcal{K} \Rightarrow \mathcal{K} \).

3. Since a cylinder

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow_{\mathcal{M}} & & \downarrow_{\mathcal{A}} \\
\end{array}
\]

is the same thing as a natural transformation

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow_{\mathcal{M}} & & \downarrow_{\mathcal{A}} \\
\end{array}
\]

(see Remark 4.3.4(3)), the identity

\[
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{(X \downarrow A)} & [X : A] \\
\downarrow 1 & & \downarrow 1 \\
[X \downarrow A] & \xrightarrow{(X \downarrow A)} & [X \uparrow A] \\
\end{array}
\]

holds.

4. Each \( (X \uparrow A) \)-arrow \( \alpha : \mathcal{K} \Rightarrow \mathcal{M} \) yields the following arrows:

- \( \uparrow \text{CYL}-\text{arrow} \alpha : 1_X \Rightarrow 1_A : \mathcal{K} \Rightarrow \mathcal{M} \)
- \( \uparrow \text{CYL}-\text{arrow} \alpha : \mathcal{K}_X \Rightarrow \mathcal{K}_A : [K] \Rightarrow \mathcal{M} \)
- \( \uparrow \text{CYL}-\text{arrow} \alpha : \mathcal{M}_X \Rightarrow \mathcal{M}_A : \mathcal{K} \Rightarrow [\mathcal{M}] \)

, defining the following faithful cells:

\[
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{(X \downarrow A)} & [X \uparrow A] \\
\downarrow \sim & & \downarrow \sim \\
\text{COM} & \xrightarrow{\text{CYL}} & \text{CLG} \\
\end{array}
\]

\[
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{(X \downarrow A)} & [X \uparrow A] \\
\downarrow [-] & & \downarrow [-] \\
\text{CAT} & \xrightarrow{\text{CYL}} & \text{CLG} \\
\end{array}
\]

\[
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{(X \downarrow A)} & [X \uparrow A] \\
\downarrow \sim & & \downarrow \sim \\
\text{COM} & \xrightarrow{\text{CYL}} & \text{CAT} \\
\end{array}
\]
Definition 9.3.3. Given an \((X \downarrow A)\)-arrow \(\alpha : K \Rightarrow M\), i.e. a natural transformation

\[
\begin{array}{ccc}
K_X & \overset{[\mathcal{K}]}{\longrightarrow} & K_A \\
\downarrow \alpha & & \downarrow \alpha \\
M_X & \overset{[\mathcal{M}]}{\longrightarrow} & M_A
\end{array}
\]

1. the comma morphism \(\alpha^\downarrow : K \rightarrow M^\downarrow : X \rightarrow A\), depicted in

\[
\begin{array}{ccc}
K_X & \overset{[\mathcal{K}]}{\longrightarrow} & K_A \\
\downarrow \alpha & & \downarrow \alpha \\
M_X & \overset{[\mathcal{M}]}{\longrightarrow} & M_A
\end{array}
\]

, is defined by the comma adjunct \([\alpha^\downarrow] : [K] \rightarrow [M^\downarrow]\) (see Definition 9.1.7(1)) of the \(\uparrow\text{CYL}\)-arrow \(\alpha : K_X \Rightarrow K_A : [K] \Rightarrow M\).

2. the collage morphism \(\alpha^\uparrow : K^\uparrow \rightarrow M : X \rightarrow A\), depicted in

\[
\begin{array}{ccc}
K_X & \overset{[\mathcal{K}]}{\longrightarrow} & K_A \\
\downarrow \alpha^\uparrow & & \downarrow \alpha^\uparrow \\
M_X & \overset{[\mathcal{M}]}{\longrightarrow} & M_A
\end{array}
\]

, is defined by the collage adjunct \([\alpha^\uparrow] : [K^\uparrow] \rightarrow [M]\) (see Definition 9.1.7(2)) of the \(\downarrow\text{CYL}\)-arrow \(\alpha : M_X \Rightarrow M_A : [K] \sim [M]\).

Proposition 9.3.4.

1. The assignment \(\alpha \mapsto \alpha^\downarrow\) yields a bijection from the set of \((X \downarrow A)\)-arrows \(K \Rightarrow M\) to the set of comma morphisms \(K \rightarrow M^\downarrow\), and the triangle

\[
\begin{array}{ccc}
& \overset{\alpha^\downarrow}{\longrightarrow} & \\\n\mathcal{M}^\downarrow & \overset{\alpha}{\longrightarrow} & M
\end{array}
\]

commutes; the unit cylinder \(1^\downarrow_M\) thus forms an inverse universal \((X \downarrow A)\)-arrow.

2. The assignment \(\alpha \mapsto \alpha^\uparrow\) yields a bijection from the set of \((X \downarrow A)\)-arrows \(K \Rightarrow M\) to the set of collage morphisms \(K^\uparrow \rightarrow M\), and the triangle

\[
\begin{array}{ccc}
\mathcal{K} & \overset{\alpha^\uparrow}{\longrightarrow} & \mathcal{K}^\uparrow \\
\downarrow \alpha & & \downarrow \alpha^\uparrow \\
\mathcal{M} & \overset{1^\uparrow_M}{\longrightarrow} & \mathcal{M}
\end{array}
\]

commutes; the unit cylinder \(1^\uparrow_K\) thus forms a direct universal \((X \downarrow A)\)-arrow.
9. Collages and Commas (continued)

Proof. By the definitions of $\alpha^\downarrow$ and $\alpha^\uparrow$, the assertion is reduced to Proposition 9.1.8. \qed

Proposition 9.3.5.

1. The functor $[X \downarrow A] \to [X \uparrow A]$ (see Remark 3.2.30(2)) and the family of unit cylinders $1^\downarrow_M : M \to M$, one for each collage $M : X \to A$, form a counit of the module $(X \uparrow A)$;

2. The functor $[X \downarrow A] \to [X \uparrow A]$ (see Remark 3.2.30(2)) and the family of unit cylinders $1^\uparrow_K : K \to K^\prime$, one for each comma $K : X \to A$, form a unit of the module $(X \uparrow A)$. Proof. We have seen in Proposition 9.3.4 the universality of each unit cylinder. It remains to show that the family of unit cylinders $1^\downarrow_M$ (resp. $1^\uparrow_K$) satisfies the naturality condition. But this follows immediately from Proposition 9.1.5. \qed

Remark 9.3.6. The counit and unit in Proposition 9.3.5 are depicted as

$$
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{\downarrow} & [X \uparrow A] \\
\xleftarrow{(X \uparrow A)} & \uparrow & \xrightarrow{1^\uparrow} \\
1^\downarrow & \xrightarrow{} & 1^\uparrow
\end{array}
$$

with $1^\downarrow$ and $1^\uparrow$ defined by

$$
1^\downarrow = (1^\downarrow_M)_{M \in [X \uparrow A]} \\
1^\uparrow = (1^\uparrow_K)_{K \in [X \downarrow A]}
$$

Theorem 9.3.7. Given a pair of categories $X$ and $A$, there exists an adjoint equivalence

$$
[X \downarrow A] \xrightarrow{\eta, \epsilon} [X \uparrow A]
$$

with the unit $\eta$ and the counit $\epsilon$ given by the isomorphisms in Theorem 3.2.29.

Proof. The assertion results by applying Theorem 8.1.10 to the counit and unit of $(X \uparrow A)$ in Remark 9.3.6 and noting Proposition 9.1.6. \qed

Corollary 9.3.8. Given a pair of categories $X$ and $A$, the module $(X \uparrow A)$ is an equivalence and each unit cylinder is a two-way universal $(X \uparrow A)$-arrow.

Proof. We have seen in Theorem 9.3.7 that the functors $[X \downarrow A] \to [X \uparrow A]$ and $[X \downarrow A] \to [X \uparrow A]$ are equivalences. Hence the assertion results by applying Corollary 8.4.7 to the counit and unit of $(X \uparrow A)$ in Remark 9.3.6. \qed

9.4. Equivalences $[X \downarrow] \simeq [X :]$ and $[\downarrow A] \simeq [: A]$

Note. The following definition is regarded as a special case of 9.1.3 where $A$ (resp. $X$) is the terminal category.

Definition 9.4.1.

1. 

276
The unit cone of a right module $\mathcal{M} : X \to \ast$ is the cone

![Diagram](image-url)

defined by

$$[1^\circ_{\mathcal{M}}]_m = m$$

for $m$ an arrow of $\mathcal{M}$.

The unit cone of a left module $\mathcal{M} : \ast \to \mathbf{A}$ is the cone

![Diagram](image-url)

defined by

$$[1^\circ_{\mathcal{M}}]_m = m$$

for $m$ an arrow of $\mathcal{M}$.

2. 

- The unit cone of a right comma $\mathbb{K} : X \to \ast$ is the cone

![Diagram](image-url)

defined by

$$[1^\circ_{\mathbb{K}}]_k = k$$

for $k$ an object of $[\mathbb{K}]$.

- The unit cone of a left comma $\mathbb{K} : \ast \to \mathbf{A}$ is the cone

![Diagram](image-url)

defined by

$$[1^\circ_{\mathbb{K}}]_k = k$$

for $k$ an object of $[\mathbb{K}]$.

Note. The following definition is regarded as a special case of Definition 9.3.1 where $\mathbf{A}$ (resp. $X$) is the terminal category.

**Definition 9.4.2.**

- Given a category $X$, the module $(X \uparrow) : [X \downarrow] \to [X :]$ is defined in the following way:
9. Collages and Commas (continued)

1. an \( \downarrow \text{X} \rightharpoonup \text{M} \) from a right comma \( \text{K} : \text{X} \rightharpoonup * \) to a right module \( \text{M} : \text{X} \rightharpoonup * \) is given by a cone

\[
\begin{array}{c}
\downarrow \\
\text{X} \xymatrix{ \text{K} \ar[r]^\Delta \ar[d]_\alpha & \text{M} \ar[d] \ar[r] & * \\
\text{K} \ar@{-->}[rr]_\Delta \ar[d]_\alpha & & \text{M} \ar@{-->}[rr]_\Delta \\
\text{X} \ar@{-->}[rr]_\alpha & & * \\
\end{array}
\]

2. for a right comma morphism \( \Phi : \text{J} \rightharpoonup \text{K} : \text{X} \rightharpoonup * \) and an \( \downarrow \text{X} \rightharpoonup \text{M} \) as in

\[
\begin{array}{c}
\downarrow \\
\text{X} \xymatrix{ \text{J} \ar[r]^\Phi \ar[d]_\Delta & \text{K} \ar[d]_\alpha \ar[r]^\Delta & \text{M} \ar@{-->}[rr]_\Delta \\
\downarrow \text{J} \ar[d]_{\downarrow \Phi} \ar[r]^\Delta & \downarrow \text{K} \ar[d]_\alpha \ar[r]^\Delta & \downarrow \text{M} \ar@{-->}[rr]_\Delta \\
\text{X} \ar@{-->}[rr]_\alpha & & * \\
\end{array}
\]

\[
\Phi \circ \alpha = [\Phi] \circ \Delta
\]

, the usual composite of a functor and a cone (see Definition 4.6.17).

3. for an \( \downarrow \text{X} \rightharpoonup \text{M} \) and a right module morphism \( \Phi : \text{M} \rightharpoonup \text{N} : \text{X} \rightharpoonup * \) as in

\[
\begin{array}{c}
\downarrow \\
\text{X} \xymatrix{ \text{K} \ar[r]^\Delta \ar[d]_\alpha & \text{M} \ar[d]_\Phi \ar[r]^\Delta & \text{N} \ar[r] & * \\
\text{K} \ar@{-->}[rr]_\Delta \ar[d]_\alpha & & \text{M} \ar@{-->}[rr]_\Delta \\
\text{X} \ar@{-->}[rr]_\alpha & & * \\
\end{array}
\]

, their composite is the \( \downarrow \text{X} \rightharpoonup \text{N} \) with the cone \( \alpha \circ \Phi : \text{K} \rightharpoonup * : [\text{K}]^\circ \rightharpoonup \text{N} \) given by the usual composition of a cone and a module morphism (see Definition 4.6.7).

Given a category \( \mathcal{A} \), the module \( \uparrow \mathcal{A} : [\downarrow \mathcal{A}] \rightharpoonup \downarrow \mathcal{A} \) is defined in the following way:

1. an \( \uparrow \mathcal{A} \) from a left comma \( \text{K} : * \rightharpoonup \mathcal{A} \) to a left module \( \text{M} : * \rightharpoonup \mathcal{A} \) is given by a cone

\[
\begin{array}{c}
\downarrow \\
* \xymatrix{ \text{K} \ar[r]^\Delta \ar[d]_\alpha & \mathcal{A} \ar[d] \ar[r] & * \\
\text{M} \ar@{-->}[rr]_\Delta \\
\end{array}
\]

2. for a left comma morphism \( \Phi : \text{J} \rightharpoonup \text{K} : * \rightharpoonup \mathcal{A} \) and an \( \uparrow \mathcal{A} \) as in

\[
\begin{array}{c}
\downarrow \\
* \xymatrix{ \text{J} \ar[r]^\Phi \ar[d]_\Delta & \text{K} \ar[d]_\alpha \ar[r]^\Delta & \mathcal{A} \ar[d]_\Delta \\
\text{J} \ar@{-->}[rr]_\Delta \ar[r]^\Delta & \text{K} \ar@{-->}[rr]_\Delta \\
* \ar@{-->}[rr]_\alpha & & \mathcal{A} \\
\end{array}
\]
2.

Proposition 9.4.4.

Their composite is the \( \downarrow A \)-arrow \( \Phi \circ \alpha : \downarrow x \Rightarrow M \) with the cone \( \Phi \circ \alpha : \downarrow x \Rightarrow M \) defined by

\[
\Phi \circ \alpha = [\Phi] \circ \alpha
\]

the usual composite of a functor and a cone (see Definition 4.6.17).

3. for an \( \downarrow A \)-arrow \( \alpha : K \Rightarrow M \) and a left module morphism \( \Phi : M \Rightarrow N \Rightarrow * \Rightarrow A \) as in

\[
\begin{array}{c}
\xymatrix{
\Delta \\
\alpha}
\end{array}
\]

\( \rightarrow \)

\[
\begin{array}{c}
A
\end{array}
\]

\( \leftarrow \)

\( [K] \)

\( \rightarrow \)

\( \leftarrow \)

, their composite is the \( \downarrow A \)-arrow \( \alpha \circ \Phi : K \Rightarrow N \) with the cone \( \alpha \circ \Phi : \downarrow x \Rightarrow K : [K] \Rightarrow N \) given by the usual composition of a cone and a module morphism (see Definition 4.6.7).

Remark 9.4.3.

1.

- The unit cone of a right module \( M : X \Rightarrow * \) forms an \( \downarrow X \)-arrow \( 1_M : M \Rightarrow \Rightarrow M \).
- The unit cone of a left module \( M : * \Rightarrow A \) forms an \( \downarrow A \)-arrow \( 1_M : M \Rightarrow \Rightarrow M \).

2.

- The unit cone of a right comma \( K : X \Rightarrow * \) forms an \( \downarrow X \)-arrow \( 1_K : K \Rightarrow K \).
- The unit cone of a left comma \( K : * \Rightarrow A \) forms an \( \downarrow A \)-arrow \( 1_K : K \Rightarrow K \).

3. The following identities hold

\[
\begin{align*}
[X \downarrow] & \mathrel{\approx} [X :] \\
[X \downarrow *] & \mathrel{\approx} [X : *]
\end{align*}
\]

\[
\begin{align*}
[\downarrow A] & \mathrel{\approx} [\downarrow A] \\
[\downarrow A] & \mathrel{\approx} [\downarrow A]
\end{align*}
\]

\( \approx \) denotes the canonical isomorphisms, giving canonical isomorphisms

\[
\downarrow X \approx X \downarrow * \quad \text{and} \quad \downarrow A \approx (\downarrow A)
\]

Proposition 9.4.4.

1.

- The functor \( X \downarrow \downarrow [X :] \) (see Remark 3.2.24) and the family of unit cones \( 1^M : M \Rightarrow \Rightarrow M \), one for each right module \( M : X \Rightarrow * \), forms a counit of the module \( \downarrow X \downarrow \);
- The functor \( \downarrow A \downarrow [A :] \) (see Remark 3.2.24) and the family of unit cones \( 1^M : M \Rightarrow \Rightarrow M \), one for each left module \( M : * \Rightarrow A \), forms a counit of the module \( \downarrow A \downarrow \);
9. Collages and Commas (continued)

- The functor \([X \downarrow] \rightarrow [X:]\) (see Remark 3.2.28) and the family of unit cones \(1^\downarrow_X : K \rightarrow K^\downarrow\), one for each right comma \(K : X \rightarrow *)\, forms a unit of the module \((X \downarrow)\).

- The functor \([\downarrow A] \rightarrow [: A]\) (see Remark 3.2.28) and the family of unit cones \(1^\downarrow_A : K \rightarrow K^\downarrow\), one for each left comma \(K : * \rightarrow A\), forms a unit of the module \((\downarrow A)\).

**Proof.** This is a special case of Proposition 9.3.5 where \(A\) (resp. \(X\)) is the terminal category.

**Remark 9.4.5.** The counit and unit in Proposition 9.4.4 are depicted as

\[
\begin{align*}
[X \downarrow] & \xrightarrow{1^\downarrow} [X:] \\
[X] & \xleftarrow{\eta} [X:] \\
\end{align*}
\]

with \(1^\downarrow\) and \(1^\uparrow\) defined by

\[
1^\downarrow = \left(1^\downarrow_{M_{\|X\|}}\right)_{M \in \|X\|} \\
1^\uparrow = \left(1^\uparrow_{K_{/\|X\|}}\right)_{K \in \|X\|}.
\]

\[
\begin{align*}
[\downarrow A] & \xrightarrow{1^\downarrow} [: A] \\
[\downarrow A] & \xleftarrow{\eta} [: A] \\
\end{align*}
\]

with \(1^\downarrow\) and \(1^\uparrow\) defined by

\[
1^\downarrow = \left(1^\downarrow_{M_{\|A\|}}\right)_{M \in \|A\|} \\
1^\uparrow = \left(1^\uparrow_{K_{/\|A\|}}\right)_{K \in \|A\|}.
\]

**Theorem 9.4.6.**

- **Given a category** \(X\), **there exists an adjoint equivalence**

\[
[X \downarrow] \xrightarrow{(\eta, \epsilon)} [X:] \\
\]

- **Given a category** \(A\), **there exists an adjoint equivalence**

\[
[\downarrow A] \xrightarrow{(\eta, \epsilon)} [: A] \\
\]

**Proof.** This is a special case of Theorem 9.3.7 where \(A\) (resp. \(X\)) is the terminal category.

**Corollary 9.4.7.**

- **Given a category** \(X\), **the module** \((X \uparrow)\) **is an equivalence and each unit cone is a two-way universal** \((X \uparrow)-arrow.\)
Given a category $A$, the module $(\uparrow A)$ is an equivalence and each unit cone is a two-way universal $(\uparrow A)$-arrow.

**Proof.** This is a special case of Corollary 9.3.8 where $A$ (resp. $X$) is the terminal category. \qed

**Note.** Since the module $(X \downarrow)$ (resp. $(\uparrow A)$) is an equivalence, Theorem 8.5.7 allows the following definition.

**Definition 9.4.8.** Let $X$ and $A$ be categories.

1. The equivalence cells

\[
\begin{array}{c}
\begin{array}{ccc}
[\downarrow A] - (\downarrow A) & \leftrightarrow & [\downarrow A] \\
\downarrow & \uparrow & \downarrow \\
[\downarrow A] - (\downarrow A) & \leftrightarrow & [\downarrow A]
\end{array}
\end{array}
\]

quasi-inverse to each other are defined by

\[
\langle \uparrow \rangle = [\downarrow A] \uparrow [1^1] \quad \langle \downarrow \rangle = \langle [\downarrow A] \uparrow [1^1] \rangle^{-1}
\]

where $[\downarrow A] \uparrow [1^1]$ and $[\downarrow A] \uparrow [1^1]$ are the module morphisms generated by $[\downarrow A]$ direct along the unit and counit of $(\downarrow A)$ (see Remark 9.4.5).

2. The equivalence cells

\[
\begin{array}{c}
\begin{array}{ccc}
[X:] - (X:) & \leftrightarrow & [X:] \\
\downarrow & \uparrow & \downarrow \\
[X:] - (X:) & \leftrightarrow & [X:]
\end{array}
\end{array}
\]

quasi-inverse to each other are defined by

\[
\langle \downarrow \rangle = [1^1] \downarrow [X:] \quad \langle \uparrow \rangle = \langle [1^1] \downarrow [X:] \rangle^{-1}
\]

where $[1^1] \downarrow [X:]$ and $[1^1] \downarrow [X:]$ are the module morphisms generated by $[X:]$ inverse along the counit and unit of $(\uparrow A)$ (see Remark 9.4.5).
The equivalence cells

\[
\begin{array}{ccc}
\left[ : A \right] - \rightharpoonup \left[ : A \right] & \quad & \left[ \downarrow A \right] - \left( \downarrow A \right) = \left[ : A \right] \\
\downarrow & \quad & \downarrow 1 \\
\left[ \downarrow A \right] - \left( \downarrow A \right) = \left[ : A \right] & \quad & \left[ : A \right] - \left( \downarrow A \right) = \left[ : A \right]
\end{array}
\]

quasi-inverse to each other are defined by

\[
\langle \downarrow \rangle = [1^\uparrow] \downarrow [: A] \quad \text{and} \quad \langle \uparrow \rangle = (1^\downarrow [[: A]])^{-1}
\]

, where \([1^\uparrow] \downarrow [: A]\) and \((1^\downarrow [[: A]])^{-1}\) are the module morphisms generated by \([[: A]]\) inverse along the counit and unit of \(\langle \downarrow A \rangle\) (see Remark 9.4.5).

Remark 9.4.9.

1. The cell \(\langle X \downarrow \rangle \rightharpoonup \langle X \uparrow \rangle\) sends each right comma morphism \(\Phi : J \rightharpoonup K : X \rightharpoonup *\) to the \(\langle X \uparrow \rangle\)-arrow \(\Phi^\uparrow : J \rightarrow K^\uparrow\), “the collage transpose of \(\Phi\)”, given by the postcomposition with the unit cone of \(K\) as indicated in

\[
\begin{array}{c}
\Phi \\
\downarrow
\end{array}
\xymatrix{
\downarrow J \\
K \ar[r]_{1_k} & K^\uparrow
}
\]

; the cone \(\Phi^\uparrow : J \rightharpoonup * : [J]^p \rightharpoonup K^\uparrow\) is defined by the composition

\[
\begin{array}{c}
[ J ] \\
\downarrow [ \Phi ] \\
J \\
\downarrow [ K ] \\
K \\
\downarrow 1_k
\end{array}
\xymatrix{
\downarrow [ \Delta ] \\
\downarrow X \\
\downarrow [ K ] \\
\downarrow [ K^\uparrow ] \\
\downarrow [ * ]
}
\]

, i.e. by

\[
[ \Phi^\uparrow ]_j = [ \Phi ] : j
\]

for \(j\) an object of \([J]\).

2. The cell \(\langle \downarrow A \rangle \rightharpoonup \langle \uparrow A \rangle\) sends each left comma morphism \(\Phi : J \rightarrow K : * \rightarrow A\) to the \(\langle \uparrow A \rangle\)-arrow \(\Phi^\uparrow : J \rightarrow K^\uparrow\), “the collage transpose of \(\Phi\)”, given by the postcomposition with the unit cone of \(K\) as indicated in

\[
\begin{array}{c}
\Phi \\
\downarrow
\end{array}
\xymatrix{
\downarrow J \\
K \ar[r]_{1_k} & K^\uparrow
}
\]


9. Collages and Commas (continued)

; the cone $\Phi^\dagger : * \to J : [J]^\dagger \to \mathbb{K}^\dagger$ is defined by the composition

\[
\begin{array}{c}
\mathbb{K} \\
\downarrow \alpha^\dagger \\
\mathbb{M}^\dagger \\
\downarrow \iota_{\mathbb{M}} \\
\mathbb{M} \\
\end{array}
\]

; i.e. by

\[
[\Phi^\dagger]_j = [\Phi] : j
\]

for $j$ an object of $[J]$.

2.

- The cell $\langle X \downarrow \rangle \downarrow \leftarrow \langle X \downarrow \rangle$ sends each $\langle X \downarrow \rangle$-arrow $\alpha : \mathbb{K} \to \mathbb{M}$ to the right comma morphism $\alpha^\dagger : \mathbb{K} \to \mathbb{M}^\dagger : X \to *$, the adjunct of $\alpha$ along the unit cone of $\mathbb{M}$ as indicated in

\[
\begin{array}{c}
\mathbb{K} \\
\downarrow \alpha^\dagger \\
\mathbb{M}^\dagger \\
\downarrow \iota_{\mathbb{M}} \\
\mathbb{M} \\
\end{array}
\]

; $\alpha^\dagger$ is given by the functor $[\alpha^\dagger] : [\mathbb{K}] \to [\mathbb{M}^\dagger]$ defined by

\[
[\alpha^\dagger] : k = \alpha_k \quad [\alpha^\dagger] : h = \mathbb{K} : h
\]

for $k$ an object and $h$ an arrow of $[\mathbb{K}]$ (cf. Definition 9.3.3(1)).

- The cell $\langle \uparrow \mathbf{A} \rangle \downarrow \leftarrow \langle \downarrow \mathbf{A} \rangle$ sends each $\langle \uparrow \mathbf{A} \rangle$-arrow $\alpha : \mathbb{K} \to \mathbb{M}$ to the left comma morphism $\alpha^\dagger : \mathbb{K} \to \mathbb{M}^\dagger : * \to \mathbf{A}$, the adjunct of $\alpha$ along the unit cone of $\mathbb{M}$ as indicated in

\[
\begin{array}{c}
\mathbb{K} \\
\downarrow \alpha^\dagger \\
\mathbb{M}^\dagger \\
\downarrow \iota_{\mathbb{M}} \\
\mathbb{M} \\
\end{array}
\]

; $\alpha^\dagger$ is given by the functor $[\alpha^\dagger] : [\mathbb{K}] \to [\mathbb{M}^\dagger]$ defined by

\[
[\alpha^\dagger] : k = \alpha_k \quad [\alpha^\dagger] : h = \mathbb{K} : h
\]

for $k$ an object and $h$ an arrow of $[\mathbb{K}]$ (cf. Definition 9.3.3(1)).

3.

- The cell $\langle X \downarrow \rangle \downarrow \leftarrow \langle X \downarrow \rangle$ sends each right module morphism $\Phi : \mathbb{M} \to \mathbb{N} : X \to *$ to the $\langle X \downarrow \rangle$-arrow $\Phi^\dagger : \mathbb{M}^\dagger \to \mathbb{N}^\dagger$, “the comma transpose of $\Phi$”, given by the precomposition with the unit cone of $\mathbb{M}$ as indicated in

\[
\begin{array}{c}
\mathbb{M}^\dagger \\
\downarrow \iota_{\mathbb{M}} \\
\mathbb{M} \\
\downarrow \Phi^\dagger \\
\mathbb{N} \\
\end{array}
\]
9. Collages and Commas (continued)

; the cone $\Phi : M^\downarrow \rightrightarrows [M^\downarrow]^\triangleright \rightrightarrows N$ is defined by the composition

$$
\begin{array}{c}
\vcenter{\hbox{\xymatrix{ & [M^\downarrow]
\ar[rr]^\Delta
\ar[rd]_\Phi
\ar[ld]^{1_M}

\ar@{..>}[rr]&&* \\
X
\ar[rr]_\Phi
\ar[rd]_\Phi
\ar[ld]^{1_M}

\ar@{..>}[rr]&&N
}}}
\end{array}
$$

, i.e. by

$$[\Phi^1]_m = \Phi \cdot m$$

for $m$ an arrow of $M$.

- The cell $\langle ; (A) \rangle \overset{\uparrow}{\rightrightarrows} \langle \uparrow (A) \rangle$ sends each left module morphism $\Phi : M \rightrightarrows N : * \rightrightarrows A$ to the $\langle \uparrow (A) \rangle$-arrow $\Phi^1 : M^\downarrow \rightrightarrows N$, “the comma transpose of $\Phi$”, given by the precomposition with the unit cone of $M$ as indicated in

$$
\begin{array}{c}
\vcenter{\hbox{\xymatrix{ & M^\downarrow
\ar[rr]^{1_M}
\ar[rd]_\Phi
\ar[ld]^{1_M}

\ar@{..>}[rr]&&A
}}}
\end{array}
$$

; the cone $\Phi^1 : * \rightrightarrows M^\downarrow : [M^\downarrow]^\triangleright \rightrightarrows N$ is defined by the composition

$$
\begin{array}{c}
\vcenter{\hbox{\xymatrix{ & [M^\downarrow]
\ar[rr]^\Delta
\ar[rd]_\Phi
\ar[ld]^{1_M}

\ar@{..>}[rr]&&M
}}}
\end{array}
$$

, i.e. by

$$[\Phi^1]_m = \Phi \cdot m$$

for $m$ an arrow of $M$.

4.

- The cell $\langle (X \uparrow) \rangle \overset{\uparrow}{\rightrightarrows} \langle (X:) \rangle$ sends each $\langle (X \uparrow) \rangle$-arrow $\alpha : (K \rightrightarrows M) \rightrightarrows (X : *)$ to the right module morphism $\alpha^\uparrow : K^\uparrow \rightrightarrows M : X \rightrightarrows *$, the adjunct of $\alpha$ along the unit cone of $K$ as indicated in

$$
\begin{array}{c}
\vcenter{\hbox{\xymatrix{ & K^\uparrow
\ar[rr]^{1_K}
\ar[rd]_\alpha
\ar[ld]^\alpha

\ar@{..>}[rr]&&M
}}}
\end{array}
$$

; $\alpha^\uparrow$ is defined by

$$\alpha^\uparrow : k \mapsto \alpha_k$$

for $k$ an object of $[K]$ (cf. Definition 9.3.3(2)).

- The cell $\langle (A \uparrow) \rangle \overset{\uparrow}{\rightrightarrows} \langle (A:) \rangle$ sends each $\langle (A \uparrow) \rangle$-arrow $\alpha : K \rightrightarrows M : * \rightrightarrows A$ to the left module morphism $\alpha^\uparrow : K^\uparrow \rightrightarrows M : * \rightrightarrows A$, the adjunct of $\alpha$ along the unit cone of $K$ as indicated in

$$
\begin{array}{c}
\vcenter{\hbox{\xymatrix{ & K^\uparrow
\ar[rr]^{1_K}
\ar[rd]_\alpha
\ar[ld]^\alpha

\ar@{..>}[rr]&&M
}}}
\end{array}
$$
; $\alpha^\dagger$ is defined by

$$\alpha^\dagger : k = \alpha_k$$

for $k$ an object of $[\mathbb{K}]$ (cf. Definition 9.3.3(2)).
10. Extensions

10.1. coYoneda lemma

Definition 10.1.1. Let $\mathcal{M} : X \to A$ be a module.

- The corepresentable module of the functor $\mathcal{M}^\triangledown : A \to [X:]$, the right exponential transpose of $\mathcal{M}$, is denoted by $\mathcal{M}^\triangledown$; that is, the module
  $$(\mathcal{M}^\triangledown) : [X:] \to A$$
is defined by the composition
  $$[X:] \xrightarrow{\mathcal{M}^\triangledown} A$$
- The representable module of the functor $\mathcal{\wedge}M : X \to [:A]^\wedge$, the left exponential transpose of $\mathcal{M}$, is denoted by $\mathcal{\wedge}M$; that is, the module
  $$(\mathcal{\wedge}M) : X \to [:A]^\wedge$$
is defined by the composition
  $$X \xrightarrow{\mathcal{\wedge}M} [:A]^\wedge \xrightarrow{(A)^\wedge} [:A]^\wedge$$

Remark 10.1.2.

1. Given a right module $\mathcal{J} : X \to *$ and an object $a \in \|A\|$, an $(\mathcal{M}^\triangledown)$-arrow $\Phi : \mathcal{J} \to a$ is a right module morphism $\Phi : \mathcal{J} \to (\mathcal{M})a : X \to *$, i.e. a conical cell
  $$\begin{array}{ccc}
  X & \xrightarrow{\mathcal{J}} & * \\
  \downarrow{1} & \Phi & \downarrow{a} \\
  X & \xrightarrow{\mathcal{M}} & A
  \end{array}$$
- This cell is denoted by $\Phi : \mathcal{J} \to a : M^\triangledown$, depicted also as
  $$\begin{array}{ccc}
  \mathcal{J} & \xrightarrow{\Phi} & * \\
  \Phi & \downarrow{a} \\
  X & \xrightarrow{\mathcal{M}} & A
  \end{array}$$
, and called a conical cell from $\mathcal{J}$ to $a$ along $\mathcal{M}$.  

10. Extensions

 Given a left module $\mathcal{J} : * \to A$ and an object $x \in \|X\|$, an $\langle \mathcal{M} \rangle$-arrow $\Phi : x \rightsquigarrow \mathcal{J}$ is a left module morphism $\Phi : \mathcal{J} \to x(M) : * \to A$, i.e. a conical cell

$$\begin{array}{c}
\ast & \overset{\mathcal{J}}{\longrightarrow} & A
\\
\downarrow \theta & & \\
X & \overset{M}{\longrightarrow} & A
\end{array}$$

This cell is denoted by $\Phi : x \rightsquigarrow \mathcal{J} : \mathcal{M}$, depicted also as

$$\begin{array}{c}
\ast
\\
\phi
\downarrow
\mathcal{J}
\\
X & \overset{M}{\longrightarrow} & A
\end{array}$$

and called a conical cell from $x$ to $\mathcal{J}$ along $\mathcal{M}$.

2. A conical cell

$$\begin{array}{c}
\ast & \overset{\mathcal{M}}{\longrightarrow} & *
\\
\downarrow \phi & & \\
Y & \overset{N}{\longrightarrow} & B
\end{array}$$

from $P$ to $b$ along $\langle M, N \rangle$ can be also depicted as a conical cell

$$\begin{array}{c}
\mathcal{M} & \overset{*}{\longrightarrow} & *
\\
\downarrow \phi & & \\
X & \overset{P(N)}{\longrightarrow} & B
\end{array}$$

from $M$ to $b$ along $P\langle N \rangle$, both being defined by a right module morphism $\Phi : M \to P\langle N \rangle b : X \to *$.

3. A conical cell

$$\begin{array}{c}
\ast & \overset{\mathcal{M}}{\longrightarrow} & A
\\
\downarrow \phi & & \\
Y & \overset{N}{\longrightarrow} & B
\end{array}$$

from $y$ to $Q$ along $\langle M, N \rangle$ can be also depicted as a conical cell

$$\begin{array}{c}
y & \overset{*}{\longrightarrow} & \mathcal{M}
\\
\downarrow \phi & & \\
Y & \overset{(N)Q}{\longrightarrow} & A
\end{array}$$

from $y$ to $M$ along $\langle N \rangle Q$, both being defined by a left module morphism $\Phi : M \to y\langle N \rangle Q : * \to A$.

3. With the notation and terminology introduced above, Theorem 5.2.11 is restated as follows:
10. Extensions

- For any pair of objects \( s \in \|X\| \) and \( t \in \|A\| \), the assignment \( m \mapsto X \uparrow m \) yields a bijection
  \[
  s (M) t \cong ((X) s) (M \triangledown) (t)
  \]
  from the set of \( M \)-arrows \( s \rightarrow t \) to the set of conical cells \((X) s \rightarrow t\) along \( M \); moreover, the bijection is natural in \( s \) and \( t \).
- For any pair of objects \( s \in \|X\| \) and \( t \in \|A\| \), the assignment \( m \mapsto m \uparrow A \) yields a bijection
  \[
  s (M) t \cong (s) (\downarrow M) (t (A))
  \]
  from the set of \( M \)-arrows \( s \rightarrow t \) to the set of conical cells \( s \rightarrow t (A) \) along \( M \); moreover, the bijection is natural in \( s \) and \( t \).

4. For any module \( M \),
  \[
  (M \triangledown)^\sim = (\downarrow M)^\sim
  \]

**Definition 10.1.3.** Let \( M : X \to A \) be a module.

- The module
  \[
  (M \triangledown) : [X \downarrow] \to A
  \]
  is defined by the composition
  \[
  [X \downarrow] \xrightarrow{(X \downarrow)} [X :] \xleftarrow{M \triangledown} A
  \]
  (see Definition 9.4.2 for \((X \downarrow))\).
- The module
  \[
  (\downarrow M) : X \to [\downarrow A]^\sim
  \]
  is defined by the composition
  \[
  X \xrightarrow{\downarrow M} [\cdot A]^\sim \xrightarrow{(\cdot A)^\sim} [\downarrow A]^\sim
  \]
  (see Definition 9.4.2 for \((\cdot A))\).

**Remark 10.1.4.**

1. Given a right comma \( K : X \to * \) and an object \( a \in \|A\| \), an \((M \triangledown)\)-arrow \( \alpha : K \sim a \) is a cone

\[
\begin{array}{c}
\begin{array}{c}
\text{[K]} \\
\downarrow
\end{array} \\
\begin{array}{c}
\text{K} \\
\downarrow \alpha
\end{array} \\
\begin{array}{c}
\text{X} \\
\downarrow M
\end{array} \\
\begin{array}{c}
\text{A}
\end{array}
\end{array} \xrightarrow{\Delta} \text{[A]}
\]

from the right comma fibration \( K : [K] \to X \) to \( a \) along \( M \).
10. Extensions

- Given a left comma \( \mathbb{K} : \ast \to A \) and an object \( x \in \| X \| \), an \( \langle \mathbb{M} \rangle \)-arrow \( \alpha : x \Rightarrow \mathbb{K} \) is a cone

\[
\begin{array}{c}
\ast \\
\downarrow \quad \Delta
\end{array}
\begin{array}{c}
\mathbb{K}
\downarrow \quad \alpha
\end{array}
\begin{array}{c}
X
\downarrow \quad \mathbb{M}
\end{array}
\begin{array}{c}
A
\end{array}
\]

from \( x \) to the left comma fibration \( \mathbb{K} : [\mathbb{K}] \to A \) along \( \mathbb{M} \).

2.

- The module \( \langle \mathbb{M} \rangle : [X \downarrow] \to A \) is corepresented by the functor \( \langle \mathbb{M} \rangle : A \to [X \downarrow] \), the right comma exponential transpose of \( \mathbb{M} \) (see Definition 3.2.31); indeed, a module isomorphism

\[
\langle \mathbb{M} \rangle \cong [X \downarrow] \langle \mathbb{M} \rangle
\]

is obtained from the equivalence cell in Definition 9.4.8(1) by the pasting composition

\[
\begin{array}{c}
[X \downarrow] \cong (X) \to [X \downarrow] \langle \mathbb{M} \rangle \to A
\end{array}
\]

is obtained from the equivalence cell in Definition 9.4.8(1) by the pasting composition

\[
\begin{array}{c}
X \cong [A \uparrow] \to [A \downarrow] \langle \mathbb{M} \rangle \to A
\end{array}
\]

(see Remark 1.2.30).

- The module \( \langle \mathbb{M} \rangle : X \to [A \downarrow] \) is represented by the functor \( \langle \mathbb{M} \rangle : X \to [A \downarrow] \), the left comma exponential transpose of \( \mathbb{M} \) (see Definition 3.2.31); indeed, a module isomorphism

\[
\langle \mathbb{M} \rangle \cong [\mathbb{M}] \langle A \rangle
\]

is obtained from the equivalence cell in Definition 9.4.8(1) by the pasting composition

\[
\begin{array}{c}
X \cong [A \downarrow] \langle \mathbb{M} \rangle \to [A \downarrow] \langle A \rangle \to [A \downarrow] \to A
\end{array}
\]

(see Remark 1.2.30).

3.

- The modules \( \langle \mathbb{M} \rangle : [X :] \to A \) and \( \langle \mathbb{M} \rangle : [X \downarrow] \to A \) are equivalent. Indeed, a pair of equivalence cells

\[
\begin{array}{c}
[X :] \cong \mathbb{M} \quad [X \downarrow] \cong \mathbb{M}
\end{array}
\]

\[
\begin{array}{c}
\downarrow \quad 1
\end{array}
\begin{array}{c}
\downarrow \quad 1
\end{array}
\begin{array}{c}
\downarrow \quad 1
\end{array}
\begin{array}{c}
\downarrow \quad 1
\end{array}
\]

\[
\begin{array}{c}
[X :] \cong \mathbb{M} \quad [X :] \cong \mathbb{M}
\end{array}
\]

\[
\begin{array}{c}
\downarrow \quad 1
\end{array}
\begin{array}{c}
\downarrow \quad 1
\end{array}
\begin{array}{c}
\downarrow \quad 1
\end{array}
\begin{array}{c}
\downarrow \quad 1
\end{array}
\]
10. Extensions

quasi-inverse to each other is obtained from the pair of equivalence cells in Definition 9.4.8(2) by the pasting compositions

\[
\begin{align*}
[X:] & \xrightarrow{(X)} [X:] \xleftarrow{\mathcal{M}^\prime} A \\
\downarrow & \quad \downarrow 1 \quad \downarrow 1 \\
[X:] & \xrightarrow{(X)} [X:] \xleftarrow{\mathcal{M}^\prime} A
\end{align*}
\]

. The cell \( \langle M \rangle \downarrow \sim \langle M \rangle \) sends each conical cell

\[
\begin{tikzpicture}
\node (X) at (0,0) {$X$};
\node (M) at (0,-2) {$M$};
\node (A) at (2,0) {$A$};
\draw[->] (X) -- (M);
\draw[->] (M) -- (A);
\draw[->] (X) -- (A);
\end{tikzpicture}
\]

to its comma transpose, i.e. the cone

\[
\begin{tikzpicture}
\node (X) at (0,0) {$X$};
\node (M) at (0,-2) {$M$};
\node (A) at (2,0) {$A$};
\draw[->] (X) -- (M);
\draw[->] (M) -- (A);
\draw[->] (X) -- (A);
\end{tikzpicture}
\]

(cf. Remark 9.4.9(3)).

The modules \( \langle \mathcal{M}, \mathcal{M} \rangle : X \rightarrow [: A]^\sim \) and \( \langle \mathcal{M}, \mathcal{M} \rangle : X \rightarrow [\downarrow A]^\sim \) are equivalent. Indeed, a pair of equivalence cells

\[
\begin{align*}
X & \xrightarrow{\mathcal{M}} [: A]^\sim \quad X & \xrightarrow{\mathcal{M}} [\downarrow A]^\sim \\
\downarrow 1 & \quad \downarrow 1 \\
X & \xrightarrow{\mathcal{M}} [: A]^\sim \quad X & \xrightarrow{\mathcal{M}} [\downarrow A]^\sim
\end{align*}
\]

quasi-inverse to each other is obtained from the pair of equivalence cells in Definition 9.4.8(2) by the pasting compositions

\[
\begin{align*}
X & \xrightarrow{\mathcal{M}} [: A]^\sim - \xrightarrow{(A)^\sim} [: A]^\sim \\
\downarrow 1 & \quad \downarrow 1 \\
X & \xrightarrow{\mathcal{M}} [: A]^\sim - \xrightarrow{(A)^\sim} [\downarrow A]^\sim
\end{align*}
\]

. The cell \( \langle \mathcal{M}, \mathcal{M} \rangle \downarrow \sim \langle \mathcal{M}, \mathcal{M} \rangle \) sends each conical cell

\[
\begin{tikzpicture}
\node (X) at (0,0) {$X$};
\node (M) at (0,-2) {$M$};
\node (A) at (2,0) {$A$};
\draw[->] (X) -- (M);
\draw[->] (M) -- (A);
\draw[->] (X) -- (A);
\end{tikzpicture}
\]

to its comma transpose, i.e. the cone

\[
\begin{tikzpicture}
\node (X) at (0,0) {$X$};
\node (M) at (0,-2) {$M$};
\node (A) at (2,0) {$A$};
\draw[->] (X) -- (M);
\draw[->] (M) -- (A);
\draw[->] (X) -- (A);
\end{tikzpicture}
\]

(cf. Remark 9.4.9(3)).
10. Extensions

4. Replacing \( \mathcal{M} : X \rightarrow A \) in (3) above with the hom of a category \( C \), we have an equivalence of modules

\[
\langle \langle C \rangle \rangle \simeq \langle \langle C \rangle \rangle
\]

given by

\[
\begin{array}{ccc}
[C :] - & \xrightarrow{\langle C \rangle} & C \\
\downarrow & & \downarrow \\
[C] - & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

This equivalence is called the coYoneda lemma in [Ma].

**Proposition 10.1.5.**

- The equivalence cell \( \langle \mathcal{M} \rangle \xrightarrow{\dashv} \langle \mathcal{M} \rangle \) in Remark 10.1.4(3) preserves, reflects and creates direct universals.

- The equivalence cell \( \langle \mathcal{M} \rangle \xrightarrow{\dashv} \langle \mathcal{M} \rangle \) in Remark 10.1.4(3) preserves, reflects and creates inverse universals.

**Proof.** Immediate from Proposition 6.2.16. \( \square \)

**Definition 10.1.6.** Let \( E \) be a category and \( \mathcal{M} : X \rightarrow A \) be a module.

- The corepresentable module of the functor \( \langle \mathcal{M} \rangle E : [E, A] \rightarrow [X : E] \), the right action of \( \mathcal{M} \) on \( [E, A] \), is denoted by \( \mathcal{M} \rangle E \); that is the module

\[
\langle \mathcal{M} \rangle E : [X : E] \rightarrow [E, A]
\]

is defined by the composition

\[
[X : E] \xrightarrow{\langle X, E \rangle} [X : E] \xrightarrow{\mathcal{M} \rangle E} [E, A]
\]

- The representable module of the functor \( \langle E \rangle \mathcal{M} : [E, X] \rightarrow [E : A] \), the left action of \( \mathcal{M} \) on \( [E, X] \), is denoted by \( E \langle \mathcal{M} \rangle \); that is the module

\[
\langle E \rangle \mathcal{M} : [E, X] \rightarrow [E : A]
\]

is defined by the composition

\[
[E, X] \xrightarrow{E \langle \mathcal{M} \rangle} [E : A] \xrightarrow{\langle E, A \rangle} [E : A]
\]

**Remark 10.1.7.**

1.
10. Extensions

- Given a module \( \mathcal{J} : X \to E \) and a functor \( K : E \to A \), an \( \langle \mathcal{M} \triangleright E \rangle \)-arrow \( \Phi : \mathcal{J} \triangleright K \) is a module morphism \( \Phi : \mathcal{J} \to \langle \mathcal{M} \rangle K : X \to E \), i.e. a cell

\[
\begin{array}{c}
X \xrightarrow{\mathcal{J}} E \\
\downarrow \Phi \quad \downarrow \mathcal{K} \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

This cell is denoted by \( \Phi : \mathcal{J} \triangleright K : \mathcal{M} \triangleright E \), depicted also as

\[
\begin{array}{c}
& \mathcal{E} \\
\mathcal{J} & \xrightarrow{\Phi} \xleftarrow{\mathcal{K}} \mathcal{A} \\
\mathcal{X} & \xrightarrow{\mathcal{M}} \xleftarrow{\mathcal{A}} \mathcal{A}
\end{array}
\]

and called a cell from \( \mathcal{J} \) to \( K \) along \( \mathcal{M} \).

- Given a module \( \mathcal{J} : E \to A \) and a functor \( K : E \to X \), an \( \langle E \triangleleft \mathcal{M} \rangle \)-arrow \( \Phi : K \triangleright \mathcal{J} \) is a module morphism \( \Phi : \mathcal{J} \to K \langle \mathcal{M} \rangle : E \to A \), i.e. a cell

\[
\begin{array}{c}
E \xrightarrow{\mathcal{J}} A \\
\downarrow \Phi \quad \downarrow \mathcal{1} \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

This cell is denoted by \( \Phi : K \triangleright \mathcal{J} : \mathcal{E} \triangleleft \mathcal{M} \), depicted also as

\[
\begin{array}{c}
& \mathcal{E} \\
\mathcal{K} & \xleftarrow{\Phi} \xrightarrow{\mathcal{J}} \mathcal{A} \\
\mathcal{X} & \xleftarrow{\mathcal{M}} \xrightarrow{\mathcal{A}} \mathcal{A}
\end{array}
\]

and called a cell from \( K \) to \( \mathcal{J} \) along \( \mathcal{M} \).

2. A cell

\[
\begin{array}{c}
X \xrightarrow{\mathcal{M}} A \\
\downarrow \Phi \quad \downarrow \mathcal{Q} \\
Y \xrightarrow{\mathcal{N}} B
\end{array}
\]

from \( P \) to \( Q \) along \( \langle \mathcal{M}, \mathcal{N} \rangle \), i.e. a module morphism \( \Phi : \mathcal{M} \to \mathcal{P} \langle \mathcal{N} \rangle Q : X \to A \), can be also depicted as

\[
\begin{array}{c}
& \mathcal{A} \\
\mathcal{M} & \xleftarrow{\Phi} \xrightarrow{\mathcal{Q}} \mathcal{B} \\
\mathcal{X} & \xleftarrow{\mathcal{P} \langle \mathcal{N} \rangle} \xrightarrow{\mathcal{B}} \mathcal{B}
\end{array}
\]

from \( \mathcal{M} \) to \( Q \) along \( \mathcal{P} \langle \mathcal{N} \rangle \).

292
10. Extensions

- a cell

```
\begin{align*}
\begin{tikzcd}
X & \ar[ld, \Phi] \ar[rd, \Psi] & M \\
Y \ar[rru, \langle N \rangle Q] & & A
\end{tikzcd}
\end{align*}
```

from \( P \) to \( M \) along \( \langle N \rangle Q \).

3.

- The right exponential transpose of a cell

```
\begin{align*}
\begin{tikzcd}
E & \ar[ld, \Phi] \ar[rd, \Psi] & K \\
\mathcal{J} \ar[rru, \Phi] & & A
\end{tikzcd}
\end{align*}
```

, i.e. a module morphism \( \Phi : \mathcal{J} \to \langle M \rangle K : X \to E \), yields the natural transformation

```
\begin{align*}
\begin{tikzcd}
\mathcal{J} & \ar[ld, \Phi] \ar[rd, \Psi] & K \\
\mathcal{M} \ar[rru, \Phi] & & A
\end{tikzcd}
\end{align*}
```

from \( \mathcal{J} \) to \( \langle M \rangle \) \( \delta K \) (see Proposition 2.1.5), i.e. the cylinder

```
\begin{align*}
\begin{tikzcd}
E & \ar[ld, \Phi] \ar[rd, \Psi] & K \\
\mathcal{J} \ar[rru, \Phi] & & A
\end{tikzcd}
\end{align*}
```

. The slice of \( \Phi \) at \( e \in \| E \| \) is the conical cell

```
\begin{align*}
\begin{tikzcd}
\mathcal{J} e & \ar[ld, \Phi e] \ar[rd, \Psi e] & K : e \\
X \ar[rru, \Phi e] & & A
\end{tikzcd}
\end{align*}
```

given by the component of the cylinder \( \Phi e \) at \( e \). The transposition \( \Phi \mapsto \Phi e \) yields the iso cell

```
\begin{align*}
\begin{tikzcd}
[X : E] \ar[rr, \langle M, \mathcal{E} \rangle] & & [E, A] \\
[E, [X :]] \ar[u, \langle E, \langle M \rangle \rangle] \ar[d, 1] & & [E, A] \ar[u, \langle M, \mathcal{E} \rangle]
\end{tikzcd}
\end{align*}
```

giving a canonical isomorphism

```
\langle M, \mathcal{E} \rangle \cong \langle E, \langle M \rangle \rangle
```

.
10. Extensions

- The left exponential transpose of a cell

\[
\begin{array}{c}
\vspace{1cm}
\end{array}
\]

, i.e. a module morphism \( \Phi : \mathcal{J} \rightarrow K(\mathcal{M}) : E \rightarrow A \), yields the natural transformation

\[
\begin{array}{c}
\vspace{1cm}
\end{array}
\]

from \( K \circ [ \cdot, \mathcal{M} ] \) to \( \cdot, \mathcal{J} \) (see Proposition 2.1.5), i.e. the cylinder

\[
\begin{array}{c}
\vspace{1cm}
\end{array}
\]

. The slice of \( \Phi \) at \( e \in \|E\| \) is the conical cell

\[
\begin{array}{c}
\vspace{1cm}
\end{array}
\]

given by the component of the cylinder \( \cdot, \Phi \) at \( e \). The transposition \( \Phi \mapsto \cdot, \Phi \) yields the iso cell

\[
\begin{array}{c}
\vspace{1cm}
\end{array}
\]

giving a canonical isomorphism

\[ \langle E \otimes \mathcal{M} \rangle \cong \langle E, \langle \cdot, \mathcal{M} \rangle \rangle \]

4. For any category \( E \) and any module \( \mathcal{M} \),

\[ \langle \mathcal{M} \triangleleft E \rangle \cong \langle E, \langle \cdot, \mathcal{M} \rangle \rangle \]

10.2. Universal cells

Note. Definition 10.2.1 and Definition 10.2.3 give two definitions of universal conical cells. We will see that they are equivalent in Proposition 10.2.5.
Definition 10.2.1.

- A conical cell

\[
\begin{array}{c}
\text{J} \\
\text{X} \\
\text{M} \\
\text{A}
\end{array}
\]

is called universal if it is a direct universal \((\mathcal{M},\mathcal{J})\)-arrow (see Definition 10.1.1). Given a right module \(\mathcal{J} : \text{X} \to \ast\), a universal conical cell \(\mathcal{Y} : \mathcal{J} \to r\) or the pair \((r,\mathcal{Y})\), or the object \(r\) itself, is called a limit of \(\mathcal{J}\) direct along \(\mathcal{M}\).

- A conical cell

\[
\begin{array}{c}
r \\
\ast \\
\text{X} \\
\text{M} \\
\text{A}
\end{array}
\]

is called universal if it is an inverse universal \((\mathcal{M},\mathcal{J})\)-arrow (see Definition 10.1.1). Given a left module \(\mathcal{J} : \ast \to \text{A}\), a universal conical cell \(\mathcal{Y} : r \to \mathcal{J}\) or the pair \((r,\mathcal{Y})\), or the object \(r\) itself, is called a limit of \(\mathcal{J}\) inverse along \(\mathcal{M}\).

Remark 10.2.2.

- A conical cell \(\Phi : \mathcal{J} \to r : \mathcal{M} \mathcal{J}\) is universal if and only if to every conical cell \(\Phi : \mathcal{J} \to a : \mathcal{M} \mathcal{J}\) there is a unique \(A\)-arrow \(\mathcal{T}\Phi : r \to a\) such that \(\Phi = \mathcal{T}\Phi \circ \Phi\).

- A conical cell \(\Phi : r \to \mathcal{J} : \mathcal{M} \mathcal{J}\) is universal if and only if to every conical cell \(\Phi : x \to \mathcal{J} : \mathcal{M} \mathcal{J}\) there is a unique \(X\)-arrow \(\Phi\mathcal{T} : x \to r\) such that \(\Phi = \Phi\mathcal{T} \circ \mathcal{T}\Phi\).

Definition 10.2.3.

- A conical cell

\[
\begin{array}{c}
\text{E} \\
\text{K} \\
\text{X} \\
\text{M} \\
\text{A}
\end{array}
\]

is called universal if it is a direct universal \((\mathcal{J},\mathcal{M})\)-arrow (see Definition 1.2.7). Given a functor \(K : \text{E} \to \text{X}\), a universal conical cell \(\mathcal{Y} : K \to r : \mathcal{J} \to \mathcal{M}\) or the pair \((r,\mathcal{Y})\), or the object \(r\) itself, is called a limit of \(K\) direct along \((\mathcal{J},\mathcal{M})\) or \(\mathcal{J}\)-weighted limit of \(K\) direct along \(\mathcal{M}\).

- A conical cell

\[
\begin{array}{c}
\text{E} \\
\text{K} \\
\text{X} \\
\text{M} \\
\text{A}
\end{array}
\]

is called universal if it is an inverse universal \((\mathcal{J},\mathcal{M})\)-arrow (see Definition 1.2.7). Given a functor \(K : \text{E} \to \text{A}\), a universal conical cell \(\mathcal{Y} : r \to K : \mathcal{J} \to \mathcal{M}\) or the pair \((r,\mathcal{Y})\), or the object \(r\) itself, is called a limit of \(K\) inverse along \((\mathcal{J},\mathcal{M})\) or \(\mathcal{J}\)-weighted limit of \(K\) inverse along \(\mathcal{M}\).
Remark 10.2.4.

- A conical cell $\Phi : K \to r : J \to M$ is universal if and only if to every conical cell $\Phi : K \to a : J \to M$ there is a unique $A$-arrow $\Upsilon/\Phi : r \to a$ such that $\Phi = \Upsilon \circ \Upsilon/\Phi$.

- A conical cell $\Phi : r \to K : J \to M$ is universal if and only if to every conical cell $\Phi : x \to K : J \to M$ there is a unique $X$-arrow $\Phi/\Upsilon : x \to r$ such that $\Phi = \Phi/\Upsilon \circ \Upsilon$.

Proposition 10.2.5.

- A conical cell

\[
\begin{array}{ccc}
E & \to & J \\
K & \downarrow & \Upsilon \\
X & \downarrow & M \\
& & A
\end{array}
\]

from $K$ to $r$ along $(J, M)$ is universal if and only if the conical cell

\[
\begin{array}{ccc}
E & \longrightarrow & J \\
& \Upsilon/\Phi \downarrow & r \\
X & \downarrow & M \\
& & A
\end{array}
\]

from $J$ to $r$ along $K(M)$ is universal; hence getting a $J$-weighted limit of $K$ direct along $M$ is the same thing as getting a limit of $J$ direct along $K(M)$. Conversely, a conical cell

\[
\begin{array}{ccc}
J & \longrightarrow & r \\
& \Upsilon \downarrow & A \\
X & \downarrow & M \\
& & A
\end{array}
\]

from $J$ to $r$ along $M$ is universal if and only if the conical cell

\[
\begin{array}{ccc}
X & \longrightarrow & J \\
1 & \downarrow & \Upsilon \\
X & \downarrow & M \\
& & A
\end{array}
\]

from the identity $X \to X$ to $r$ along $(J, M)$ is universal; hence getting a limit of $J$ direct along $M$ is the same thing as getting a $J$-weighted limit of the identity $X \to X$ direct along $M$.

- A conical cell

\[
\begin{array}{ccc}
* & \to & E \\
& \Upsilon \to & K \\
X & \downarrow & M \\
& & A
\end{array}
\]

from $r$ to $K$ along $(J, M)$ is universal if and only if the conical cell

\[
\begin{array}{ccc}
X & \longrightarrow & (M)K \\
& \Upsilon \downarrow & M \\
& & A
\end{array}
\]

from $X$ to $(M)K$ is universal.
10. Extensions

from \(r\) to \(J\) along \(\langle M \rangle K\) is universal; hence getting a \(J\)-weighted limit of \(K\) inverse along \(M\) is the same thing as getting a limit of \(J\) inverse along \(\langle M \rangle K\). Conversely, a conical cell

\[
\begin{array}{c}
\text{x} \\
\downarrow \gamma \\
\downarrow \downarrow \downarrow \downarrow \\
M \\
\downarrow \gamma \\
\downarrow \downarrow \downarrow \downarrow \\
\text{A}
\end{array}
\]

from \(r\) to \(J\) along \(M\) is universal if and only if the conical cell

\[
\begin{array}{c}
\ast \\
\downarrow \gamma \\
\downarrow \downarrow \downarrow \downarrow \\
1 \\
\downarrow \gamma \\
\downarrow \downarrow \downarrow \downarrow \\
\text{A}
\end{array}
\]

from \(r\) to the identity \(A \to A\) along \(\langle J, M \rangle\) is universal; hence getting a limit of \(J\) inverse along \(M\) is the same thing as getting a \(J\)-weighted limit of the identity \(A \to A\) inverse along \(M\).

Proof. The left slice of the module \(\langle J, M \rangle\) at \(K\) and the left slice of the module \(\langle K(M) \rangle \mathcal{G}\) at \(J\) are the same left module over \(A\) given by

\[
a \mapsto (J) \langle E : (K(M) a) \rangle
\]

Hence \(\Upsilon : K \Rightarrow r\) is a direct universal \(\langle J, M \rangle\)-arrow iff \(\Upsilon : J \Rightarrow r\) is a direct universal \(\langle (K(M)) \mathcal{G} \rangle\)-arrow. The second assertion is verified similarly. The left slice of the module \(M \mathcal{G}\) at \(J\) and the left slice of the module \(\langle J, M \rangle\) at \(1_X\) are the same left module over \(A\) given by

\[
a \mapsto (J) \langle X : (M a) \rangle
\]

Hence \(\Upsilon : J \Rightarrow r\) is a direct universal \(\langle M \mathcal{G} \rangle\)-arrow iff \(\Upsilon : 1_X \Rightarrow r\) is a direct universal \(\langle J, M \rangle\)-arrow.

Remark 10.2.6.

1. Limits subsume weighted limits. Indeed, by Proposition 10.1.5,

- a conical cell

\[
\begin{array}{c}
\text{x} \\
\downarrow \gamma \\
\downarrow \downarrow \downarrow \downarrow \\
M \\
\downarrow \gamma \\
\downarrow \downarrow \downarrow \downarrow \\
\text{A}
\end{array}
\]

is universal if and only if its comma transpose

\[
\begin{array}{c}
\text{X} \\
\downarrow \gamma_i \\
\downarrow \downarrow \downarrow \downarrow \\
\downarrow \gamma_i \\
\downarrow \downarrow \downarrow \downarrow \\
\text{A}
\end{array}
\]

is a universal cone; that is, getting a limit of \(J\) direct along \(M\) is the same thing as getting a limit of \(J^\dagger\) direct along \(M\).
10. Extensions

- a conical cell

\[ \begin{array}{ccc} & \ast & \\
\downarrow & \gamma & \downarrow \\
X & \longrightarrow_{\mathcal{M}} & A \\
\end{array} \]

is universal if and only if its comma transpose

\[ \begin{array}{ccc} \ast & \leftarrow & \Delta [\mathcal{J}^i] \\
\downarrow & & \downarrow \\
X & \rightarrow_{\mathcal{M}} & A \\
\end{array} \]

is a universal cone; that is, getting a limit of \( \mathcal{J} \) inverse along \( \mathcal{M} \) is the same thing as getting a limit of \( \mathcal{J}^i \) inverse along \( \mathcal{M} \).

2. The converse is also the case: weighted limits subsume limits. Indeed, by the isomorphism in Corollary 5.5.5,

- a cone

\[ \begin{array}{ccc} E & \rightarrow & \ast \\
\downarrow & \mu & \downarrow \\
X & \rightarrow_{\mathcal{M}} & A \\
\end{array} \]

is universal if and only if the conical cell

\[ \begin{array}{ccc} E & \rightarrow & \ast \leftrightarrow \ast \\
\downarrow & \mu^{-1} & \downarrow \\
X & \rightarrow_{\mathcal{M}} & A \\
\end{array} \]

is universal; that is, getting a limit of \( K \) direct along \( \mathcal{M} \) is the same thing as getting a \( \langle E^* \rangle \)-weighted limit of \( K \) direct along \( \mathcal{M} \).

- a cone

\[ \begin{array}{ccc} \ast & \leftarrow & E \\
\downarrow & \mu & \downarrow \\
X & \rightarrow_{\mathcal{M}} & A \\
\end{array} \]

is universal if and only if the conical cell

\[ \begin{array}{ccc} \ast & \leftarrow & \ast E \\
\downarrow & \ast \mu & \downarrow \\
X & \rightarrow_{\mathcal{M}} & A \\
\end{array} \]

is universal; that is, getting a limit of \( K \) inverse along \( \mathcal{M} \) is the same thing as getting a \( \langle \ast E \rangle \)-weighted limit of \( K \) inverse along \( \mathcal{M} \).

Note. Definition 10.2.7 and Definition 10.2.9 give two definitions of universal cells. We will see that they are equivalent in Proposition 10.2.11.
10. Extensions

**Definition 10.2.7.**

- A cell

\[
\begin{array}{c}
X \\
\downarrow M \\
\mapright{\gamma} \\
A \\
\end{array}
\]

along \( M \) is called

1. universal if it is an inverse universal \((M \not\rightarrow E)\)-arrow (see Definition 10.1.6);
2. pointwise universal if the right slice

\[
\begin{array}{c}
X \\
\downarrow M \\
\mapright{\gamma} \\
A \\
\end{array}
\]

(see Remark 10.1.7(3)) of \( \gamma \) at each \( e \in \|E\| \) is a universal conical cell, i.e. a direct universal \((M \not\rightarrow)\)-arrow.

Given a module \( J : X \rightarrow E \), a universal (resp. pointwise universal) cell \( \gamma : J \sim R : M \not\rightarrow E \) or the pair \((R, \gamma)\), or the functor \( R \) itself, is called an extension (resp. pointwise extension) of \( J \) direct along \( M \).

- A cell

\[
\begin{array}{c}
X \\
\downarrow M \\
\mapright{\gamma} \\
A \\
\end{array}
\]

along \( M \) is called

1. universal if it is an inverse universal \((E \not\rightarrow M)\)-arrow (see Definition 10.1.6);
2. pointwise universal if the left slice

\[
\begin{array}{c}
X \\
\downarrow M \\
\mapright{\gamma} \\
A \\
\end{array}
\]

(see Remark 10.1.7(3)) of \( \gamma \) at each \( e \in \|E\| \) is a universal conical cell, i.e. an inverse universal \((\not\rightarrow M)\)-arrow.

Given a module \( J : E \rightarrow A \), a universal (resp. pointwise universal) cell \( \gamma : R \sim J : E \not\rightarrow M \) or the pair \((R, \gamma)\), or the functor \( R \) itself, is called an extension (resp. pointwise extension) of \( J \) inverse along \( M \).

**Remark 10.2.8.**

- A cell \( \Phi : J \sim R : M \not\rightarrow E \) is universal if and only if to every cell \( \Phi : J \sim K : M \not\rightarrow E \) there is a unique natural transformation \( \gamma \downarrow \Phi : R \rightarrow K \) such that \( \Phi = \gamma \downarrow \gamma \downarrow \Phi \).

- A cell \( \Phi : R \sim J : E \not\rightarrow M \) is universal if and only if to every cell \( \Phi : K \sim J : E \not\rightarrow M \) there is a unique natural transformation \( \Phi / \gamma : K \rightarrow R \) such that \( \Phi = \Phi / \gamma \downarrow \gamma \).
10. Extensions

Definition 10.2.9.

- A cell

\[
\begin{array}{c}
E \xrightarrow{\gamma} D \\
\downarrow \gamma \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

along \((\mathcal{J}, \mathcal{M})\) is called

1. direct universal if it is a direct universal \((\mathcal{J}, \mathcal{M})\)-arrow (see Definition 1.2.7);
2. pointwise direct universal if the right slice

\[
\begin{array}{c}
E \xrightarrow{(\mathcal{J})^d} * \\
\downarrow (\gamma)^d \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

(see Definition 2.1.7) of \(\gamma\) at each \(d \in \|D\|\) is a universal conical cell, i.e. a direct universal \((d, \mathcal{M})\)-arrow.

Given a functor \(K: E \to X\), a direct universal (resp. pointwise direct universal) cell

\[
\begin{array}{c}
E \xrightarrow{\gamma} D \\
\downarrow \gamma \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

along \((\mathcal{J}, \mathcal{M})\) is called

1. inverse universal if it is an inverse universal \((\mathcal{J}, \mathcal{M})\)-arrow (see Definition 1.2.7);
2. pointwise inverse universal if the left slice

\[
\begin{array}{c}
* \xrightarrow{d(\mathcal{J})} E \\
\downarrow d(\gamma) \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

(see Definition 2.1.7) of \(\gamma\) at each \(d \in \|D\|\) is a universal conical cell, i.e. an inverse universal \((d, \mathcal{M})\)-arrow.

Given a functor \(K: E \to A\), an inverse universal (resp. pointwise inverse universal) cell

\[
\begin{array}{c}
E \xrightarrow{\gamma} D \\
\downarrow \gamma \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

along \((\mathcal{J}, \mathcal{M})\) is called

1. inverse universal if it is an inverse universal \((\mathcal{J}, \mathcal{M})\)-arrow (see Definition 1.2.7);
2. pointwise inverse universal if the left slice

\[
\begin{array}{c}
* \xrightarrow{d(\mathcal{J})} E \\
\downarrow d(\gamma) \\
X \xrightarrow{\mathcal{M}} A
\end{array}
\]

(see Definition 2.1.7) of \(\gamma\) at each \(d \in \|D\|\) is a universal conical cell, i.e. an inverse universal \((d, \mathcal{M})\)-arrow.

Remark 10.2.10.

- A cell \(\Phi: K \to R: \mathcal{J} \to \mathcal{M}\) is inverse universal if and only if to every cell \(\Phi: K \to L: \mathcal{J} \to \mathcal{M}\) there is a unique natural transformation \(\gamma\) such that \(\Phi = \gamma \circ \Phi\).
A cell $\Phi : R \rightsquigarrow K : J \to M$ is direct universal if and only if to every cell $\Phi : L \rightsquigarrow K : J \to M$ there is a unique natural transformation $\Phi/\Upsilon : L \to R$ such that $\Phi = \Phi/\Upsilon \circ \Upsilon$.

**Proposition 10.2.11.**

- A cell

$$
\begin{array}{c}
E \xrightarrow{J} D \\
K \downarrow \Upsilon \downarrow R \\
\downarrow \downarrow \\
\downarrow \downarrow \\
X \xrightarrow{M} A
\end{array}
$$

from $K$ to $R$ along $(J, M)$ is direct universal (resp. pointwise direct universal) if and only if the cell

$$
\begin{array}{c}
D \\
J \xrightarrow{\Upsilon} R \\
\downarrow \\
\downarrow \\
E \xrightarrow{\Upsilon_{(M)}} A
\end{array}
$$

from $J$ to $R$ along $K(M)$ is universal (resp. pointwise universal); hence getting an extension (resp. pointwise extension) of $K$ direct along $(J, M)$ is the same thing as getting an extension (resp. pointwise extension) of $J$ direct along $K(M)$. Conversely, a cell

$$
\begin{array}{c}
E \\
J \xrightarrow{\Upsilon} R \\
\downarrow \\
\downarrow \\
X \xrightarrow{M} A
\end{array}
$$

from $J$ to $R$ along $M$ is universal (resp. pointwise universal) if and only if the cell

$$
\begin{array}{c}
X \xrightarrow{J} E \\
I \downarrow \Upsilon \downarrow R \\
\downarrow \downarrow \\
\downarrow \downarrow \\
X \xrightarrow{M} A
\end{array}
$$

from the identity $X \to X$ to $R$ along $(J, M)$ is direct universal (resp. pointwise direct universal); hence getting an extension (resp. pointwise extension) of $J$ direct along $M$ is the same thing as getting an extension (resp. pointwise extension) of the identity $X \to X$ direct along $(J, M)$.

- A cell

$$
\begin{array}{c}
D \xrightarrow{J} E \\
R \downarrow \Upsilon \downarrow K \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\downarrow \downarrow \\
X \xrightarrow{M} A
\end{array}
$$

from $R$ to $K$ along $(J, M)$ is inverse universal (resp. pointwise inverse universal) if and only if the cell

$$
\begin{array}{c}
D \\
R \xrightarrow{\Upsilon} J \\
\downarrow \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\downarrow \downarrow \\
X \xrightarrow{\Upsilon_{(M)K}} A
\end{array}
$$
from $R$ to $J$ along $(M)K$ is universal (resp. pointwise universal); hence getting an extension (resp. pointwise extension) of $K$ inverse along $(J,M)$ is the same thing as getting an extension (resp. pointwise extension) of $J$ inverse along $(M)K$. Conversely, a cell

$$
\begin{array}{c}
\text{E} \\
\text{X} \overset{\gamma}{\longrightarrow} M \overset{\text{R}}{\longrightarrow} A
\end{array}
$$

from $R$ to $J$ along $M$ is universal (resp. pointwise universal) if and only if the cell

$$
\begin{array}{c}
\text{E} \overset{J}{\longrightarrow} A \\
\text{X} \overset{\gamma}{\longrightarrow} M \overset{\text{1}}{\longrightarrow} A
\end{array}
$$

from $R$ to the identity $A \to A$ along $(J,M)$ is inverse universal (resp. pointwise inverse universal); hence getting an extension (resp. pointwise extension) of $J$ inverse along $M$ is the same thing as getting an extension (resp. pointwise extension) of the identity $A \to A$ inverse along $(J,M)$.

**Proof.** The left slice of the module $(J,M)$ at $K$ and the left slice of the module $(K(M))\not\sim D$ at $J$ are the same left module over $[D,A]$ given by

$$
R \mapsto (J)\{E:D\}(K(M)R)
$$

Hence $\Upsilon : K \sim R$ is a direct universal $(J,M)$-arrow iff $\Upsilon : J \sim R$ is a direct universal $(K(M))\not\sim D$-arrow. By Proposition 10.2.5, the slice

$$
\begin{array}{c}
\text{E} \overset{(J)d}{\longrightarrow} \ast \\
\text{K} \overset{(\Upsilon)d}{\longrightarrow} M \overset{R:d}{\longrightarrow} A
\end{array}
$$

of $\Upsilon : K \sim R : J \to M$ at $d \in \| D \|$ is universal iff the slice

$$
\begin{array}{c}
\text{E} \overset{(J)d}{\longrightarrow} \ast \\
\text{K(M)} \overset{(\Upsilon)d}{\longrightarrow} A
\end{array}
$$

of $\Upsilon : J \sim R : (K(M))\not\sim D$ at $d \in \| D \|$ is universal. The second assertion is verified similarly and more easily. $\square$

**Proposition 10.2.12.**

- A cell

$$
\begin{array}{c}
\text{E} \\
\text{X} \overset{\gamma}{\longrightarrow} M \overset{\text{R}}{\longrightarrow} A
\end{array}
$$
10. Extensions

is universal (resp. pointwise universal) if and only if its right exponential transpose

\[
\begin{array}{c}
\mathbf{J} \\
\mathbf{E} \\
\mathbf{A}
\end{array}
\]

forms a left Kan lift (resp. pointwise left Kan lift) of \( \mathbf{J} \) along \( \mathbf{M} \); that is, if and only if the cylinder

\[
\begin{array}{c}
\mathbf{J} \\
\mathbf{E} \\
\mathbf{A}
\end{array}
\]

is direct universal (resp. pointwise direct universal).

\[ A \text{ cell} \]

\[
\begin{array}{c}
\mathbf{X} \\
\mathbf{E} \\
\mathbf{A}
\end{array}
\]

is universal (resp. pointwise universal) if and only if its left exponential transpose

\[
\begin{array}{c}
\mathbf{X} \\
\mathbf{E} \\
\mathbf{[A]}^\sim
\end{array}
\]

forms a right Kan lift (resp. pointwise right Kan lift) of \( \mathbf{J} \) along \( \mathbf{M} \); that is, if and only if the cylinder

\[
\begin{array}{c}
\mathbf{X} \\
\mathbf{E} \\
\mathbf{[A]}^\sim
\end{array}
\]

is inverse universal (resp. pointwise inverse universal).

Proof. By the isomorphisms in Remark 10.1.7(3), \( \Upsilon \) is universal iff the cylinder \( \Upsilon \mathbf{J} \) is direct universal. Since the slice of \( \Upsilon \) at each \( e \in |E| \) is given by the component of \( \Upsilon \mathbf{J} \) at \( e \), \( \Upsilon \) is pointwise universal iff the cylinder \( \Upsilon \mathbf{J} \) is pointwise direct universal.

Remark 10.2.13. Kan lifts (resp. pointwise Kan lifts) thus subsume extensions (resp. pointwise extensions). We will see in Corollary 10.2.21 that the converse is also the case.

Proposition 10.2.14. If a cell is pointwise universal, then it is universal.

Proof. The assertion is reduced to Proposition 6.4.5 by Proposition 10.2.12.

Theorem 10.2.15.
10. Extensions

- Consider a pair of modules

\[ \begin{array}{ccc}
E & \sim & J \\
\downarrow & & \downarrow \\
X & \sim & M \\
\downarrow & & \downarrow \\
A & \sim & E
\end{array} \]

. If \( X \) is small and \( M \) is direct complete, then there is a pointwise extension of \( J \) direct along \( M \).

- Consider a pair of modules

\[ \begin{array}{ccc}
X & \sim & M \\
\downarrow & & \downarrow \\
A & \sim & J \\
\downarrow & & \downarrow \\
E & \sim & \end{array} \]

. If \( A \) is small and \( M \) is inverse complete, then there is a pointwise extension of \( J \) inverse along \( M \).

**Proof.** By Proposition 10.2.12, getting a pointwise extension of \( J \) direct along \( M \) is getting a pointwise lift of \( J \) \( \downarrow \) direct along \( M \). Hence, by Corollary 6.4.8, it suffices to show that for every \( e \in \parallel E \parallel \) the right module \( (J) e \) has a limit direct along \( M \), or, equivalently (see Remark 10.2.6(1)), the right comma fibration \([J \downarrow e] : [J \downarrow e] \to X\) has a limit direct along \( M \). But this is the case since \( M \) is direct complete and the smallness of \( X \) implies the smallness of the category \([J \downarrow e]\). \( \square \)

**Theorem 10.2.16.**

- Given a conical cell

\[ \begin{array}{ccc}
M & \sim & * \\
\downarrow & & \downarrow \\
X & \sim & X \\
\downarrow & & \downarrow \\
(X) & \sim & \end{array} \]

along the hom of a category \( X \), the following conditions are equivalent:

1. the right module morphism \( \Upsilon : M \to (X) \) is iso; that is, the pair \((r, \Upsilon)\) forms a representation of \( M \);
2. \( \Upsilon \) is a universal conical cell and \( M \) preserves direct limits.

Hence, if a right module \( M : X \to * \) preserves direct limits, getting a representation of \( M \) is the same thing as getting a limit of \( M \) direct along \( (X) \).

- Given a conical cell

\[ \begin{array}{ccc}
r & \sim & * \\
\downarrow & & \downarrow \\
A & \sim & A \\
\downarrow & & \downarrow \\
(A) & \sim & \end{array} \]

along the hom of a category \( A \), the following conditions are equivalent:

1. the left module morphism \( \Upsilon : M \to r(A) \) is iso; that is, the pair \((r, \Upsilon)\) forms a representation of \( M \);
2. \( \Upsilon \) is a universal conical cell and \( M \) preserves inverse limits.

Hence, if a left module \( M : * \to A \) preserves inverse limits, getting a representation of \( M \) is the same thing as getting a limit of \( M \) inverse along \( (A) \).

**Proof.**
10. Extensions

(1) ⇒ (2) \( \mathcal{M} \) preserves direct limits by Corollary 7.4.8. Since the Yoneda functor \( \mathbf{X} \Rightarrow \) is fully faithful, the universality of \( \Upsilon \) follows from Corollary 6.2.12.

(2) ⇒ (1) By Remark 10.2.6(1), the conical cell \( \Upsilon \) is universal iff the cone

\[
\begin{array}{c}
\Delta \\
\mathcal{M} \\
\Upsilon \\
X \\
(\mathbf{X}) \\
X
\end{array}
\]

is universal. Hence \( \mathcal{M} \) is representable by Theorem 8.6.15; that is, there is an isomorphism \( \Upsilon' : \mathcal{M} \to (\mathbf{X}) \Rightarrow \). Since \( \Upsilon' \) is universal by what we have just seen, \( \Upsilon : \mathcal{M} \to (\mathbf{X}) \Rightarrow \) is iso by Corollary 6.2.7.

\[\square\]

Remark 10.2.17. A representation is always a limit. The converse does not hold in general. For example, consider a conical cell

\[
\begin{array}{c}
* \\
\Delta \\
\{0,1\} \\
X
\end{array}
\]

given by the unique function \( \Delta \) from the set \( \{0,1\} \) to the hom \( \{\ast\} \) of the terminal category. The cell, being the only cell \( \{0,1\} \to \{\ast\} \), forms a limit of \( \{0,1\} \) but not a representation.

Corollary 10.2.18.

- Given a cell

\[
\begin{array}{c}
A \\
\mathcal{M} \\
\Upsilon \\
X \\
(\mathbf{X}) \\
X
\end{array}
\]

along the hom of a category \( \mathbf{X} \), the following conditions are equivalent:

1. the module morphism \( \Upsilon : \mathcal{M} \to (\mathbf{X}) \Rightarrow \) is iso; that is, the pair \( (\mathcal{R}, \Upsilon) \) forms a corepresentation of \( \mathcal{M} \);

2. \( \Upsilon \) is a pointwise universal cell and the right slice \( (\mathcal{M}) \mathbf{a} : \mathbf{X} \Rightarrow \ast \) of \( \mathcal{M} \) at each \( \mathbf{a} \in \|\mathbf{A}\| \) preserves direct limits.

- Given a cell

\[
\begin{array}{c}
\mathbf{X} \\
\mathcal{R} \\
\Upsilon \\
\mathcal{M} \\
\mathbf{A}
\end{array}
\]

along the hom of a category \( \mathbf{A} \), the following conditions are equivalent:

1. the module morphism \( \Upsilon : \mathcal{M} \to \mathcal{R}(\mathbf{A}) \) is iso; that is, the pair \( (\mathcal{R}, \Upsilon) \) forms a representation of \( \mathcal{M} \);
2. \( \Upsilon \) is a pointwise universal cell and the left slice \( \mathbf{x}(M) : * \to A \) of \( M \) at each \( \mathbf{x} \in \mathbf{X} \) preserves inverse limits.

**Proof.** By Proposition 2.3.11, \( \Upsilon : M \to (X)R \) is iso iff the right slice

\[
\xymatrix{ \mathbf{x}(M)a & (\Upsilon)a \\
X & (X)a \ar[ll]_R}
\]

is iso for every \( a \in \| A \| \). The assertion is thus reduced to Theorem 10.2.16. \( \square \)

**Note.** The following is a special case of Theorem 10.2.20 where \( K \) is given by the identity.

**Theorem 10.2.19.**

- Given a left cylinder

\[
\xymatrix{ D & \mu \\
X & M \ar[ll]_F \ar[uul]_R}
\]

, assume that \( F \) is fully faithful, and consider the cell

\[
\xymatrix{ D & \mu^D \\
X & M \ar[ll]_F \ar[uul]_R}
\]

generated by \( D \) inverse along \( \mu \) (see Theorem 5.5.2). In this situation, if \( \mu^D \) is pointwise universal, then \( \mu \) is a unit of \( M \). The converse holds if we assume in addition that \( F \) is surjective.

- Given a right cylinder

\[
\xymatrix{ D & \mu \\
X & M \ar[ll]_F \ar[uul]_R}
\]

, assume that \( F \) is fully faithful, and consider the cell

\[
\xymatrix{ D & \mu^D \\
X & M \ar[ll]_R \ar[uul]_{(D)F}
\]

generated by \( D \) direct along \( \mu \) (see Theorem 5.5.2). In this situation, if \( D^\mu \) is pointwise universal, then \( \mu \) is a counit of \( M \). The converse holds if we assume in addition that \( F \) is surjective.

**Proof.** Let \( \mathbf{x} \in \| \mathbf{X} \| \) and \( d \in \| D \| \) be such that \( \mathbf{x}^\ast F = d \). It suffices to show that the right slice
of \(\mu|D\) at \(d\) is a universal conical cell iff the \(M\)-arrow \(\mu_x : x \sim R \cdot d\) is direct universal. Depicting \(\mu\) as

\[
\begin{array}{c}
X \xrightarrow{\langle M \rangle R} \mu \xrightarrow{F} D
\end{array}
\]

, we have the commutative diagram

\[
\begin{array}{c}
\langle X \rangle x \xrightarrow{(F)x} F\langle D \rangle d \\
\langle X \rangle \xrightarrow{\langle X \rangle \mu x} \langle X \rangle \xrightarrow{(\mu|D)d} \\
\langle M \rangle \langle R \cdot d \rangle \xrightarrow{\langle (M) R \rangle d}
\end{array}
\]

by Example 5.3.5(1) and Corollary 5.3.10. Since \(F\) is fully faithful, \(\langle F \rangle x\) is iso. Hence, \((\mu|D) d : F\langle D \rangle d \sim R \cdot d\) is a direct universal \(\langle M \rangle \langle R \rangle\)-arrow iff so is \(\langle X \rangle \mu x : \langle X \rangle x \sim R \cdot d\). But, by Remark 10.1.2(3), \(X \xrightarrow{\mu x} \langle X \rangle x \sim R \cdot d\) is a direct universal \(\langle M \rangle \langle R \rangle\)-arrow iff \(\mu_x : x \sim R \cdot d\) is a direct universal \(M\)-arrow.

**Theorem 10.2.20.**

- Given a cylinder

\[
\begin{array}{c}
E \xrightarrow{F} D \\
\downarrow \mu \downarrow R \\
X \xrightarrow{\langle M \rangle} A
\end{array}
\]

, assume that \(F\) is fully faithful, and consider the cell

\[
\begin{array}{c}
E \xrightarrow{\langle F \rangle d} D \\
\downarrow \mu|D \downarrow R \\
X \xrightarrow{\langle M \rangle} A
\end{array}
\]

generated by \(D\) inverse along \(\mu\) (see Theorem 5.5.2). In this situation, if \(\mu|D\) is pointwise direct universal, so is \(\mu\). The converse holds if we assume in addition that \(F\) is surjective.

- Given a cylinder

\[
\begin{array}{c}
D \xleftarrow{F} E \\
\downarrow \mu \downarrow K \\
X \xrightarrow{\langle M \rangle} A
\end{array}
\]

, assume that \(F\) is fully faithful, and consider the cell

\[
\begin{array}{c}
D \xleftarrow{\langle D \rangle F} E \\
\downarrow D|\mu \downarrow K \\
X \xrightarrow{\langle M \rangle} A
\end{array}
\]

generated by \(D\) direct along \(\mu\) (see Theorem 5.5.2). In this situation, if \(D|\mu\) is pointwise inverse universal, so is \(\mu\). The converse holds if we assume in addition that \(F\) is surjective.
Proof. The assertion is reduced to Theorem 10.2.19 on noting that $\mu$ is pointwise direct universal iff the left cylinder

\[
\begin{array}{ccc}
D & \xrightarrow{\mu} & A \\
\downarrow F & & \downarrow \mu \\
E & \xrightarrow{\mu(M)} & A
\end{array}
\]

is a unit of $\mathbb{K}(\mathcal{M})$ (see Remark 6.4.2(2)) and $\mu|D$ is pointwise direct universal iff the cell

\[
\begin{array}{ccc}
D & \xrightarrow{\mu|D} & A \\
\downarrow F(D) & & \downarrow \mu|D \\
E & \xrightarrow{\mu(M)} & A
\end{array}
\]

is pointwise universal (see Proposition 10.2.11). \qed

Corollary 10.2.21.

- A cylinder

\[
\begin{array}{ccc}
E & \xrightarrow{\mu} & A \\
\downarrow K & & \downarrow R \\
X & \xrightarrow{\mu(M)} & A
\end{array}
\]

is direct universal (resp. pointwise direct universal) if and only if so is the cell

\[
\begin{array}{ccc}
E & \xrightarrow{(E)} & E \\
\downarrow K & & \downarrow (\mu) \\
X & \xrightarrow{\mu(M)} & A
\end{array}
\]

(see Corollary 5.5.4).

- A cylinder

\[
\begin{array}{ccc}
E & \xleftarrow{\mu} & A \\
\downarrow K & & \downarrow R \\
X & \xleftarrow{\mu(M)} & A
\end{array}
\]

is inverse universal (resp. pointwise inverse universal) if and only if so is the cell

\[
\begin{array}{ccc}
E & \xleftarrow{(E)} & E \\
\downarrow K & & \downarrow (\mu) \\
X & \xleftarrow{\mu(M)} & A
\end{array}
\]

(see Corollary 5.5.4).

Proof. By the isomorphism in Corollary 5.5.4, the cylinder $\mu$ is universal iff so is the cell $\langle \mu \rangle$. The pointwise version follows from Theorem 10.2.20 by replacing the functor $F : D \to E$ with the identity $E \to E$. \qed
10.3. Kan extensions

*Note.* Theorem 5.5.2 allows the following definition.

**Definition 10.3.1.** Given a pair of functors

\[
\begin{array}{c}
C \xrightarrow{K} E \xrightarrow{F} D
\end{array}
\]

- a natural transformation

\[
\begin{array}{c}
E \xrightarrow{F} D \\
\mu \downarrow \downarrow \\
C \xrightarrow{(C)} C
\end{array}
\]

from \(K\) to \(R \circ F\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a left Kan extension (resp. pointwise left Kan extension) of \(K\) along \(F\) if the cell

\[
\begin{array}{c}
E \xrightarrow{F(D)} D \\
\mu \downarrow \downarrow \\
C \xrightarrow{(C)} C
\end{array}
\]

generated by \(D\) inverse along \(\mu\) is direct universal (resp. pointwise direct universal).

- a natural transformation

\[
\begin{array}{c}
D \xleftarrow{F} E \\
\mu \downarrow \downarrow \\
C \xrightarrow{(C)} C
\end{array}
\]

from \(F \circ R\) to \(K\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a right Kan extension (resp. pointwise right Kan extension) of \(K\) along \(F\) if the cell

\[
\begin{array}{c}
D \xleftarrow{(D)F} E \\
\mu \downarrow \downarrow \\
C \xrightarrow{(C)} C
\end{array}
\]

generated by \(D\) direct along \(\mu\) is inverse universal (resp. pointwise inverse universal).

**Remark 10.3.2.** By the bijective correspondences in Theorem 5.5.2, the following remarks hold.

1. A Kan extension is regarded as a special instance of an extension as defined in Definition 10.2.9 where \(J\) is representable and \(M\) is a hom, or a special instance of an extension as defined in Definition 10.2.7 where both \(J\) and \(M\) are representable. Specifically,

- getting a left Kan extension (resp. pointwise left Kan extension) of \(K\) along \(F\) is getting an extension (resp. pointwise extension)

\[
\begin{array}{c}
E \xrightarrow{F(D)} D \\
\Upsilon \downarrow \downarrow \\
C \xrightarrow{(C)} C
\end{array}
\]
10. Extensions

of $K$ direct along $(F(D), (C))$, or, equivalently, getting an extension (resp. pointwise extension)

$$
\begin{array}{c}
\text{D} \\
\downarrow F(D) \\
\text{E} \\
\downarrow K(C) \\
\text{C}
\end{array}
\xrightarrow{T} \xleftarrow{R} \begin{array}{c}
\text{C} \\
\downarrow K(C) \\
\text{E} \\
\downarrow F(D) \\
\text{D}
\end{array}
$$

of $F(D)$ direct along $K(C)$.

• getting a right Kan extension (resp. pointwise right Kan extension) of $K$ along $F$ is getting an extension (resp. pointwise extension)

$$
\begin{array}{c}
\text{D} \\
\downarrow (D)F \\
\text{E} \\
\downarrow K(C) \\
\text{C}
\end{array}
\xrightarrow{T} \xleftarrow{R} \begin{array}{c}
\text{C} \\
\downarrow (C)K \\
\text{C} \\
\downarrow (D)F \\
\text{D}
\end{array}
$$

of $K$ inverse along $(D,F,(C))$, or, equivalently, getting an extension (resp. pointwise extension)

$$
\begin{array}{c}
\text{D} \\
\downarrow (D)F \\
\text{E} \\
\downarrow K(C) \\
\text{C}
\end{array}
\xrightarrow{T} \xleftarrow{R} \begin{array}{c}
\text{C} \\
\downarrow (C)K \\
\text{C} \\
\downarrow (D)F \\
\text{D}
\end{array}
$$

of $(D,F)$ inverse along $(C,K)$.

2. A (not necessarily pointwise) left Kan extension of $K$ along $F$ may be defined, as in the literature, more directly as a universal of $K$ direct along the module $(F^*, C)$ (see Remark 4.5.4(2)). Dually a right Kan extension of $K$ along $F$ may be defined as a universal of $K$ inverse along the module $(F^*, C)$.

**Proposition 10.3.3.** If $E$ is small and $C$ is direct (resp. inverse) complete, then any functor $K : E \to C$ has a pointwise left (resp. right) Kan extension along any functor $F : E \to D$.

**Proof.** By Remark 10.3.2(1), getting a pointwise left Kan extension of $K$ along $F$ is getting a pointwise extension of the module $F(D)$ direct along the module $K(C)$. Hence the assertion follows from Theorem 10.2.15 on noting that $K(C)$ is direct complete by Corollary 7.3.10. □

**Proposition 10.3.4.** If $F : E \to D$ is a fully faithful functor, then a pointwise left (resp. right) Kan extension along $F$ is a natural isomorphism.

**Proof.** Immediate from Theorem 10.2.20. □

**Theorem 10.3.5.** Consider a pair of functors

$$
\begin{array}{c}
\text{X} \\
\downarrow F \\
\text{A}
\end{array}
\xrightarrow{G} \begin{array}{c}
\text{A} \\
\downarrow G \\
\text{X}
\end{array}
$$

• Given a natural transformation $\eta : 1_X \to G \circ F$, the following conditions are equivalent:

1. the module morphism $\eta_1 : A \to (X)G : X \to A$ is iso; that is, $\eta$ is the unit of an adjunction between $F$ and $G$. 

310
10. Extensions

2. \( \eta \) forms a pointwise left Kan extension

\[
\begin{array}{c}
X \xrightarrow{F} A \\
\downarrow \eta \downarrow G \\
\underbrace{X}_{(X)} \xrightarrow{} X
\end{array}
\]

of the identity \( X \to X \) along \( F \) and \( F \) preserves direct limits;

- Given a natural transformation \( \epsilon : G \circ F \to 1_A \), the following conditions are equivalent:
  1. the module morphism \( X|\epsilon : (X) G \to F(A) : X \to A \) is iso; that is, \( \epsilon \) is the counit of an adjunction between \( G \) and \( F \).
  2. \( \epsilon \) forms a pointwise right Kan extension

\[
\begin{array}{c}
X \xleftarrow{G} A \\
F \downarrow \epsilon \downarrow 1 \\
\underbrace{A}_{(A)} \xleftarrow{} A
\end{array}
\]

of the identity \( A \to A \) along \( G \) and \( G \) preserves inverse limits;

Proof. By definition, \( \eta \) forms a pointwise left Kan extension iff the cell

\[
\begin{array}{c}
X \xrightarrow{F(A)} A \\
\downarrow \eta|A \downarrow G \\
\underbrace{X}_{(X)} \xrightarrow{} X
\end{array}
\]

is pointwise universal. The assertion thus follows from Corollary 10.2.18 and noting Corollary 7.4.10. \qed

Problem. In Proposition 10.2.12, we saw that pointwise Kan lifts subsume pointwise extensions, hence in particular pointwise Kan extensions. Does the converse hold? Do pointwise Kan extensions subsume pointwise Kan lifts?

10.4. Density

Definition 10.4.1.

1. A module \( M : X \to A \) is called
   - dense if its right exponential transpose \( M \bowtie A : A \to [X:] \) is fully faithful.
   - codense if its left exponential transpose \( \bowtie M : X \to [: A]^\ast \) is fully faithful.

2. A functor \( F : E \to D \) is called
   - dense if its representable module \( F(D) : E \to D \) is dense.
   - codense if its corepresentable module \( (D) F : D \to E \) is codense.

3. A subcategory \( E \) of a category \( D \) is called
dense in $D$ if the inclusion $E \to D$ is dense.

- codense in $D$ if the inclusion $E \to D$ is codense.

**Proposition 10.4.2.** Given a module $M : X \to A$,

- the following conditions are equivalent:
  1. $M$ is dense;
  2. the identity $M \to M$ forms a pointwise universal cell

\[
\begin{array}{c}
\text{A} \\
\text{X} \begin{array}{c}
\text{M} \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \text{A}
\end{array}
\]

3. for every $a \in \|A\|$, the identity $\langle M \rangle a \to \langle M \rangle a$ forms a universal conical cell

\[
\begin{array}{c}
\langle M \rangle a \begin{array}{c}
\ast \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \begin{array}{c}
a \\
\end{array} \\
\text{X} \begin{array}{c}
\text{M} \\
\end{array} \text{A}
\end{array}
\]

4. for every $a \in \|A\|$, the unit cone

\[
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{X} \begin{array}{c}
\text{M} \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \text{A}
\end{array}
\]

(see Definition 9.4.1) of $\langle M \rangle a$ forms a universal cone.

- the following conditions are equivalent:
  1. $M$ is codense;
  2. the identity $M \to M$ forms a pointwise universal cell

\[
\begin{array}{c}
\text{X} \\
\text{X} \begin{array}{c}
\text{M} \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \text{A}
\end{array}
\]

3. for every $x \in \|X\|$, the identity $x \langle M \rangle \to x \langle M \rangle$ forms a universal conical cell

\[
\begin{array}{c}
\times \begin{array}{c}
\ast \\
\end{array} \begin{array}{c}
\text{1} \\
\end{array} \begin{array}{c}
x \langle M \rangle \\
\end{array} \\
\text{X} \begin{array}{c}
\text{M} \\
\end{array} \text{A}
\end{array}
\]

312
10. Extensions

4. for every $x \in \mathbb{X}$, the unit cone

$$
\begin{array}{c}
\Delta \downarrow \\
\downarrow 1 \downarrow \\
\mathbb{X} \rightarrow \mathbb{M} \rightarrow A
\end{array}
$$

(see Definition 9.4.1) of $\mathbb{x}(M)$ forms a universal cone.

Proof.

(1) $\Leftrightarrow$ (2) By Proposition 10.2.12, the cell

$$
\begin{array}{c}
\mathbb{M} \rightarrow A \\
\downarrow 1 \\
\mathbb{X} \rightarrow \mathbb{M} \rightarrow A
\end{array}
$$

is pointwise universal iff its right exponential transpose

$$
\begin{array}{c}
\mathbb{M} \rightarrow A \\
\downarrow 1 \\
[\mathbb{X}] \leftarrow \mathbb{M} \rightarrow A
\end{array}
$$

forms a pointwise left Kan lift of $\mathbb{M} \rightarrow$ along $\mathbb{M} \leftarrow$. By Theorem 6.5.3, this is the case iff $\mathbb{M} \leftarrow$ is fully faithful.

(2) $\Leftrightarrow$ (3) Evident by the definition of pointwise universality.

(3) $\Leftrightarrow$ (4) Immediate from Remark 10.2.6(1).

Theorem 10.4.3. The right (resp. left) Yoneda module is dense (resp. codense).

Proof. Immediate from Proposition 5.1.3.

Corollary 10.4.4. The right (resp. left) Yoneda functor is dense (resp. codense).

Proof. Since the Yoneda module is represented by the Yoneda functor (see Theorem 5.2.15), the assertion follows from Proposition 10.4.3.

Note. If $F$ in Corollary 6.2.12 is dense (resp. codense), then we have the following.

Theorem 10.4.5. Let $F : E \rightarrow D$ be functor.

- If $F$ is fully faithful and dense, then a $D$-arrow $u : r : F \rightarrow d$ is invertible if and only if it is universal from $F$ to $d$, i.e. the $F(D)$-arrow $u : r \rightarrow d$ is inverse universal.

- If $F$ is fully faithful and codense, then a $D$-arrow $u : d \rightarrow F \cdot r$ is invertible if and only if it is universal from $d$ to $F$, i.e. the $(D)F$-arrow $u : d \rightarrow r$ is direct universal.
10. Extensions

**Proof.** By Example 5.2.9(2), the diagram

\[
\begin{array}{c}
\xymatrix{(E) r & F \downarrow (D) \rightarrow & d \\
(F) r & F(D) r & F(D)(F \cdot r) \\
\end{array}
\]

commutes. Since \( F \) is fully faithful, \( (F) r \) is iso. Hence \( E \mid u \) is iso iff \( F(D) u \) is iso. By definition, \( E \mid u \) is iso iff \( u \) is inverse universal. Since \( (F(D)) r \) is fully faithful by the denseness of \( F \), \( F(D) u \) is iso iff \( u \) is invertible.

**Theorem 10.4.6.**

- If a module \( M : X \to A \) is dense and has a representation \((R, \Upsilon)\), then, for any \( a \in \| A \| \), the composition

\[
\begin{array}{c}
\xymatrix{[M \downarrow a] \ar[r]^\Delta & * \\
M_{(a)} \ar[r]^{1^1_{(M)a}} & A \\
X \ar[r] & A \\
R \ar[r] & A \\
\Upsilon \ar[r] & A \\
\end{array}
\]

yields a universal cone \( [M \downarrow a] \circ R \simeq a \) in \( A \).

- If a module \( M : X \to A \) is co-dense and has a corepresentation \((R, \Upsilon)\), then, for any \( x \in \| X \| \), the composition

\[
\begin{array}{c}
\xymatrix{* \ar[r]^\Delta & [X \downarrow M] \\
x \ar[r]^{1^1'_{(X)a}} & x_{\downarrow M} \\
X \ar[r] & A \\
1 \ar[r] & A \\
\Upsilon \ar[r] & A \\
\end{array}
\]

yields a universal cone \( x \simeq R \circ [X \downarrow M] \) in \( X \).

**Proof.** The unit cone \( 1^1_{(M)a} \) is universal by Proposition 10.4.2, and \( \Upsilon \) preserves it by Theorem 7.3.8.

**Corollary 10.4.7.**

- Any right module is a direct limit of representables. Specifically, given a right module \( M : X \to * \), the composition

\[
\begin{array}{c}
\xymatrix{[M] \ar[r]^\Delta & * \\
M \ar[r]^{1_M} & M \\
X \ar[r] & [X:] \\
x_\cdot \ar[r] & x_{\cdot} \ar[r] & [X:] \\
\end{array}
\]

of the unit cone of \( M \) and the Yoneda representation yields a universal cone \( M^\downarrow \hat{\circ} [X_\cdot] \simeq M \) in \( [X:] \).
10. Extensions

- Any left module is a direct limit of representables. Specifically, given a left module \( M : * \to A \), the composition

\[
\begin{array}{c}
\ast \quad \xrightarrow{\Delta} [M^\dagger] \\
\downarrow M \quad \downarrow 1_M \quad \downarrow M^\dagger \\
[: A]^- \quad \xrightarrow{\vdash (\cdot, A)} \quad [\cdot] \quad \xrightarrow{\vdash A}
\end{array}
\]

of the unit cone of \( M \) and the Yoneda corepresentation yields a universal cone \( M^\dagger \circ [\vdash A] \sim M \) in \( [: A] \).

Proof. Since the Yoneda module \( X.\star \) is dense (see Proposition 10.4.3), the assertion follows from Theorem 10.4.6 on noting that \( (X.\star) M = M \) (see Proposition 5.1.3).

Theorem 10.4.8.

- Getting a pointwise left Kan extension

\[
\begin{array}{c}
X \quad \xrightarrow{X.\star} [X:] \\
\downarrow F \quad \downarrow \mu \quad \downarrow R \\
\downarrow \downarrow \quad \downarrow \quad \downarrow \\
A \quad \xrightarrow{\vdash (A)} \quad A
\end{array}
\]

of a functor \( F : X \to A \) along the right Yoneda functor \( X.\star \) is the same thing as getting a left adjoint of the functor \( [\vdash (F(A)) \star] : A \to [X:] \), the right exponential transpose of the representable module of \( F \).

- Getting a pointwise right Kan extension

\[
\begin{array}{c}
[: A]^- \quad \xrightarrow{\vdash A} \quad A \\
\downarrow R \quad \downarrow \mu \quad \downarrow G \\
\downarrow \downarrow \quad \downarrow \quad \downarrow \\
X \quad \xrightarrow{\vdash (X)} \quad X
\end{array}
\]

of a functor \( G : A \to X \) along the left Yoneda functor \( \vdash A \) is the same thing as getting a right adjoint of the functor \( [\vdash (X G)] : X \to [: A]^\dagger \), the left exponential transpose of the corepresentable module of \( G \).

Proof. By Remark 10.3.2(1) and the Yoneda representation (Theorem 5.2.15), getting a pointwise left Kan extension of \( F \) along \( X.\star \) is the same thing as getting a pointwise extension

\[
\begin{array}{c}
[X:] \\
\downarrow X.\star \quad \downarrow Y \quad \downarrow R \\
X \quad \xrightarrow{\vdash F(A)} \quad A
\end{array}
\]

of the Yoneda module \( X.\star \) direct along \( F(A) \). By Proposition 10.2.12 and noting that \( [(X.\star) \star] = 1 \) (see Proposition 5.1.3), this in turn is the same thing as getting a pointwise
10. Extensions

left Kan lift

\[
\begin{array}{c}
[X:] \\
\downarrow \mu \\
\downarrow R \\
\downarrow \langle F(A) \rangle \\
\end{array}
\]

of the identity \([X:] \to [X:]\) along \(\langle F(A) \rangle \), and, by Remark 8.1.4(3), this is the same thing as getting a left adjoint of \(\langle F(A) \rangle \).

**Theorem 10.4.9.**

- If \(F : X \to A\) is a functor with \(X\) small and \(A\) direct complete, then \(F\) has a pointwise left Kan extension

\[
X \xrightarrow{X'} [X:] \\
\downarrow F \\
\downarrow \mu \\
A \xleftarrow{\langle A \rangle} A
\]

along the right Yoneda functor \(X \langle A \rangle\), and \(R\) is characterized by the following properties:

1. \(R\) preserves small direct limits;
2. \(F \cong R \circ \langle A \rangle\);

that is, if a functor \(S : [X:] \to A\) has the properties above, then \(S \cong R\).

- If \(G : A \to X\) is a functor with \(A\) small and \(X\) inverse complete, then \(G\) has a pointwise right Kan extension

\[
\begin{array}{c}
[A]^\sim \\
\downarrow R \\
\downarrow \langle X \rangle \\
X \xleftarrow{\langle A \rangle} X
\end{array}
\]

along the left Yoneda functor \(\langle A \rangle\), and \(R\) is characterized by the following properties:

1. \(R\) preserves small inverse limits;
2. \([A]^\sim \circ R \cong G\);

that is, if a functor \(S : [A]^\sim \to X\) has the properties above, then \(S \cong R\).

**Proof.** By Proposition 10.3.3, a pointwise left Kan extension \((R, \mu)\) exists. By Theorem 10.4.8, \(R\) is a left adjoint of the functor \(\langle F(A) \rangle : A \to [X:]\); hence, by Corollary 8.3.24, \(R\) preserves direct limits. Since the Yoneda functor is fully faithful, by Proposition 10.3.4, \(\mu\) is a natural isomorphism; hence \(F \cong R \circ \langle X \rangle\). Now suppose that a functor \(S : [X:] \to A\) satisfies these properties. \(R\) and \(S\) coincide at each representable right module over \(X\) up to isomorphism. Since \(R\) and \(S\) preserve small direct limits, \(R \cong S\) by Corollary 10.4.7.

**Corollary 10.4.10.** If \(F : E \to D\) is a functor between small categories, then the precomposition functor \([F:] : [D:] \to [E:]\) (resp. \([F : [D] \to [E]]\)) has both left and right adjoints.
10. Extensions

Proof. Consider the Yoneda functors as in

\[
\begin{array}{ccc}
E & \overset{E^\mu}{\longrightarrow} & [E:] \\
\downarrow F & & \downarrow \mu \\
D & \overset{D^\mu}{\longrightarrow} & [D:] \\
\end{array}
\]

Since \(E\) is small and \([D:]\) is complete (see Example 7.3.15), by Theorem 10.4.9, \(F \circ [D^\mu]\) has a pointwise left Kan extension along \(E^\mu\) as depicted in

\[
\begin{array}{ccc}
E & \overset{E^\mu}{\longrightarrow} & [E:] \\
\downarrow F & \mu & \downarrow R \\
D & \overset{D^\mu}{\longrightarrow} & [D:] \\
\end{array}
\]

Hence, by Theorem 10.4.8, \(R\) is a left adjoint of the functor \(\langle [F \circ [D^\mu]] (D:) \rangle\). But

\[
\langle [F \circ [D^\mu]] (D:) \rangle = \langle F ([D^\mu] (D:)) \rangle \\
\cong \langle F (D^\mu) \rangle \\
= [F:] \circ ([D^\mu] (D:)) \\
= [F:] \\
(1^* \text{ by Theorem 5.2.15; } 2^* \text{ by Proposition 2.1.6; } 3^* \text{ by Proposition 5.1.3}).
\]

\([F:]\) thus has a left adjoint. Now consider the composition

\[
\begin{array}{ccc}
D & \overset{D^\mu}{\longrightarrow} & [D:] \\
\downarrow [D^\mu] \circ [F:] & & \downarrow [F:] \\
[E:] & & \\
\end{array}
\]

Since \([F:]\) preserves small direct limits (see Example 7.3.17), by Theorem 10.4.9, \([F:]\) is a pointwise left Kan extension of the composite \([D^\mu] \circ [F:]\) along \(D^\mu\), and thus has a right adjoint by Theorem 10.4.8. \(\square\)

10.5. Ends

Definition 10.5.1. Let \(E\) be a category and \(M : X \to A\) be a module.

- An outer cylinder \(\mu : r \leadsto K : E^r \leadsto M\) from an object \(r \in \|X\|\) to a bifunctor \(K : E^r \times E \to A\) along \(M\) is called universal if it is an inverse universal \((E^r,M)-\text{arrow}\) (see Definition 4.4.5). Given a bifunctor \(K : E^r \times E \to A\), a universal cylinder \(\mu : r \leadsto K : E^r \leadsto M\) or the pair \((r,\mu)\), or the object \(r\) itself, is called an end of \(K\) inverse along \(M\).

- An outer cylinder \(\mu : K \leadsto r : E^r \leadsto M\) from a bifunctor \(K : E \times E^r \to X\) to an object \(r \in \|A\|\) along \(M\) is called universal if it is a direct universal \((E^r,M)-\text{arrow}\) (see Definition 4.4.5). Given a bifunctor \(K : E \times E^r \to X\), a universal cylinder \(\mu : K \leadsto r : E^r \leadsto M\) or the pair \((r,\mu)\), or the object \(r\) itself, is called an end of \(K\) direct along \(M\).

Remark 10.5.2.
10. Extensions

1. An outer cylinder $\mu : r \rightarrow K : E^+ \rightarrow M$ is universal if and only if to every cylinder $\alpha : x \rightarrow K : E^+ \rightarrow M$ there is a unique $\mathbf{X}$-arrow $\alpha / \mu : x \rightarrow r$ such that $\alpha = \alpha / \mu \circ \mu$.

An outer cylinder $\mu : K \rightarrow r : E^+ \rightarrow M$ is universal if and only if to every cylinder $\alpha : K \rightarrow a : E^+ \rightarrow M$ there is a unique $\mathbf{A}$-arrow $\mu / \alpha : r \rightarrow a$ such that $\alpha = \mu \circ \mu / \alpha$.

2. Weighted limits subsume ends. Indeed, by the isomorphism in Theorem 5.5.8,

- a cylinder $\mu : r \rightarrow K : E^+ \rightarrow M$ is universal if and only if the conical cell

$$
\begin{array}{c}
\ast \\
\downarrow r \\
X \\
\downarrow \mu \\
M \\
\downarrow \alpha \\
A
\end{array}
\xrightarrow{(E)}
\begin{array}{c}
E^+ \times E \\
\downarrow \alpha \\
E \\
\downarrow \mu \\
C \\
\downarrow \beta \\
A
\end{array}
$$

is universal.

- a cylinder $\mu : K \rightarrow r : E^+ \rightarrow M$ is universal if and only if the conical cell

$$
\begin{array}{c}
E \times E^- \\
\downarrow \mu \\
K \\
\downarrow \alpha \\
C \\
\downarrow \beta \\
M \\
\downarrow \gamma \\
A
\end{array}
\xrightarrow{(E)}
\begin{array}{c}
\ast \\
\downarrow r \\
\ast \\
\downarrow a
\end{array}
$$

is universal.

3. As a special case where $M$ is the hom of a category $C$,

- an extranatural transformation $\mu : r \rightarrow K : E^+ \rightarrow C$ from an object $r \in \|C\|$ to a bifunctor $K : E^+ \times E \rightarrow C$ is called universal if it is an inverse universal $(E^+, C)$-arrow (see Remark 4.4.6(4)). Given a bifunctor $K : E^+ \times E \rightarrow C$, a universal extranatural transformation $\mu : r \rightarrow K : E^+ \rightarrow C$ or the pair $(r, \mu)$, or the object $r$ itself, is called an end of $K$ in $C$.

- an extranatural transformation $\mu : K \rightarrow r : E^+ \rightarrow C$ from a bifunctor $K : E \times E^- \rightarrow C$ to an object $r \in \|C\|$ is called universal if it is a direct universal $(E^+, C)$-arrow (see Remark 4.4.6(4)). Given a bifunctor $K : E \times E^- \rightarrow C$, a universal extranatural transformation $\mu : K \rightarrow r : E^+ \rightarrow C$ or the pair $(r, \mu)$, or the object $r$ itself, is called a coend of $K$ in $C$.

**Definition 10.5.3.** Let $E$ be a small category. Given an endomodule $M : E \rightarrow E$, i.e. a bifunctor $E^+ \times E \rightarrow \text{Set}$, the extranatural transformation

$$
E^+_M : \prod_E^* M \rightarrow M : E^+ \rightarrow \text{Set}
$$

is defined by

$$
\alpha : [E^+_M]_e = \alpha_e
$$

for $e \in \|E\|$ and $\alpha$ a frame of $M$.

**Remark 10.5.4.**

1. The smallness of $E$ guarantees the smallness of $\prod_E M$. 

1. 

318
2. The component

\[ [E_M^e]^e : \prod_E M \to e(M) e \]

of \( E_M^e \) at \( e \in \| E \| \) maps each frame \( \alpha \) of \( M \) to its component at \( e \).

**Proposition 10.5.5.** The extranatural transformation \( E_M^e : \prod_E M \to M \) defined above forms an end of \( M \) in \( \text{Set} \).

**Proof.** Given a set \( S \) and an extranatural transformation \( \alpha : S \to M \), the unique function \( \alpha' : S \to \prod_E M \) making the triangle

\[
\begin{array}{c}
S \\
\alpha' \\
\downarrow \\
\prod_E M \\
\downarrow \\
M
\end{array}
\]

commute is defined by

\[(s \cdot \alpha')_e = s \cdot \alpha_e\]

for \( s \in S \) and \( e \in \| E \| \). \( \square \)
## A. List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Usage</th>
<th>Meaning</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>[:]</td>
<td>[X:]</td>
<td>category of right modules over X</td>
<td>(1.1.2)</td>
</tr>
<tr>
<td>[: A]</td>
<td>category of left modules over A</td>
<td>(1.1.2)</td>
<td></td>
</tr>
<tr>
<td>[K:]</td>
<td>precomposition with K</td>
<td>(1.1.2)</td>
<td></td>
</tr>
<tr>
<td>[K]</td>
<td>precomposition with K</td>
<td>(1.1.2)</td>
<td></td>
</tr>
<tr>
<td>[X: A]</td>
<td>category of modules X → A</td>
<td>(1.1.6)</td>
<td></td>
</tr>
<tr>
<td>[S: T]</td>
<td>precomposition with S × T</td>
<td>(1.1.6)</td>
<td></td>
</tr>
<tr>
<td>[M: N]</td>
<td>category of cells M → N</td>
<td>(1.3.3)</td>
<td></td>
</tr>
<tr>
<td>[↑]</td>
<td>[X ↑ A]</td>
<td>category of collages X → A</td>
<td>(3.1.4)</td>
</tr>
<tr>
<td>[M ↑ N]</td>
<td>category of collage cells M → N</td>
<td>(3.1.9)</td>
<td></td>
</tr>
<tr>
<td>[↓]</td>
<td>[X ↓ A]</td>
<td>category of commas X → A</td>
<td>(3.2.11)</td>
</tr>
<tr>
<td>[M]</td>
<td>category of a collage M</td>
<td>(3.1.1)</td>
<td></td>
</tr>
<tr>
<td>[K]</td>
<td>category of a comma K</td>
<td>(3.2.7)</td>
<td></td>
</tr>
<tr>
<td>[Φ]</td>
<td>frame of Φ</td>
<td>(5.5.6)</td>
<td></td>
</tr>
<tr>
<td>⟨C⟩</td>
<td>hom of a category C</td>
<td>(1.1.11)</td>
<td></td>
</tr>
<tr>
<td>⟨H⟩</td>
<td>hom of a functor H</td>
<td>(1.2.25)</td>
<td></td>
</tr>
<tr>
<td>⟨τ⟩</td>
<td>hom of a natural transformation τ</td>
<td>(1.3.7)</td>
<td></td>
</tr>
<tr>
<td>⟨X⟩x</td>
<td>representable right module of x</td>
<td>(2.3.2)</td>
<td></td>
</tr>
<tr>
<td>a ⟨A⟩</td>
<td>representable left module of a</td>
<td>(2.3.2)</td>
<td></td>
</tr>
<tr>
<td>⟨X⟩G</td>
<td>corepresentable module of G: A → X</td>
<td>(2.3.6)</td>
<td></td>
</tr>
<tr>
<td>F ⟨A⟩</td>
<td>representable module of F: X → A</td>
<td>(2.3.6)</td>
<td></td>
</tr>
<tr>
<td>⟨E, C⟩</td>
<td>hom of category [E, C]</td>
<td>(1.1.11)</td>
<td></td>
</tr>
<tr>
<td>⟨J, M⟩</td>
<td>module of cells J → M</td>
<td>(1.2.7)</td>
<td></td>
</tr>
<tr>
<td>⟨J, Φ⟩</td>
<td>postcomposition with Φ</td>
<td>(1.2.12) (1.2.21)</td>
<td></td>
</tr>
<tr>
<td>⟨E, M⟩</td>
<td>module of cylinders E ∼ M</td>
<td>(4.3.6)</td>
<td></td>
</tr>
<tr>
<td>⟨E, Φ⟩</td>
<td>postcomposition with Φ</td>
<td>(4.3.11) (4.3.15)</td>
<td></td>
</tr>
<tr>
<td>⟨F, M⟩</td>
<td>precomposition with F</td>
<td>(4.3.26)</td>
<td></td>
</tr>
<tr>
<td>⟨X:⟩</td>
<td>hom of category [X:]</td>
<td>(1.2.25)</td>
<td></td>
</tr>
<tr>
<td>⟨: A⟩</td>
<td>hom of category [: A]</td>
<td>(1.2.25)</td>
<td></td>
</tr>
<tr>
<td>⟨X: A⟩</td>
<td>hom of category [X: A]</td>
<td>(1.2.25)</td>
<td></td>
</tr>
<tr>
<td>⟨X: M⟩</td>
<td>right hom of M: X → A</td>
<td>(5.2.1)</td>
<td></td>
</tr>
<tr>
<td>⟨M: A⟩</td>
<td>left hom of M: X → A</td>
<td>(5.2.1)</td>
<td></td>
</tr>
<tr>
<td>⟨↑⟩</td>
<td>cylinder module [X ↑ A] → [X ↑ A]</td>
<td>(9.3.1)</td>
<td></td>
</tr>
<tr>
<td>⟨↑⟩</td>
<td>cone module [X ↑] → [X:]</td>
<td>(9.4.2)</td>
<td></td>
</tr>
<tr>
<td>⟨↑ A⟩</td>
<td>cone module [↑ A] → [: A]</td>
<td>(9.4.2)</td>
<td></td>
</tr>
<tr>
<td>⟨↑ A⟩</td>
<td>module of right slant cells X → A</td>
<td>(8.2.1)</td>
<td></td>
</tr>
<tr>
<td>⟨M ↑ N⟩</td>
<td>module of right slant cells M → N</td>
<td>(8.3.5)</td>
<td></td>
</tr>
<tr>
<td>⟨A ↑ X⟩</td>
<td>module of left slant cells A → X</td>
<td>(8.2.1)</td>
<td></td>
</tr>
<tr>
<td>⟨N ↑ M⟩</td>
<td>module of left slant cells N → M</td>
<td>(8.3.5)</td>
<td></td>
</tr>
</tbody>
</table>
### A. List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Usage</th>
<th>Meaning</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>${}$</td>
<td>$(\alpha)$</td>
<td>module morphism of $\alpha$</td>
<td>(5.5.6)</td>
</tr>
<tr>
<td>$||$</td>
<td>$|C|$</td>
<td>set of objects of $C$</td>
<td>(3)</td>
</tr>
<tr>
<td>*</td>
<td>$X \ast A$</td>
<td>join $X \rightarrow A$</td>
<td>(1.1.18)</td>
</tr>
<tr>
<td>*</td>
<td>$X \ast$</td>
<td>join $X \rightarrow \ast$</td>
<td>(1.1.18)</td>
</tr>
<tr>
<td>*A</td>
<td>$\ast A$</td>
<td>join $\ast \rightarrow \ast$</td>
<td>(1.1.18)</td>
</tr>
<tr>
<td>$\langle \ast : E' \rangle$</td>
<td>module of cones $\ast \rightarrow E'$</td>
<td>(4.9.1)</td>
<td></td>
</tr>
<tr>
<td>$\langle E' : \ast \rangle$</td>
<td>module of cones $E' \rightarrow \ast$</td>
<td>(4.9.1)</td>
<td></td>
</tr>
<tr>
<td>$\langle X : E' \rangle$</td>
<td>module of wedges $X \rightarrow E'$</td>
<td>(4.9.1)</td>
<td></td>
</tr>
<tr>
<td>$\langle E' : A \rangle$</td>
<td>module of wedges $E' \rightarrow A$</td>
<td>(4.9.1)</td>
<td></td>
</tr>
<tr>
<td>$\sim$</td>
<td>$\sim C$</td>
<td>opposite module of $C$</td>
<td>(6)</td>
</tr>
<tr>
<td>$\sim M$</td>
<td>opposite module of $M$</td>
<td>(1.1.21)</td>
<td></td>
</tr>
<tr>
<td>$\triangleright M$</td>
<td>right exponential transpose of $M$</td>
<td>(2.1.1)</td>
<td></td>
</tr>
<tr>
<td>$\triangleright M \triangleright E$</td>
<td>right action of $M : X \rightarrow A$ on $[E, A]$</td>
<td>(2.2.1)</td>
<td></td>
</tr>
<tr>
<td>$X \triangleright$</td>
<td>right Yoneda functor for $X$</td>
<td>(2.3.1)</td>
<td></td>
</tr>
<tr>
<td>$X \triangleright A$</td>
<td>right general Yoneda functor for $[A, X]$</td>
<td>(2.3.5)</td>
<td></td>
</tr>
<tr>
<td>$\langle M</td>
<td>A \rangle \triangleright$</td>
<td>right Yoneda functor for $M$</td>
<td>(5.2.3)</td>
</tr>
<tr>
<td>$\langle M</td>
<td>A \rangle \triangleright$</td>
<td>right Yoneda morphism for $M$</td>
<td>(5.2.5)</td>
</tr>
<tr>
<td>$\langle X, M \rangle \triangleright E$</td>
<td>right general Yoneda functor for $[E, M]$</td>
<td>(5.3.1)</td>
<td></td>
</tr>
<tr>
<td>$\langle X, M \rangle \triangleright E$</td>
<td>right general Yoneda morphism for $[E, M]$</td>
<td>(5.3.3)</td>
<td></td>
</tr>
<tr>
<td>$\triangleleft M$</td>
<td>left exponential transpose of $M$</td>
<td>(2.1.1)</td>
<td></td>
</tr>
<tr>
<td>$E \triangleleft M$</td>
<td>left action of $M : X \rightarrow A$ on $[E, X]$</td>
<td>(2.2.1)</td>
<td></td>
</tr>
<tr>
<td>$\triangleleft A$</td>
<td>left Yoneda functor for $A$</td>
<td>(2.3.1)</td>
<td></td>
</tr>
<tr>
<td>$X \triangleleft A$</td>
<td>left general Yoneda functor for $[X, A]$</td>
<td>(2.3.5)</td>
<td></td>
</tr>
<tr>
<td>$\langle M</td>
<td>A \rangle \triangleleft$</td>
<td>left Yoneda functor for $M$</td>
<td>(5.2.3)</td>
</tr>
<tr>
<td>$\langle M</td>
<td>A \rangle \triangleleft$</td>
<td>left Yoneda morphism for $M$</td>
<td>(5.2.5)</td>
</tr>
<tr>
<td>$E \triangleleft \langle M</td>
<td>A \rangle$</td>
<td>left general Yoneda functor for $[E, M]$</td>
<td>(5.3.1)</td>
</tr>
<tr>
<td>$E \triangleleft \langle M</td>
<td>A \rangle$</td>
<td>left general Yoneda morphism for $[E, M]$</td>
<td>(5.3.3)</td>
</tr>
<tr>
<td>$\triangleright \triangleright M$</td>
<td>right comma exponential transpose of $M$</td>
<td>(3.2.31)</td>
<td></td>
</tr>
<tr>
<td>$\triangleright \triangleright M$</td>
<td>left comma exponential transpose of $M$</td>
<td>(3.2.31)</td>
<td></td>
</tr>
<tr>
<td>$X \triangleright$</td>
<td>right Yoneda module for $X$</td>
<td>(5.1.1)</td>
<td></td>
</tr>
<tr>
<td>$X \triangleright A$</td>
<td>right general Yoneda module for $[A, X]$</td>
<td>(5.1.6)</td>
<td></td>
</tr>
<tr>
<td>$\triangleright A$</td>
<td>left Yoneda module for $A$</td>
<td>(5.1.1)</td>
<td></td>
</tr>
<tr>
<td>$X \triangleright A$</td>
<td>left general Yoneda module for $[X, A]$</td>
<td>(5.1.6)</td>
<td></td>
</tr>
<tr>
<td>$M \triangleright$</td>
<td>corepresentable module of $M \triangleright$</td>
<td>(10.1.1)</td>
<td></td>
</tr>
<tr>
<td>$M \triangleright E$</td>
<td>corepresentable module of $M \triangleright E$</td>
<td>(10.1.6)</td>
<td></td>
</tr>
<tr>
<td>$\triangleleft \triangleleft M$</td>
<td>representable module of $\triangleleft M$</td>
<td>(10.1.1)</td>
<td></td>
</tr>
<tr>
<td>$E \triangleleft \triangleleft M$</td>
<td>representable module of $E \triangleleft M$</td>
<td>(10.1.6)</td>
<td></td>
</tr>
<tr>
<td>$\triangleright \triangleright M$</td>
<td>corepresentable module of $M \triangleright$</td>
<td>(10.1.3)</td>
<td></td>
</tr>
<tr>
<td>$\triangleleft \triangleleft M$</td>
<td>representable module of $\triangleleft M$</td>
<td>(10.1.3)</td>
<td></td>
</tr>
<tr>
<td>$\downarrow M$</td>
<td>comma of $M$</td>
<td>(3.2.21)</td>
<td></td>
</tr>
<tr>
<td>$\downarrow M \downarrow a$</td>
<td>right comma slice of $M$ at $a$</td>
<td>(3.2.32)</td>
<td></td>
</tr>
<tr>
<td>$x \downarrow M$</td>
<td>left comma slice of $M$ at $x$</td>
<td>(3.2.32)</td>
<td></td>
</tr>
<tr>
<td>$F \downarrow a$</td>
<td>right comma slice of $F$ at $a$</td>
<td>(3.2.32)</td>
<td></td>
</tr>
<tr>
<td>$x \downarrow G$</td>
<td>left comma slice of $G$ at $x$</td>
<td>(3.2.32)</td>
<td></td>
</tr>
<tr>
<td>$1_M^\downarrow$</td>
<td>unit cylinder of $M$</td>
<td>(9.1.3)</td>
<td></td>
</tr>
<tr>
<td>$\Phi^\downarrow$</td>
<td>unit cone of $M$</td>
<td>(9.4.1)</td>
<td></td>
</tr>
<tr>
<td>$\Phi^\downarrow$</td>
<td>comma transpose of $\Phi$</td>
<td>(9.4.9)</td>
<td></td>
</tr>
</tbody>
</table>
### A. List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Usage</th>
<th>Meaning</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>†</td>
<td>∫_K</td>
<td>module of K</td>
<td>(3.2.25)</td>
</tr>
<tr>
<td>†</td>
<td>1_K</td>
<td>unit cylinder of K</td>
<td>(9.1.3)</td>
</tr>
<tr>
<td>†</td>
<td>Φ^†</td>
<td>collage transpose of Φ</td>
<td>(9.4.9)</td>
</tr>
<tr>
<td>†</td>
<td>&lt;E^+, M&gt;</td>
<td>module of outer cylinders E^+ ∼ M</td>
<td>(4.4.5)</td>
</tr>
<tr>
<td>†</td>
<td>K^†</td>
<td>right exponential transpose of K</td>
<td>(11)</td>
</tr>
<tr>
<td>†</td>
<td>α^†</td>
<td>left exponential transpose of α</td>
<td>(4.7.3)</td>
</tr>
<tr>
<td>†</td>
<td>X^†</td>
<td>simple transpose of K</td>
<td>(11)</td>
</tr>
<tr>
<td>†</td>
<td>α^†</td>
<td>simple transpose of α</td>
<td>(4.7.4)</td>
</tr>
<tr>
<td>†</td>
<td>F^†, M</td>
<td>module of cylinders left weighted by F</td>
<td>(4.5.3)</td>
</tr>
<tr>
<td>†</td>
<td>F^†, Φ</td>
<td>postcomposition with Φ</td>
<td>(4.5.5)</td>
</tr>
<tr>
<td>†</td>
<td>F^†, M</td>
<td>module of cylinders right weighted by F</td>
<td>(4.5.3)</td>
</tr>
<tr>
<td>†</td>
<td>F^†, Φ</td>
<td>postcomposition with Φ</td>
<td>(4.5.5)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, M</td>
<td>module of cones E^+ ∼ M</td>
<td>(4.6.5)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, Φ</td>
<td>postcomposition with Φ</td>
<td>(4.6.8)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, M</td>
<td>precomposition with F</td>
<td>(4.6.19)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, M</td>
<td>module of outer cylinders E^+ ∼ M</td>
<td>(4.6.5)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, Φ</td>
<td>postcomposition with Φ</td>
<td>(4.6.8)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, M</td>
<td>precomposition with F</td>
<td>(4.6.19)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, M</td>
<td>module of wedges E × D^+ ∼ M</td>
<td>(4.7.7)</td>
</tr>
<tr>
<td>†</td>
<td>E^+, M</td>
<td>universal cone of a left module M</td>
<td>(4.9.5)</td>
</tr>
<tr>
<td>†</td>
<td>X</td>
<td>m</td>
<td>module morphism generated by X direct along m</td>
</tr>
<tr>
<td>†</td>
<td>X</td>
<td>α</td>
<td>module morphism generated by X direct along α</td>
</tr>
<tr>
<td>†</td>
<td>X</td>
<td>Yoneda representation</td>
<td>(5.2.16)</td>
</tr>
<tr>
<td>†</td>
<td>A</td>
<td>general Yoneda representation</td>
<td>(5.3.16)</td>
</tr>
<tr>
<td>†</td>
<td>m</td>
<td>A</td>
<td>module morphism generated by A inverse along m</td>
</tr>
<tr>
<td>†</td>
<td>α</td>
<td>A</td>
<td>module morphism generated by A inverse along α</td>
</tr>
<tr>
<td>†</td>
<td>A</td>
<td>Yoneda corepresentation</td>
<td>(5.2.16)</td>
</tr>
<tr>
<td>†</td>
<td>X</td>
<td>A</td>
<td>general Yoneda corepresentation</td>
</tr>
<tr>
<td>\</td>
<td>m/u</td>
<td>adjunct of m inverse along u</td>
<td>(6.1.2)</td>
</tr>
<tr>
<td>\</td>
<td>u</td>
<td>m</td>
<td>adjunct of m direct along u</td>
</tr>
<tr>
<td>\</td>
<td>Π</td>
<td>set of frames of M : E → E</td>
<td>(4.1.1)</td>
</tr>
<tr>
<td>\</td>
<td>Π</td>
<td>postcomposition with Φ</td>
<td>(4.1.5)</td>
</tr>
<tr>
<td>\</td>
<td>F</td>
<td>precomposition with F</td>
<td>(4.1.8)</td>
</tr>
<tr>
<td>\</td>
<td>Π</td>
<td>set of frames of M : E → *</td>
<td>(4.2.1)</td>
</tr>
<tr>
<td>\</td>
<td>Π</td>
<td>postcomposition with Φ</td>
<td>(4.2.5)</td>
</tr>
<tr>
<td>\</td>
<td>F</td>
<td>precomposition with F</td>
<td>(4.2.9)</td>
</tr>
<tr>
<td>\</td>
<td>Π</td>
<td>set of frames of a M : * → E</td>
<td>(4.2.1)</td>
</tr>
<tr>
<td>\</td>
<td>Π</td>
<td>precomposition with Φ</td>
<td>(4.2.5)</td>
</tr>
<tr>
<td>\</td>
<td>F</td>
<td>precomposition with F</td>
<td>(4.2.9)</td>
</tr>
<tr>
<td>\</td>
<td>Δ</td>
<td>funtor D → *</td>
<td>(15)</td>
</tr>
<tr>
<td>\</td>
<td>Δc</td>
<td>constant functor on c</td>
<td>(16)</td>
</tr>
</tbody>
</table>
Bibliography


