# The logical difference in quantum mathematics separating pure states from mixed states

## context: quantum randomness

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**Abstract** I give a short explanation of how quantum mathematics representing pure states, is logically distinct from the mathematics of mixed states. And further: how standard quantum theory easily shows itself to contain logical independence. This work is part of a project researching logical independence in quantum mathematics, for the purpose of advancing a complete theory of quantum randomness.

**Keywords** quantum mechanics, quantum indeterminacy, quantum information, prepared state, wave packet, unitary, orthogonal, scalar product, mathematical logic, arithmetic, formal system, axioms, Soundness Theorem, Completeness Theorem, logical independence, mathematical undecidability, semantics, syntax.

### 1 Introduction

Quantum physics suffers from logical problems and yet classical physics does not. The fault lays with logical discrepancies where quantum theory doesn't tell the full story of quantum experiments. Specifically, the machinery of *quantum indeterminacy* is 'missing'.

Actually, that machinery *is* to be found, already present, as *mathematical information* making up the mathematics of quantum theory.

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. The 'randomness' stems from ignorance of *physical information* in the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [4], the inequalities of John Bell [3], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, Tomasz Paterek et al offer explanation in *mathematical information.* They demonstrate a link between quantum randomness and *logical independence* in (Boolean) mathematical propositions [5,6]. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. In experiments measuring photon polarisation, Paterek et al demonstrate statistics correlating *predictable* outcomes with logically dependent mathematical propositions, and *random* outcomes with propositions that are logically independent.

While those Boolean propositions *do* convey definitive information about quantum randomness, any insight they offer is obscure. In order to advance a full and complete theory of quantum randomness and indeterminacy, understanding is needed of logical independence, inherent in *standard textbook quantum theory.*

In the Paterek et al experiments, the predictable, logically dependent outcomes correspond to measurement aligned parallel with the prepared state; whereas the random, logically independent outcomes correspond to measurement aligned across, orthogonal to the prepared states. Hence logical dependence is associated with pure states and logical independence with mixed states.

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**Possible versus necessary information**

In another quantum system – that of the *free particle* – the mathematics outwardly exhibits the logical difference between pure and mixed states.

It is helpful to understand the difference between *syntactical information* versus *semantical information.* Syntax concerns rules used for constructing or transforming symbols and formulae – here, in this case – the rules of Elementary Algebra. Semantics, on the other hand, concerns interpretation. Here, *interpretation* does not refer to *physical* meaning, but to *mathematical* meaning: whether symbols might be understood to mean: complex scalars, real scalars, or rational. Such interpretation has null connectivity with the rules of algebra — the syntax. Indeed, typically, the interpretation may be only in the theorist's mind and not asserted by the mathematics, at all.

A most relevant example is the comparison of syntax versus semantics in the mathematics representing pure eigenstates, set against mixed states, in system of the quantum free particle. Consider the *eigenformulae pair:*

$$\frac{d}{dx}\left[\Phi\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right] = +i\mathsf{k}\left[\Phi\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right] \tag{1}$$

$$\frac{d}{dk}\left[\Psi\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right] = -i\mathsf{x}\left[\Psi\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right] \tag{2}$$

This pair of formulae is true, irrespective of any interpretation placed on the variable $i$. But in contrast, the *superposition pair:*

$$\Psi\left(x\right) = \int \left[\Phi\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right] d\mathsf{k} \tag{3}$$

$$\Phi\left(k\right) = \int \left[\Psi\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right] d\mathsf{x} \tag{4}$$

is true, only if we interpret $i$ as *pure imaginary.* (And if $k$ is restricted to real or rational $\mathsf{k}$; and if $x$ is restricted to real or rational $\mathsf{x}$.) In the case of the eigenvalue pair (1) & (2) the imaginary interpretation is purely in the mind of the theorist, but for the superposition pair (3) & (4), the imaginary interpretation is implied by the mathematics. Whilst for the superposition pair (3) & (4), specific interpretation is *necessary*, for the eigenvalue pair (1) & (2), interpretation is *possible*, but not necessary.

In Mathematical Logic, *'necessary information* versus *possible information'* is recognised as constituting what is known as a 'modal logic'. However, in textbook quantum theory, the distinction separating possible from necessary is not noticeable, nor is it recognised; and the logical distinction between pure states and mixed states is lost. The crucial difference in expressing pure states is that their information derives from pure syntax. The transition in forming mixed states from pure states demands the creation of new information[1]. That creation goes unopposed.

The important point is that the logical status of pure states and mixed is distinct, not only in experiments, but also in Theory.

The fact is that quantum theory for pure states need not be unitary (or self-adjoint); whereas, for mixed states, unitarity is necessary. The jump between pure states and mixed states represents a logical jump between *possible unitarity* and *necessary unitarity.*

Historically, the distinction between necessary and possible unitarity has not been noticed, as any point of significance. No doubt, standard quantum theory ignores the fact for reasons of consistency. But, re-writing (1) – (4) as formulae in *first order logic* shows there is no contradiction, and that the possible / necessary information can be conveyed by a single theory. Thus, for pure states:

$$\forall\eta \mid \; \frac{d}{dx}\left[\Phi\left(\mathsf{k}\right)\exp\left(+\eta\mathsf{k}x\right)\right] \; = \; +\eta\mathsf{k}\left[\Phi\left(\mathsf{k}\right)\exp\left(+\eta\mathsf{k}x\right)\right] \tag{5}$$

$$\forall\eta \mid \; \frac{d}{dk}\left[\Psi\left(\mathsf{x}\right)\exp\left(-\eta\mathsf{x}k\right)\right] \; = \; -\eta\mathsf{x}\left[\Psi\left(\mathsf{x}\right)\exp\left(-\eta\mathsf{x}k\right)\right] \tag{6}$$

And for mixed:

$$\exists\eta \mid \; \Psi\left(x\right) = \int \left[\Phi\left(\mathsf{k}\right)\exp\left(+\eta\mathsf{k}x\right)\right] d\mathsf{k} \tag{7}$$

$$\exists\eta \mid \; \Phi\left(k\right) = \int \left[\Psi\left(\mathsf{x}\right)\exp\left(-\eta\mathsf{x}k\right)\right] d\mathsf{x} \tag{8}$$

[1] In some way, yet to be understood, this information is lost again during measurement.

## 2 The Unitary (or self-adjointness) Postulate is redundant

But having rewritten formulae as (5) – (8), these new formulae are inconsistent with the Postulates of Quantum Mechanics. Specifically, (5) & (6) disagree with unitarity (or self-adjointness) – imposed *by Postulate.* Whilst (5) – (8) represent a mathematical system that is logically self-consistent, and conveys the *whole information* of unitarity, that conveyance of whole information is gained at the expense of quantum theory's most treasured fact.

Not to worry. The postulated unitarity (or self-adjointness) is not needed. Unitarity is implied where it is needed – in the mathematics of the mixed states. Elsewhere, unitarity (or self-adjointness) is redundant.

## 3 Logical Independence

Crucially, once free of the Unitary Postulate, the imaginary unit no longer exists axiomatically, but is implied in (7) & (8). And because it is not axiomatic, and neither does its existence contradict the algebra, but is consistent with it, the imaginary unit is logically independent in the mathematics. This opens up the possibility that quantum randomness (or indeterminacy) in the quantum free particle may have origins in logical independence.

## References

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