# The logical difference in quantum mathematics separating pure states from mixed states

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**Abstract** I give a short explanation of how mathematics representing pure states is logically distinct from mixed states. This is intended as understandable to the undergraduate.

**Keywords** quantum mechanics, quantum indeterminacy, quantum information, prepared state, wave packet, unitary, orthogonal, scalar product, mathematical logic, arithmetic, formal system, axioms, Soundness Theorem, Completeness Theorem, logical independence, mathematical undecidability, semantics, syntax.

## 1 Introduction

Quantum physics suffers from logical problems and yet classical physics does not. The fault lays with logical discrepancies where quantum theory doesn't tell the full story of quantum experiments. Specifically, the machinery of *quantum indeterminacy* is missing.

Nonetheless, that machinery *is* to be found as mathematical information making up the mathematics of quantum theory. The story of that machinery is told through knowledge of : where mathematical information originates, how it is conveyed, where it flows and how one item of information relates logically with another. In that story, different classes of information play their role:

- ▷ axiomatic, implied, true–logically dependent, consequential, true–provable;
- ▷ consistent, compliant, satisfying, logically independent, true–non-provable, non-contradictory;
- ▷ self-referent, accidental, inadvertent, coincidental, spontaneous;
- ▷ inconsistent, contradictory, false–logically dependent, disprovable, false–provable.

Quantum indeterminacy is information, whose existence we *infer*, that we deduce is ontological in single measurement experiments — that is implied in quantum randomness accumulated when that same experiment is repeated many times over. Adopting the viewpoint that this indeterminate ontology exists *in fact*, and accepting the evidence of Tomasz Paterek et al [1] that links quantum randomness with logical independence, we should expect to find logical independence within quantum mathematics, corresponding to indeterminacy in experiments that prepare mixed-states.

This present paper traces through the mathematics of the quantum system we know as the *free particle*. In that mathematics, logical independence is found located in probability amplitude, with the suggestion that the same should extend, generally, to all quantum systems in quantum physics.

*Logical independence* refers to an information structure, in a system where an item or region of information is logically disconnect. This is the complimentary opposite to items of information that *imply* or *negate* one another. Logical connectivity between logically independent information can be regarded, as *null*.

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*Logical Independence in Physics. Information flow and self-reference in Arithmetic.*
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It is helpful to understand the difference between *syntactical information* versus *semantical information*. Syntax concerns rules used for constructing or transforming symbols and formulae – the rules of elementary algebra in the context of this enquiry. Semantics, on the other hand, concerns interpretation. Here, *interpretation* does not refer to *physical* meaning, but to *mathematical* meaning: whether symbols might be understood to mean: complex scalars, real scalars, or rational. Such interpretation has null connectivity with the rules of algebra — the syntax. Indeed, typically, the interpretation may be only in the theorist's mind and not asserted by the mathematics, at all.

A most relevant example of syntax versus semantics, is the comparison of pure eigenstates against mixed states. Consider the *eigenformulae pair:*

$$\frac{d}{dx}\left[\Phi\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right] = +i\mathsf{k}\left[\Phi\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right] \tag{1}$$

$$\frac{d}{dk}\left[\Psi\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right] = -i\mathsf{x}\left[\Psi\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right] \tag{2}$$

This pair of formulae is true, irrespective of any interpretation placed on the variable $i$. But in contrast, the *superposition pair:*

$$\Psi\left(x\right) = \int\left[\Phi\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right]d\mathsf{k} \tag{3}$$

$$\Phi\left(k\right) = \int\left[\Psi\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right]d\mathsf{x} \tag{4}$$

is true, only if we interpret $i$ as *pure imaginary*. In the case of the eigenvalue pair (1) & (2) the imaginary interpretation is purely in the mind of the theorist, but for the superposition pair (3) & (4), the imaginary interpretation is implied. Whilst for the superposition pair (3) & (4), specific interpretation is *necessary*, for the eigenvalue pair (1) & (2), interpretation is *possible*, but not necessary.

In Mathematical Logic, *'necessary information* versus *possible information'* is recognised as constituting what is known as a 'modal logic'. However, in textbook quantum theory, the distinction separating possible from necessary is not noticeable, nor is it recognised; and the logical distinction between pure states and mixed states is lost. The crucial difference in expressing pure states is that their information derives from pure syntax. The transition in forming mixed states from pure states demands the creation of new information[1]. That creation goes unopposed.

The important point is that the logical status of pure states and mixed is distinct, not only in experiments, but in theory also.

## References

1. Tomasz Paterek, Johannes Kofler, Robert Prevedel, Peter Klimek, Markus Aspelmeyer, Anton Zeilinger, and Caslav Brukner, *Logical independence and quantum randomness*, New Journal of Physics **12** (2010), no. 013019, 1367–2630.

[1] In some way, yet to be understood, this information is lost again during measurement.